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Article

The Deterministic Statistical Feedback Law in Finite Dimensions: Residuals, Channels, and Nonlocal Correlations

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Abstract

We develop the Deterministic Statistical Feedback Law (DSFL) as a concrete finite-dimensional framework for quantum information. DSFL works in a single calibrated Hilbert “room”, compares statistical blueprints to physical responses through one residual of sameness, and declares as lawful exactly those updates that contract this residual in a fixed instrument norm. We prove that admissibility of a channel is equivalent to a spectral data-processing inequality and that iterated admissible dynamics admit a Lyapunov-style envelope, with a canonical DSFL clock making the decay of the residual a straight line in semilog coordinates. On bipartite systems we reinterpret standard entanglement proxies, such as negativity, sandwiched Rényi divergences, and trace distance to product form, as correlation residuals that cannot increase under local completely positive maps. Embedding CHSH and GHZ or Mermin tests into the same room, we recover the usual quantum violations while all local hidden-variable models remain confined to their classical bounds.

Keywords: DSFL; admissibility; data-processing inequality; Lyapunov envelope; quantum channels; entanglement; bell inequalities; Hilbert-space geometry

1. Introduction

Any serious attempt to understand entanglement has to separate the *correlations we actually observe* from the *formal machinery* we use to compute them. It is not obvious that the wave function should be treated as a physical object rather than as a compact encoding of conditional statistics. Likewise, the Born rule can be read either as a primitive axiom—“outcome frequencies are given by $|\psi|^2$ ”—or as a hint of a deeper structure: perhaps $|\psi|^2$ is the unique way a single underlying degree of freedom can distribute a finite “sameness budget” across outcomes, so as to reduce a calibrated mismatch between a statistical blueprint and a physical response. The aim of the present paper is to push this second reading as far as we can in finite dimensions, and to ask whether a single residual law can simultaneously govern channels, entanglement, and Bell nonlocality.

The standard formalism of quantum mechanics assigns to every system a state, usually represented by a density operator or wave function, and to every observable a Hermitian operator acting on a Hilbert space. From these data one computes expectation values, correlation functions, and violations of Bell inequalities. In practice this machinery is extraordinarily successful: entangled states are routinely prepared and used as resources for quantum teleportation of unknown states, device-independent cryptography, and quantum communication in ways that cannot be reproduced by shared classical randomness alone [1–4]. From a logical point of view, this success still leaves open a question that goes back to Einstein, Podolsky, and Rosen [5]: is the quantum formalism merely empirically correct, or is it also complete?

Bell’s theorem and the experiments that followed show that entangled correlations are incompatible with any *local hidden-variable* description [4,6–9]. In that narrow sense, the search for a classical completion has failed: no assignment of pre-existing local values reproduces all quantum predictions.

But the completeness question is older and more basic. It asks whether there is a *single, transparent law* that explains why these correlations have the structure they do, and how they should evolve under general dynamics, rather than just a collection of separate calculation rules and monotonicity theorems.

In its sharpest form, the Einstein–Podolsky–Rosen question can be rephrased as follows. Given a theory we accept as empirically correct, does every element of physical reality have a clear counterpart in its mathematical description, and does that description come with a well-defined arrow of time? For entanglement, this would mean more than having many inequivalent measures and criteria. It would mean that there is a *single quantity*, or a tightly controlled family of quantities, whose behaviour under admissible dynamics captures what is meant by “nonclassical correlation”. At present the situation is more fragmented. In quantum information, entanglement is organised into resource theories and manipulated by LOCC maps [3]. In open quantum systems, the natural language is that of completely positive maps and Lindblad or GKSL equations [10–12], with contractive norms as a key property. Bell tests are phrased in terms of probability polytopes and correlation functions [4], largely decoupled from the dynamical structure of channels and instruments. Negativity, sandwiched Rényi divergences, and trace distance to product states reappear in the context of hypothesis testing, thermodynamics, and information theory [13–15]. Each viewpoint is powerful, but they do not obviously form a single, law-like whole.

The Deterministic Statistical Feedback Law (DSFL) is an attempt to provide such a law. The basic idea is to work in *one Hilbert room* and to compare a *statistical blueprint* with a *physical response* by means of a single *residual of sameness*. Informally, one thinks of the blueprint as “what the device thinks it is doing” (settings, control parameters, intended state) and the response as “what actually happens in the apparatus or in the world”. DSFL then tracks, in one fixed norm, how far the response is from the calibrated image of the blueprint, and declares as lawful exactly those evolutions that do not increase this discrepancy.

Concretely, we fix:

- a statistical space of blueprints $\mathcal{H}_s$ (statistical degrees of freedom, sDoF), carrying the variables that describe preparations, control knobs, and intended states;
- a physical space of responses $\mathcal{H}_p$ (physical degrees of freedom, pDoF), carrying the variables that describe actual readouts, fields, or detector states;
- a calibration map $\mathcal{I} : \mathcal{H}_s \rightarrow \mathcal{H}_p$, which tells us, for each blueprint $s \in \mathcal{H}_s$, what the *ideal* response $\mathcal{I}s$ in $\mathcal{H}_p$ would be if the device were perfectly implemented; and
- an instrument weight $W \succ 0$ on $\mathcal{H}_p$, which encodes how the instrument “measures distance” between two responses via the norm $\|x\|_W^2 = \langle Wx, x \rangle$.

Given a blueprint–response pair (s, p) , the DSFL residual of sameness is the single scalar

$$\mathcal{R}(s, p) := \|p - \mathcal{I}s\|_W^2,$$

i.e. the squared distance, in the instrument norm, between the actual response p and the calibrated ideal response $\mathcal{I}s$. In DSFL this quantity plays the central rôle: it is the only observable we track for lawfulness (via a data–processing inequality), for dynamics (via Lyapunov decay), and for locality (via cone bounds), and all sectors of the theory are required to respect the same residual $\mathcal{R}$.

Admissible evolutions are precisely those that *contract* this residual in the instrument norm. In finite dimension this is equivalent to a spectral data–processing inequality of the form $\Phi^* W \Phi \preceq W$, and $\mathcal{R}$ plays the rôle of a Lyapunov function with a printed decay rate. In this sense DSFL proposes a very simple arrow of time: in an admissible world the calibrated mismatch between blueprint and response is nonincreasing and, where a local margin exists, decays exponentially. At the level of broad ideas, DSFL has been suggested as a common geometric framework in which quantum mechanics, thermodynamics, and even gravitational sectors can be expressed in terms of *one residual*, *one admissibility (DPI) gate*, *one Lyapunov envelope*, and *one causal cone*.

The aim of the present paper is to confront this proposal with some of the sharpest tests from *finite-dimensional quantum information theory*: entanglement monotones and Bell inequalities for small systems (a few qubits).[2–4] We deliberately restrict attention to finite-dimensional Hilbert spaces and familiar scenarios such as CHSH[4,7] and GHZ/Mermin experiments.[16,17] Within this setting we ask three concrete questions:

- (Q1) **Channels.** Can DSFL be realised as a genuine Lyapunov law for finite-dimensional quantum channels, with admissibility exactly equivalent to a *spectral* data-processing condition in a fixed instrument norm?
- (Q2) **Entanglement.** Can standard entanglement proxies— negativity,[13] sandwiched Rényi divergences [14,15], and trace distance to the product of marginals—be interpreted as *DSFL-style correlation residuals* that are nonincreasing under local completely positive maps?
- (Q3) **Bell nonlocality.** Can Bell tests be embedded in a DSFL room so that the usual quantum bounds for CHSH and GHZ/Mermin inequalities [7,17,18] are recovered, while all classical local hidden-variable strategies remain confined to their standard limits [4,6]?

Our main conclusion is that, in this finite-dimensional setting, the answer to all three questions is *yes*. Once a qubit sector is calibrated in a DSFL room, the residual $\mathcal{R}$ can be treated as the relevant “element of reality” for correlations in that sector: its contraction expresses admissibility, its level sets distinguish separable from entangled states (via correlation residuals), and its geometry supports Bell violations up to the Tsirelson bounds [18]. In particular, DSFL reproduces the usual quantum CHSH and GHZ/Mermin values for suitably chosen channels and measurements, while a brute-force enumeration of deterministic local hidden-variable assignments confirms that no classical model within the same framework exceeds the classical bounds [4].

On the Einstein–Podolsky–Rosen question, our answer is therefore twofold. The bare wave-function description, by itself, is not complete in the DSFL sense: it does not single out a residual or a law of admissible contraction. But once the wave function is supplemented by a calibration map, an instrument norm, and the DSFL admissibility condition, we obtain a simple, explicit residual law that is compatible with known entanglement phenomena and Bell nonlocality [3,4], without reverting to local hidden variables. In this qualified sense, DSFL offers a candidate for a more complete description of the physical reality underlying entangled quantum systems. DSFL is not a local hidden-variable theory and does not alter any finite-dimensional quantum predictions; it reorganises the standard formalism into a single residual/DPI/Lyapunov/cone structure.

1.1. Main Idea and Contributions

Informally, the main idea is simple: we take the DSFL residual

$$\mathcal{R}(s, p) = \|p - \mathcal{I}s\|_W^2$$

seriously as *the* dynamical object and show that, in small finite dimension, it can simultaneously (a) act as a Lyapunov function for quantum channels; (b) behave like an entanglement/correlation monotone under local CP maps; and (c) host Bell tests that separate quantum predictions from all classical local hidden-variable (LHV) strategies [4].

Formally, our contributions are:

- C1 DSFL as a Lyapunov/DPI law for channels.** In a finite-dimensional quantum sector $\mathcal{H} \cong \mathbb{C}^d$ we fix an SPD instrument weight W on $\mathcal{B}(\mathcal{H})$ and define admissibility of a channel Φ via the induced norm

$$\|\Phi\|_{W \rightarrow W}^2 = \lambda_{\max}(W^{-1/2}\Phi^*W\Phi W^{-1/2}).$$

We prove that

$$\Phi \text{ admissible} \iff \mathcal{R}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}(s, p) \quad \forall (s, p),$$

i.e. admissibility is exactly equivalent to a *spectral* data-processing inequality for the single residual, and we show that in DSFL time the semilog envelope of $\mathcal{R}$ has unit slope.

- C2 Entanglement proxies as DSFL correlation residuals.** On bipartite systems $\mathcal{H}_A \otimes \mathcal{H}_B$ we interpret: (i) negativity,[13] (ii) sandwiched Rényi divergences $D_\alpha^*(\rho \| \rho_A \otimes \rho_B)$, [14,15] and (iii) trace distance $\|\rho - \rho_A \otimes \rho_B\|_1$, as DSFL-style correlation residuals. We recall their monotonicity under local CPTP maps and support this with numerical audits using random states and random local channels.[3,4] In DSFL language this becomes a “no hidden gain in correlations” law for local admissible dynamics.
- C3 Bell tests inside a DSFL room.** We embed CHSH and GHZ/Mermin inequalities [7,17,18] into DSFL-calibrated qubit sectors. On the classical side we explicitly enumerate all deterministic LHV assignments in the CHSH scenario and confirm the classical bound $|S| \leq 2$ [4]. On the DSFL side we calibrate a two-qubit sector to a Bell state and Pauli measurements and recover the quantum value $S_{\text{DSFL}} = 2\sqrt{2}$. We further show that a three-qubit GHZ sector yields a Mermin value $|\langle M \rangle_{\text{DSFL}}| = 4 > 2$, again in agreement with standard quantum predictions and strictly beyond all LHV models [17,18]. In this sense the DSFL residual law is strictly *beyond* classical local hidden-variable theories in the calibrated finite-dimensional sector.

1.2. Outline

Section 3 introduces the core DSFL framework in a finite-dimensional Hilbert room: ambient geometry, calibration, instrument norm, the single residual of sameness, and admissible updates, including the equivalence between admissibility and a one-line DPI, together with a brief two-loop (continuous-time) Lyapunov law.

Section 4 specialises DSFL to finite-dimensional quantum channels, proving the Lyapunov/DPI characterisation for CPTP maps and introducing the discrete DSFL clock on which the residual decays with unit semilog slope. Section 5 treats negativity, sandwiched Rényi divergences, trace distance to product form, and a DSFL-native quadratic residual as correlation measures; it recalls or proves their monotonicity under local CP maps and interprets them as DSFL correlation residuals.

Section 6 embeds CHSH and GHZ/Mermin experiments in DSFL-calibrated two- and three-qubit sectors and compares the DSFL predictions with classical LHV bounds obtained by brute-force enumeration, showing that the DSFL residual law reproduces Tsirelson violations while remaining strictly beyond local hidden-variable models. Section 7 explains how Born weights and Lüders projection arise geometrically from the single residual and compares DSFL with the textbook postulates of finite-dimensional quantum mechanics, drawing on Gleason-type arguments and standard measurement theory.

Section 8 presents numerical audits of the DSFL law in a small finite-dimensional testbed, probing DPI/spectral admissibility, Lyapunov envelopes, cone locality, local CP monotonicity of correlation residuals, and the CHSH bounds. Section 9 then collects two global structural results: a DSFL Lyapunov theorem for finite-dimensional channels and an equivalence theorem showing how the standard finite-dimensional quantum formalism is recovered from the DSFL postulates. Finally, Section 10 summarises the finite-dimensional picture and sketches extensions to field-theoretic and gravitational settings.

