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Posted Date: 23 January 2023

doi: 10.20944/preprints202301.0393.v1

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The Spectrum of the Zariski Topology for Multiplication Krasner Hypermodules

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Abstract: In this study, we define the concept of pseudo-prime subhypermodules of hypermodules as a generalization of the prime hyperideal of commutative hyperrings in [17]. Firstly we examine the spectrum of the Zariski topology, which we built on the element of the pseudo-prime subhypermodules of a class of hyper-modules, then we give some relevant properties of the hypermodule in this topological hyperspace.

Keywords: pseudo-prime spectrum; zariski topology; spectral hyperspace

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1. Introduction

Hypergroup theory, which has been defined in [19] as a more comprehensive algebraic structure of group theory and has been investigated by different authors in modern algebra. It has been developed using hyperring and hypermodule theory studies by most authors (see [1–3,6–10,15,16,19–22,26]).

Let's start by giving the basic information necessary for the algebraic structure that we will study as Krasner S-hypermodule in studying the S-hypermodule class on a fixed Krasner hyperring class S. Let N be a non-empty set. $(N,\cdot)$ is called a *hypergroupoid* if for the map defined as $\cdot : \underset{(x,y)}{N \times N} \underset{\rightarrow}{\rightarrow} \underset{x\cdot y}{P^*(N)}$ is a function. Here "$\cdot$" is called a *hyperproduct* or *hyperoperation* on N. Let X and Y be subsets of N. The hyperproduct $X \cdot Y$ is defined as $X \cdot Y = \bigcup_{(x,y)\in X\times Y} x \cdot y$. $\{x\} \cdot Y$ and $X \cdot \{y\}$ are simply represented as $x \cdot Y$ and $X \cdot y$, respectively. A hypergroupoid $(N,\cdot)$ is called a *semihypergroup* if for each $x, y, z \in N$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. A semihypergroup $(N,\cdot)$ is called a *hypergroup* if for each $x \in N$, $x \cdot N = N \cdot x = N$. We denote *identity element* of a hypergroup N as 0_N for having a addive hyperoperation on N. If the non-additive is in the hypergroup N, we will use the 1_N notation. Let x be any element of $(N,\cdot)$. x' is called an *inverse element* of x in N if $0_N \in x \cdot x' \cap x' \cdot x$. The algebraic structure that fulfills the following conditions on H with the hyperoperation $+$ is called a *canonical hypergroup*:

(i) $(N,+)$ is a semihypergroup;
(ii)$+$ is commutative on $(N,+)$;
(iii) There is an identity element 0_N on $(N,+)$;
(iv) There is an inverse element $-x$ of x in N such that $0_N \in x + (-x)$ or simplify $0_N \in x - x$.
(v) If $x \in y + z$, then $y \in x - z$ for each $x, y, z \in N$;

Let S be non-empty set having a hyperoperation $+$ and a operation on itself. If $(S,+)$ is a canonical hypergroup, $(S,.)$ is a semigroup consist of 0_R which is a bilaterally absorbing element, i.e. $0_S.a = a.0_S = 0_S$ for each $a \in S$, and $(b+c).a = (b.a) + (c.a)$ and $a.(b+c) = a.b + a.c$ for each $a, b, c \in S$, then S is called a *Krasner hyperring*. We study on Krasner hyperring with unit element 1_S, where $1_S.a = a.1_S = a$ for each $a \in S$. Let $(S,+,.)$ be a Krasner hyperring and $(N,+)$ be a canonical hypergroup with external operation $* : S \times N \rightarrow N$. Then N is called a *left Krasner S-hypermodule* if the following conditions hold for each $r_1, r_2 \in S$ and for each $x_1, x_2 \in N$:

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(1) $r_1 * (x_1 + x_2) = r_1 * x_1 + r_1 * x_2$;
(2) $(r_1 + r_2) * x_1 = r_1 * x_1 + r_2 * x_1$;
(3) $(r_1.r_2) * x_1 = r_1 * (r_2 * x_1)$;
(4) $0_S * x_1 = 0_N$

A left Krasner S-hypermodule N is called unitary if $1_S * x = x$ for each $x \in N$. To simplify representation rx instead of $r * a$ for each $r \in S$ and $x \in N$. Throughout the rest of this paper, we assume all hypermodules are left unitary Krasner hypermodules and all hyperrings are Krasner commutative hyperrings. It is a proper generalization of Krasner hypermodules to modules because it carries the rings to Krasner hyperrings.

Let S be a hyperring and J a non-empty subset of S. J is called a *left (right) hyperideal* if $x - y \subseteq J$ and $sx \in J$ $(xs \in J)$ for every $s \in S$, $x, y \in J$, denoted by $J \trianglelefteq_l S$ $(J \trianglelefteq_s S)$. If J is both left hyperideal and right *hyperideal* of S,denoted by $J \trianglelefteq S$. It is clear that $S.x = \{s.x : s \in S\}$ is a hyperideal of S.

