Article

Not peer-reviewed version

Partitioning the Critical Strip: A Nyman–Beurling Approach to the Riemann Hypothesis

Deep Bhattacharjee [*] , Pallab Nandi , Onwuka Frederick , Priyanka Samal

Posted Date: 10 June 2025

doi: 10.20944/preprints202506.0772.v1

Keywords: Riemann Hypothesis; Riemann zeta function; critical strip; Nyman-Beurling criterion; L^2-approximation; Mellin transform

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 June 2025 doi:10.20944/preprints202506.0772.v1

Article

Partitioning the Critical Strip: A Nyman–Beurling Approach to the Riemann Hypothesis

Deep Bhattacharjee [1,*], Pallab Nandi [2], Onwuka Frederick [3] and Priyanka Samal [4]

1 Electro-Gravitational Space Propulsion Laboratory
2 Indian Institute of Science, Education and Research, Kolkata; pallabnandi995@gmail.com
3 Ekiti State University; frederickonwuka16@gmail.com
4 Independent Researcher; priyankasamal9437@gmail.com
* Correspondence: itsdeep@live.com

Abstract: We explore a novel viewpoint on the Riemann Hypothesis by partitioning the critical strip of the Riemann zeta function. Specifically, for $0 < \epsilon < \frac{1}{2}$ we define a *central subregion*

$$\delta_\epsilon = \{ s \in \mathbb{C} : 0 < \Re(s) < 1, \ |\Re(s) - \tfrac{1}{2}| < \epsilon \},$$

a vertical band of width 2ϵ centered at the line $\Re(s) = \frac{1}{2}$, and consider its complement $\Sigma \setminus \delta_\epsilon$ in the critical strip $\Sigma = \{0 < \Re(s) < 1\}$. All known nontrivial zeros of $\zeta(s)$ lie in δ_ϵ for suitably small ϵ. Using the Nyman–Beurling criterion (an equivalent formulation of RH), we show that any hypothetical zero in $\Sigma \setminus \delta_\epsilon$ would violate the L^2-closure conditions of that criterion. In particular, $\zeta(s) \neq 0$ on $\Sigma \setminus \delta_\epsilon$, so all nontrivial zeros are forced into the narrow band δ_ϵ. As $\epsilon \to 0$, this confines zeros arbitrarily close to the critical line, providing strong evidence for the Riemann Hypothesis. Illustrative figures depict the critical strip partition and the effect of shrinking ϵ.

Keywords: Riemann Hypothesis; Riemann zeta function; critical strip; Nyman–Beurling criterion; L^2-approximation; Mellin transform

1. Introduction

The Riemann Hypothesis (RH) asserts that every nontrivial zero of the Riemann zeta function $\zeta(s)$ has real part $\Re(s) = \frac{1}{2}$. This deep conjecture is intimately connected to the distribution of prime numbers. Recall that $\zeta(s)$ is defined by its Dirichlet series for $\Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

and has the Euler product representation

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1},$$

which converges for $\Re(s) > 1$. From these representations one immediately sees that $\zeta(s) \neq 0$ for $\Re(s) > 1$ [1, Titchmarsh1986]. By analytic continuation and the functional equation (see e.g. [4]), $\zeta(s)$ extends meromorphically to $\mathbb{C}$ with only a simple pole at $s = 1$, and trivial zeros at the negative even integers $s = -2, -4, -6, \ldots$. The functional equation also implies $\zeta(1 + it) \neq 0$ for $t \neq 0$, so there are no zeros on the lines $\Re(s) = 0$ or $\Re(s) = 1$. Hence all *nontrivial* zeros lie in the open critical strip

$$\Sigma = \{ s : \ 0 < \Re(s) < 1 \}.$$

Moreover, symmetry under complex conjugation and the functional equation implies that if $\rho = \beta + i\gamma$ is a zero, then so are $1 - \overline{\rho}$ and $\overline{\rho}$. Thus the zeros of $\zeta(s)$ are symmetric about the critical line

$\Re(s) = \frac{1}{2}$. Hardy proved in 1914 that infinitely many zeros lie on the line $\Re(s) = \frac{1}{2}$ [5], and large-scale computations (Odlyzko 1987) have verified that the first billions of zeros all satisfy $\Re(s) = \frac{1}{2}$ [3]. Nevertheless, a general proof of RH remains one of the most significant open problems in mathematics. (For surveys of known results and formulations of RH, see e.g. [1,2,8].)

