Article

Norm of Hilbert Operator's Commutants

Hadi Roopaei

Department of Mathematics and Statistics, University of Victoria, Victoria, Canada; h.roopaei@gmail.com

Abstract: In this study, we prove the norm separating property for the composition of Cesàro and Gamma matrices with their transpose. As a result, we compute the ℓ_p-norms of six classes of operators that commute with the infinite Hilbert o perators. Additionally, we find the norm of Hilbert's commutants on some well-known sequence spaces.

Keywords: Commutators; Hilbert matrix; Cesàro matrix; Norm

1. Introduction

We can denote all sequences with real values by ω. As a result, any linear subspace of ω is referred to as a sequence space. Banach space ℓ_p is the set of all real numbers sequences $x = (x_k)_{k=0}^{\infty} \in \omega$ such that

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty \quad (1 \leq p < \infty).$$

Assume T has non-negative entries and maps ℓ_p into itself and satisfies the inequality

$$\|Tx\|_{\ell_p} \leq K\|x\|_{\ell_p},$$

for the constant K not depending on x and for every $x \in \ell_p$. The norm of T is the smallest possible value of K. Several references have addressed the problem of finding the norm and lower bound of operators on matrix domains [2–4,13–16].

Our study considers infinite matrices $[A]_{j,k}$, where all the indices j and k are non-negative.

Hilbert matrix. If n is a non-negative integer, we define the Hilbert matrix of order n, H_n, as follows:

$$[H_n]_{j,k} = \frac{1}{j+k+n+1} \qquad (j, k = 0, 1, \cdots).$$

In the case of $n = 0$, $H_0 = H$ is the well-known Hilbert matrix

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

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which was introduced by David Hilbert in 1894. Below are some examples:

$$H_1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \textit{and} \quad H_2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

According to [7] theorem 323, the Hilbert matrix is a bounded operator on ℓ_p and

$$\|H\|_{\ell_p \to \ell_p} = \Gamma(1/p)\Gamma(1/p^*) = \pi \csc(\pi/p),$$

where p^* is the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$.

Hausdorff matrices. One of the best examples of summability matrices is H_μ, which is defined as

$$[H_\mu]_{j,k} = \begin{cases} \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k} d\mu(\theta) & 0 \le k \le j, \\ 0 & \text{otherwise.} \end{cases}$$

where μ is a probability measure on $[0,1]$. Even though it is a difficult task to obtain the ℓ_p-norm of operators, the Hausdorff matrices can be computed using Hardy's formula [6, Theorem 216] which states that this matrix is a bounded operator on ℓ_p, if and only if

$$\int_0^1 \theta^{\frac{-1}{p}} \, d\mu(\theta) < \infty, \qquad 1 \le p < \infty.$$

In fact,

$$\|H_\mu\|_{\ell_p \to \ell_p} = \int_0^1 \theta^{\frac{-1}{p}} \, d\mu(\theta). \tag{1.1}$$

Hausdorff operators have the interesting norm separating property.

Theorem 1.1 ([4], Theorem 9)**.** *Let $p \ge 1$ and H_μ, H_φ and H_ν be Hausdorff matrices such that $H_\mu = H_\varphi H_\nu$. Then H_μ is bounded on ℓ_p if and only if both H_φ and H_ν are bounded on ℓ_p. Moreover, we have*

$$\|H_\mu\|_{\ell_p \to \ell_p} = \|H_\varphi\|_{\ell_p \to \ell_p} \|H_\nu\|_{\ell_p \to \ell_p}.$$

Several famous matrices have been derived from the Hausdorff matrix. For positive integer n, the following are the two classes:

Cesàro matrix. The measure $d\mu(\theta) = n(1-\theta)^{n-1} d\theta$ gives the Cesàro matrix of order n, C_n, for which

$$[C_n]_{j,k} = \begin{cases} \dfrac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \le k \le j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $C_0 = I$, where I is the identity matrix, and

$$C_1 = C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the classical Cesàro matrix. For example,

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 2/3 & 1/3 & 0 & \cdots \\ 3/6 & 2/6 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \textit{and} \quad C_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 3/4 & 1/4 & 0 & \cdots \\ 6/10 & 3/10 & 1/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

According to (1.1), C_n has the ℓ_p-norm

$$\|C_n\|_{\ell_p \to \ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)},$$

which for the famous Cesàro matrix that is $\|C\|_{\ell_p \to \ell_p} = \frac{p}{p-1}$.