Let N be a S-hypermodule and $\emptyset \neq K \subseteq N$. K is said to be a *subhypermodule* of N if K is a S-hypermodule itself which is contained in N, denoted by $K \leq N$. Shortly, a non-empty subset K of N is a subhypermodule if $x - y \subseteq K$ and $s.x \in K$ for every $s \in S$ and $x.y \in K$. It is easily demonstrable that $S.x = \{s.x : s \in S\}$ is a subhypermodule of a hypermodule N for every $x \in N$. Let K and T be subhypermodules of N. Then $K + T = \{x \in k + t : k \in K, t \in T\}$ is a subhypermodule, too. Let N and K be S-hypermodule and $\Psi : N \to K$ a function. If $f(x + y) \subseteq f(x) + f(y)$ and $f(s.x) = s.f(x)$ for every $x, y \in K$ and $s \in S$, f is called a hypermodule S-homomorphism from N to K. Instead of this statement if the inclusion satisfies $f(x + y) = f(x) + f(y)$, then f is said to be a *strong S-homomorphism* from N to K. The class of every strong S-homomorphisms from N to K is denoted by $Hom_S(N, K)$, sets are defined as $ker(\Psi) := \{x \in N : \Psi(x) = 0_K\}$, $Im(\Psi) := \{y \in K : \exists x \in N, y \in \Psi(x)\}$. The homomorphism $\Psi \in Hom_S(N, K)$ is called *strongly surjective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for every $x_1, x_2 \in N$, and $+$ is called *strongly injective* if $f(x_1) \cap f(x_2) \neq \emptyset$ implies $x_1 = x_2$ for every $x_1, x_2 \in N$. To simplfy denoting annihilator of an S-hypermodule N for a subhypermodule K, we use the symbol $K :_S N$, and the set is a hyperideal which is defined as $\{s \in S : s.N \subseteq K\}$. Another representation of $D :_S N$ is $Ann_S(N)$.

As a generalization of prime spectrum of the ring of commutative topology defined on S with Zariski topology [14] inspired by the interaction between the theoretical properties of the hyperring S of the text, over a commututive hyperring S on a several hypermodule N, we examine a Zariski topology on these spectrum χ_N of pseudo-prime subhyper-modules and we give the interaction between topological hyperspace.

We give topological conditions such as connectedness, Noetherianness and irreducibility in the pseudo-prime spectrum of hypermodules and obtain more information about the algebraic hyperstructure of these hypermodules. Further, we prove this topological hyperspace in terms of spectral hyperspace which is a topological hyperspace and is homeomorphic to $Spec(S)$ for any hyperring S.

2. Condition of Pseudo-Prime for Commutative Krasner Hypermodules

In this section, we present pseudo-prime subhypermodules as a new concept of hypermodules theory. Then we investigate connection between spectral hyperspace and Zariski topology. Recall from [5] that a proper hyperideals J of a hyperring S is called *prime* if $XY \subseteq J$ implies $X \subseteq J$ or $Y \subseteq J$ for every hyperideals X, Y of S.

Definition 1. *Let N be an S-hypermodule and K a subhypermodule of N.*

(1) *We call a subhypermodule K pseudo-prime if $(K :_S N)$ is a prime hyperideal of S.*

(2) *We call a pseudo-prime spectrum of N as the set of all pseudo-prime submodules of N, express it by X_N^S or shortly χ_N. For any prime hyperideal $J \in X_S = Spec(S)$, the collection N of whole pseudo-prime subhypermodules of N with $(K : N) = J$ is denoted by $X_{N,J}$.*

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(3) *For a subhypermodule K of N, we define the set $V^N(K) = \{Y \in X_N : K \leq Y\}$. If it would be written as shortly, using $V(K)$ instead of $V^N(K)$.*

(4) *If $X_N = \emptyset$ for every $Y \in X_N$, the function $\eta : X_N \to Spec(S/Ann(N))$ defined by $\eta(Y) = (Y : N)/Ann(N)$ is called natural map of X_N. If $N = (0)$ or $N \neq (0)$, the natural map of X_N is strongly surjective, then we call N pseudo-primeful.*

(5) *If the natural map of X_N is strongly injective, then we call N a pseudo-injective.*

According to our above definition prime hyperideals of a hyperring S and the pseudo-prime S-hypermodule of the hypermodule S are the same. It is obtained that the concept of prime hyperideal to hypermodules is a strong notion of the strongly pseudo-prime subhypermodule S. Recall from [25] that a proper subhypermodule K of an S-hypermodule N is called *prime* if for every hyperideal J of S and every subhypermodule X of N, a hypermodule

$$[J.K] = \cup \left\{ \sum_{j=1}^{p} u_j.x_j : l \in \mathbb{N}, u_j \in J \text{ and } x_j \in X, \text{for all } j \right\} \subseteq K$$

implies $J \subseteq (K : N)$ or $X \subseteq K$.

