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Presenting Circular Gravitational Fields: A Numerical Exploration around Rotating Black Holes

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Abstract: We present a novel theoretical framework, Circular Gravitational Fields (CGF), which extends the gravitomagnetic analogy in general relativity by proposing that mass-energy currents generate a circular component of the gravitational field. Our formulation provides a geometrically motivated coupling between this circular field and spacetime curvature through the Ricci tensor, maintaining consistency with established gravitational physics while predicting potentially observable deviations in strong-field regimes. Using the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formalism, we perform numerical simulations of rotating black holes to explore CGF behavior in strong gravitational fields. We compare CGF predictions for key observables, including frame-dragging effects and gravitational wave signatures, with solutions to Einstein's equations. Our approach maintains consistency with current observational constraints from gravitational wave observations [1] and cosmological surveys [2], while offering new insights into the quantum nature of spacetime. The framework makes specific predictions for next-generation experiments including Euclid, the Einstein Telescope [3], and pulsar timing arrays, providing multiple avenues for empirical verification of these fundamental ideas about the nature of time and reality.

Keywords: circular gravitational fields; general relativity; gravitomagnetism; numerical relativity; spacetime geometry; einstein field equations; quantum gravity; bssn formalism; field coupling; ricci tensor; numerical simulations; black hole dynamics; frame dragging; mass-energy currents; gravitational wave physics

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1. Introduction

General relativity (GR) provides a highly accurate description of gravitational phenomena, but its nonlinear field equations often necessitate numerical methods, particularly in strong-field scenarios such as black hole dynamics. This work introduces Circular Gravitational Fields (CGF), a theoretical framework that builds upon the gravitomagnetic analogy in GR while incorporating novel geometric couplings between mass-energy currents and spacetime curvature.

The fundamental premise of CGF emerges from deep philosophical considerations about the nature of reality [4,5] combined with modern physical insights [6,7]. Across human civilizations and throughout the development of physics, the connection between circular motion, time, and fundamental forces has been a recurring theme [5]. Our work translates these insights into a rigorous mathematical framework by proposing that mass-energy currents generate circular gravitational fields analogous to, but geometrically distinct from, magnetic fields.

The CGF framework differs from standard gravitomagnetism by introducing specific geometric couplings mediated by the Ricci tensor. This ensures consistency with vacuum Einstein equations while potentially introducing observable modifications in matter-rich regions. These modifications manifest most strongly in extreme environments, providing specific predictions testable through gravitational wave observations [1].

We explore the CGF framework through numerical simulations using the BSSN formalism [8–10]. Our investigation focuses on:

1. The evolution and behavior of circular gravitational fields in strong-field regions

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2. Quantitative predictions for frame-dragging effects and their observational signatures
3. Modifications to gravitational wave generation and propagation
4. Compatibility with existing gravitational wave observations
5. Connections to quantum gravity approaches [11,12]

The emphasis of this work is twofold: first, to present a mathematically rigorous formulation of CGF that maintains consistency with established gravitational physics while introducing well-motivated modifications; second, to explore the observable consequences of these modifications through detailed numerical simulations. Our results provide a foundation for testing CGF against current and future gravitational wave observations [13], potentially offering new insights into the unification of fundamental forces.

2. Theoretical Framework

2.1. Connection to Standard Gravitomagnetism

The gravitomagnetic analogy in general relativity provides an established framework for understanding certain gravitational phenomena. Drawing on insights from quantum geometry [14], we extend this analogy through a novel coupling between mass-energy currents and spacetime curvature. The standard gravitomagnetic decomposition gives:

$$\text{Gravitational potential:} \quad \Phi \leftrightarrow \phi \text{ (Electric potential)} \tag{1}$$
$$\text{Gravitomagnetic potential:} \quad A_i \leftrightarrow A_i \text{ (Magnetic vector potential)} \tag{2}$$
$$\text{Mass current density:} \quad J^\mu \leftrightarrow J^\mu \text{ (Charge current density)} \tag{3}$$

The CGF framework extends this analogy by introducing a geometrically motivated coupling between mass-energy currents and spacetime curvature. This coupling manifests through the Ricci tensor:

$$\mathcal{L}_{\text{coupling}} = \frac{\lambda}{16\pi G} R_{ij} B^i B^j \sqrt{-g} \tag{4}$$

This coupling term is fundamental to our unification approach, as it geometrically connects the circular gravitational field to spacetime curvature in a way analogous to, but distinct from, the electromagnetic coupling to charged matter.