2. Positioning and Relation to Prior Work

This section situates the DSFL framework within existing work on Hilbert-space geometry, data-processing inequalities, entanglement theory, and Bell nonlocality. Our goal is not to claim new quantum predictions, but to show how a single residual in one calibrated Hilbert room reorganises a broad toolkit—from projections and DPI to entanglement monotones and CHSH bounds—into a unified geometric pipeline.

2.1. Synthesis, Not Reinvention

Our contribution is a synthesis of classical ingredients—principal (Friedrichs) angles and subspace geometry [19,20], orthogonal projections and the Lüders post-measurement map [21],

contractivity/data-processing principles [11,22–25], and $\sin \theta$ -type perturbation/stability theory for projectors [26]—organised into a single-budget, single-geometry pipeline for preparation, update, and locality. The novelty lies in the framing: (i) a single observable in one comparison geometry, the residual of sameness $R = \|p - \mathcal{I}s\|_W^2$; (ii) uncertainty recast (in the broader DSFL programme) as a *principal-angle remainder* on a conserved norm, with commutators fixing only the numeric floor; (iii) collapse as the *unique* budget-preserving nearest-point projection in that norm (sharp case); (iv) *dimension-free* stability envelopes driven by a single DPI; and (v) a deterministic, rate-bearing law (DSFL) that makes locality and an arrow of time *auditable* rather than axiomatic.

2.2. Geometry of Information: Angles, Projections, and Alternating Schemes

The use of principal angles to capture frame incompatibility and to parameterise single- and two-frame remainders goes back to Jordan and Friedrichs [19,20]; robustness under tilt via the Davis–Kahan $\sin \theta$ -theorems is standard [26]. Classical results on alternating projections (von Neumann, Halperin, Deutsch) and modern convergence refinements provide a geometric mechanism for nearest-point enforcement and cyclic constraint tightening [27–31]. In DSFL these tools are applied to a single calibrated residual, so “update” (Lüders/projection) and “compatibility” (angle laws) live in the same inner product and can in principle be tested directly by projection ratios and $\Pi_B \Pi_A$ -loops.

2.3. Data Processing and Lyapunov Structure

Quantum and classical DPI/monotonicity (contractivity under admissible maps) are well-developed [11,22–25]. DSFL singles out a specific L^2 -type residual $\mathcal{R}_{\text{sameness}} = \|p - \mathcal{I}s\|_W^2$ for which *admissible* (intertwining, nonexpansive) evolutions obey a one-line DPI

$$\mathcal{R}_{\text{sameness}}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}_{\text{sameness}}(s, p).$$

In the finite-dimensional channel setting of this paper this is equivalent to a simple spectral condition on $W^{-1/2}\Phi^*W\Phi W^{-1/2}$ and turns static geometric bands (principal-angle bounds) into *deterministic tightening* statements. Iterating an admissible step yields a discrete Lyapunov envelope with an explicit contraction rate and a canonical DSFL clock in which $\log R$ has unit slope.

2.4. Uncertainty: Variance, Entropic, and Geometric Viewpoints

Variance-based uncertainty relations of Heisenberg–Robertson–Schrödinger [32–34] and entropic/Fourier refinements (Hirschman, Beckner–Babenko, Białynicki–Birula–Mycielski) [35–37] express incompatibility via overlap constants; modern work separates preparation uncertainty from measurement disturbance and joint-measurement noise [38–41]. Within the broader DSFL programme, these ideas are rephrased geometrically: *meaning* (what can be simultaneously sharp) is captured by principal-angle geometry in a fixed norm; *scale* (commutators, entropies) enters only to set numerical floors on DSFL distances. In the present finite-dimensional paper we do not develop full uncertainty theorems; we use the same geometric language mainly to organise channel contractivity, entanglement, and Bell tests.

2.5. Collapse, Instruments and POVMs

For sharp measurements, projective (Lüders) update is textbook [21]; POVMs and instruments, together with Naimark–Stinespring dilations, extend the story [38,39,42,43]. In our setting, the sharp update is *forced* to be the orthogonal projection that uniquely minimises the post-update residual in the same comparison geometry: collapse $p \mapsto \Pi_V p$ is the unique idempotent, nonexpansive map with range V . General POVMs can be treated as *effective frames* via dilations, so that the geometry-vs-scale split persists also for nonprojective measurements.

2.6. Locality and Finite Speed

Finite-speed locality for lattice and quasi-local evolutions originates with Lieb–Robinson [44] and has been significantly refined [45–47]. In algebraic quantum field theory, locality is encoded in net structures and modular properties [48–52]. DSFL formulates causality directly in the instrument norm: the relay kernel is required to have geometric cone support with speed c and an exponential margin, and the immediate loop is non-signalling on unconditional marginals. In this paper we illustrate exactly this structure on a discretised one-dimensional lattice relay, as a finite-dimensional proxy for the general cone picture.

2.7. Contextuality, EPR/Bell, and Tsirelson Bounds

Kochen–Specker and Bell theorems preclude noncontextual hidden variables and constrain local hidden-variable strategies [6,53]. Tsirelson bounds describe the quantum region of Bell/CHSH correlations [54]. DSFL is explicitly contextual: joint outcomes are orthogonal projections onto chosen frames of the same calibrated vector. In the finite-dimensional sector studied here we show that the DSFL residual law reproduces Tsirelson’s $2\sqrt{2}$ CHSH value once a qubit sector is calibrated, while purely local DSFL-admissible updates remain confined to the classical LHV bound $|S| \leq 2$. Thus DSFL is deterministic and norm-contractive but still strictly beyond local hidden-variable models in the calibrated two-qubit sector.

2.8. Operator–Algebraic and Spectral Backdrop

The comparison geometry we use is compatible with standard spectral calculus and representation theory (Stone–von Neumann, von Neumann algebras) [55–57]. Gleason’s theorem [58] underlies the uniqueness of Born weights when one demands frame-covariant probability assignments. In this sense the DSFL cause (one room, one ruler, one residual) dovetails with the usual units/probabilities provided by the operator–algebraic formalism.

2.9. Holography and Boundary Perspectives (Motivation Only)

The AdS/CFT correspondence views bulk gravity as encoded in a boundary quantum theory, with entanglement and modular flow playing central roles in the geometric dictionary [59–61]. From this angle, a DSFL “room” is naturally read as a boundary Hilbert space equipped with a single instrument norm and residual that constrain admissible dynamics and information flow. The present paper does not attempt a holographic construction; all results are strictly finite-dimensional. The holographic viewpoint is included here only as motivation: the one-room, one-residual, one-cone structure suggested by DSFL could in principle be explored first in toy scrambling models and later in genuine boundary QFTs,

2.10. Summary

DSFL rearranges well-known structures into a single calibrated mechanism: principal angles supply the *meaning* (one geometry, one budget), commutators and entropies set the *scale*, the Lüders map is the *mechanism* (unique nearest point), DPI provides *dynamics and robustness*, and cone-limited relay encodes *locality*. For finite-dimensional quantum information this yields a single residual that simultaneously governs channel contractivity, entanglement proxies, and Bell nonlocality, with all of these properties *geometrically transparent and operationally testable* in one room with one ruler.

3. DSFL Framework in a Single Hilbert Room

In this section we summarise the *room-level* DSFL structure that underlies all later constructions. The aim is to isolate a minimal, finite-dimensional framework in which quantum channels, entanglement measures and Bell tests are all governed by a single quadratic residual of sameness

$$\mathcal{R}_{\text{sameness}}(s, p) = \|p - \mathcal{I}s\|_W^2$$

in a fixed, calibrated Hilbert norm. The construction is entirely standard from the point of view of Hilbert–space geometry and nonexpansive maps (see, e.g., [62,63]), but we organise it around one observable and one admissibility condition.

3.1. Ambient Geometry, Calibration, and Instrument Norm

We begin by fixing the common “room” in which statistical blueprints (sDoF) and physical responses (pDoF) are compared.

Definition 1 (Ambient DSFL room). *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a finite–dimensional complex Hilbert space and let $\mathcal{S}, \mathcal{P} \subset \mathcal{H}$ be closed subspaces, interpreted as statistical and physical arenas. Fix:*

- *a bounded linear map $\mathcal{I} : \mathcal{S} \rightarrow \mathcal{P}$ (the calibration or interchangeability map), which sends blueprints $s \in \mathcal{S}$ to ideal responses $\mathcal{I}s \in \mathcal{P}$;*
- *a bounded, selfadjoint, strictly positive operator $W : \mathcal{P} \rightarrow \mathcal{P}$ (the instrument weight), inducing the W –inner product and norm*

$$\langle x, y \rangle_W := \langle Wx, y \rangle, \quad \|x\|_W^2 := \langle x, x \rangle_W, \quad x, y \in \mathcal{P}.$$

We refer to $(\mathcal{H}, \mathcal{S}, \mathcal{P}, \mathcal{I}, W)$ as a DSFL room.

For any bounded map $\Phi : \mathcal{P} \rightarrow \mathcal{P}$ we write

$$\|\Phi\|_{W \rightarrow W} := \sup_{\|x\|_W=1} \|\Phi x\|_W$$

for the induced operator norm in the instrument metric. In finite dimension every $W \succ 0$ defines an equivalent Hilbert norm, so the geometry is fully controlled by the choice of calibration $\mathcal{I}$ and weight W .

3.2. Residual of Sameness and Positivity

A first consistency check on the DSFL framework is that the single residual of sameness

$$\mathcal{R}_{\text{sameness}}(s, p) := \|p - \mathcal{I}s\|_W^2$$

really behaves like a squared distance between the physical configuration $p \in \mathcal{H}_p$ and its calibrated blueprint $\mathcal{I}s \in \mathcal{H}_p$. In particular, it should never be negative and should vanish only when the two coincide.

Theorem (Positivity and zero set of the residual of sameness)

Theorem 1 (Positivity and zero set of the residual of sameness). *Let $(\mathcal{H}_s, \langle \cdot, \cdot \rangle_s)$ and $(\mathcal{H}_p, \langle \cdot, \cdot \rangle_p)$ be Hilbert spaces, let $\mathcal{I} : \mathcal{H}_s \rightarrow \mathcal{H}_p$ be a bounded linear map (the calibration), and let $W : \mathcal{H}_p \rightarrow \mathcal{H}_p$ be a bounded, selfadjoint, strictly positive operator (the instrument weight). Define*

$$\|x\|_W := \sqrt{\langle Wx, x \rangle_p}, \quad \mathcal{R}_{\text{sameness}}(s, p) := \|p - \mathcal{I}s\|_W^2, \quad (s, p) \in \mathcal{H}_s \times \mathcal{H}_p.$$

Then:

1. $\mathcal{R}_{\text{sameness}}(s, p) \geq 0$ for all $(s, p) \in \mathcal{H}_s \times \mathcal{H}_p$;
2. $\mathcal{R}_{\text{sameness}}(s, p) = 0$ if and only if $p = \mathcal{I}s$.

Equivalently, $\mathcal{R}_{\text{sameness}}(s, p)$ is exactly the square of the distance from p to the affine subspace $\mathcal{I}(\mathcal{H}_s) \subset \mathcal{H}_p$ in the instrument norm $\|\cdot\|_W$.

Proof. Fix $s \in \mathcal{H}_s$ and $p \in \mathcal{H}_p$, and set $e := p - \mathcal{I}s \in \mathcal{H}_p$. By definition of the weighted norm,

$$\mathcal{R}_{\text{sameness}}(s, p) = \|e\|_W^2 = \langle We, e \rangle_p.$$

Since W is selfadjoint and strictly positive, one has $\langle Wx, x \rangle_p \geq 0$ for all x , with equality if and only if $x = 0$ (see, e.g., [62]). Thus $\mathcal{R}_{\text{sameness}}(s, p) \geq 0$ for all (s, p) , and $\mathcal{R}_{\text{sameness}}(s, p) = 0$ if and only if $e = 0$, i.e. $p = \mathcal{I}s$. Moreover, the map $p \mapsto \|p - \mathcal{I}s\|_W$ is exactly the distance from p to the affine subspace $\mathcal{I}(\mathcal{H}_s) \subset \mathcal{H}_p$ in the norm $\|\cdot\|_W$, so $\mathcal{R}_{\text{sameness}}(s, p)$ is the square of that distance. $\square$

Thus, at the room level, $\mathcal{R}_{\text{sameness}}$ is a bona fide squared distance to the calibrated image of $\mathcal{H}_s$ in the instrument norm.

3.3. Admissible Updates and DPI Equivalence

In DSFL, *lawful* (admissible) updates are precisely those that preserve the calibration map and do not expand distances in the instrument metric. This is the usual notion of a nonexpansive mapping in a weighted Hilbert norm [63], enriched by the intertwining condition. The key point is that this geometric condition is *equivalent* to a single data-processing inequality (DPI) for the residual $\mathcal{R}_{\text{sameness}}$.

Definition 2 (Admissible update). Let $(\mathcal{H}, \mathcal{S}, \mathcal{P}, \mathcal{I}, W)$ be a DSFL room. A pair of bounded linear maps

$$\tilde{\Phi} : \mathcal{S} \rightarrow \mathcal{S}, \quad \Phi : \mathcal{P} \rightarrow \mathcal{P}$$

is called DSFL-admissible (with respect to $\mathcal{I}$ and W) if

$$\Phi \mathcal{I} = \mathcal{I} \tilde{\Phi} \quad \text{and} \quad \|\Phi\|_{W \rightarrow W} \leq 1. \quad (1)$$

The first condition expresses calibration preservation (intertwining), and the second is nonexpansiveness in the instrument norm.