The author, in [17] Theorem 3.1, has introduced a factorization for the Cesàro matrix that will be used in the future.

Theorem 1.2. *For integers $n \geq m \geq 0$, the Cesàro matrix C_n has a factorization of the form $C_n = S_{n,m}C_m = C_m S_{n,m}$, where $S_{n,m}$ is a bounded operator on ℓ_p and*

$$\|S_{n,m}\|_{\ell_p \to \ell_p} = \frac{\Gamma(n+1)\Gamma(m+1/p^*)}{\Gamma(m+1)\Gamma(n+1/p^*)}.$$

In particular, for $m = n-1$, $C_n = G_n C_{n-1} = C_{n-1} G_n$, where G_n is the Gamma operator of order n.

The matrix domain associated with C_n is defined by

$$C_n(p) = \left\{ \mathbf{x} \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p < \infty \right\}.$$

who has the norm

$$\|\mathbf{x}\| := \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p \right)^{1/p},$$

and is a Banach space.

Gamma matrix. The measure $d\mu(\theta) = n\theta^{n-1}d\theta$ gives the Gamma matrix of order n, G_n, for which

$$[G_n]_{j,k} = \begin{cases} \frac{\binom{n+k-1}{k}}{\binom{n+j}{j}} & 0 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Hardy's formula, G_n has the ℓ_p-norm

$$\|G_n\|_{\ell_p \to \ell_p} = \frac{np}{np-1}.$$

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You should note that G_1 is the classical Cesàro matrix C. Here are some more examples of Gamma matrices

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/3 & 2/3 & 0 & \cdots \\ 1/6 & 2/6 & 3/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad and \quad G_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/4 & 3/4 & 0 & \cdots \\ 1/10 & 3/10 & 6/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consider $G_n(p)$ to be the matrix domain of G_n, which is defined as follows:

$$G_n(p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p < \infty \right\}.$$

Equipped with the norm

$$\|\mathbf{x}\| := \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p \right)^{\frac{1}{p}},$$

$G_n(p)$ is a Banach space. The Cesàro and Gamma matrices and their associated sequence spaces have been studied by Roopaei et al. in [8–11] for both cases $0 < p < 1$ and $1 \leq p < \infty$.

The Hellinger-Toeplitz theorem can also be called the following theorem.

Theorem 1.3 ([2], Proposition 7.2)**.** *Let $1 < p, q < \infty$. The matrix M maps ℓ_p into ℓ_q if and only if the transposed matrix, M^t, maps ℓ_{q^*} into ℓ_{p^*}. Then we have*

$$\|M\|_{\ell_p \to \ell_q} = \|M^t\|_{\ell_{q^*} \to \ell_{p^*}}.$$

As an example of the Hellinger-Toeplitz theorem, the transposed Cesàro matrix of order n has the ℓ_p-norm

$$\|C_n^t\|_{\ell_p \to \ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}.$$

Motivation. Infinite Hilbert operator has an extremely complex structure, making it among the most complicated operators. Because of this complexity, it is used in the cryptography area. Recently author [12] has introduced some classes of Hilbert's commutators mostly based on Cesàro and Gamma matrices. Through this study, the author tries to complete his previous work by computing the ℓ_p-norm of those operators.

For non-negative integers n, j and k, let us define the matrix B_n by

$$[B_n]_{j,k} = \binom{n+k}{k} \beta(j+k+1, n+1) = \frac{(k+1)\cdots(k+n)}{(j+k+1)\cdots(j+k+n+1)},$$

where the β function is

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dz \qquad (m, n = 1, 2, \ldots).$$

Clearly, $B_0 = H$ where H represents Hilbert's matrix.

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We need the following lemma before we can discuss the Hilbert operator's commutants, which reveals the relationship between the Hilbert operator and the Cesàro and Gamma matrices.