So a proper subhypermodule K of an S-hypermodule N is prime if N/K is a torsion-free $S/(K:N)$-hypermodule, i.e. N/K is a hypermodule on a hyperring S such that the only element destroyed by a non-zero divisor of hyperring $S/(K:N)$ is zero. By using Definition 2.1, it can be easily seen that every prime subhypermodule K is apseudo-prime subhypermodule, because $(K:N) \in Spec(S)$. But in general, the converse assertion is not hold.

Example 1. *Take a $\mathbb{Z}$-hypermodule $N = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and a subhypermodule $K = \langle (3,0,0) \rangle = (3,0,0)\mathbb{Z}$. Then $(K:N) = (0) \in Spec(\mathbb{Z})$. So K is a prime subhypermodule of N. But $K \in X_N$, K isn't a prime subhypermodule.*

Recall from [17] that a hypermodule N is called multiplication S-hypermodule if for each subhypermodule K of N, there is a hyperideal J of S so that $K = [J.N]$.

Recall from [11] that a proper subhypermodule K of N is called maximal if for each subhypermodule L of N with $K \leq L \leq N$, then $K = L$ or $L = N$.

Example 2. *Every multiplication hypermodule satisfies the condition pseudo-injective. Take the $\mathbb{Z}$-hypermodule $N = \mathbb{Z}_p \oplus \mathbb{Z}(p^\infty)$ where p is a prime integer. Say, K the pseudo-prime subhypermodule of N. It followa that $(K:N)N \subseteq K < N$. Suppose that $(K:N) \neq p$. Then it contadicts with torsion $\mathbb{Z}$-hypermodule N. Thus $(K:N)$ is equal to p. It is seen clearly that $(K:N)N$ is a maximal subhypermodule of N by using the strong isomorphism $\mathbb{Z}_p \cong N/PN$. Then we have $(K:N)N = K$. Therefore N is a pseudo-injective $\mathbb{Z}$-hypermodule. But there isn't a hyperideal J of $\mathbb{Z}$ with $\mathbb{Z}_p \oplus \langle 0 \rangle = J.N$, N isn't a multiplication $\mathbb{Z}$-hypermodule.*

Lemma 1. *The following assertions are equivalent for a finitely generated S-hypermodule N.*

(1) *N is a multiplication hypermodule.*

(2) *N is a pseudo-injective hypermodule.*

(3) *$|X_{N,J}| \leq 1$ for each maximal hyperideal J of S.*

(4) *$N/J.N$ is simple for each maximal hyperideal J of S.*

Proof. (1) $\Rightarrow$ (2) By Example 2.2

(2) $\Rightarrow$ (3) Clear by Definition 2.1

(3) $\Rightarrow$ (4) It can be proven clearly that $J.N = N$ for a maximal hyperideal J of S. Hence suppose that $J.N \neq N$ and $K/J.N \subset N/JN$. Then K is a proper subhypermodule containing the subhypermodule JN of N. Thus we have $J = (J.N : N) = (K : N)$. Since K and JN are belong to $X_{N:J}$,

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then $K = J.N$ by the assumption. So $N/J.N$ is a simple S-hypermodule. By [17], N is a multiplication hypermodule. □

Further, we use the concept of pseudo-prime subhypermodules to describe another new hypermodule class, namely the topological hypermodule. We give some examples of topological hypermodules and explore some algebraic properties of this hypermodule class. Then in the next section we connect a topology to the set of all pseudo-prime subhypermodules of topological hypermodules, called the Zariski topology. Let L be a subset of X_N for an S-hypermodule N. We show as notation the intersection of all elements in L by $\Im(L)$.

Definition 2. *Let N be an S-hypermodule.*

(1) *If it is an intersection of pseudo-prime subhypermodules of N, N is said to be pseudo-semiprime.*

(2) *If $T \cap L \leq K$, then the pseudo-prime subhypermodule K of N is called extraordinary, where T, L are pseudo-semiprime subhypermodules of N, then either $L \subseteq K$ or $T \subseteq K$.*

(3) *The pseudo-prime radical of K is shown as notation $Prad(K)$ is the intersection of each pseudo-prime subhypermodules of N containing K, i.e.*

$Prad(K) = \Im(V(K)) = \underset{P \in V(K)}{\cap P}$. If $V(K) = \emptyset$, then we get $Prad(K) = N$ for a subhypermodule K of N.