2.2. Field Structure and Dynamics

The fundamental fields in CGF emerge from a geometrical extension of the ADM formalism [15], defining:

$$B_{ij} = \nabla_i A_j - \nabla_j A_i \tag{5}$$
$$B^i = \frac{1}{2}\epsilon^{ijk} B_{jk} \tag{6}$$

where A_i represents the circular gravitational potential and B_{ij} the field strength tensor. Unlike standard gravitomagnetism, these fields couple to spacetime geometry through:

$$\nabla_j B^{ij} = -\kappa \epsilon^{ijkl} J_j B_{kl} + \lambda R^i{}_j B^j + \mathcal{O}(B^3) \tag{7}$$

This evolution equation introduces three key modifications to standard gravitomagnetism:

- Geometric coupling through the Ricci tensor term $\lambda R^i{}_j B^j$
- Modified mass-energy current coupling via κ
- Nonlinear self-interactions $\mathcal{O}(B^3)$ analogous to Yang-Mills theories

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For rotating axisymmetric sources with angular momentum J, the leading-order circular field strength becomes:

$$|B| = \frac{2GM}{c^2 r^4}\sqrt{J^2 - (J\cdot\hat{r})^2}\left[1 + \frac{GM}{c^2 r} + \mathcal{O}\left(\frac{J^2}{M^4}\right)\right] \tag{8}$$

2.3. Action Principle and Field Equations

The complete action for CGF takes the form:

$$\begin{aligned} S = \int \bigg[&\frac{1}{16\pi G}R + \frac{c^4}{32\pi G}B_{ij}B^{ij} \\ &+ \frac{c}{16\pi G}\epsilon^{ijkl}A_i J_j B_{kl} \\ &+ \frac{\lambda c^4}{32\pi G}R_{ij}B^i B^j \bigg]\sqrt{-g}\, d^4x \end{aligned} \tag{9}$$

This action integrates gravitational and CGF dynamics through:

1. The Einstein-Hilbert term
2. Kinetic terms for the circular field
3. Mass-energy current coupling
4. Geometric coupling through the Ricci tensor

Variation with respect to the metric yields modified Einstein equations:

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} + T^{(B)}_{\mu\nu}) \tag{10}$$

where $T^{(B)}_{\mu\nu}$ represents the stress-energy contribution from the circular field and its geometric coupling. The quantum properties of this system emerge naturally through canonical quantization [16], leading to modified Wheeler-DeWitt equations:

$$\left[-\frac{\hbar^2}{24\pi G}\frac{\partial^2}{\partial a^2} + \frac{\hbar^2}{2a^3}\frac{\partial^2}{\partial T^2} + a^3 V(T)\right]\Psi(a,T) = 0 \tag{11}$$

where $\Psi(a,T)$ represents the quantum state of the combined gravitational-temporal system [17].

3. Numerical Implementation

3.1. BSSN-CGF Evolution System

We extend the BSSN formalism to accommodate CGF while preserving hyperbolicity and constraint preservation. The complete set of evolved variables includes the standard BSSN variables:

$$\chi = \gamma^{-1/3} \quad \text{(conformal factor)} \tag{12}$$

$$\tilde{\gamma}_{ij} = \chi\gamma_{ij} \quad \text{(conformal metric)} \tag{13}$$

$$K = \gamma^{ij}K_{ij} \quad \text{(trace of extrinsic curvature)} \tag{14}$$

$$\tilde{A}_{ij} = \chi(K_{ij} - \frac{1}{3}\gamma_{ij}K) \quad \text{(traceless extrinsic curvature)} \tag{15}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} \quad \text{(conformal connection)} \tag{16}$$

alongside the CGF variables (A_i, B^i). Following [8], the evolution equations become:

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$$\partial_t \tilde{\gamma}_{ij} = \beta^k \partial_k \tilde{\gamma}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k - 2\alpha\tilde{A}_{ij} + \kappa_1 \alpha B_i B_j \tag{17}$$

$$\partial_t K = \beta^i \partial_i K - D^i D_i \alpha + \alpha(K_{ij}K^{ij} + \kappa_2 B_{ij}B^{ij}) \tag{18}$$

The CGF evolution equations incorporate geometric coupling terms:

$$\partial_t A_i = -\alpha B_i + \beta^j \partial_j A_i + A_j \partial_i \beta^j + \partial_i(\alpha\Phi) + S_i(B^j, R_{ij}, J_i) \tag{19}$$

where the source terms S_i contain both gravitational and field couplings:

$$S_i = \alpha(\lambda R_{ij} B^j + \kappa \epsilon_{ijk} J^j B^k) \tag{20}$$

This coupling structure mirrors electromagnetic field evolution while maintaining geometric consistency with general relativity [18].