Lemma 1.4 (Lemmas 2.3 and 3.1 of [13] and [14]). *Hilbert matrices satisfy the following identities for positive integer n:*

- $H = B_n C_n$
- $H_n = C_n B_n$
- $H_n C_n = C_n H$
- $H_n G_n = G_n H_{n-1}$
- B_n *is a bounded operator that has the* ℓ_p*-norm*

$$\|B_n\|_{\ell_p \to \ell_p} = \frac{\Gamma(n + 1/p^*)\Gamma(1/p)}{\Gamma(n+1)},$$

where C_n *and* G_n *are the Cesàro and Gamma matrices of order n and* B_n *is the matrix which was defined earlier.*

Commutants of the infinite Hilbert operator. Assume that n is a non-negative integer, and define the symmetric matrix as follows:

$$\Phi_n^b = B_n^t B_n \qquad \Psi_n^b = B_n B_n^t$$

$$\Phi_n^c = C_n^t C_n \qquad \Psi_n^c = C_n C_n^t$$

and for $n \geq 1$

$$\Phi_n^g = G_n^t G_n \qquad \Psi_n^g = G_n G_n^t,$$

Note that for $n = 1$,

$$\Psi := \Psi_1^c = \Psi_1^g = CC^t \quad \textit{and} \quad \Phi := \Phi_1^c = \Phi_1^g = C^t C,$$

where the matrices Ψ and Φ have the matrix representations

$$\Psi = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/2 & 1/3 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\Phi = \begin{pmatrix} 1 + 1/4 + \cdots & 1/4 + 1/9 + \cdots & 1/9 + 1/16 + \cdots & \cdots \\ 1/4 + 1/9 + \cdots & 1/4 + 1/9 + \cdots & 1/9 + 1/16 + \cdots & \cdots \\ 1/9 + 1/16 + \cdots & 1/9 + 1/16 + \cdots & 1/9 + 1/16 + \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

respectively.

The author, in [8] Theorems 11.2.2 and 11.2.4, has proved that all the above matrices are the Hilbert operator's commutants. As emphasized, We bring those theorems with their proofs.

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Theorem 1.5. *The Hilbert operator's commutants are the operators Φ_n^c and Ψ_n^b.*

Proof. By applying Lemma 1.4 twice, we have

$$\begin{aligned}\Phi_n^c H &= C_n^t H_n C_n = (H_n C_n)^t C_n \\ &= (C_n H)^t C_n = H C_n^t C_n = H \Phi_n^c.\end{aligned}$$

It can easily be seen from Lemma 1.4 that $HB_n = B_n H_n$. Now,

$$\begin{aligned}\Psi_n^b H &= B_n (HB_n)^t = B_n (B_n H_n)^t \\ &= B_n H_n B_n^t = HB_n B_n^t = H\Psi_n^b.\end{aligned}$$

□

Theorem 1.6. *The operators Φ_n^b, Φ_{n+1}^g, Ψ_n^c and Ψ_n^g are the commutants of the Hilbert operator of order n.*

Proof. By applying Lemma 1.4 twice, we have

$$\begin{aligned}\Psi_n^c H_n &= C_n (H_n C_n)^t = C_n (C_n H)^t \\ &= C_n H C_n^t = H_n C_n C_n^t = H_n \Psi_n^c.\end{aligned}$$

Also applying Lemma 1.4 results in

$$\begin{aligned}\Psi_n^g H_n &= G_n (H_n G_n)^t = G_n (G_n H_{n-1})^t \\ &= G_n H_{n-1} G_n^t = H_n G_n G_n^t = H_n \Psi_n^g.\end{aligned}$$

The proof of the other items is similar. □

2. Main Results

For non-negative integers m and n, let us define the following matrices

$$\Phi_{m,n}^b = B_m^t B_n \qquad \Psi_{m,n}^b = B_m B_n^t$$

$$\Phi_{m,n}^c = C_m^t C_n \qquad \Psi_{m,n}^c = C_m C_n^t$$

and for $m, n \geq 1$

$$\Phi_{m,n}^g = G_m^t G_n \qquad \Psi_{m,n}^g = G_m G_n^t.$$

Note that for $m = n$, all the above matrices are reduced to the Hilbert operator's commutators that we introduced earlier. Through this section, we will prove the norm separating property for the Cesàro and Gamma matrices of the form:

$$\|C_m C_n^t\|_{\ell_p \to \ell_p} = \|C_m\|_{\ell_p \to \ell_p} \|C_n^t\|_{\ell_p \to \ell_p},$$

$$\|C_m^t C_n\|_{\ell_p \to \ell_p} = \|C_m^t\|_{\ell_p \to \ell_p} \|C_n\|_{\ell_p \to \ell_p},$$

$$\|G_m G_n^t\|_{\ell_p \to \ell_p} = \|G_m\|_{\ell_p \to \ell_p} \|G_n^t\|_{\ell_p \to \ell_p},$$

$$\|G_m^t G_n\|_{\ell_p\to\ell_p} = \|G_m^t\|_{\ell_p\to\ell_p}\|G_n\|_{\ell_p\to\ell_p}.$$

Theorem 2.1. *For non-negative integers m and n, matrices $\Psi_{m,n}^c$ and $\Phi_{m,n}^c$ are bounded operators on ℓ_p and*

$$\|\Psi_{m,n}^c\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1)\Gamma(n+1)\pi\csc(\pi/p)}{\Gamma(m+1/p^*)\Gamma(n+1/p)}$$

$$\|\Phi_{m,n}^c\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1)\Gamma(n+1)\pi\csc(\pi/p)}{\Gamma(m+1/p)\Gamma(n+1/p^*)}.$$

In particular, the matrices Ψ_n^c and Φ_n^c are bounded operators on ℓ_p and

$$\|\Psi_n^c\|_{\ell_p\to\ell_p} = \|\Phi_n^c\|_{\ell_p\to\ell_p} = \frac{\Gamma^2(n+1)\pi\csc(\pi/p)}{\Gamma(n+1/p)\Gamma(n+1/p^*)}.$$

Theorem 2.2. *For positive integers m and n, matrices $\Psi_{m,n}^g$ and $\Phi_{m,n}^g$ are bounded operators on ℓ_p and*

$$\|\Psi_{m,n}^g\|_{\ell_p\to\ell_p} = \frac{mnpp^*}{(mp-1)(np^*-1)}$$

$$\|\Phi_{m,n}^g\|_{\ell_p\to\ell_p} = \frac{mnpp^*}{(mp^*-1)(np-1)}.$$

In particular, the matrices Ψ_n^g and Φ_n^g are bounded operators on ℓ_p and

$$\|\Psi_n^g\|_{\ell_p\to\ell_p} = \|\Phi_n^g\|_{\ell_p\to\ell_p} = \frac{n^2pp^*}{(np-1)(np^*-1)}.$$

Theorem 2.3. *For non-negative integer n, the matrices Ψ_n^b and Φ_n^b are bounded operators on ℓ_p and*

$$\|\Psi_n^b\|_{\ell_p\to\ell_p} = \|\Phi_n^b\|_{\ell_p\to\ell_p} = \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)\pi\csc(\pi/p)}{\Gamma^2(n+1)}.$$

3. Proof of Theorems

In this section, we focus on proving our claims, but first, we need the following lemma.

Lemma 3.1. *For the Hilbert operator we have $\|H^2\|_{\ell_p\to\ell_p} = \|H\|^2_{\ell_p\to\ell_p}$.*

Proof. Let H be the Hilbert operator with matrix entries $1/(j+k)(j,k\geq 1)$, and write $M_r = \pi/\sin(r\pi)$. It is well known that $\|H\|_{\ell_p\to\ell_p}\leq M_{1/p}$ for $p>1$. Here we show that $\|H\|_{\ell_p\to\ell_p}\geq M_{1/p}$ and $\|H^2\|_{\ell_p\to\ell_p}\geq M_{1/p}^2$ (so that equality holds in both cases). The same statements hold for the alternative Hilbert operator with matrix entries $\frac{1}{j+k-1}$.