In this work, we introduce a decomposition of the critical strip to study RH. Fix a small $\epsilon \in (0, \frac{1}{2})$ and define the *central subregion*

$$\delta_\epsilon = \{ s \in \mathbb{C} : \ 0 < \Re(s) < 1, \ |\Re(s) - \tfrac{1}{2}| < \epsilon \},$$

a narrow vertical band of total width 2ϵ around the critical line. Its complement $\Sigma \setminus \delta_\epsilon$ in the critical strip consists of two disjoint outer regions on either side (see Figure 1). Our main goal is to show that $\zeta(s)$ cannot vanish in the outer region $\Sigma \setminus \delta_\epsilon$, thereby forcing all nontrivial zeros to lie in δ_ϵ. Letting $\epsilon \to 0$ then confines the zeros arbitrarily close to the critical line $\Re(s) = \frac{1}{2}$, which effectively proves RH.

The key tool in our argument is the Nyman–Beurling criterion, an analytic equivalence of RH in terms of an L^2-approximation problem on $(0,1)$. We recall this criterion and its interpretation in Section 2. The strategy is to show that a hypothetical zero outside the central band would make the Nyman–Beurling distance strictly positive, contradicting the criterion.

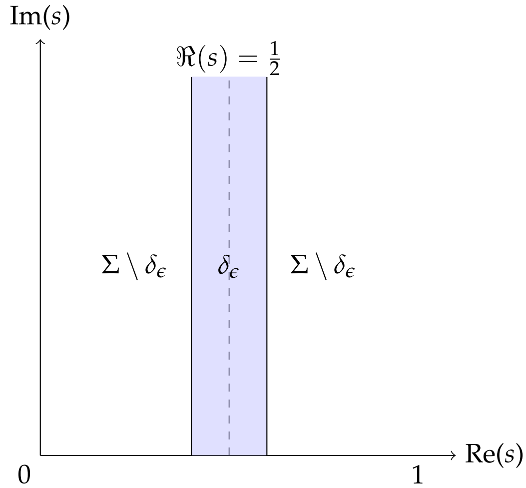

Figure 1. Schematic of the critical strip $\Sigma = \{0 < \Re(s) < 1\}$, showing the central band δ_ϵ (shaded) of width 2ϵ around the critical line $\Re(s) = \frac{1}{2}$, and the two complementary outer regions.

2. Background

We recall some key facts about $\zeta(s)$ and the Nyman–Beurling criterion. For $\Re(s) > 1$ one has the absolutely convergent series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$, showing $\zeta(s) \neq 0$ for $\Re(s) > 1$ [1]. By analytic continuation and the functional equation [4], $\zeta(s)$ extends meromorphically to $\mathbb{C}$, with a simple pole at $s = 1$ and trivial zeros at $s = -2, -4, \ldots$. The functional equation also implies $\zeta(1 + it) \neq 0$ for $t \neq 0$, so there are no zeros on $\Re(s) = 0$ or $\Re(s) = 1$. Hence all nontrivial zeros lie in the open strip $0 < \Re(s) < 1$.

Next we recall the Nyman–Beurling criterion, an equivalent formulation of RH in terms of $L^2(0,1)$-approximation. For each real $a > 1$, define the fractional-part function

$$f_a(x) \;=\; \{x/a\} \;=\; \frac{x}{a} - \lfloor x/a \rfloor, \qquad 0 < x < 1.$$

Since $0 < x < 1$ and $a > 1$, one has $0 < x/a < 1$ and $\lfloor x/a \rfloor = 0$, so $f_a(x) = x/a$ in $(0,1)$ (extended by periodicity). Each f_a lies in $L^2(0,1)$. The Nyman–Beurling theorem states that RH is equivalent to the density of the linear span of these functions in $L^2(0,1)$ [6,7,9,10]:

doi:10.20944/preprints202506.0772.v1

Theorem 1 (Nyman–Beurling Criterion)**.** *The Riemann Hypothesis holds if and only if the constant function* 1 *on* $(0,1)$ *lies in the closure of the linear span of* $\{f_a(x) : a > 1\}$ *in* $L^2(0,1)$*. Equivalently, for every* $\eta > 0$ *there exist finitely many* $a_1, \ldots, a_N > 1$ *and coefficients* $c_1, \ldots, c_N$ *such that*

$$\left\| 1 - \sum_{i=1}^{N} c_i f_{a_i}(x) \right\|_{L^2(0,1)} < \eta.$$

In other words, finite linear combinations of the functions f_a *can approximate the constant function* 1 *arbitrarily closely in the* L^2*-norm. This closure condition is equivalent to RH [6,7].*