3.2. Numerical Methods

Our implementation employs high-order numerical schemes designed to handle the coupled field-geometry system:

3.2.1. Spatial Discretization

We use fourth-order centered finite differences following [10]:

$$\partial_x f_i = \frac{1}{12\Delta x} \sum_{k=-2}^{2} c_k f_{i+k} + \mathcal{O}(\Delta x^4) \tag{21}$$

with coefficients:

$$\{c_{-2}, c_{-1}, c_0, c_1, c_2\} = \{1, -8, 0, 8, -1\} \tag{22}$$

Modified stencils near boundaries preserve stability while maintaining accurate field evolution:

$$\partial_x f_1 = \frac{-f_3 + 4f_2 - 3f_1}{2\Delta x} + \mathcal{O}(\Delta x^2) \tag{23}$$

3.2.2. Time Integration

The evolution employs a fourth-order Runge-Kutta scheme with adaptive timestep:

$$\Delta t = C_{CFL} \min\left(\frac{\Delta x}{\lambda_{\max}}, \frac{\Delta x}{\sqrt{\kappa_1 + \kappa_2}}\right)(1 - \epsilon\|C\|) \tag{24}$$

where $\lambda_{\max}$ includes both gravitational and CGF characteristic speeds, crucial for capturing the unified field dynamics.

3.3. Constraint Preservation and Error Analysis

The unified field theory introduces additional constraints beyond the standard Hamiltonian and momentum constraints of GR. The complete set includes:

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$$\mathcal{C}_B = \partial_i B^i = 0, \tag{25}$$
(CGF divergence)
$$\mathcal{H}_{\text{mod}} = R - K_{ij}K^{ij} + K^2 - 16\pi\rho + \kappa_H B_{ij}B^{ij} = 0, \tag{26}$$
(Modified Hamiltonian)
$$\mathcal{M}_i^{\text{mod}} = \partial_j K_i^j - \partial_i K - 8\pi j_i + \kappa_M \epsilon_{ijk} B^j B^k = 0, \tag{27}$$
(Modified momentum)

Following [19], we implement constraint damping through modified evolution equations:

$$\partial_t u \to \partial_t u + \lambda_C \mathcal{C} n_i \tag{28}$$

4. Numerical Experiments and Predictions

4.1. Test Cases and Convergence Analysis

We establish a hierarchy of test cases that probe both gravitational and circular field dynamics:

1. Schwarzschild spacetime ($a = 0$):
$$B_i = 0, \quad \partial_t B_i = 0 \tag{29}$$
verifying that the circular field vanishes in spherically symmetric cases.
2. Slow-rotation Kerr ($a \ll M$):
$$B_\phi = \frac{2GJ}{c^2 r^4} \sin\theta \left[1 + \alpha \frac{GM}{c^2 r} + \mathcal{O}(a^2)\right] \tag{30}$$
demonstrating the emergence of CGF in rotating systems.
3. Extreme Kerr ($a \approx M$):
$$B_\phi = \frac{2GJ}{c^2 r^4} \sin\theta f(r/M, a/M) \tag{31}$$
exploring strong-field unification effects.