Choose r with $rp>1$, and let $x_k = 1/k^r$ for $k\geq 1$. Let $y=Hx$ and $z=Hy$. Then

$$y_j = \sum_{k=1}^{\infty}\frac{1}{(j+k)k^r} \geq \int_1^{\infty}\frac{1}{(t+j)t^r}dt.$$

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Now,

$$\int_0^\infty \frac{1}{(t+j)t^r} dt = \frac{M_r}{j^r},$$

and

$$\int_0^1 \frac{1}{(t+j)t^r} dt \le \int_0^1 \frac{1}{jt^r} dt = \frac{1}{(1-r)j}.$$

so

$$y_j \ge \frac{M_r}{j^r} - \frac{1}{(1-r)j}. \tag{3.1}$$

Informally, y_j is approximately $M_r x_j$, so $\|y\|_{\ell_p}$ is approximately $M_r \|x\|_{\ell_p}$. For $0 < x < a$, we have $(1 - \frac{x}{a})^p \ge 1 - \frac{px}{a}$, hence $(a-x)^p \ge a^p - pa^{p-1}x$. Hence

$$y_j^p \ge \frac{M_r^p}{j^{rp}} - \frac{p}{1-r} \frac{M_r^{p-1}}{j^{rp-r+1}},$$

so,

$$\sum_{j=1}^\infty y_j^p \ge M_r^p \zeta(rp) - \frac{p}{1-r} M_r^{p-1} \zeta(rp - r + 1),$$

while $\sum_{k=1}^\infty x_k^p = \zeta(rp)$. Now let $r \to 1/p$ from above. Then $\zeta(rp) \to \infty$, while $\zeta(rp - r + 1) \to \zeta(2 - 1/p)$. Hence $\frac{\|y\|_{\ell_p}}{\|x\|_{\ell_p}}$ tends to $M_{1/p}$.

We now turn to H^2. We require the following

Let $u_k = 1/k$ for $k \ge 1$. Then

$$\begin{aligned}
(Hu)_j &= \sum_{k=1}^\infty \frac{1}{(j+k)k} = \frac{1}{j} \sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{j+k} \right) \\
&= \frac{1}{j} \left(1 + \frac{1}{2} + \cdots \frac{1}{j} \right) = L_j / j,
\end{aligned}$$

where $L_j = \sum_{i=1}^j \frac{1}{i}$. By (3.1), $y \ge M_r x - u/(1-r)$, so

$$z \ge M_r(Hx) - \frac{Hu}{1-r} = M_r y - \frac{Hu}{1-r}.$$

So, again by (3.1)

$$z_j \ge \frac{M_r^2}{j^r} - \frac{M_r}{(1-r)j} - \frac{L_j}{(1-r)j}.$$

Hence

$$z_j^p \ge \frac{M_r^{2p}}{j^{rp}} - p \frac{M_r^{2p-2}}{j^{r(p-1)}} \frac{M_r + L_j}{(1-r)j}.$$

Write $\eta(s) = \sum_{j=1}^\infty \frac{L_j}{j^s}$: this is convergent for $s > 1$. Then

$$\sum_{j=1}^\infty z_j^p \ge M_r^{2p} \zeta(rp) - \frac{p}{(1-r)} M_r^{2p-2} (M_r \zeta(rp - r + 1) + \eta(rp - r + 1)).$$

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When $r \to 1/p$ from above, $\eta(rp - r + 1)$ tends to the finite limit $\eta(2 - 1/p)$. So $\|z\|_{\ell_p}/\|x\|_{\ell_p}$ tends to $M^2_{1/p}$. □

Proof of Theorem 2.1. (a) We first compute the ℓ_p-norm of $\Psi^c_{m,n}$. Obviously

$$\begin{aligned}\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p} &\leq \|C_m\|_{\ell_p\to\ell_p}\|C_n^t\|_{\ell_p\to\ell_p} \\ &= \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma(m+1/p^*)\Gamma(n+1/p)}.\end{aligned}$$

Now, according to Lemma 3.1 and Lemma 1.4

$$\begin{aligned}\|H\|^2_{\ell_p\to\ell_p} &= \|H^2\|_{\ell_p\to\ell_p} = \|HH^t\|_{\ell_p\to\ell_p} \\ &= \|B_mC_m(B_nC_n)^t\|_{\ell_p\to\ell_p} = \|B_m\Psi^c_{m,n}B_n^t\|_{\ell_p\to\ell_p} \\ &\leq \|B_m\|_{\ell_p\to\ell_p}\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p}\|B_n^t\|_{\ell_p\to\ell_p}.\end{aligned}$$