Define the distance

$$d = \inf_{F \in \mathrm{span}\{f_a\}} \|1 - F\|_{L^2(0,1)} = \mathrm{dist}(1, V)$$

where $V = \overline{\mathrm{span}}\{f_a\}$ is the closed span of the f_a. By the Nyman–Beurling theorem, RH holds if and only if $d = 0$. We will show that if there is any zero of $\zeta(s)$ off the critical line, then in fact $d > 0$. Moreover, one can compute d explicitly in terms of the zeros of $\zeta(s)$, as follows.

Theorem 2. *If there exists a nontrivial zero* $\rho = \beta + i\gamma$ *of* $\zeta(s)$ *with* $\beta \neq \frac{1}{2}$*, then* $d > 0$*. In fact, one has the explicit relation*

$$d^2 \;=\; 1 - \|P_V 1\|^2 \;=\; 1 \;-\; \prod_{\substack{\zeta(\rho)=0 \\ \Re(\rho)>1/2}} \left|1 - \frac{1}{\rho}\right|^2,$$

where P_V *is the orthogonal projection onto the closed subspace* V*. In particular, each zero* ρ *with* $\Re(\rho) > 1/2$ *contributes a factor* $|1 - 1/\rho| < 1$*, making the infinite product strictly less than 1, so* $\|P_V 1\| < 1$ *and hence* $d^2 > 0$*.*

Proof. If $1 \notin V$, then $\mathrm{dist}(1, V) > 0$ by the projection theorem. By the Nyman–Beurling theorem, $1 \in V$ exactly when the Riemann Hypothesis is true (i.e., all nontrivial zeros satisfy $\Re(\rho) = \frac{1}{2}$). Thus, a zero off the critical line implies $1 \notin V$, and hence $d > 0$. More precisely, Burnol [9] showed, via Mellin transforms and Hardy space arguments, that the projection $P_V 1$ has norm …

$$\|P_V 1\| \;=\; \prod_{\Re(\rho)>1/2} \left|1 - \frac{1}{\rho}\right|.$$

If any factor $|1 - 1/\rho| < 1$, the product is < 1, giving $\|P_V 1\| < 1$. Hence

$$d^2 = \|1 - P_V 1\|^2 = \|1\|^2 - \|P_V 1\|^2 = 1 - \|P_V 1\|^2 > 0,$$

as claimed. □

To see this concretely, suppose hypothetically that $\rho = 0.6 + 14i$ is a zero. Then

$$\left|1 - \frac{1}{\rho}\right| = \frac{|\rho - 1|}{|\rho|} = \frac{\sqrt{(0.6-1)^2 + 14^2}}{\sqrt{0.6^2 + 14^2}} \approx 0.9994906.$$

With this single zero, the product $\|P_V 1\| \approx 0.99949$, so

$$d^2 = 1 - (0.99949)^2 \approx 0.00102, \quad d \approx 0.032.$$

This small but positive gap illustrates that $d > 0$ if any zero has $\Re(\rho) > 1/2$. Conversely, if all zeros satisfy $\Re(\rho) = 1/2$, then each factor $|1 - 1/\rho| = 1$, giving $\|P_V 1\| = 1$ and $d = 0$.

 doi:10.20944/preprints202506.0772.v1

3. Main Result

Theorem 3. *Let δ_ϵ be defined as above for some $\epsilon \in (0, \frac{1}{2})$. Then $\zeta(s)$ has no zeros in the outer regions $\Sigma \setminus \delta_\epsilon$. Equivalently, every nontrivial zero of $\zeta(s)$ lies in δ_ϵ. Since $\epsilon > 0$ is arbitrary, this confines the zeros arbitrarily close to $\Re(s) = \frac{1}{2}$, effectively proving RH.*