Theorem (Admissibility $\iff$ DPI for the single residual)

Theorem 2 (Admissibility $\iff$ DPI for the single residual). Let $(\mathcal{H}, \mathcal{S}, \mathcal{P}, \mathcal{I}, W)$ be a DSFL room and let $\mathcal{R}_{\text{sameness}}(s, p) = \|p - \mathcal{I}s\|_W^2$ be the residual of sameness. For bounded linear maps $\tilde{\Phi} : \mathcal{S} \rightarrow \mathcal{S}$ and $\Phi : \mathcal{P} \rightarrow \mathcal{P}$, the following are equivalent:

- (i) Admissibility: $\Phi \mathcal{I} = \mathcal{I} \tilde{\Phi}$ and $\|\Phi\|_{W \rightarrow W} \leq 1$;
- (ii) Data-processing inequality (DPI) for $\mathcal{R}_{\text{sameness}}$:

$$\mathcal{R}_{\text{sameness}}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}_{\text{sameness}}(s, p) \quad \forall (s, p) \in \mathcal{S} \times \mathcal{P}.$$

Proof. (i) $\Rightarrow$ (ii): Set $e := p - \mathcal{I}s$. If $\Phi \mathcal{I} = \mathcal{I} \tilde{\Phi}$, then

$$\Phi p - \mathcal{I} \tilde{\Phi}s = \Phi(p - \mathcal{I}s) = \Phi e.$$

Hence

$$\mathcal{R}_{\text{sameness}}(\tilde{\Phi}s, \Phi p) = \|\Phi e\|_W^2 \leq \|\Phi\|_{W \rightarrow W}^2 \|e\|_W^2 \leq \|e\|_W^2 = \mathcal{R}_{\text{sameness}}(s, p),$$

whenever $\|\Phi\|_{W \rightarrow W} \leq 1$.

(ii) $\Rightarrow$ (i): Assume $\mathcal{R}_{\text{sameness}}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}_{\text{sameness}}(s, p)$ for all (s, p) . Taking $p = \mathcal{I}s$ yields

$$0 \leq \mathcal{R}_{\text{sameness}}(\tilde{\Phi}s, \Phi \mathcal{I}s) \leq \mathcal{R}_{\text{sameness}}(s, \mathcal{I}s) = 0,$$

so $\mathcal{R}_{\text{sameness}}(\tilde{\Phi}s, \Phi\mathcal{I}s) = 0$ and hence $\Phi\mathcal{I}s = \mathcal{I}\tilde{\Phi}s$ for all s , i.e. $\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}$. Now fix $e \in \mathcal{P}$ and write $p := \mathcal{I}s + e$ for some s (possible since $\mathcal{I}$ has dense range and we are in finite dimension). Then

$$\mathcal{R}_{\text{sameness}}(\tilde{\Phi}s, \Phi p) = \|\Phi e\|_W^2, \quad \mathcal{R}_{\text{sameness}}(s, p) = \|e\|_W^2.$$

The DPI inequality gives $\|\Phi e\|_W^2 \leq \|e\|_W^2$ for all e , hence $\|\Phi\|_{W \rightarrow W} \leq 1$. $\square$

In finite dimension, nonexpansiveness in the W -norm admits a clean spectral characterisation, familiar from the theory of similarity transforms and operator norms.

Theorem (Spectral characterisation of DSFL-admissibility)

Theorem 3 (Spectral characterisation of DSFL-admissibility). *In finite dimension, let $W \succ 0$ on $\mathcal{P}$ and $\Phi : \mathcal{P} \rightarrow \mathcal{P}$ be bounded. Then*

$$\|\Phi\|_{W \rightarrow W} \leq 1 \iff \Phi^* W \Phi \preceq W \iff \lambda_{\max}(W^{-1/2} \Phi^* W \Phi W^{-1/2}) \leq 1.$$

In particular, DSFL-admissibility can be checked by the single spectral inequality $\Phi^ W \Phi \preceq W$.*

Proof. Define the W -inner product $\langle x, y \rangle_W = \langle Wx, y \rangle$ and norm $\|x\|_W^2 = \langle Wx, x \rangle$. Then

$$\|\Phi\|_{W \rightarrow W} = \sup_{\|x\|_W=1} \|\Phi x\|_W = \sup_{\|x\|_W=1} \sqrt{\langle \Phi^* W \Phi x, x \rangle}.$$

Thus $\|\Phi\|_{W \rightarrow W} \leq 1$ if and only if $\langle \Phi^* W \Phi x, x \rangle \leq \langle Wx, x \rangle$ for all x , i.e. $\Phi^* W \Phi \preceq W$. Conjugating by $W^{1/2}$ gives

$$\Phi^* W \Phi \preceq W \iff W^{-1/2} \Phi^* W \Phi W^{-1/2} \preceq I,$$

which is equivalent to $\lambda_{\max}(W^{-1/2} \Phi^* W \Phi W^{-1/2}) \leq 1$. $\square$

3.4. Two-Loop Continuous Law and DSFL Time

In time-continuous settings, DSFL dynamics are organised into a fast *immediate* loop, which trims the defect locally, and a slower *relay* loop, which transports any remaining mismatch along causal cones. Under mild coercivity and positivity assumptions, the residual of sameness satisfies a Lyapunov inequality with an explicit rate, and this induces the DSFL notion of time. The argument is a standard Lyapunov-functional estimate for linear Volterra equations (see, e.g., [64]), specialised to the residual $R(t) = \|e(t)\|_W^2$.

Theorem (Lyapunov inequality and DSFL time for two-loop dynamics)

Theorem 4 (Lyapunov inequality and DSFL time for two-loop dynamics). *Let $e(t) \in \mathcal{H}_p$ solve the abstract two-loop evolution*

$$\dot{e}(t) = -K_{\text{imm}}(t)e(t) - \int_0^t M(t-\tau)e(\tau) d\tau + r(t),$$

where $K_{\text{imm}}(t) \succeq \kappa(t)I$ with $\kappa(t) \geq 0$, $M(\sigma) \succeq 0$ for $\sigma \geq 0$, and $|\langle r(t), x \rangle_W| \leq \varepsilon_{\text{rem}}(t)\|x\|_W^2$ for all $x \in \mathcal{H}_p$. Then the residual $R(t) := \|e(t)\|_W^2$ satisfies

$$\dot{R}(t) \leq -2\lambda(t)R(t), \quad \lambda(t) := \kappa(t) - \varepsilon_{\text{rem}}(t) \geq 0.$$

Defining DSFL time by $d\hat{\tau} = 2\lambda(t) dt$, one has

$$\frac{d}{d\hat{\tau}} \log R(\hat{\tau}) \leq -1,$$

so that $R(\hat{\tau})$ decays at least as fast as a unit-slope semilog line in DSFL time.

Proof sketch. Differentiating $R(t) = \langle e(t), We(t) \rangle$ and using the evolution equation yields

$$\dot{R}(t) = -2\langle K_{\text{imm}}(t)e(t), We(t) \rangle - 2\text{Re}\left\langle \int_0^t M(t-\tau)e(\tau) d\tau, We(t) \right\rangle + 2\text{Re}\langle r(t), We(t) \rangle.$$

The coercivity $K_{\text{imm}}(t) \succeq \kappa(t)I$ gives $-2\langle K_{\text{imm}}(t)e, We \rangle \leq -2\kappa(t)\|e(t)\|_W^2$; the Loewner-positivity of M makes the memory term nonpositive; and the remainder bound yields $2|\text{Re}\langle r(t), We(t) \rangle| \leq 2\varepsilon_{\text{rem}}(t)\|e(t)\|_W^2$. Altogether,

$$\dot{R}(t) \leq -2(\kappa(t) - \varepsilon_{\text{rem}}(t)) R(t) = -2\lambda(t)R(t).$$

The inequality for $\log R$ in DSFL time follows by dividing by $2\lambda(t)R(t)$ on intervals where $\lambda(t) > 0$ and integrating; see also the analogous Lyapunov arguments for semigroups in [64]. $\square$

In the rest of the paper we specialise this room-level DSFL structure to finite-dimensional quantum systems, entanglement measures, and Bell tests. The same formalism extends, with additional functional-analytic work, to field-theoretic and gravitational sectors, as discussed in the outlook.

4. Finite-Dimensional Quantum Channels

We now specialise the general DSFL room to the case of finite-dimensional quantum channels. The aim is to show that, once a channel sector is calibrated into a single instrument norm, the DSFL residual plays the rôle of a Lyapunov functional for completely positive trace-preserving (CPTP) maps, with an associated discrete DSFL clock on which $\log R$ has a unit-slope envelope. We also explain how this structure is used in numerical audits. Throughout we work in the standard finite-dimensional operator setting; see, for example, [2,65] for background on quantum channels and [63,66] for weighted norms and induced operator norms.

4.1. DSFL Room for Channels and Admissible CPTP Maps

We begin by fixing the finite-dimensional DSFL room for channels and formalising DSFL-admissibility in this setting.

Setup 4.1 (DSFL room for channels). Let $\mathcal{H} \cong \mathbb{C}^d$ be a d -dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the space of $d \times d$ complex matrices with the Hilbert-Schmidt inner product $\langle X, Y \rangle_{\text{HS}} := \text{Tr}(X^\dagger Y)$. Set

$$\mathcal{H}_s = \mathcal{H}_p = \mathcal{B}(\mathcal{H}), \quad \mathcal{I} = \text{id},$$

and fix a strictly positive definite operator $W : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ (the instrument weight), inducing the norm

$$\|X\|_W^2 := \langle X, WX \rangle_{\text{HS}}.$$

For $\rho, p \in \mathcal{B}(\mathcal{H})$ define the residual of sameness

$$R(\rho, p) := \|p - \rho\|_W^2.$$

A quantum channel is a completely positive trace-preserving (CPTP) map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, equivalently a map admitting a Kraus decomposition $\Phi(X) = \sum_i K_i X K_i^\dagger$ with $\sum_i K_i^\dagger K_i = I$ [2,65,67].

Definition 3 (DSFL–admissible quantum channel). *In the setting of Setup 4.1, a quantum channel Φ is called DSFL–admissible if it is nonexpansive in the instrument norm:*

$$\|\Phi\|_{W \rightarrow W} := \sup_{\|X\|_W=1} \|\Phi X\|_W \leq 1. \quad (2)$$

Equivalently, by the spectral characterisation (Theorem 3), DSFL–admissibility is equivalent to the operator inequality

$$\Phi^* W \Phi \preceq W \iff \lambda_{\max}(W^{-1/2} \Phi^* W \Phi W^{-1/2}) \leq 1,$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue. This is the usual similarity–transform description of induced operator norms in a weighted Hilbert space [66].

Since here $\mathcal{I} = \text{id}$, the general admissibility–DPI equivalence (Theorem 2) reduces to a single–functional DPI for R :

Theorem 5 (Single–residual DPI for channels). *In the finite–dimensional DSFL room of Setup 4.1, a channel Φ is DSFL–admissible if and only if*

$$R(\Phi\rho, \Phi p) \leq R(\rho, p) \quad \text{for all } (\rho, p) \in \mathcal{B}(\mathcal{H})^2.$$

Proof. With $\mathcal{I} = \text{id}$ the residual is $R(\rho, p) = \|p - \rho\|_W^2$, and Theorem 2 applied to $(\mathcal{H}_s, \mathcal{H}_p) = (\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ and $\mathcal{I} = \text{id}$ gives exactly the stated equivalence between $\|\Phi\|_{W \rightarrow W} \leq 1$ and $R(\Phi\rho, \Phi p) \leq R(\rho, p)$ for all (ρ, p) . $\square$

Thus each admissible CPTP map is exactly a DSFL “one–step” update that cannot increase the single residual of sameness in the chosen instrument norm.

4.2. Discrete Lyapunov Envelope and DSFL Clock

We next show that iterates of a DSFL–admissible channel admit a Lyapunov–type envelope in a discrete DSFL clock on which $\log R$ has unit slope. This is the discrete analogue of the continuous two–loop Lyapunov law.

Definition 4 (DSFL iterates and discrete clock). *Let Φ be DSFL–admissible on $\mathcal{H}_p$ and let $e_0 \in \mathcal{H}_p$ be an initial defect. Define*

$$e_{k+1} := \Phi e_k, \quad R_k := \|e_k\|_W^2,$$

and the discrete DSFL time by

$$\hat{\tau}_0 := 0, \quad \hat{\tau}_{k+1} := \hat{\tau}_k - \log\left(\frac{R_{k+1}}{R_k}\right), \quad k \geq 0, \quad (3)$$

whenever $R_k > 0$.

Theorem 6 (Unit–slope envelope in discrete DSFL time). *For the sequence $\{R_k\}$ and clock $\{\hat{\tau}_k\}$ defined above one has*

$$\hat{\tau}_k = \log R_0 - \log R_k, \quad \log R_k = \log R_0 - \hat{\tau}_k,$$

so that the graph of $\hat{\tau} \mapsto \log R(\hat{\tau})$ is an exact straight line of slope -1 .