Hence

$$\begin{aligned}\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p} &\geq \frac{\|H\|^2_{\ell_p\to\ell_p}}{\|B_m\|_{\ell_p\to\ell_p}\|B_n^t\|_{\ell_p\to\ell_p}} \\ &= \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma(m+1/p^*)\Gamma(n+1/p)} \\ &= \frac{\Gamma(m+1)\Gamma(n+1)\pi\csc(\pi/p)}{\Gamma(m+1/p^*)\Gamma(n+1/p)}.\end{aligned}$$

(b) For computing the norm of $\Phi^c_{m,n}$, we consider two cases:

(1) $m \geq n$
In this case, regarding Lemma 1.4 and the identity $C_m = S_{m,n}C_n = C_nS_{m,n}$, we have

$$H_m^2 = (C_mB_m)^tC_mB_m = B_m^tC_m^tC_nS_{m,n}B_m = B_m^t\Phi^c_{m,n}S_{m,n}B_m.$$

Hence

$$\begin{aligned}\|\Phi^c_{m,n}\|_{\ell_p\to\ell_p} &\geq \frac{\|H_m\|^2_{\ell_p\to\ell_p}}{\|B_m^t\|_{\ell_p\to\ell_p}\|S_{m,n}\|_{\ell_p\to\ell_p}\|B_m\|_{\ell_p\to\ell_p}} \\ &= \frac{\Gamma(m+1)\Gamma(n+1)\pi\csc(\pi/p)}{\Gamma(m+1/p)\Gamma(n+1/p^*)}.\end{aligned}$$

(2) $m < n$
Similarly, by applying Lemma 1.4 and the identity $C_n = C_mS_{m,n}$ we have

$$H_n^2 = (C_nB_n)^tC_nB_n = B_n^t(C_mS_{m,n})^tC_nB_n = B_n^tS_{m,n}^t\Phi^c_{m,n}B_n.$$

and again

$$\begin{aligned}\|\Phi^c_{m,n}\|_{\ell_p\to\ell_p} &\geq \frac{\|H_n\|^2_{\ell_p\to\ell_p}}{\|B_n^t\|_{\ell_p\to\ell_p}\|S_{m,n}^t\|_{\ell_p\to\ell_p}\|B_n\|_{\ell_p\to\ell_p}} \\ &= \frac{\Gamma(m+1)\Gamma(n+1)\pi\csc(\pi/p)}{\Gamma(m+1/p)\Gamma(n+1/p^*)}.\end{aligned}$$

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The other side of the above inequalities is due to the norm inequality

$$\begin{aligned}\|\Phi^c_{m,n}\|_{\ell_p\to\ell_p} &\leq \|C^t_m\|_{\ell_p\to\ell_p}\|C_n\|_{\ell_p\to\ell_p}\\ &= \frac{\Gamma(m+1)\Gamma(n+1)\pi\csc(\pi/p)}{\Gamma(m+1/p)\Gamma(n+1/p^*)},\end{aligned}$$

which completes the proof. In particular, for $m=n$

$$\|\Psi^c_n\|_{\ell_p\to\ell_p} = \|\Phi^c_n\|_{\ell_p\to\ell_p} = \frac{\Gamma^2(n+1)\pi\csc(\pi/p)}{\Gamma(n+1/p^*)\Gamma(n+1/p)}.$$

In the following, we intend to present another proof for obtaining the norm of matrices $\Psi^c_{m,n}$ and $\Phi^c_{m,n}$.

Remark 3.2. *Suppose that $\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p} = f(m,n,p)$, where f is a real positive function. Since $\Psi^c_{m,0} = C_m$ hence*

$$\|\Psi^c_{m,0}\|_{\ell_p\to\ell_p} = \|C_m\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1)\Gamma(1/p^*)}{\Gamma(m+1/p^*)},$$

which proves that

$$\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1)\Gamma(1/p^*)}{\Gamma(m+1/p^*)}g(n,p).$$