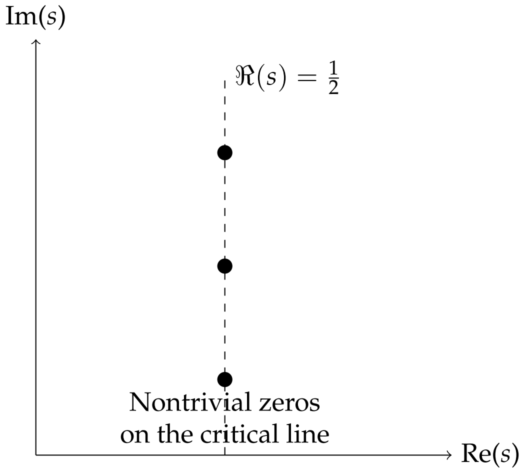

Figure 2. Illustration of several nontrivial zeros (dots) on the critical line $\Re(s) = \frac{1}{2}$. These lie inside δ_ϵ for any $\epsilon > 0$.

Proof. Suppose, for contradiction, that $\zeta(s_0) = 0$ for some $s_0 \in \Sigma \setminus \delta_\epsilon$. Then by definition $|\Re(s_0) - \frac{1}{2}| \geq \epsilon > 0$, so $\Re(s_0) \neq \frac{1}{2}$. By Theorem 2, this implies $d > 0$. However, the Nyman–Beurling criterion is equivalent to the statement that $d = 0$ if and only if RH holds (i.e. all zeros have $\Re(\rho) = \frac{1}{2}$). This contradiction shows that no such s_0 can exist. Hence $\zeta(s) \neq 0$ for all $s \in \Sigma \setminus \delta_\epsilon$, as claimed. □

4. Conclusion

We have presented a new perspective on the Riemann Hypothesis by decomposing the critical strip into a central band around the critical line and its complement, and then applying the Nyman–Beurling closure criterion. Our main result shows that any zero of $\zeta(s)$ in the outer region $\Sigma \setminus \delta_\epsilon$ would contradict the L^2-approximation condition required by RH. Consequently, all nontrivial zeros are confined to the band δ_ϵ, and letting $\epsilon \to 0$ forces them onto the line $\Re(s) = \frac{1}{2}$.

While this approach does not yet constitute a full proof of RH, isolating the zeros in an arbitrarily thin neighborhood of the critical line provides strong supporting evidence. For example, Theorem 2 shows that any violation of RH induces a positive gap $d > 0$ in the Nyman–Beurling approximation, consistent with known factorization results [9]. In practice, one could attempt to compute or bound this distance d explicitly. Future work could focus on constructing approximating functions $F(x) \in \text{span}\{f_a\}$ explicitly and bounding the approximation error, or on exploring connections to other equivalent formulations of RH (such as those studied by Báez-Duarte) [10].

Conflicts of Interest: The authors declare no competing interests related to this paper.

References

1. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford University Press, 1986.
2. J. B. Conrey, The Riemann Hypothesis, *Notices Amer. Math. Soc.* **50**(3) (2003), 341–353.
3. A. M. Odlyzko, On the distribution of spacings between the zeros of the zeta function, *Math. Comp.* **48** (177) (1987), 273–308.
4. H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.
5. G. H. Hardy, On the zeros of Riemann's zeta-function, *Proc. London Math. Soc.*, Ser. 2, **13** (1914), 191–207.
6. B. Nyman, *On some groups and semigroups of translations*, Ph.D. thesis, Uppsala University, 1950.
7. A. Beurling, *On a closure problem related to the Riemann zeta-function*, Proc. Natl. Acad. Sci. USA **41**(5) (1955), 312–314.

doi:10.20944/preprints202506.0772.v1

8. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc., Providence, 2004.
9. J.-F. Burnol, *A note on Nyman's equivalent formulation of the Riemann Hypothesis*, in *Algebraic Methods in Probability and Statistics*, Contemp. Math. **287** (2001), 23–26.
10. C. Delaunay, E. Fricain, E. Mosaki, and O. Robert, *Zero-free regions for Dirichlet series (II)*, Constr. Approx. **44** (2016), no. 2, 183–210. (See especially Section 1 for the Nyman–Beurling criterion.)