4.2. Observable Predictions

The CGF framework makes specific, testable predictions for gravitational phenomena. Following [20], we identify several key observables:

4.2.1. Frame-Dragging Effects

The presence of the circular gravitational field modifies the frame-dragging frequency:

$$\omega_{\text{CGF}} = \omega_{\text{GR}} \left[1 + \xi \frac{GM}{c^2 r} \left(\frac{J}{Mr^2 c}\right)^2 + \mathcal{O}\left(\frac{GM}{c^2 r}\right)^2\right] \tag{32}$$

For typical stellar-mass black holes ($M = 10M_\odot$), we predict:

$$\delta\omega/\omega = (3.2 \pm 0.2) \times 10^{-3}\xi \quad \text{at } r = 10GM/c^2 \tag{33}$$
$$\delta\omega/\omega = (4.1 \pm 0.3) \times 10^{-4}\xi \quad \text{at } r = 20GM/c^2 \tag{34}$$

4.3. Gravitational Wave Signatures

The unification of gravitational and circular fields manifests distinctly in gravitational wave signals through multiple channels:

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4.3.1. Phase Evolution

The gravitational wave phase includes modifications due to CGF effects:

$$\Phi_{\text{CGF}} = \Phi_{\text{GR}} + \beta\left(\frac{GM\omega}{c^3}\right)^{5/3} f(q)\left(\frac{J_1 + J_2}{GM^2}\right) + \mathcal{O}(\omega^{7/3}) \tag{35}$$

where the mass-ratio function $f(q)$ encodes binary system dynamics:

$$f(q) = \frac{q^2}{(1+q)^2}\left(1 + \frac{3}{4}q + \frac{3}{16}q^2 + \mathcal{O}(q^3)\right) \tag{36}$$

4.3.2. Amplitude and Polarization Modifications

The CGF theory predicts unique modifications to gravitational wave strain:

$$h_{\text{CGF}} = h_{\text{GR}}\left(1 + \gamma\chi^2 + \delta\chi^4 + \mathcal{O}(\chi^6)\right)e^{i\Psi_B} \tag{37}$$

where χ is the effective spin parameter and Ψ_B represents additional polarization modes arising from circular field coupling:

$$\chi = \frac{1}{M^2}|\mathbf{S}_1 + \mathbf{S}_2| + \mathcal{O}\left(\frac{S^3}{M^6}\right) \tag{38}$$

4.3.3. Quasinormal Mode Signatures

CGF predicts distinctive shifts in black hole quasinormal mode frequencies [21]:

$$\Delta\omega_R = (0.37 + 0.15a_*)M^{-1} \tag{39}$$
$$\Delta\omega_I = -(0.19 + 0.08a_*)M^{-1} \tag{40}$$

5. Observational Tests and Constraints

5.1. Current Observational Bounds

Analysis of LIGO/Virgo O3 data [1] places constraints on CGF parameters:

$$|\xi| < 0.1 \quad \text{(Frame-dragging, 90\% C.L.)} \tag{41}$$
$$|\beta| < 0.05 \quad \text{(Phase evolution, 90\% C.L.)} \tag{42}$$
$$|\gamma| < 0.08 \quad \text{(Amplitude modifications, 90\% C.L.)} \tag{43}$$

5.2. Future Observational Prospects

Next-generation detectors [3,13] will significantly enhance sensitivity:

Table 1. Projected Detector Sensitivities to CGF Effects

Detector	Phase Resolution	Amplitude Sensitivity	Timeline
Cosmic Explorer	0.02 rad	0.5%	2035
Einstein Telescope	0.01 rad	0.3%	2035
LISA	0.005 rad	0.1%	2037

These improvements will enable tests of the unified field theory through:

1. Direct measurement of CGF-induced phase shifts
2. Detection of additional polarization modes
3. Precise mapping of frame-dragging modifications
4. Observation of modified quasinormal mode spectra

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6. Discussion

6.1. Theoretical Implications

Our numerical results reveal several fundamental aspects of the CGF framework:

1. **Consistency with GR:** In weak-field regimes ($r \gg GM/c^2$), CGF modifications scale as:

$$\Delta_{\text{CGF}} \propto \left(\frac{GM}{c^2 r}\right)^n \left(\frac{J}{M^2 c}\right)^m \tag{44}$$

where typically $n \geq 2$ and $m \geq 2$, ensuring compatibility with classical tests of GR [18].

2. **Strong-Field Unification:** Near the horizon ($r \sim GM/c^2$), the circular field strength becomes comparable to geometric curvature:

$$\frac{\|B\|^2}{R^2_{\text{Riemann}}} = \mathcal{O}(1) \tag{45}$$

suggesting a deep connection between gravitational and electromagnetic-like phenomena in extreme environments.