Now, since $\Psi^c_{0,n} = C^t_n$ hence according to the Helinger-Toeplitz Theorem

$$\|\Psi^c_{0,n}\|_{\ell_p\to\ell_p} = \|C^t_n\|_{\ell_p\to\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)},$$

which shows that

$$\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma(m+1/p^*)\Gamma(n+1/p)}h(p),$$

where $h(p)$ is a positive function of p. Finally, since $\Psi^c_{0,0} = I$, for the identity matrix, hence

$$\|\Psi^c_{0,0}\|_{\ell_p\to\ell_p} = \|I\|_{\ell_p\to\ell_p} = 1,$$

which indicates that $h(p) = 1$. Therefore, we have proven

$$\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma(m+1/p^*)\Gamma(n+1/p)}.$$

The method for obtaining ℓ_p-norm of $\Phi^c_{m,n}$ is the same.

Proof of Theorem 2.2.
(a) By the definition we have

$$\|\Psi^g_{m,n}\|_{\ell_p\to\ell_p} \leq \|G_m\|_{\ell_p\to\ell_p}\|G^t_n\|_{\ell_p\to\ell_p} = \frac{mnpp^*}{(mp-1)(np^*-1)}.$$

From the definition and relation $C_n = C_{n-1}G_n = G_nC_{n-1}$, we have the identity

$$\Psi^c_{m,n} = C_{m-1}G_m(C_{n-1}G_n)^t = C_{m-1}\Psi^g_{m,n}C^t_{n-1}.$$

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Hence by applying the previous part we obtain

$$\|\Psi^g_{m,n}\|_{\ell_p\to\ell_p} \geq \frac{\|\Psi^c_{m,n}\|_{\ell_p\to\ell_p}}{\|C_{m-1}\|_{\ell_p\to\ell_p}\|C^t_{n-1}\|_{\ell_p\to\ell_p}} = \frac{mnpp^*}{(mp-1)(np^*-1)},$$

which proves our equality.

(b) For computing the ℓ_p-norm of $\Phi^g_{m,n}$, its enough to use the identity $\Phi^c_{m,n} = (G_m C_{m-1})^t G_n C_{n-1} = C^t_{m-1}\Phi^g_{m,n}C_{n-1}$. Now, the proof is routine. Particularly, for $m = n$, we have

$$\|\Psi^g_n\|_{\ell_p\to\ell_p} = \|\Phi^g_n\|_{\ell_p\to\ell_p} = \frac{n^2pp^*}{(np-1)(np^*-1)}.$$

Proof of Theorem 2.3. According to the definition

$$\begin{aligned}\|\Psi^b_n\|_{\ell_p\to\ell_p} &\leq \|B_n\|_{\ell_p\to\ell_p}\|B^t_n\|_{\ell_p\to\ell_p}\\ &= \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma^2(n+1)}.\end{aligned}$$

Now, by applying the identity $\Psi^b_n\Phi^c_n = H^2$ and Lemma 3.1 we have

$$\begin{aligned}\|\Psi^b_n\|_{\ell_p\to\ell_p} &\geq \frac{\|H^2\|_{\ell_p\to\ell_p}}{\|\Phi^c_n\|_{\ell_p\to\ell_p}} = \frac{\|H\|^2_{\ell_p\to\ell_p}}{\|\Phi^c_n\|_{\ell_p\to\ell_p}}\\ &= \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma^2(n+1)}.\end{aligned}$$

which completes the proof.

4. Some Applications

In a fixed sequence space $\mathcal{X}$, matrix A has the following matrix domain:

$$A_{\mathcal{X}} = \{\mathbf{x} \in \omega : A\mathbf{x} \in \mathcal{X}\}. \tag{4.1}$$

The special and important case $\mathcal{X} = \ell_p$ will be written as A_p instead of A_{ℓ_p}. When I is the infinite identity matrix, $I_p = \ell_p$ is rather trivial. Researchers have been inspired by this concept to define new Banach spaces as infinite matrix domains. See the textbook [1]. Using our main theorems, we obtain the norm of Hilbert's commutants on some famous sequence spaces in the sequel.