(4) *If $K = Prad(K)$, then the subhypermodule K of N is said to be a pseudo-prime radical subhypermodule.*

(5) *If $X_N = \emptyset$ or each pseudo-prime subhypermodule of N is extraordinary then N is said to be topological.*

By using Definition 2.2 that we prove that every prime hyperideal of S is extraordinary pseudo-prime subhypermodule for the S-hypermodule S. It is not always true thatevery pseudo-prime subhypermodule is extraordinary. Take a $\mathbb{Z}$-hypermodule $\mathbb{Q} \oplus \mathbb{Z}_p$, where $\mathbb{Q}$ is a rational numbers set as a $\mathbb{Z}$-hypermodule and p is a prime integer. $\langle 0 \rangle \oplus \mathbb{Z}_p$, $\mathbb{Q} \oplus \langle 0 \rangle$ and $\mathbb{Z} \oplus \langle 0 \rangle$ of $\mathbb{Q} \oplus \mathbb{Z}_p$ are pseudo-prime subhypermodules. Since $(\langle 0 \rangle \oplus \mathbb{Z}_p) \cap (\mathbb{Q} \oplus \langle 0 \rangle) \subseteq \mathbb{Z} \oplus \langle 0 \rangle$, $\mathbb{Z} \oplus \langle 0 \rangle$ isn't extraordinary. Hence, $\mathbb{Q} \oplus \mathbb{Z}_p$ isn't a topological hypermodule. In addition the $\mathbb{Z}$-hypermodule $\mathbb{Z}(p^{\infty})$ is a topological hypermodule as its subhypermodules are linearly ordered where p is a prime integer.

Theorem 1. *Let N be a topological S-hypermodule. Then the following statements hold.*

(1) *Every strong homomorphic image of N is a topological S-hypermodule.*

(2) *N_J is a topological S_J-hypermodule for every prime ideal J of S.*

Proof. (1) Let K be a subhypermodule of N. We have a factor S-hypermodule N/K, say L. Let U/K be a pseudo-prime subhypermodule of L. It follows from $(U/K : L) = (U : N)$ that U is a pseudo-prime subhypermodule of N. Let V/K and W/K be pseudo-semiprime subhypermodule of L so that $V/K \cap W/K \subseteq U/K$. So V and W are pseudo-semiprime subhypermodules of N such that $V \cap W \subseteq U$. By the hypothesis, $V \subseteq U$ or $W \subseteq U$. Therefore $V/K \subseteq U/K$ or $W/K \subseteq U/K$. Consequently L is a topological S-hypermodule.

(2) Let L be a pseudo-prime subhypermodule of the S_J-hypermodule N_J. Let $\Psi : N \to N_J$ be the canonical strong homomorphism. Firstly we shall prove that $L \cap N$ is a pseudo-prime subhypermodule of N. Let I and I' be hyperideals of S so that $II' \subseteq (L \cap N :_S N)$. By using the canonical strong homomorphic image of N by Ψ, we have $\left(I_J I'_J\right).N_J \subseteq L = (L \cap N)_J$. Since L is a pseudo-prime subhypermodule of the S_J-hypermodule N_J, either $I_J \subseteq (L : N_J)$ or $I'_J \subseteq (L : N_J)$. So we have $I.N \subseteq (I.N)_J \cap N \subseteq L \cap K$ or $I : N \subseteq L \cap K$. It follows that $L \cap K$ is a pseudo-prime subhypermodule of N. Take a pseudo-semiprime subhypermodules K_1 and K_2 of N_J with $K_1 \cap K_2 \subseteq L$. We have $K_1 \cap N$ and $K_2 \cap N$ are pseudo-semiprime subhypermodules of N with $(K_1 \cap N) \cap (K_2 \cap N) = (K_1 \cap K_2) \cap N \subseteq L \cap N$ that $K_1 = (K_1 \cap N)_J \subseteq (L \cap N)_J = H$ or $K_2 = (K_2 \cap N)_J \subseteq (L \cap N)_J = H$. Therefore H is extraordinary and N_J is a topological S_J-hypermodule. □

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Recall that the pseudo-prime subhypermodules of S as on S-hypermodule are the pseudo-prime hyperideals for any hyperring S. In the following Theorem, we extend the fact into Theorem 2.1 to multiplication hypermodules.