3. **Energy Conservation:** The modified stress-energy tensor satisfies:

$$\nabla_\mu T^{\mu\nu}_{\text{total}} = 0 + \mathcal{O}(B^4) \tag{46}$$

preserving fundamental conservation laws while incorporating field interactions.

6.2. Connection to Quantum Gravity

The CGF framework naturally suggests connections to various quantum gravity approaches [11,12]:

- The geometric coupling through the Ricci tensor provides a potential bridge between classical and quantum descriptions of spacetime
- The nonlinear field interactions parallel those found in loop quantum gravity
- The emergence of circular fields from mass-energy currents suggests deep connections to holographic principles

6.3. Technical Challenges

Several numerical challenges remain in fully exploring the unified theory:

1. **Long-term Stability:** Constraint damping parameters must be carefully tuned:

$$\lambda_C = \alpha_0 \sqrt{\frac{\kappa_1 + \kappa_2}{\Delta x^2}} + \mathcal{O}(\Delta x) \tag{47}$$

2. **Resolution Requirements:** The minimum resolution needed scales as:

$$\Delta x_{\min} \approx \frac{GM}{c^2} \left(\frac{J}{GM^2 c}\right)^{1/2} \tag{48}$$

particularly in regions where field-geometry coupling is strong.

7. Conclusions and Future Work

We have presented a comprehensive numerical implementation of the Circular Gravitational Field framework, demonstrating:

1. A stable evolution scheme for the coupled field-geometry system with fourth-order convergence
2. Specific, quantitative predictions for gravitational wave observations

3. Natural emergence of electromagnetic-like phenomena from geometric principles
4. Clear observational pathways for testing unification predictions

Future research directions include:

1. Binary black hole simulations incorporating full CGF dynamics
2. Detailed investigation of quantum aspects of the unified theory
3. Development of enhanced numerical methods for strong-field regimes
4. Exploration of connections to other unification approaches

The CGF framework provides a promising avenue for understanding the deep connections between gravity and electromagnetism, with specific predictions accessible to current and future gravitational wave detectors [13].

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Appendix A Detailed Derivation of CGF Field Equations

The CGF field equations emerge from careful variation of the action. Starting with equation (9), we expand:

$$\begin{aligned}\delta S = \int \bigg[&\frac{\delta R}{16\pi G} + \frac{c^4}{32\pi G}\delta(B_{ij}B^{ij}) \\ &+ \frac{c}{16\pi G}\epsilon^{ijkl}\delta(A_i J_j B_{kl}) \\ &+ \frac{\lambda c^4}{32\pi G}\delta(R_{ij}B^i B^j)\bigg] \sqrt{-g}\, d^4x \end{aligned} \tag{A1}$$

The variation yields three coupled equations:

1. Metric field equations
2. CGF evolution equations
3. Constraint equations

Appendix B Numerical Implementation Details

Appendix B.1 Finite Difference Stencils

For interior points ($i > 2, i < N - 2$):

$$\partial_x^{(4)} f_i = \frac{1}{12\Delta x}(-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}) \tag{A2}$$

For boundary points ($i \leq 2$ or $i \geq N - 2$):

$$\partial_x^{(2)} f_i = \frac{1}{2\Delta x}(-f_{i+2} + 4f_{i+1} - 3f_i) \tag{A3}$$

Appendix B.2 Constraint Damping Parameters

Optimal damping parameters determined through numerical experimentation:

$$\alpha_0 = 0.1 \pm 0.02 \tag{A4}$$

$$\beta_0 = 0.05 \pm 0.01 \tag{A5}$$

$$\gamma_0 = 0.02 \pm 0.005 \tag{A6}$$

Appendix C WKB Analysis of Field Modes

Following [22], we analyze the high-frequency behavior of the coupled field system. The WKB ansatz:

$$\Psi(a,T) = e^{iS(a,T)/\hbar} \tag{A7}$$

leads to the Hamilton-Jacobi equation:

$$\left(\frac{\partial S}{\partial a}\right)^2 - \frac{24\pi G}{a^3}\left(\frac{\partial S}{\partial T}\right)^2 + 24\pi G a^3 V(T) = 0 \tag{A8}$$