Proof. By definition,

$$\hat{\tau}_k = \sum_{j=0}^{k-1} (\hat{\tau}_{j+1} - \hat{\tau}_j) = - \sum_{j=0}^{k-1} \log \frac{R_{j+1}}{R_j} = - \log \frac{R_k}{R_0} = \log R_0 - \log R_k.$$

Rearranging gives $\log R_k = \log R_0 - \hat{\tau}_k$. $\square$

When Φ is strictly contractive in the W -norm, one also obtains an exponential bound in the physical iteration index k .

Theorem 7 (Discrete Lyapunov envelope for channels). *Let Φ be DSFL-admissible and suppose $\|\Phi\|_{W \rightarrow W} \leq 1 - \eta_{\min}$ for some $\eta_{\min} \in (0, 1]$. With e_k , R_k and $\hat{\tau}_k$ as in Definition 4, one has*

$$R_{k+1} \leq (1 - \eta_{\min})^2 R_k \quad \text{for all } k \in \mathbb{N},$$

and hence

$$R_k \leq e^{-2\alpha_* k} R_0, \quad \alpha_* := \eta_{\min}(2 - \eta_{\min}) > 0.$$

Moreover $\log R_k = \log R_0 - \hat{\tau}_k$, so that in DSFL time the envelope is a unit-slope semilog line.

Proof. Nonexpansiveness and $\|\Phi\|_{W \rightarrow W} \leq 1 - \eta_{\min}$ give

$$R_{k+1} = \|\Phi e_k\|_W^2 \leq \|\Phi\|_{W \rightarrow W}^2 \|e_k\|_W^2 \leq (1 - \eta_{\min})^2 R_k.$$

Iterating the inequality yields $R_k \leq (1 - \eta_{\min})^{2k} R_0$. Using $(1 - \eta_{\min})^2 \leq e^{-2\eta_{\min}(2 - \eta_{\min})}$ gives

$$R_k \leq e^{-2\alpha_* k} R_0, \quad \alpha_* := \eta_{\min}(2 - \eta_{\min}) > 0.$$

The identity $\log R_k = \log R_0 - \hat{\tau}_k$ is exactly Theorem 6. $\square$

Thus, for finite-dimensional channels, DSFL provides both: a spectral criterion for admissibility via the single matrix $W^{-1/2}\Phi^*W\Phi W^{-1/2}$, and a Lyapunov/clock structure in which the decay of the residual has an exact unit-slope semilog envelope. This complements the more traditional use of trace-norm or relative-entropy contractivity for quantum channels [22,25,52].

Small-gain bound in the instrument norm

Theorem 8 (Small-gain theorem). *Let $W : \mathcal{H}_p \rightarrow \mathcal{H}_p$ be a strictly positive, selfadjoint operator and let $\|\cdot\|_W$ be the associated norm $\|x\|_W^2 = \langle Wx, x \rangle$. Let $\Phi : \mathcal{H}_p \rightarrow \mathcal{H}_p$ be a bounded linear map and suppose that there exists $\eta \geq 0$ such that*

$$\sup_{\|x\|_W=1} \|(\Phi - I)x\|_W \leq \eta. \quad (4)$$

Then:

(i) (Operator-norm bound) One has

$$\|\Phi\|_{W \rightarrow W} := \sup_{\|x\|_W=1} \|\Phi x\|_W \leq 1 + \eta. \quad (5)$$

(ii) (Spectral small-gain bound) The associated quadratic form satisfies

$$\Phi^*W\Phi \preceq (1 + \eta)^2 W, \quad (6)$$

i.e. $\langle \Phi^*W\Phi x, x \rangle \leq (1 + \eta)^2 \langle Wx, x \rangle$ for all $x \in \mathcal{H}_p$.

In particular, if $\eta < 1$ then Φ is strictly contractive in the instrument norm: $\|\Phi\|_{W \rightarrow W} < 2$.

Proof. For any $x \in \mathcal{H}_p$ with $\|x\|_W = 1$, the hypothesis gives

$$\|\Phi x\|_W \leq \|(\Phi - I)x\|_W + \|x\|_W \leq \eta + 1,$$

so $\|\Phi\|_{W \rightarrow W} \leq 1 + \eta$, proving (i). For general $x \in \mathcal{H}_p$,

$$\langle \Phi^* W \Phi x, x \rangle = \|\Phi x\|_W^2 \leq (1 + \eta)^2 \|x\|_W^2 = (1 + \eta)^2 \langle Wx, x \rangle,$$

which is equivalent to $\Phi^* W \Phi \preceq (1 + \eta)^2 W$. If $\eta < 1$ then $1 + \eta < 2$ and hence $\|\Phi\|_{W \rightarrow W} < 2$, so Φ is strictly contractive in $\|\cdot\|_W$. $\square$

4.3. Numerical Audits for Channels

The finite-dimensional channel framework above is particularly convenient for numerical audits of the DSFL law. In practice we use it in three ways:

- **DPI / spectral audits.** For a given instrument weight W and channel Φ , we compute the largest eigenvalue $\lambda_{\max}(W^{-1/2} \Phi^* W \Phi W^{-1/2})$. If $\lambda_{\max} > 1$ the map is not DSFL-admissible and one can find explicit defects e with $R(\Phi e) > R(e)$; if $\lambda_{\max} \leq 1$ (up to numerical tolerance) the DPI for R is satisfied. This is the basic “admissible / nonadmissible” test for finite-dimensional channels.
- **Envelope fits in DSFL time.** For a DSFL-admissible channel, we generate a sequence $e_{k+1} = \Phi e_k$, record $R_k = \|e_k\|_W^2$, and form the discrete DSFL time $\hat{\tau}_k = \log R_0 - \log R_k$. A linear regression of $\log R_k$ versus $\hat{\tau}_k$ confirms, to numerical precision, the unit-slope relation $\log R_k = \log R_0 - \hat{\tau}_k$ implied by Theorem 6.
- **Small-gain / robustness estimates.** In applications one often has a numerical bound of the form $\sup_{\|x\|_W=1} \|(\Phi - I)x\|_W \leq \eta$. The small-gain theorem (Theorem 8) then yields the quantitative estimates $\|\Phi\|_{W \rightarrow W} \leq 1 + \eta$ and $\Phi^* W \Phi \preceq (1 + \eta)^2 W$, which quantify how close Φ is to being DSFL-admissible and how tight the Lyapunov envelope remains. This is what we use to interpret noisy or approximate implementations of DSFL channels.

Altogether, the finite-dimensional channel module (Setup 4.1–Theorem 7) provides a compact DSFL testbed: admissibility is checked spectrally, the single residual R serves as a Lyapunov functional with an intrinsic clock, and numerical experiments can directly probe the DPI and envelope structure encoded by the DSFL law.

A simple triangle-inequality argument in $\|\cdot\|_W$ gives $\|\Phi\|_{W \rightarrow W} \leq 1 + \eta$ and $\Phi^* W \Phi \preceq (1 + \eta)^2 W$, so Φ is approximately DSFL-admissible whenever η is small (see, e.g., [63, Prop. 3.3] for a general treatment).

5. Entanglement and Correlation Residuals

In this section we specialise the DSFL room to bipartite finite-dimensional quantum systems. We first recall the usual notion of separability, then reinterpret standard entanglement quantities—negativity, sandwiched Rényi divergences, and trace-distance to product form—as *correlation residuals* compatible with the single DSFL residual $R(s, p) = \|p - \mathcal{I}s\|_W^2$. We then introduce a purely DSFL correlation residual R_{corr} and show that it is nonincreasing under local CPTP maps, with numerical audits supporting this behaviour. For general background on bipartite entanglement and correlation measures, see e.g. the reviews [3,4].

5.1. Bipartite DSFL Room, Separable vs. Entangled

Setup 5.1 (Bipartite DSFL room). Let $\mathcal{H}_A \cong \mathbb{C}^{d_A}$ and $\mathcal{H}_B \cong \mathbb{C}^{d_B}$, and set $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$. Denote by $\mathcal{S}(\mathcal{H})$ the set of density matrices on $\mathcal{H}$. For $\rho_{AB} \in \mathcal{S}(\mathcal{H})$, write

$$\rho_A := \text{Tr}_B(\rho_{AB}), \quad \rho_B := \text{Tr}_A(\rho_{AB}).$$

We embed this sector in a DSFL room by taking $\mathcal{H}_s = \mathcal{H}_p = \mathcal{B}(\mathcal{H})$, calibration $\mathcal{I} = \text{id}$, and a fixed SPD instrument weight $W : \mathcal{H}_p \rightarrow \mathcal{H}_p$, with $\|X\|_W^2 = \langle X, WX \rangle_{\text{HS}}$.

Definition 5 (Separable and entangled states). A state $\rho_{AB} \in \mathcal{S}(\mathcal{H})$ is separable if there exist $p_k \geq 0$ with $\sum_k p_k = 1$ and local states $\rho_A^{(k)} \in \mathcal{S}(\mathcal{H}_A)$, $\rho_B^{(k)} \in \mathcal{S}(\mathcal{H}_B)$ such that

$$\rho_{AB} = \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)}.$$

Otherwise ρ_{AB} is entangled.[3]

5.2. Standard Correlation Functionals

We recall three widely used correlation/entanglement functionals which will be treated as DSFL correlation residuals.

Definition 6 (Negativity). For a bipartite state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, the negativity is

$$N(\rho_{AB}) := \frac{\|\rho_{AB}^{T_B}\|_1 - 1}{2},$$

where T_B is partial transposition on B and $\|\cdot\|_1$ the trace norm.[13]

Negativity as an entanglement monotone

Theorem 9. For any bipartite state ρ_{AB} , $N(\rho_{AB}) \geq 0$ and $N(\rho_{AB}) = 0$ for all separable states. Moreover, for any local CPTP maps Λ_A and Λ_B ,

$$N((\Lambda_A \otimes \text{id}_B)(\rho_{AB})) \leq N(\rho_{AB}), \quad N((\text{id}_A \otimes \Lambda_B)(\rho_{AB})) \leq N(\rho_{AB}).$$

Proof sketch. Nonnegativity and vanishing on separable states follow from the PPT criterion (Peres–Horodecki) in 2×2 and 2×3 dimensions and from the fact that N measures the sum of absolute values of negative eigenvalues of $\rho_{AB}^{T_B}$. [3,13] Monotonicity under local CPTP maps is a standard property of negativity as an entanglement monotone, derived from the contractivity of the trace norm under CPTP maps and the structure of the partial transpose on Kraus decompositions.[13] $\square$

Definition 7 (Sandwiched Rényi divergence). For $\alpha \in (0, 1) \cup (1, \infty)$ and positive ρ, σ with $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ when $\alpha > 1$, the sandwiched Rényi divergence is

$$D_\alpha^*(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right].$$

DPI for sandwiched Rényi divergences

Theorem 10. For $\alpha \in (0, 1) \cup (1, \infty)$ and any CPTP map Φ ,

$$D_\alpha^*(\Phi(\rho) \parallel \Phi(\sigma)) \leq D_\alpha^*(\rho \parallel \sigma).$$

In particular, for $\rho_{AB} \in \mathcal{S}(\mathcal{H})$,

$$D_\alpha^*(\rho_{AB} \parallel \rho_A \otimes \rho_B)$$

is nonincreasing under all local CPTP maps $\Lambda_A \otimes \text{id}_B$ and $\text{id}_A \otimes \Lambda_B$.

Proof sketch. The data-processing inequality for D_α^* is proved in [14,15,68]. Setting $\sigma = \rho_A \otimes \rho_B$ and $\Phi = \Lambda_A \otimes \text{id}_B$ or $\Phi = \text{id}_A \otimes \Lambda_B$ gives the claimed monotonicity of the correlation divergence under local maps. $\square$

Definition 8 (Trace distance to product form). For $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ set

$$\Delta_{\text{prod}}(\rho_{AB}) := \|\rho_{AB} - \rho_A \otimes \rho_B\|_1.$$

Trace distance to product as a correlation residual

Theorem 11. For any local CPTP map Λ_A (resp. Λ_B),

$$\Delta_{\text{prod}}((\Lambda_A \otimes \text{id}_B)(\rho_{AB})) \leq \Delta_{\text{prod}}(\rho_{AB}),$$

and similarly for $\text{id}_A \otimes \Lambda_B$.

Proof. Write $\sigma_{AB} := \rho_A \otimes \rho_B$ and let $\Lambda = \Lambda_A \otimes \text{id}_B$. Then

$$\Lambda(\rho_{AB}) - \Lambda(\sigma_{AB}) = \Lambda(\rho_{AB} - \sigma_{AB}),$$

and CPTP maps are contractive in trace norm:[2,65] $\|\Lambda(X)\|_1 \leq \|X\|_1$. Hence

$$\Delta_{\text{prod}}(\Lambda(\rho_{AB})) = \|\Lambda(\rho_{AB} - \sigma_{AB})\|_1 \leq \|\rho_{AB} - \sigma_{AB}\|_1 = \Delta_{\text{prod}}(\rho_{AB}).$$

The argument for $\text{id}_A \otimes \Lambda_B$ is identical. $\square$

Each of $N(\rho_{AB})$, $D_\alpha^*(\rho_{AB} \|\rho_A \otimes \rho_B)$ and $\Delta_{\text{prod}}(\rho_{AB})$ vanishes on product states and cannot increase under local CPTP maps. In DSFL language they are *correlation parts* of the global mismatch.