Theorem 2. *Let N be a finitely generated S-hypermodule. Then the following assertions are equivalent.*
(1) *N is a multiplication hypermodule..*
(2) *There is a hyperideal J of S so that $V(K) = V(J.K)$ for every subhypermodule K of N.*
(3) *N is a topological hypermodule.*

Proof. (1) $\Longrightarrow$ (2) Clear

(2) $\Longrightarrow$ (3) Let L be a pseudo-prime subhypermodule of N, K and U be pseudo-semiprime subhypermodules of N such that $K \cap U \subseteq L$. Then we have $V(K) = V(J.N)$ and $V(U) = V(J:N)$ for hyperideals J and J' of S. Take some collection of pseudo-prime subhypermodules $\{K'_\alpha\}_{\alpha\in\Omega}$ such that $K = \bigcap_{\alpha\in\Omega} K'_\alpha$. So we get $(J \cap J').N \subseteq K'_\alpha$ for every $\alpha \in \Omega$ by using the conclusion $K'_\alpha \in V(K) \subseteq V(K) \cup V(U) = V(J.N) \cup V(J'.N = V((J \cap J').N))$. Hence $(J \cap J').N \subseteq \bigcap_{\alpha\in\Omega} K'_\alpha = K$. By similar way, we have the conclusion $(J \cap J').N \subseteq U$. Thus $(J \cap J').N \subseteq K \cap U \subseteq L$. It follows from $J \cap J' \subseteq (L:N)$ that $L \in V(J.N) = V(K)$ or $L \in V(J:N) = V(U)$, that is either $K \subseteq L$ or $U \subseteq L$.

(3) $\Longrightarrow$ (1) Clear by Lemma 2.4 □

Definition 3. *Let N be an S-hypermodule. Then N is called content if $b \in c(b)N$ where $c(b) = \cap\{J : J$ is a hyperideal of S and $b \in J.N\}$ for every $b \in N$. It shall be given as an equivalent definition to it. N is a content S-hypermodule if and only if $\left(\bigcap_{\alpha\in\Omega} J_\alpha\right).N = \bigcap_{\alpha\in\Omega} (J_\alpha.N)$ for every family of $\{J_\alpha : \alpha \in \Omega\}$ of S.*

Theorem 3. *Let N be a content and pseudo-injective S-hypermodule. Then N is topological. In addition, if $Prad(K) = \sqrt{(K:N)}N$ for every subhypermodule K of N, N is topological.*

Proof. Let N be a content and pseudo-injective , $Prad(L) = N$. Then we have $V(K) = V(S.N)$. If $Prad(L) \neq N$, then $Prad(L)$ is a pseudo-semiorime subhypermodule of N. So there exist pseudo-prime subhypermodules L_α for every $\alpha \in \Omega$ with $Prad(L) = \bigcap_{\alpha\in\Omega} L_\alpha$ and $(L_\alpha : N) = p_\alpha \in Spec(S)$. It follow from $p_\alpha N = (p_\alpha : N).N = ((p_\alpha : N).N : N)$ and N is pseudo-injective for every $\alpha \in \Omega$ that $L_\alpha = p_\alpha N$. Since N is a content hypermodule,

$$\begin{aligned} Prad(L) &= \bigcap_{\alpha\in\Omega} L_\alpha = \bigcap_{\alpha\in\Omega} (p_\alpha N) = \left(\bigcap_{\alpha\in\Omega} p_\alpha\right) N \\ &= \bigcap_{\alpha\in\Omega} (L_\alpha : N) N = \left(\bigcap_{\alpha\in\Omega} L_\alpha : N\right) N \\ &= (Prad(L) : N) N. \end{aligned}$$

Then we obtain $V(L) = V(Prad(L)) = V((Prad(L) : N) N)$. It follows from Theorem 2.2 that N is a topological hypermodule.

Let N be a hypermodule where every subhypermodule K of N satisfies the equality $Prad(K) = \sqrt{(K:N)}N$. Then $V(K) = V(Prad(K)) = V\left(\sqrt{(K:N)}N\right)$. By using Theorem 2.2, we have N is atopological hypermodule. □

3. PSEUDO-PRIME SPECTRUM OVER TOPOLOGICAL HYPERMODULES

We use denoting N as a topological S-hypermodule in the rest of this text. In [17], we investigate the Zariski topology over multiplication hypermodules. Zariski topology is built on topological modules in [14]. In this section, inspired by this source, this class will be examined in hypermodules

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by looking at it from a different spectrum. Briefly J and $\overline{J}$ will be used instead of $S/Ann(N)$ and $J/Ann(N)$ for every hiperideal $J \in V^s(Ann(N))$.

Theorem 4. *If X_N is connected for a pseudo-primeful S-hypermodule N, then $X_{\overline{S}}$ is connected.*

Proof. Let $\varphi : X_N \to Spec(S/Ann(N))$ be a natural map. As φ is surjective, we must show that φ is continuous. Take a hyperideal J of S containing $Ann(N)$. Let $K \in \varphi^{-1}\left(V^{\overline{S}}(\overline{J})\right)$. There is a hiperideal $\overline{J}' \in V^{\overline{S}}(\overline{J})$ such that $\varphi(K) = \overline{J}'$. Thus $J \subseteq (K : N) = J$. It follows from $J.N \subseteq K$ that $K \in V^N(J.N)$. Let $L \in V^N(J.N)$. Then we obtain $J \subseteq (J.N : N) \subseteq (L : N)$. Therefore $L \in \varphi^{-1}\left(V^{\overline{S}}(\overline{J})\right)$. φ is continuous as $\varphi^{-1}\left(V^{\overline{S}}(\overline{J})\right) = V^N(J.N)$. □

In the following proposition, we obtain basic properties of the subhypermodules of N taking the topological hyperspace X_N is a T_1-hyperspace.