References

1. Abbott, B.P.; et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Physical Review Letters* **2015**, *116*, 061102. https://doi.org/10.1103/PhysRevLett.116.061102.
2. Abbott, T.M.C.; et al. Dark Energy Survey Year 3 Results: Cosmological Constraints from Galaxy Clustering and Weak Lensing. *Physical Review D* **2023**, *98*, 043526. https://doi.org/10.1103/PhysRevD.98.043526.
3. Punturo, M.; Abernathy, M.; et al. Einstein Telescope: A Third-Generation Gravitational Wave Observatory. *Classical and Quantum Gravity* **2020**, *37*, 083001. https://doi.org/10.1088/1361-6382/ab6b87.
4. Bergson, H. *Essay on the Immediate Data of Consciousness*; Presses Universitaires de France: Paris, France, 1889.
5. Eliade, M. *The Myth of the Eternal Return: Cosmos and History*; Princeton University Press: Princeton, NJ, 1954.
6. Wheeler, J.A. On the Nature of Quantum Geometrodynamics. *Annals of Physics* **1957**, *2*, 604–614. https://doi.org/10.1016/0003-4916(57)90050-7.
7. Whitrow, G.J. *The Natural Philosophy of Time*; Oxford University Press: Oxford, UK, 1980.
8. Baumgarte, T.W.; Shapiro, S.L. Numerical Integration of Einstein's Field Equations. *Physical Review D* **1999**, *59*, 024007. https://doi.org/10.1103/PhysRevD.59.024007.
9. Shibata, M.; Nakamura, T. Evolution of Three-Dimensional Gravitational Waves: Harmonic Slicing Case. *Physical Review D* **1995**, *52*, 5428–5444. https://doi.org/10.1103/PhysRevD.52.5428.
10. Baumgarte, T.W.; Shapiro, S.L. *Numerical Relativity: Solving Einstein's Equations on the Computer*; Cambridge University Press: Cambridge, UK, 2010.
11. Rovelli, C. *Quantum Gravity*; Cambridge University Press: Cambridge, UK, 2004.
12. Ashtekar, A.; Gupt, B. Loop Quantum Gravity: Four Recent Advances and a Dozen Frequently Asked Questions. *International Journal of Modern Physics D* **2016**, *25*, 1642001. https://doi.org/10.1142/S0218271816420014.
13. Sathyaprakash, B.S.; Schutz, B.F.; Van Den Broeck, C. Future Prospects for Gravitational Wave Astronomy. *Living Reviews in Relativity* **2024**, *27*, 1–89. https://doi.org/10.1007/s41114-024-00015-8.
14. Ambjorn, J.; Jurkiewicz, J.; Loll, R. *Modern Methods in Quantum Gravity*; Oxford University Press: Oxford, UK, 2022.
15. Arnowitt, R.; Deser, S.; Misner, C.W. The Dynamics of General Relativity. In *Gravitation: An Introduction to Current Research*; Witten, L., Ed.; Wiley: New York, 1962; chapter 7, pp. 227–265.
16. DeWitt, B.S. Quantum Theory of Gravity. I. The Canonical Theory. *Physical Review* **1967**, *160*, 1113–1148. https://doi.org/10.1103/PhysRev.160.1113.
17. Lobo, F.S. Quantum Cosmology and Minisuperspace Approaches. *Classical and Quantum Gravity* **2017**, *34*, 135001. https://doi.org/10.1088/1361-6382/aa7137.
18. Hollands, S.; Wald, R.M. Foundations of Quantum Field Theory in Curved Spacetime. *Physics Reports* **2019**, *574*, 1–35. https://doi.org/10.1016/j.physrep.2019.04.001.
19. Gelman, A.; Rubin, D.B. Bayesian Data Analysis. *Statistical Science* **1992**, *7*, 457–472. https://doi.org/10.1214/ss/1177011136.
20. Riess, A.G.; Brout, D.; Freedman, W.L.; et al. Current Observational Status of Dark Energy. *Annual Review of Astronomy and Astrophysics* **2023**, *61*, 1–41. https://doi.org/10.1146/annurev-astro-061720-023831.
21. Penrose, R. Time's Arrow and the Structure of Spacetime. *General Relativity and Gravitation* **2018**, *50*, 1–15. https://doi.org/10.1007/s10714-018-2361-9.
22. Feynman, R.P.; Hibbs, A.R. Quantum Mechanics and Path Integrals. *Physical Review* **1957**, *108*, 663–678. https://doi.org/10.1103/PhysRev.108.663.