5.3. DSFL Correlation Residual and Local CP Monotonicity

The DSFL framework suggests a single quadratic correlation residual in the instrument norm.

Definition 9 (DSFL correlation residual). In the bipartite DSFL room of Setup 5.1, the DSFL correlation residual of a state ρ_{AB} is

$$R_{\text{corr}}(\rho_{AB}) := \|\rho_{AB} - \rho_A \otimes \rho_B\|_W^2. \quad (7)$$

This is just the DSFL residual $R(s, p)$ with $s = \rho_A \otimes \rho_B$ and $p = \rho_{AB}$, and vanishes if and only if ρ_{AB} is a product state.

Local CP monotonicity of the DSFL correlation residual

Theorem 12 (DSFL correlation law for bipartite systems). In the setting of Setup 5.1, let Λ_A and Λ_B be CPTP maps on $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$, respectively, such that $\|\Lambda_A\|_{W \rightarrow W} \leq 1$ and $\|\Lambda_B\|_{W \rightarrow W} \leq 1$ in the instrument norm. Then, for all $\rho_{AB} \in \mathcal{S}(\mathcal{H})$,

$$R_{\text{corr}}((\Lambda_A \otimes \text{id}_B)(\rho_{AB})) \leq R_{\text{corr}}(\rho_{AB}), \quad (8)$$

$$R_{\text{corr}}((\text{id}_A \otimes \Lambda_B)(\rho_{AB})) \leq R_{\text{corr}}(\rho_{AB}). \quad (9)$$

Proof. Set $\sigma_{AB} := \rho_A \otimes \rho_B$ and consider $\Lambda := \Lambda_A \otimes \text{id}_B$. Then

$$\Lambda\rho_{AB} - \Lambda\sigma_{AB} = \Lambda(\rho_{AB} - \sigma_{AB}),$$

and

$$R_{\text{corr}}(\Lambda\rho_{AB}) = \|\Lambda(\rho_{AB} - \sigma_{AB})\|_W^2.$$

By hypothesis, Λ is DSFL-admissible in the instrument norm: $\|\Lambda\|_{W \rightarrow W} \leq 1$, so

$$\|\Lambda(\rho_{AB} - \sigma_{AB})\|_W^2 \leq \|\rho_{AB} - \sigma_{AB}\|_W^2 = R_{\text{corr}}(\rho_{AB}).$$

The proof for $\text{id}_A \otimes \Lambda_B$ is identical. $\square$

5.4. Numerical Local-CP Audits

To complement these analytic monotonicity results we performed numerical audits of local-CP channels on 2×2 systems. The details are as follows.

Setup 5.2 (Random states and local channels). Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We generate:

- random states ρ_{AB} by sampling $G \in \mathbb{C}^{4 \times 4}$ with i.i.d. complex Gaussian entries and setting $\rho_{AB} = GG^\dagger / \text{Tr}(GG^\dagger)$;
- random local CPTP maps Λ_A, Λ_B by sampling Kraus operators $\{K_i\}$ and normalising $\sum_i K_i^\dagger K_i = I$.

For each ρ_{AB} and local Λ we compute

$$\delta E := E(\Lambda(\rho_{AB})) - E(\rho_{AB})$$

for $E \in \{N, D_\alpha^*(\cdot \| \rho_A \otimes \rho_B), \Delta_{\text{prod}}, R_{\text{corr}}\}$.

Local-CP audits: no observed increase of correlation residuals

Theorem 13 (Empirical monotonicity under local CPTP maps). In 2×2 tests with 10^2 random states and 10^2 random local channels on each side, all measured variations δE for $E \in \{N, D_\alpha^*, \Delta_{\text{prod}}, R_{\text{corr}}\}$ satisfy

$$\delta E \leq \varepsilon,$$

with ε at the level of numerical roundoff ($\sim 10^{-13}$ – 10^{-15}). No statistically significant positive excursions are observed.

These numerical audits are fully consistent with the analytical monotonicity theorems and support the interpretation of N , Rényi correlation divergences, Δ_{prod} and R_{corr} as DSFL correlation residuals: in a calibrated bipartite DSFL room, local admissible dynamics can at best redistribute nonclassical correlations, never create them from nothing.

6. Bell Tests in a DSFL Room

In this section we embed CHSH-type Bell tests into a DSFL room. The aim is to show, in a clean finite-dimensional setting, that (i) classical local hidden-variable (LHV) strategies are confined to the usual CHSH bound $|S| \leq 2$; [4,6,7] (ii) a DSFL-calibrated two-qubit sector reproduces the Tsirelson value $2\sqrt{2}$ while respecting DSFL admissibility; [4,18] and (iii) numerical audits confirm these statements to machine precision.

6.1. Classical CHSH Bound

Definition 10 (CHSH combination). Let A_0, A_1, B_0, B_1 be real-valued random variables on a common probability space, each taking values in $\{\pm 1\}$. The CHSH combination is

$$S := A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1. \quad (10)$$

Classical CHSH bound

Theorem 14 (Classical CHSH bound). Let A_0, A_1, B_0, B_1 be $\{\pm 1\}$ -valued random variables on a common probability space. Then

$$|\mathbb{E}[S]| \leq 2.$$

Equivalently, any local hidden-variable (LHV) model for the CHSH scenario satisfies $|S| \leq 2$. [4,7]

Proof. For a deterministic assignment $(a_0, a_1, b_0, b_1) \in \{\pm 1\}^4$ one has

$$S = a_0b_0 + a_0b_1 + a_1b_0 - a_1b_1 = a_0(b_0 + b_1) + a_1(b_0 - b_1).$$

Since $b_0, b_1 \in \{\pm 1\}$, the pair $(b_0 + b_1, b_0 - b_1)$ is one of $(2, 0)$, $(0, 2)$, $(0, -2)$ or $(-2, 0)$, so $S \in \{-2, 2\}$ and $|S| \leq 2$ for each deterministic strategy. An arbitrary LHV model is a convex combination of deterministic assignments, and linearity of expectation yields $|\mathbb{E}[S]| \leq 2$. $\square$

6.2. DSFL Calibration and Tsirelson Value

We now calibrate a two-qubit sector into a DSFL room so that the instrument norm coincides (up to scale) with the Hilbert–Schmidt norm and the channel and measurement structure match the standard CHSH setting.[2,7,18]

Setup 6.1 (Two-qubit DSFL room for CHSH). Let $\mathcal{H}_A \cong \mathbb{C}^2$, $\mathcal{H}_B \cong \mathbb{C}^2$ and $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$. Consider the Bell state

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad \rho_\Phi := |\Phi^+\rangle\langle\Phi^+|.$$

Fix the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ on $\mathbb{C}^2$ and define local dichotomic observables

$$A_0 := \sigma_z, \quad A_1 := \sigma_x, \quad B_0 := \frac{\sigma_z + \sigma_x}{\sqrt{2}}, \quad B_1 := \frac{\sigma_z - \sigma_x}{\sqrt{2}}. \quad (11)$$

We embed this sector in a DSFL room by taking $\mathcal{H}_s = \mathcal{H}_p = \mathcal{B}(\mathcal{H})$, calibration $\mathcal{I} = \text{id}$, and W proportional to the identity, so that $\|X\|_W$ is equivalent to the Hilbert–Schmidt norm.

The CHSH operator is

$$C := A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1,$$

and the CHSH value in state ρ_{AB} is $S(\rho_{AB}) = \text{Tr}(\rho_{AB}C)$.

Tsirelson value in a DSFL-calibrated two-qubit sector

Theorem 15 (Tsirelson value in the DSFL two-qubit room). In the setting of Setup 6.1, the Bell state ρ_Φ and Pauli settings (A_0, A_1, B_0, B_1) satisfy

$$S(\rho_\Phi) = \text{Tr}(\rho_\Phi C) = 2\sqrt{2},$$

i.e. they achieve the Tsirelson bound for CHSH.[4,18]

Proof. The Bell state satisfies the standard two-qubit correlations

$$\langle\sigma_z \otimes \sigma_z\rangle_{\rho_\Phi} = 1, \quad \langle\sigma_x \otimes \sigma_x\rangle_{\rho_\Phi} = 1, \quad \langle\sigma_z \otimes \sigma_x\rangle_{\rho_\Phi} = \langle\sigma_x \otimes \sigma_z\rangle_{\rho_\Phi} = 0.$$

From the definitions of A_j, B_k ,

$$\begin{aligned} \langle A_0 \otimes B_0 \rangle_{\rho_\Phi} &= \frac{1}{\sqrt{2}}(\langle\sigma_z \otimes \sigma_z\rangle + \langle\sigma_z \otimes \sigma_x\rangle) = \frac{1}{\sqrt{2}}, \\ \langle A_0 \otimes B_1 \rangle_{\rho_\Phi} &= \frac{1}{\sqrt{2}}(\langle\sigma_z \otimes \sigma_z\rangle - \langle\sigma_z \otimes \sigma_x\rangle) = \frac{1}{\sqrt{2}}, \\ \langle A_1 \otimes B_0 \rangle_{\rho_\Phi} &= \frac{1}{\sqrt{2}}(\langle\sigma_x \otimes \sigma_z\rangle + \langle\sigma_x \otimes \sigma_x\rangle) = \frac{1}{\sqrt{2}}, \\ \langle A_1 \otimes B_1 \rangle_{\rho_\Phi} &= \frac{1}{\sqrt{2}}(\langle\sigma_x \otimes \sigma_z\rangle - \langle\sigma_x \otimes \sigma_x\rangle) = -\frac{1}{\sqrt{2}}. \end{aligned}$$

Therefore

$$S(\rho_\Phi) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2},$$

which matches the Tsirelson bound.[18] $\square$

6.3. DSFL vs. Local Hidden-Variable Models

DSFL is strictly beyond LHV in the calibrated CHSH sector

Theorem 16 (DSFL vs. LHV in the CHSH scenario). *In the two-qubit DSFL room of Setup 6.1, any DSFL model whose admissible updates factor as local maps $\Phi_A \otimes \Phi_B$ with $\|\Phi_A\|_{W \rightarrow W}, \|\Phi_B\|_{W \rightarrow W} \leq 1$ must satisfy the classical CHSH bound $|S| \leq 2$, whereas the calibrated Bell sector realises $S = 2\sqrt{2} > 2$. Thus, in this sector, the DSFL residual law cannot be reduced to a classical local hidden-variable theory.[4,6]*

Proof. Local DSFL-admissible updates $\Phi_A \otimes \Phi_B$ act separately on the two wings and are contractive in the instrument norm on each subsystem. Any statistics generated by such maps from separable or product blueprints can be represented as convex mixtures of deterministic assignments $(a_0, a_1, b_0, b_1) \in \{\pm 1\}^4$, hence are constrained by Theorem 14 and satisfy $|S| \leq 2$. [4] On the other hand, Theorem 15 provides a DSFL-calibrated choice of blueprint ρ_Φ and local observables (A_j, B_k) for which $S = 2\sqrt{2}$, with all steps DSFL-admissible in the instrument norm. No purely local DSFL model can simultaneously obey $|S| \leq 2$ and reproduce $S = 2\sqrt{2}$, so DSFL is strictly beyond LHV in this sector. $\square$

6.4. Numerical CHSH Audit

Finally we report a minimal numerical audit confirming the classical and DSFL/quantum CHSH values.

Setup 6.2 (Brute-force LHV and DSFL evaluations). • LHV side: *enumerate all $2^4 = 16$ deterministic assignments $(a_0, a_1, b_0, b_1) \in \{\pm 1\}^4$ and compute $S(a_0, a_1, b_0, b_1) = a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1$.*

- DSFL/quantum side: *build the 4×4 matrices for A_0, A_1, B_0, B_1 and ρ_Φ in the computational basis and evaluate $S_{\text{DSFL}} = \text{Tr}(\rho_\Phi C)$.*

Numerical CHSH audit: LHV vs. DSFL

Theorem 17 (Numerical confirmation of CHSH bounds). *Using the protocol of Setup 6.2, one finds:*

- For all 16 deterministic assignments, $S(a_0, a_1, b_0, b_1) \in \{-2, 2\}$ and $\max |S| = 2$ exactly, in agreement with Theorem 14.*
- For the DSFL-calibrated Bell sector, $S_{\text{DSFL}} = 2.828427124746\dots$, agreeing with $2\sqrt{2}$ up to machine precision, in agreement with Theorem 15.*

Proof sketch. Part (a) is a direct enumeration of all deterministic LHV strategies, reproducing Theorem 14. Part (b) follows from a straightforward matrix calculation of $\text{Tr}(\rho_\Phi C)$; the numerical value matches $2\sqrt{2}$ to within 10^{-15} , confirming Theorem 15.[2] $\square$

Taken together, Theorems 14, 15, 16 and 17 show that a DSFL room calibrated to a two-qubit sector reproduces the standard quantum nonlocality of CHSH tests[4] while remaining strictly more general than classical LHV models, all within a single residual-based law.

7. Born Rule, Projection, and Comparison with Standard QM

In this section we collect the DSFL statements about measurement and probabilities, and compare them with the textbook postulates of finite-dimensional quantum mechanics.[2,55] The key points are:

- Born weights arise as *geometric fractions* of a single quadratic norm in the DSFL room, in line with Gleason's characterisation of frame-covariant probability measures.[58]
- Lüders projection is the *unique* sharp update that is both idempotent and DSFL-admissible (nearest-point in the instrument norm), consistent with the standard post-measurement rule.[39, 69]

- In finite dimension, DSFL reproduces all standard quantum structures (unitary dynamics, projection, Born rule), while adding a global Lyapunov/DPI law and a single residual as the common “meter” for admissibility and locality.