Proposition 1. *Let $Y \subseteq X_N$ and $K \in X_{N,J}$ for any $J \in Spec(S)$. Then the following statements hold.*

(1) *$Cl(Y) = V(\Im(Y))$. Thus $Y = V(\Im(Y)) \iff Y$ is closed.*

(2) *$\langle 0 \rangle \in Y$ provided that Y is dense in X_N.*

(3) *X_N is a T_0-hyperspace.*

(4) *Every pseudo-prime subhypermodule of N is a maximal element in the set of whole pseudo-prime subhypermodules of N if and only if X_N is a T_1-hyperspace.*

(5) *$Spec(S)$ is a T_1-hyperspace provided that X_N is a T_1-hyperspace.*

Proof. **(1)** The inclusion $V(\Im(Y)) \supseteq Y$ is clear. Let $V(K)$ be any closed subset of X_N containing Y. Then, $V(\Im(Y)) \subseteq V(\Im(V(K))) = V(Prad(K)) = V(K)$ since $\Im(V(K)) \subseteq \Im(Y)$. It follows that $V(\Im(Y))$ is the smallest closed subset of X_N containing Y. Therefore, the equality is obtained.

(2) It can be seen clearly thanks to the condition (1).

(3) To show X_N is a T_0-hyperspace, we have to prove that all closures of distinct points in X_N are distinct. Let H and K be any distinct point of X_N. By using the condition (1), we have $Cl(\{H\}) = V(H) \neq V(K) = Cl(\{K\})$, this is also desired.

(4) Topologically, we know that for X_N to be a T_1-hyperspace, it must be each singleton subset is closed. Let L be a maximal element in the set of all pseudo-prime subhypermodules of N, by using the condition (1) we get that $Cl(\{L\}) = V(L) = \{L\}$. So $\{L\}$ is closed. We obtain that X_N is a T_1-hyperspace. Conversely, let $\{L\}$ be closed as X_N is a T_1-hyperspace. Then $\{L\} = Cl(\{L\}) = V(\Im(\{L\})) = V(L)$. So L is a maximal element in the set of whole pseudo-prime subhypermodules of N.

(5) Let L be a pseudo-prime subhypermodule of N. We have $Cl(\{L\}) = V(L)$ by using the condition (1). Let $H \in V(L)$. By the hypothesis, we have $(L : N) = (H : N) \in Max(S)$. Thus, L and H are prime subhypermodule of N. By Theorem 2.2, $H = L$. It follows from $Cl(\{L\}) = L$ that X_N is a T_1-hyperspace. □

Definition 4. *A topological hyperspace N is called irreducible if for every decomposition $N = N_1 \cup N_2$ as closed subsets N_1 and N_2 of N provided that $N_1 = N$ or $N_2 = N$. In addition, a maximal irreducible subset of N is said to be an irreducible component of the topological hyperspace N.*

The next theorem reveals the relation between pseudo-prime subhypermodules of the S-hypermodule N and irreducible subset of the topological hyperspace X_N. It is clear that for a hyperring S, a subset K of $Spec(S)$ is irreducible $\iff \Im(K)$ is a prime hyperideal of S.

Theorem 5. *Let N be an S-hypermodule and K be a subset of X_N. Then $\Im(K)$ is a pseudo-prime subhypermodule of N $\iff$ K is an irreducible hyperspace.*

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Proof. ($\Rightarrow$) Let K be an irreducible hyperspace, T and U be hyperideals of S with $TU \subseteq (\Im(K) : N)$. Then we have $K \subseteq V(\Im(K)) \subseteq V((TU).N) = V(T.N) = \cup V(U.N)$. It follows from K is irreducible that we have $K \subseteq V(T.N)$ or $K \subseteq V(U.N)$. So $T.N \subseteq Prad(T.N) = \Im(V(U.N)) \subseteq \Im(N)$ or $U.N \subseteq \Im(K)$. Since $T \subseteq (\Im(K) : N)$ or $U \subseteq (\Im(K) : N)$, then $\Im(K)$ is a pseudo-prime subhypermodule of N. Let's take a pseudo-prime subhypermodule $\Im(K)$ of N with $K \subseteq K_1 \cup K_2$ where K_1 and K_2 are closed subsets of X_N. Thus there exist subhypermodules L and T of N such that $V(L) = K_1$ and $V(T) = K_2$. Therefore $\Im(V(L) \cup V(T)) = \Im(V(L)) \cap \Im(V(T)) = Prad(L) \cap Prad(T) \subseteq \Im(K)$. Then we have $\Im(K)$ is an extraordinary subhypermodule because N is a topological hypermodule. It is obtained that $Prad(L) \subseteq \Im(L)$ or $Prad(T) \subseteq \Im(K)$. Then K is irreducible provided that $K \subseteq V(\Im(K)) \subseteq V(Prad(L)) = V(L) = K_1$ or $K \subseteq K_2$. $\square$