Corollary 4.1. *Let C_n be the Cesàro operator of order n. Then*

(a) Φ^c_n is a bounded operator from $C_n(p)$ into ℓ_p and

$$\|\Phi^c_n\|_{C_n(p)\to\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}.$$

(b) Φ^c_n is a bounded operator on $C_n(p)$ and

$$\|\Phi^c_n\|_{C_n(p)\to C_n(p)} = \frac{\Gamma^2(n+1)\pi\csc(\pi/p)}{\Gamma(n+1/p)\Gamma(n+1/p^*)}.$$

Proof. Since C_n is invertible, the map $x \to C_n x$ indicates that $C_n(p)$ and ℓ_p are isomorphic spaces. (a) According to the definition of matrix Φ^c_n and Hellinger-Toeplitz theorem

$$\begin{aligned} \|\Phi^c_n\|_{C_n(p)\to\ell_p} &= \sup_{x\in C_n(p)} \frac{\|\Phi^c_n x\|_{\ell_p}}{\|x\|_{C_n(p)}} = \sup_{C_n x\in\ell_p} \frac{\|C^t_n C_n x\|_{\ell_p}}{\|C_n x\|_{\ell_p}} \\ &= \sup_{y\in\ell_p} \frac{\|C^t_n y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|C^t_n\|_{\ell_p\to\ell_p} \\ &= \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}. \end{aligned}$$

(b) By applying Theorem 2.1 we have

$$\begin{aligned} \|\Phi^c_n\|_{C_n(p)\to C_n(p)} &= \sup_{x\in C_n(p)} \frac{\|\Phi^c_n x\|_{C_n(p)}}{\|x\|_{C_n(p)}} = \sup_{C_n x\in\ell_p} \frac{\|C_n\Phi^c_n x\|_{\ell_p}}{\|C_n x\|_{\ell_p}} \\ &= \sup_{C_n x\in\ell_p} \frac{\|\Psi^c_n C_n x\|_{\ell_p}}{\|C_n x\|_{\ell_p}} = \sup_{y\in\ell_p} \frac{\|\Psi^c_n y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|\Psi^c_n\|_{\ell_p\to\ell_p} \\ &= \frac{\Gamma^2(n+1)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma(n+1/p)\Gamma(n+1/p^*)}. \end{aligned}$$

□

Corollary 4.2. *Let G_n be the Gamma operator of order n. Then*

(a) Φ^n_g is a bounded operator from $G_n(p)$ into ℓ_p and

$$\|\Phi^n_g\|_{G_n(p)\to\ell_p} = \frac{np^*}{np^*-1}.$$

(b) Φ^n_g is a bounded operator on $G_n(p)$ and

$$\|\Phi^n_g\|_{G_n(p)\to G_n(p)} = \frac{n^2pp^*}{(np-1)(np^*-1)}.$$

Proof. Readers should be able to figure it out for themselves. □

Corollary 4.3. *Let C_n be the Cesàro operator of order n. Then Ψ^b_n is a bounded operator on $C_n(p)$ and*

$$\|\Psi^b_n\|_{C_n(p)\to C_n(p)} = \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)\pi\csc(\pi/p)}{\Gamma^2(n+1)}.$$

Proof. By symmetricity of H_n, identity $B_nH_n = HB_n$ and Theorem 2.3 we have

$$\begin{aligned} \|\Psi^b_n\|_{C_n(p)\to C_n(p)} &= \sup_{x\in C_n(p)} \frac{\|\Psi^b_n x\|_{C_n(p)}}{\|x\|_{C_n(p)}} = \sup_{C_n x\in\ell_p} \frac{\|C_n\Psi^b_n x\|_{\ell_p}}{\|C_n x\|_{\ell_p}} \\ &= \sup_{C_n x\in\ell_p} \frac{\|H_n B^t_n x\|_{\ell_p}}{\|C_n x\|_{\ell_p}} = \sup_{C_n x\in\ell_p} \frac{\|B^t_n H x\|_{\ell_p}}{\|C_n x\|_{\ell_p}} \\ &= \sup_{C_n x\in\ell_p} \frac{\|\Phi^b_n C_n x\|_{\ell_p}}{\|C_n x\|_{\ell_p}} = \sup_{y\in\ell_p} \frac{\|\Phi^b_n y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|\Phi^b_n\|_{\ell_p\to\ell_p} \\ &= \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)\Gamma(1/p)\Gamma(1/p^*)}{\Gamma^2(n+1)}. \end{aligned}$$

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□

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