7.1. Born Weights as Geometric Fractions

In a DSFL room, probabilities are not postulated; they are extracted from the geometry of a single quadratic norm on a calibrated Hilbert space. The starting point is a Gleason–type uniqueness statement for probability assignments on projective measurements.[58]

Born weights as the unique geometric probabilities

Theorem 18 (Gleason–type geometric uniqueness of Born weights). *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 3$ and let $\{P_k\}$ be a finite projective measurement, i.e. a family of mutually orthogonal projections with $\sum_k P_k = \mathbf{1}$. Suppose that to each nonzero $u \in \mathcal{H}$ we assign a probability vector $p(u) = (p_k(u))_k$ with $p_k(u) \in [0, 1]$ and $\sum_k p_k(u) = 1$, such that:*

- (i) Ray dependence: $p_k(u)$ depends only on the ray $[u] = \{\lambda u : \lambda \neq 0\}$;
- (ii) Unitary covariance: for every unitary U on $\mathcal{H}$, $p_k(Uu)$ is obtained from $p_k(u)$ by the induced permutation of the projectors $P_k \mapsto UP_kU^*$;
- (iii) Finite additivity on commuting projectors: if $\{Q_j\}$ is a finite family of pairwise orthogonal projectors with $\sum_j Q_j = P_k$ and $[Q_j, Q_\ell] = 0$ for all j, ℓ , then $p_k(u) = \sum_j p_{Q_j}(u)$, where $p_{Q_j}(u)$ denotes the weight assigned to the outcome corresponding to Q_j ;
- (iv) Continuity: $u \mapsto p_k(u)$ is continuous in the norm topology on $\mathcal{H}$.

Then there exists a density matrix ρ on $\mathcal{H}$ such that

$$p_k(u) = \frac{\langle u, P_k \rho u \rangle}{\langle u, u \rangle} \quad \text{for all nonzero } u \in \mathcal{H} \text{ and all } k. \quad (12)$$

In particular, for pure $\rho = |\psi\rangle\langle\psi|$ and rank-one $P_k = |a_k\rangle\langle a_k|$ one recovers the Born rule

$$p_k = |\langle a_k | \psi \rangle|^2. \quad (13)$$

Idea of proof. The assumptions (ii)–(iv) ensure that for each fixed unit vector u the map $P \mapsto p_P(u)$ defines a finitely additive, normalised measure on the lattice of projections in $\mathcal{B}(\mathcal{H})$, invariant under unitary conjugations. By Gleason’s theorem,[58] in dimension ≥ 3 such a probability measure arises from a unique density matrix ρ , i.e. $p_P(u) = \langle u, P \rho u \rangle / \langle u, u \rangle$ for all projectors P . Restricting to the finite family $\{P_k\}$ gives the stated formula. For rank-one $P_k = |a_k\rangle\langle a_k|$ and pure $\rho = |\psi\rangle\langle\psi|$ one finds $p_k = \langle \psi, P_k \psi \rangle = |\langle a_k | \psi \rangle|^2$, the standard Born rule.[2,55] $\square$

The DSFL reading is that, once one fixes a unitarily invariant quadratic instrument norm on the room, the only consistent way to assign measurement weights which are compatible with “one residual in all frames” is to use squared projection norms in that norm.

DSFL reading of Born weights as geometric fractions

Theorem 19 (DSFL reading of Born weights). *Let $\mathcal{H}_s = \mathcal{H}$ and $\mathcal{H}_p$ be a Hilbert space, and let $\mathcal{I} : \mathcal{H} \rightarrow \mathcal{H}_p$ be an isometric embedding. Let $W \succ 0$ be a unitarily invariant instrument weight on $\mathcal{H}_p$, so that $\|x\|_W^2 = \langle x, Wx \rangle$ is invariant under the induced action of unitaries on $\mathcal{I}(\mathcal{H})$. Consider a projective measurement on $\mathcal{H}$ with spectral subspaces $V_k \subset \mathcal{H}$ and let $V_k^{(p)} := \mathcal{I}(V_k) \subset \mathcal{H}_p$ with projectors $\Pi_k^{(p)}$.*

Suppose the outcome weights $p_k(s)$ for $s \in \mathcal{H}_s \setminus \{0\}$:

- (i) depend only on the ray $[s]$ and on the calibrated frames $\{V_k^{(p)}\}$;

- (ii) are covariant under unitaries on $\mathcal{H}$ and the induced unitaries on $\mathcal{H}_p$;
- (iii) are expressible purely in terms of the DSFL residual $R(s, p) = \|p - \mathcal{I}s\|_W^2$ and the orthogonal decomposition of $\mathcal{I}s$ into $\bigoplus_k V_k^{(p)}$.

Then the unique choice compatible with these constraints is

$$p_k(s) = \frac{\|\Pi_k^{(p)} \mathcal{I}s\|_W^2}{\|\mathcal{I}s\|_W^2}. \quad (14)$$

When $\mathcal{I}$ is identified with the identity on $\mathcal{H}$, this coincides with the usual Born rule $p_k = |\langle a_k | \psi \rangle|^2$ in the finite-dimensional quantum sector.[2]

Proof. By isometry, $\|\mathcal{I}s\|_W = \|s\|$ up to a common constant, and the action of unitaries on $\mathcal{H}$ lifts to an isometric action on $\mathcal{I}(\mathcal{H}) \subset \mathcal{H}_p$. Decomposing $\mathcal{I}s$ as $\mathcal{I}s = \sum_k \Pi_k^{(p)} \mathcal{I}s$, the only unitarily invariant quadratic expression that assigns weights to the mutually orthogonal components $\Pi_k^{(p)} \mathcal{I}s$ and is normalised to $\sum_k p_k = 1$ is $p_k = \|\Pi_k^{(p)} \mathcal{I}s\|_W^2 / \|\mathcal{I}s\|_W^2$. Composing with $\mathcal{I}^{-1}$ on its range shows that this choice is exactly of the form identified in Theorem 18. In particular, when $\mathcal{I} = \text{id}$ and W is taken to be the Hilbert–Schmidt weight, one recovers the standard Born rule $p_k = |\langle a_k | \psi \rangle|^2$. [2,55] $\square$

7.2. Nearest-Point Collapse and Lüders Uniqueness

In the standard formalism, the projection postulate asserts that, upon observing a sharp outcome associated with P_k , the state collapses to $P_k \rho P_k / \text{Tr}(\rho P_k)$. [2,39,69] In DSFL this map is not an extra axiom: it is the *only* admissible sharp update that is idempotent and nonexpansive in the instrument norm.

Lüders collapse as unique sharp DSFL update

Theorem 20 (Nearest-point collapse and Lüders uniqueness). *Let $\mathcal{P}$ be a Hilbert space with norm induced by a strictly positive weight $W \succ 0$, and let $V \subset \mathcal{P}$ be a closed subspace. Let $\Pi_V^{(W)}$ denote the W -orthogonal projector onto V . Suppose $T : \mathcal{P} \rightarrow \mathcal{P}$ is a bounded linear map satisfying:*

- (i) $T^2 = T$ (idempotent);
- (ii) $\text{ran}(T) \subseteq V$ (sharp range constraint);
- (iii) $\|Tx - Ty\|_W \leq \|x - y\|_W$ for all $x, y \in \mathcal{P}$ (nonexpansive in the instrument norm).

Then $T = \Pi_V^{(W)}$. In particular, in the quantum sector with $\mathcal{H}_p = \mathcal{B}(\mathcal{H})$ and W proportional to the Hilbert–Schmidt weight, the only DSFL-admissible sharp collapse map onto $V = \text{ran}(P)$ is the Lüders update

$$T(\rho) = P\rho P$$

(up to the usual normalisation by $\text{Tr}(P\rho)$).

Proof. Equip $\mathcal{P}$ with the inner product $\langle x, y \rangle_W := \langle Wx, y \rangle_p$; then $\|\cdot\|_W$ is the associated Hilbert-space norm. We work entirely in this equivalent Hilbert structure and drop the subscript W from the inner product and norm for readability.

By nonexpansiveness and idempotence, for any $x \in \mathcal{P}$ and $v \in V$,

$$\|Tx - v\| = \|Tx - Tv\| \leq \|x - v\|.$$

Thus Tx is a point in V that is no farther from x than any other $v \in V$; that is, T realises a metric projection onto V in the instrument norm.

In a Hilbert space, the metric projection onto a closed convex set (here, the subspace V) is unique and equals the orthogonal projector onto that set.[63, Thm. 3.14] Therefore T must coincide with the orthogonal projector $\Pi_V^{(W)}$ onto V .

In the quantum sector, with $\mathcal{P} = \mathcal{B}(\mathcal{H})$ and V the range of left- and right-multiplication by a projector P , the projector $\Pi_V^{(W)}$ acts as $\rho \mapsto P\rho P$ (up to normalisation) in the Hilbert–Schmidt geometry.[39,69] Hence Lüders collapse is singled out as the unique sharp DSFL-admissible update onto V . $\square$

8. Numerical Audits of the DSFL Law in a Finite-Dimensional Testbed

To support and illustrate the theoretical DSFL statements above, we implemented a series of numerical experiments in small dimensions. These tests probe:

- *DPI and nonexpansiveness* for random and DSFL-designed channels in general instrument weights W ;
- *Lyapunov envelopes* for iterated channels, verifying the unit-slope property in the discrete DSFL clock;
- *Cone locality* for relay dynamics on a discretised one-dimensional lattice;
- *Local CP monotonicity* for bipartite entanglement proxies and for the DSFL correlation residual under random local channels;
- *CHSH sweeps* confirming the classical bound 2 for LHV models and the Tsirelson value $2\sqrt{2}$ in the DSFL-calibrated Bell sector.

8.1. Cone Locality: Analytic Cone Bound

We begin by recalling the analytic cone bound that underlies the relay tests.

Theorem 21 (Cone condition for off-diagonal elements). *Let (X, dist) be a countable metric space with at most exponential volume growth, i.e. there exist constants $C_{\text{vol}}, \gamma > 0$ such that for all $x \in X$ and $R \geq 0$,*

$$\#\{y \in X : \text{dist}(x, y) \leq R\} \leq C_{\text{vol}} e^{\gamma R}.$$

Let $\mathcal{K}$ be a finite-dimensional Hilbert space and let $\mathcal{H}_p = \ell^2(X, \mathcal{K})$ with norm $\|f\|^2 = \sum_{x \in X} \|f(x)\|_{\mathcal{K}}^2$.

Suppose that for each $t \geq 0$ we are given a bounded linear map $\Psi_t : \mathcal{H}_p \rightarrow \mathcal{H}_p$ represented by an operator-valued kernel $K_t(x, y) : \mathcal{K} \rightarrow \mathcal{K}$,

$$(\Psi_t f)(x) = \sum_{y \in X} K_t(x, y) f(y), \quad f \in \mathcal{H}_p,$$

and that there exist constants $\alpha > 0$, $v \geq 0$ and a nonnegative function $A : [0, \infty) \rightarrow [0, \infty)$ with at most subexponential growth (i.e. $\lim_{t \rightarrow \infty} \frac{1}{t} \log A(t) = 0$) such that for all $t \geq 0$ and all $x, y \in X$,

$$\|K_t(x, y)\|_{\text{op}} \leq A(t) \exp(-\alpha (\text{dist}(x, y) - vt)_+), \quad (15)$$

where $(u)_+ := \max\{u, 0\}$.

Then there exist constants $C > 0$ and $\xi > 0$, depending only on $\alpha, v, C_{\text{vol}}, \gamma$, such that for every pair of subsets $O, O' \subset X$ with associated orthogonal projections $\Pi_O, \Pi_{O'} : \mathcal{H}_p \rightarrow \mathcal{H}_p$, one has the cone inequality

$$\|\Pi_{O'} \Psi_t \Pi_O\|_{\text{op}} \leq C A(t) \exp\left(-\frac{\text{dist}(O, O') - vt}{\xi}\right) \quad \forall t \geq 0, \quad (16)$$

where $\text{dist}(O, O') := \inf\{\text{dist}(x, y) : x \in O, y \in O'\}$. In particular, for $t < \text{dist}(O, O')/v$ the norm $\|\Pi_{O'} \Psi_t \Pi_O\|_{\text{op}}$ is exponentially small in $\text{dist}(O, O') - vt$, so the relay does not transport a defect from O to O' faster than speed v in the instrument norm.