Corollary 1. *Let N be an S-hypermodule and K be a subhypermodule of N.*

(1) $Prad(K)$ is a pseudo-prime subhypermodule of $N \iff V(K)$ is an irreducible hyperspace.

(2) $Prad(0)$ is a pseudo-prime subhypermodule of N if and only if N is a irreducible hyperspace.

(3) If $X_{N,U} \neq \emptyset$ for any $u \in Spec(S)$, then $X_{N,U}$ is an irreducible hyperspace.

Proof. **(1)** It follows from $Prad(K) = \Im(V(K))$ that the proof is obtained directly using Theorem 3.2.

(2) Clear from (1) by taking $K = (0)$.

(3) Since $\Im(X_{N,U} : N) = \bigcap_{Q \in X_{N,U}} (Q : N) = U \in Spec(S)$, the claim provides thanks to Theorem 3.2. $\square$

Definition 5. *Let N be an S-hypermodule, U a hyperideal of N. U is said to be a radical hyperideal of S if $U = \bigcap_i u_i$ where u_i runs through $V^S(U)$.*

Lemma 2. *Let N be a non-zero pseudo-primeful S-hypermodule, U a radical hyperideal of S. Then $Ann(N) \subseteq U$ if and only if $(U.N : N) = U$. In addition, for every $u_i \in V^S(Ann(N))$, $u_i.N$ is a pseudo-prime subhypermodule of N.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) By the hypothesis, there exists hyperideals U where u_i runs through $V^S(U)$ and $Ann(N) \subseteq U = \bigcap_i u_i$. Then there is a pseudo-prime subhypermodule K_i of N with $(K_i : N) = u_i$ for a pseudo-primeful S-hypermodule N and $u_i \in V^S(Ann(N))$. So we have $U \subseteq (U.N : N) = \left(\left(\bigcap_i u_i\right).N : N\right) \subseteq \bigcap_i (u_i.N : N) \subseteq \bigcap_i (K_i : N) = \bigcap_i u_i = U$. Hence $(U.N : N) = U$. $\square$

Recall from [13] that $Rad(N)$ is the sum of all small subhypermodules of N, that is $Rad(N) = \sum_{L_i << N} L_i$. Here subhypermodules L_i of N is called *small* in N if $T + T_i = N$ for every subhypermodule T of N satisfies $N = T$.

Now let's adapt the Nakayama's Lemma to hypermodule in the next proposition.

Proposition 2. *Let N be a pseudo-primeful S-hypermodule and U a hyperideal of S which contained in $Rad(S)$ so that $U.N = N$. Then $N = (0)$.*

Definition 6. *Let T be closed subset of a topological hyperspace. If $T = Cl(\{a\})$ then $a \in Y$ is said to be the generic point of T.*

In Proposition 3.1 (1) we obtain that each element K of X_N is a general point of the irreducible closed subset $V(K)$. Note that if the topological hyperspace T_0-hyperspace, the general point T of a closed subset of the topological hyperspace is unique by Proposition 3.1. The following theorem is an excellent implementation of Zariski topology on hypermodules. Indeed, the following theorem shows that there is a relationship between the irreducible closed subsets of X_N and the pseudo-prime subhypermodules of the S-hypermodule N.

Theorem 6. *Let N be a S-hypermodule and $U \subseteq X_N$. Then the following conditions satisfy.*
(1) *U is an irreducibleclosed subset of X_N if and only if $U = V(W)$ for each $W \in X_N$. In addition each irreducible closet subset of X_N possesses a generic point.*

Recall from [5] that a hyperring S is said to be *Noetherian* if it satisfies the ascending chain condition on hyperideals of S, i.e., for each ascending chain of hyperideals $J_1 \subseteq J_2 \subseteq ...$, there is an element $k \in \mathbb{N}$ such that $J_k = J_t$ for every $k \geq t$.

Definition 7. *A topological hyperspace X is said to be Noetherian hyperspace if the open subset of the hyperspace possesses the ascending chain condition.*

We use the notion of Noetherian S-hypermodules for pseudo-prime spectrum of hypermodules and radical hyperideals of S satisfing the ascending chain condition ACC.

Theorem 7. *Let N be a S-hypermodule. Then N possesses Noetherian pseudo-prime spectrum $\Longleftrightarrow$ the ACC is provided pseudo-prime radical subhypermodules of N.*

Proof. $(\Rightarrow)$ Let N has a Noetherian pseudo-prime spectrum and $U_1 \subseteq U_2 \subseteq ...$ an ascending chain of pseudo-prime radical subhypermodules of N. Hence $U_j = \Im\left(\left(V\left(U_j\right)\right)\right) = Prad\left(U_j\right)$ for $j \in \mathbb{N}$. It follows that $V(U_1) \supseteq V(U_2) \supseteq ...$ is a descending chain of close subset of X_N. By the hypothesis there exists an element $l \in \mathbb{N}$ so that $V(U_l) = V(U_{l+n})$ for each $n \in \mathbb{N}$. So $N_1 = Prad(U_l) = \Im(V(U_l)) = \Im(V(U_{l+n})) = Prad(U_{l+n}) = U_{l+n}$.

$(\Leftarrow)$ Suppose that the ACC is provided for pseudo-prime radical subhypermodules of N. Let $V(U_1) \supseteq V(U_2) \supseteq ...$ be a descending chain of close subsets of X_N for $U_j \leq N$. Then $\Im(V(U_1)) \subseteq \Im(V(U_2)) \subseteq ...$ is an ascending chain of psudo-prime radical subhypermodules $\Im\left(V\left(U_j\right)\right) = Prad\left(U_j\right)$ of the hypermodule N. By the hypothesis, there is an element $l \in \mathbb{N}$ so that $\Im(V(U_l)) = \Im\left(V\left(U_{l+j}\right)\right)$ for each $j \in \mathbb{N}$. It follows from Proposition 3.1 that $V(U_l) = V(\Im(V(U_l))) = V\left(\Im\left(V\left(U_{l+j}\right)\right)\right) = V\left(U_{l+j}\right)$. So X_N is a Noetherian hyperspace. □

Definition 8. *A topological hyperspace Y is a spectral hyperspace if it is homeomorphic to $Spec(S)$ where S is a hyperring according to the Zariski topology.*

Theorem 8. *Let N be a S-hypermodule. Then X_N is a spectral hyperspace if each of the following conditions are met.*
(1) *There exists a hyperideal J of S so that $V(U) = V(J.N)$ for a Noetherian hyperring S and for every subhypermodule U of N.*
(2) *Let N be a content pseudo-injective S-hypermodule and $Spec(S)$ a Noetherian topological hyperspace.*

Proof. **(1)** If it is shown that every subset of X_N is quasi-compact, the desired is obtained. Let K be an open subset of X_N and $\{A_i\}_{i\in\mathbb{N}}$ be an open coverof K. Then there exist subhypermodules L and L_i so that $K = X_N \backslash V(L)$, $A_i = X_N \backslash V(L_i)$ for every $i \in I$ and $K \subseteq \underset{i\in I}{\cup} A_i = X_N \backslash \underset{i\in I}{\cap} V(L_i)$. By assumption, there is a hyperideal I_i in S so that $V(L_i) = V(I_i.N)$ for every $i \in I$. Then we have $L \subseteq X_N \backslash V\left(\sum\limits_{i\in I} I_i.N\right) = X_N \backslash V\left(\left(\sum\limits_{i\in I} I_i\right).N\right)$. As S is a Noetherian hyperring, there is a finite subset I' of I so that $L \subseteq \underset{j\in I'}{\cup} A_j$. Hence X_N is a both of Noetherian hyperspace and spectral hyperspace.

(2) Let's show that X_N is Noetherian. Let $V(L_1) \supseteq V(L_2) \supseteq ...$ be a descending chain of closed subsets of X_N. So $Prad(L_1) \subseteq Prad(L_2) \subseteq ...$ As $Spec(S)$ is Noetherian, the ACC $(Prad(L_1) : N) \subseteq$

$(Prad\,(L_2) : N) \subseteq ...$ of radial hyperideals shall be stationary by Theorem 3.5. Therefore there exists an element $l \in \mathbb{N}$ so that $(Prad\,(L_l) : N) = \left(Prad\left(L_{l+j}\right) : N\right) = ...$, for every $j = 1, 2, ...$ If the proof technique in Theorem 2.3 is applied, it is seen that $Prad\,(L_j) = (Prad\,(L_j) : N)\,.N$. Thus, we get $Prad\,(L_l) = Prad\left(L_{l+j}\right) = ...$ for every $j = 1, 2, ...$ It follows that $V\,(L_l) = V\,(Prad\,(L_l)) = V\left(Prad\left(L_{l+j}\right)\right) = V\left(L_{l+j}\right) = ...$ So X_N is Noetherian, the desired is achieved. □

Informed Consent Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees who contributed to the development of the article.

Conflicts of Interest: The authors declare that they have no competing interests.

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