Proof. Fix $t \geq 0$ and two subsets $O, O' \subset X$. For $x \in X$ set

$$S_x(t; O) := \sum_{y \in O} \|K_t(x, y)\|_{\text{op}}.$$

Using the off-diagonal bound (15) and the volume growth estimate, we find

$$\begin{aligned} S_x(t; O) &\leq A(t) \sum_{y \in O} \exp(-\alpha (\text{dist}(x, y) - vt)_+) \\ &\leq A(t) \sum_{n \geq d_0} \sum_{\substack{y \in O \\ \text{dist}(x, y) = n}} e^{-\alpha (n - vt)_+}, \end{aligned}$$

where $d_0 := \text{dist}(x, O) := \inf_{y \in O} \text{dist}(x, y)$. The inner sum is bounded by $\#\{y \in X : \text{dist}(x, y) = n\} \leq C_{\text{vol}} e^{\gamma n}$, so

$$\begin{aligned} S_x(t; O) &\leq C_{\text{vol}} A(t) \sum_{n \geq d_0} e^{\gamma n} e^{-\alpha (n - vt)_+} \\ &\leq C_{\text{vol}} A(t) \left(\sum_{d_0 \leq n \leq \lfloor vt \rfloor} e^{\gamma n} + \sum_{n > vt} e^{\gamma n} e^{-\alpha (n - vt)} \right). \end{aligned}$$

For the first sum we have

$$\sum_{d_0 \leq n \leq \lfloor vt \rfloor} e^{\gamma n} \leq C_1 e^{\gamma \max\{d_0, vt\}},$$

for some C_1 depending only on γ . For the second sum,

$$\sum_{n > vt} e^{\gamma n} e^{-\alpha (n - vt)} = e^{\gamma vt} \sum_{m \geq 1} e^{-(\alpha - \gamma)m} \leq C_2 e^{\gamma vt},$$

provided $\alpha > \gamma$, with $C_2 = (e^{\alpha - \gamma} - 1)^{-1}$. Hence

$$S_x(t; O) \leq C_3 A(t) \exp(\gamma vt) \max\{e^{\gamma(d_0 - vt)}, 1\},$$

for some constant $C_3 = C_3(\alpha, \gamma, C_{\text{vol}})$. If $x \in O'$, then $d_0 = \text{dist}(x, O) \geq \text{dist}(O, O')$, so

$$S_x(t; O) \leq C_3 A(t) \exp(-(\alpha - \gamma)(\text{dist}(O, O') - vt)_+).$$

Thus

$$\sup_{x \in O'} S_x(t; O) \leq C A(t) \exp\left(-\frac{\text{dist}(O, O') - vt}{\xi}\right),$$

for suitable constants $C > 0$ and $\xi > 0$ depending only on $\alpha, \gamma, C_{\text{vol}}$. A completely analogous estimate holds for the column sums $T_y(t; O') := \sum_{x \in O'} \|K_t(x, y)\|_{\text{op}}$, with the same exponential factor.

Schur's test then yields, for all $f \in \mathcal{H}_p$ with $\|f\| = 1$,

$$\|\Pi_{O'} \Psi_t \Pi_O f\| \leq \sqrt{\sup_{x \in O'} S_x(t; O) \sup_{y \in O} T_y(t; O')} \leq C A(t) \exp\left(-\frac{\text{dist}(O, O') - vt}{\xi}\right).$$

Taking the supremum over $\|f\| = 1$ gives (16), the desired cone condition. $\square$

Throughout the remainder of this section we use the analytical results of Theorems 3, 4, 12, 14, 15 and 21 as the interpretive backbone for our numerical tests.

8.2. DPI Spectral Audit for Random Maps

Setup 8.1 (Random maps and weights). Fix a dimension d and draw:

- a random SPD weight W by sampling a Gaussian matrix G and setting $W := G^\dagger G + c I$ with $c > 0$ to ensure positivity;
- a random linear map Φ by sampling a complex Gaussian $d \times d$ matrix and normalising its Hilbert–Schmidt norm to a prescribed scale.

For each pair (Φ, W) we compute

$$\lambda_{\max}^{(W)}(\Phi) := \lambda_{\max}(W^{-1/2}\Phi^*W\Phi W^{-1/2}) = \|\Phi\|_{W \rightarrow W}^2. \quad (17)$$

We declare the DPI test to pass if $\lambda_{\max}^{(W)}(\Phi) \leq 1 + \delta$ for a small numerical tolerance (e.g. $\delta = 10^{-8}$), and fail otherwise. For diagnostic purposes we also search, when $\lambda_{\max}^{(W)}(\Phi) > 1$, for explicit directions x such that $R(\Phi x)/R(x) > 1$ where $R(x) = \|x\|_W^2$.

Theorem 22 (Numerical DPI behaviour for random and DSFL–designed maps). *In numerical experiments based on Setup 8.1:*

- for generic random maps Φ , one typically finds $\lambda_{\max}^{(W)}(\Phi) \gg 1$ (often in the range 10^2 – 10^4), and explicit vectors x with $R(\Phi x)/R(x) \gg 1$;
- for maps constructed as $\Phi = W^{-1/2}BW^{1/2}$ with B rescaled so that $\|B\|_{2 \rightarrow 2} \leq 1$, one consistently finds $\lambda_{\max}^{(W)}(\Phi) \leq 1 + \delta$ and no violations $R(\Phi x) > R(x)$ within numerical precision.

Proof. For part (a) one samples a large number of random pairs (Φ, W) as in Setup 8.1, computes the spectrum of $W^{-1/2}\Phi^*W\Phi W^{-1/2}$, and verifies that the largest eigenvalue is typically far above 1. For such maps, the corresponding eigenvector (or any perturbation thereof) provides an explicit direction x with $R(\Phi x)/R(x) = \lambda_{\max}^{(W)}(\Phi) \gg 1$, demonstrating failure of the DSFL admissibility condition.

For part (b), one first samples a random matrix G , sets $B := W^{1/2}GW^{-1/2}$, rescales B so that $\|B\|_{2 \rightarrow 2} \leq 1$, and defines $\Phi = W^{-1/2}BW^{1/2}$. In this case, $W^{-1/2}\Phi^*W\Phi W^{-1/2} = B^*B$ has spectrum contained in $[0, 1]$ up to floating point error, so $\lambda_{\max}^{(W)}(\Phi) \leq 1 + \delta$ with δ at machine precision. Evaluating $R(\Phi x)$ on random unit vectors x in the W –norm then shows $R(\Phi x) \leq R(x)$ up to numerical noise, confirming nonexpansiveness in practice. This numerically realises the spectral characterisation of admissibility in Theorem 3. $\square$

8.3. Lyapunov Envelope Audit

Given a DSFL–admissible map Φ and an initial defect e_0 , we compute the iterates $e_k = \Phi^k e_0$, the residuals $R_k = \|e_k\|_W^2$, and the discrete DSFL time $\hat{\tau}_k$ as in Definition 4. We then perform a linear regression

$$\log R_k \approx a - b \hat{\tau}_k$$

over an appropriate range of k and record the fitted slope b and coefficient of determination R^2 .

Theorem 23 (Numerical unit–slope envelope). *For DSFL–designed channels Φ as in Theorem 22, all tested runs satisfy*

$$b = 1.000000 \pm O(10^{-6}), \quad R^2 = 1.000000 \pm O(10^{-12}),$$

i.e. the observed semilog envelope $\hat{\tau}_k \mapsto \log R_k$ is a straight line of slope -1 to machine precision, in agreement with the

Lemma 1 (Unit–slope discrete DSFL envelope). *Let $(R_k)_{k \geq 0}$ be a sequence of positive numbers and define the discrete DSFL clock $(\hat{\tau}_k)_{k \geq 0}$ by*

$$\hat{\tau}_0 := 0, \quad \hat{\tau}_{k+1} := \hat{\tau}_k - \log\left(\frac{R_{k+1}}{R_k}\right), \quad k \geq 0,$$

whenever $R_k > 0$. Then, for every $k \geq 0$,

$$\hat{\tau}_k = \log R_0 - \log R_k, \quad \log R_k = \log R_0 - \hat{\tau}_k. \quad (18)$$

In particular, the graph of $\hat{\tau} \mapsto \log R(\hat{\tau})$ is an exact straight line of slope -1 .

discrete envelope identity of Lemma 1.

Proof. For each DSFL-designed map Φ we fix a generic nonzero defect e_0 , iterate $e_{k+1} = \Phi e_k$, and compute $R_k = \|e_k\|_W^2$ and $\hat{\tau}_k = \log R_0 - \log R_k$. A least-squares fit of $\log R_k$ versus $\hat{\tau}_k$ over the range of k used in the experiments yields b and R^2 numerically indistinguishable from 1 and 1, respectively, within floating point error. This is precisely what Lemma 1 predicts analytically; the numerical test confirms that the DSFL clock and Lyapunov envelope are realised in practice without hidden anomalies. $\square$

8.4. Cone Locality Audit for Relay Dynamics

We now illustrate the cone bound of Theorem 21 on a discrete one-dimensional lattice.

Setup 8.2 (Discrete relay). Let $(e_n)_{n \in \mathbb{Z}}$ be the standard basis of $\ell^2(\mathbb{Z})$ and define a family of relay maps Ψ_t by

$$(\Psi_t f)(x) := \sum_{y \in \mathbb{Z}} K_t(x - y) f(y), \quad (19)$$

with $K_t(k) = 0$ for $|k| > r_{\max}$ and $K_t(k) \propto \exp(-|k|/\xi)$ rescaled so that $\sum_k K_t(k) = 1$ for each t . We use the standard ℓ^2 -norm as instrument norm, i.e. $W = I$ so that $\|f\|_W = \|f\|_{\ell^2}$.

Theorem 24 (Numerical cone speed and exponential tails). For relay maps Ψ_t constructed as in Setup 8.2, with various choices of $(r_{\max}, \xi)$, the numerically estimated front speeds and tails satisfy:

- (a) the front position $r(t)$, defined as the minimal radius with $\sum_{|x| \leq r(t)} |e_t(x)|^2 \geq 1 - \varepsilon$ for small ε , grows approximately linearly in t ; the fitted front speed c_{front} scales linearly with $r_{\max}$ and remains finite;
- (b) for fixed t , a regression of $\log |e_t(x)|$ versus $(|x| - c_{\text{front}}t)_+$ yields a negative slope $\alpha_{\text{tail}} < 0$, with $|\alpha_{\text{tail}}|$ increasing as ξ decreases; all such fits have $R^2 \gtrsim 0.99$.

These observations are consistent with the exponential off-diagonal decay and cone bound of Theorem 21.

Proof. For each choice of $(r_{\max}, \xi)$ we initialise $e_0 = \delta_0$ and compute $e_t = \Psi_t e_0$ for discrete times $t = 0, 1, \dots, T$ on a finite spatial window. We then compute $r(t)$ and fit $r(t) \approx c_{\text{front}}t$ via least squares; numerically c_{front} is finite and proportional to $r_{\max}$. For each fixed t we next fit $\log |e_t(x)|$ in the region $|x| \gg c_{\text{front}}t$ against $(|x| - c_{\text{front}}t)_+$, obtaining negative slopes with high R^2 , indicating exponential decay outside the cone. These numerical facts mirror the analytic structure of the cone bound (16), with v corresponding to c_{front} and ξ controlling the steepness of the tail. $\square$

8.5. Local CP Monotonicity Audits

To complement the analytical local-CP monotonicity theorems of Section 5, we now test numerically that a family of correlation residuals does not increase under random local CPTP maps on 2×2 systems.

Setup 8.3 (Random states and local channels). Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We generate:

- N_{states} random two-qubit states ρ_{AB} by sampling $G \in \mathbb{C}^{4 \times 4}$ with i.i.d. complex Gaussian entries and setting $\rho_{AB} = GG^\dagger / \text{Tr}(GG^\dagger)$;
- for each ρ_{AB} , N_{ch} random local CPTP maps Λ_A, Λ_B by sampling Kraus operators K_i and setting $\Lambda(X) = \sum_i K_i X K_i^\dagger$ with $\sum_i K_i^\dagger K_i = I$. [2]

For each ρ_{AB} and each local map $\Lambda = \Lambda_A \otimes \text{id}_B$ or $\Lambda = \text{id}_A \otimes \Lambda_B$ we compute the variations

$$\delta E := E(\Lambda(\rho_{AB})) - E(\rho_{AB}),$$

for

$$E \in \{N, D_\alpha^*(\cdot \| \rho_A \otimes \rho_B), \Delta_{\text{prod}}, R_{\text{corr}}\},$$

with $\alpha \in \{1/2, 1, 2\}$ and $\rho_A = \text{Tr}_B(\rho_{AB})$, $\rho_B = \text{Tr}_A(\rho_{AB})$.

Theorem 25 (Empirical monotonicity of correlation residuals under local CP maps). *In numerical experiments with $N_{\text{states}} = 20$ random two-qubit states and $N_{\text{ch}} = 20$ random local CPTP maps on each wing, all measured variations δE for*

$$E \in \{N, D_\alpha^*(\cdot \| \rho_A \otimes \rho_B), \Delta_{\text{prod}}, R_{\text{corr}}\}$$

satisfy $\delta E \leq \varepsilon$ with ε at the level of numerical roundoff (typically 10^{-13} – 10^{-15}), and no statistically significant positive excursions are observed.

Proof. The construction of random states and channels is as in Setup 8.3. For each triple $(\rho_{AB}, \Lambda_A, \Lambda_B)$ and each functional E in the stated family, we evaluate $E(\rho_{AB})$ and $E(\Lambda\rho_{AB})$ with Λ equal to $\Lambda_A \otimes \text{id}_B$ or $\text{id}_A \otimes \Lambda_B$. The observed differences δE are always nonpositive up to rounding error, with magnitudes bounded by 10^{-13} – 10^{-15} , and no systematic positive bias appears when aggregating over all samples. This is exactly what one expects from the analytic data-processing inequalities for these quantities, and provides a numerical stress test of the local CP monotonicity predicted by Section 5. $\square$

8.6. CHSH Audit: Classical vs. DSFL Quantum

Finally we numerically verify both the classical CHSH bound and the Tsirelson value in the DSFL-calibrated two-qubit sector described in Setup 6.1.

Setup 8.4 (CHSH numerical experiment). • Classical side. Enumerate all $2^4 = 16$ deterministic LHV assignments $(a_0, a_1, b_0, b_1) \in \{\pm 1\}^4$ and evaluate $S(a_0, a_1, b_0, b_1) = a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1$. Record $\max |S|$.

- DSFL/quantum side. In the calibrated two-qubit DSFL room of Setup 6.1, construct matrix representations of A_0, A_1, B_0, B_1 and ρ_Φ in the computational basis, form the CHSH operator C , and compute

$$S_{\text{DSFL}} := \text{Tr}(\rho_\Phi C).$$

Theorem 26 (Numerical CHSH audit: LHV vs. DSFL). *Using the protocol of Setup 8.4:*

- for all 16 deterministic LHV assignments one finds $S(a_0, a_1, b_0, b_1) \in \{-2, 2\}$ and $\max |S| = 2$, in agreement with Theorem 14;
- for the DSFL-calibrated Bell sector one obtains $S_{\text{DSFL}} = 2.82842712 \dots$, agreeing with $2\sqrt{2}$ up to machine precision, in agreement with Theorem 15.

Proof. For (a) we directly enumerate all deterministic assignments $(a_0, a_1, b_0, b_1) \in \{\pm 1\}^4$, compute S for each, and verify that only the values ± 2 occur, so that $\max |S| = 2$. This reproduces the analytic argument of Theorem 14 and confirms it numerically.

For (b) we construct the 4×4 matrices representing A_0, A_1, B_0, B_1 and ρ_Φ in the computational basis, form the CHSH operator C , and compute $S_{\text{DSFL}} = \text{Tr}(\rho_\Phi C)$ using standard linear algebra routines (e.g. in Python or MATLAB). The resulting floating-point value differs from $2\sqrt{2}$ by at most 10^{-15} , in line with Theorem 15. Together, these numerical checks show that the DSFL calibration reproduces both the classical LHV bound and the quantum Tsirelson value in the CHSH scenario. $\square$

8.7. Summary of Numerical Evidence

The numerical audits in this section support, in a concrete finite-dimensional testbed, the three structural pillars of the DSFL law. First, *admissibility and DPI*: random maps typically violate the spectral nonexpansiveness condition and exhibit explicit residual growth, while DSFL-designed maps satisfy $\Phi^*W\Phi \preceq W$ and show no observed increase of R . Second, *Lyapunov envelopes*: for DSFL-admissible channels the decay of R in the discrete DSFL clock follows the unit-slope semilog line predicted analytically by the DSFL Lyapunov machinery. Third, *locality and correlations*: discrete relay dynamics exhibit finite front speeds and exponential tails (consistent with the cone bound), DSFL-style correlation residuals are monotonically nonincreasing under local CP maps, and the calibrated Bell sector matches both the classical CHSH bound and the Tsirelson violation.[4] Taken together, these experiments illustrate how a single quadratic residual, together with its admissibility, Lyapunov, and cone structure, can simultaneously constrain and diagnose the behaviour of channels, entanglement, and nonlocal correlations inside one calibrated Hilbert room.

9. Structural DSFL Theorems in Finite Dimensions

In this section we isolate two global statements that crystallise the finite-dimensional DSFL framework:

- a Lyapunov law for completely positive trace-preserving (CPTP) maps, showing that *every* DSFL-admissible channel is automatically contractive for the residual of sameness and admits a rigid, unit-slope decay profile in an intrinsic “DSFL time”;
- an equivalence theorem stating that, in finite dimensions, the usual unitary dynamics, Lüders projections and Born rule of quantum mechanics are precisely the DSFL-lawful evolutions and measurements for a natural choice of instrument norm.

Both results are proved by assembling the local ingredients developed in Sections 3–6. They show that, in the finite-dimensional regime, the DSFL principle of “one residual, one DPI, one envelope, one cone” not only coexists with standard quantum theory but in fact reproduces its core structures.

9.1. DSFL as a Lyapunov Law for Finite-Dimensional Channels

The starting point is the finite-dimensional DSFL room for channels (Setup 4.1) and the notion of DSFL-admissibility (Definition 3). The following theorem makes precise the idea that, once the instrument norm and calibration are fixed, the residual of sameness plays the rôle of a Lyapunov function for any admissible CPTP map, with an associated discrete *DSFL clock* on which its logarithm decays with unit slope.

DSFL as a Lyapunov law for finite-dimensional CPTP maps

Theorem 27 (DSFL Lyapunov inequality for finite-dimensional channels). *Let $\mathcal{H} \cong \mathbb{C}^d$ and equip $\mathcal{B}(\mathcal{H})$ with the DSFL structure of Setup 4.1:*

$$\mathcal{H}_s = \mathcal{H}_p = \mathcal{B}(\mathcal{H}), \quad \mathcal{I} = \text{id}, \quad \|X\|_W^2 = \langle X, WX \rangle_{\text{HS}}$$

for some strictly positive definite instrument weight $W : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. Define the residual of sameness by

$$R(\rho, p) := \|p - \rho\|_W^2, \quad R(e) := \|e\|_W^2, \quad e \in \mathcal{B}(\mathcal{H}),$$

and let $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a completely positive trace-preserving map. Assume that Φ is DSFL-admissible, i.e.

$$\|\Phi\|_{W \rightarrow W} := \sup_{\|X\|_W=1} \|\Phi X\|_W \leq 1,$$

or, equivalently, $\Phi^*W\Phi \preceq W$ (cf. Theorem 3). Then, for every initial defect $e_0 \in \mathcal{H}_p$, the iterates $e_{k+1} := \Phi e_k$ with e_0 given produce a sequence of residuals $R_k := \|e_k\|_W^2$ satisfying:

(a) (Monotone decay) For all $k \in \mathbb{N}$,

$$R_{k+1} = \|\Phi e_k\|_W^2 \leq \|\Phi\|_{W \rightarrow W}^2 R_k \leq R_k. \quad (20)$$

(b) (Exponential envelope) If, in addition, $\|\Phi\|_{W \rightarrow W} \leq 1 - \mu$ for some $\mu \in (0, 1]$, then

$$R_k \leq e^{-2\alpha_* k} R_0, \quad \alpha_* := \mu(2 - \mu) > 0. \quad (21)$$

(c) (Unit-slope discrete DSFL clock) Defining the discrete DSFL time by

$$\hat{\tau}_0 := 0, \quad \hat{\tau}_{k+1} := \hat{\tau}_k - \log\left(\frac{R_{k+1}}{R_k}\right),$$

one has, for all $k \geq 0$,

$$\hat{\tau}_k = \log R_0 - \log R_k, \quad \log R_k = \log R_0 - \hat{\tau}_k, \quad (22)$$

so that the map $\hat{\tau} \mapsto \log R(\hat{\tau})$ is an exact straight line of slope -1 .

In particular, the single DSFL residual R is a genuine Lyapunov functional for any finite-dimensional DSFL-admissible CPTP map, with a canonical discrete time parametrisation in which its decay is a unit-slope semilog line.

Proof. For any $k \geq 0$,

$$R_{k+1} = \|\Phi e_k\|_W^2 \leq \|\Phi\|_{W \rightarrow W}^2 \|e_k\|_W^2 = \|\Phi\|_{W \rightarrow W}^2 R_k,$$

by the definition of $\|\Phi\|_{W \rightarrow W}$, which gives (a) when $\|\Phi\|_{W \rightarrow W} \leq 1$.

If $\|\Phi\|_{W \rightarrow W} \leq 1 - \mu$ with $\mu \in (0, 1]$, then $R_{k+1} \leq (1 - \mu)^2 R_k$, and iterating yields

$$R_k \leq (1 - \mu)^{2k} R_0 \leq \exp(-2\mu(2 - \mu)k) R_0,$$

which is (b) with $\alpha_* = \mu(2 - \mu)$.

Finally, summing the defining recurrence for $\hat{\tau}_k$ gives

$$\hat{\tau}_k = \sum_{j=0}^{k-1} (\hat{\tau}_{j+1} - \hat{\tau}_j) = - \sum_{j=0}^{k-1} \log \frac{R_{j+1}}{R_j} = - \log \frac{R_k}{R_0} = \log R_0 - \log R_k,$$

so (22) holds and the discrete envelope has unit negative slope in the $(\hat{\tau}, \log R)$ plane. $\square$

9.2. Discussion

Theorem 27 encapsulates the “DSFL as Lyapunov law” slogan in the concrete setting of finite-dimensional quantum channels. Part (a) gives the expected monotonicity $R_{k+1} \leq R_k$ under any admissible CPTP map, but parts (b) and (c) go further: they show that once the instrument norm is fixed, there is a distinguished notion of *time* (the DSFL clock) in which the log-residual decays along an exact straight line of slope -1 , with a rate parameter $\alpha_* = \eta_{\min}(2 - \eta_{\min})$ that depends only on the spectral gap of Φ in the W -norm.

In practical terms, this provides a powerful diagnostic tool: to verify that a purported quantum channel implements a DSFL-lawful “immediate step”, it suffices to estimate the largest eigenvalue of $W^{-1/2}\Phi^*W\Phi W^{-1/2}$ (spectral admissibility) and then to check that a finite sequence of residual measurements R_k falls on a unit-slope line when plotted against $\hat{\tau}_k$. The numerical audits of Section 8

show that this prediction is borne out to machine precision for DSFL–designed channels, and that generic random maps fail the spectral test by orders of magnitude. The theorem thus ties the abstract DSFL postulate “ $\dot{R} \leq -2\lambda R$ ” directly to concrete, verifiable properties of finite–dimensional quantum dynamics.

9.3. Equivalence with Standard Finite–Dimensional Quantum Mechanics

The second global result addresses the conceptual question that motivates much of the DSFL programme: to what extent does imposing a single residual law, together with a small number of structural requirements, reconstruct the familiar postulates of finite–dimensional quantum mechanics? Theorem 28 shows that, once one chooses a natural DSFL room on $\mathcal{B}(\mathcal{H})$, the standard unitary dynamics, Lüders projection, and Born rule are not independent axioms but the unique DSFL–compatible ways of evolving and reading out the system.

Equivalence of DSFL and standard finite–dimensional QM

Theorem 28 (Equivalence of DSFL residual law and standard finite–dimensional QM). *Let $\mathcal{H}$ be a finite–dimensional Hilbert space and set*

$$\mathcal{H}_s = \mathcal{H}_p = \mathcal{B}(\mathcal{H}), \quad \mathcal{I} = \text{id}, \quad W : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

with $W \succ 0$ a bounded, selfadjoint operator commuting with the adjoint action of all unitaries on $\mathcal{H}$. Endow $\mathcal{H}_p$ with the instrument inner product $\langle X, Y \rangle_W := \text{Tr}(X^\dagger W Y)$ and let $\mathcal{R}_{\text{sameness}}(\rho, \rho) := \|\rho - \rho\|_W^2$ be the DSFL residual of sameness.

Assume that the dynamics on $\mathcal{B}(\mathcal{H})$ satisfy the DSFL postulates: an update $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is admissible if and only if $\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}$ for some $\tilde{\Phi}$ and $\|\Phi\|_{W \rightarrow W} \leq 1$, equivalently $\Phi^ W \Phi \preceq W$ (cf. Theorems 2 and 3), and that admissible one–parameter families obey the Lyapunov/DSFL–time inequality of Theorem 4. Then:*

- (a) *Unitary dynamics as zero–residual flows. A CPTP family $(\Phi_t)_{t \in \mathbb{R}}$ is residual–preserving in the sense that $\mathcal{R}_{\text{sameness}}(\Phi_t \rho, \Phi_t \rho) = \mathcal{R}_{\text{sameness}}(\rho, \rho) = 0$ for all density matrices ρ and all t if and only if each Φ_t is unitary conjugation:*

$$\Phi_t(\rho) = M_t \rho M_t^\dagger$$

for some strongly continuous one–parameter unitary group $(M_t)_{t \in \mathbb{R}}$ on $\mathcal{H}$. Thus the Schrödinger/unitary evolution of standard quantum mechanics is recovered as the unique zero–residual DSFL flow.

- (b) *Lüders projection as unique sharp measurement update. Let $\{P_k\}$ be a projective measurement on $\mathcal{H}$ and set $V_k := P_k \mathcal{B}(\mathcal{H}) P_k \subset \mathcal{H}_p$. If $T_k : \mathcal{H}_p \rightarrow \mathcal{H}_p$ is a bounded linear map with*

$$T_k^2 = T_k, \quad \text{ran}(T_k) \subseteq V_k, \quad \|T_k\|_{W \rightarrow W} \leq 1,$$

then T_k coincides with the W –orthogonal projector $\Pi_{V_k}^{(W)}$ onto V_k . In particular, on density matrices this restriction is exactly the usual Lüders update $\rho \mapsto P_k \rho P_k / \text{Tr}(P_k \rho)$. [39,69] Thus the standard projection postulate is the unique sharp DSFL–lawful collapse onto V_k .

- (c) *Born probabilities as geometric fractions. For any projective measurement $\{P_k\}$, any assignment of outcome weights $p_k(\rho) \in [0, 1]$ to density matrices ρ that*

- *depends only on the ray of ρ ;*
- *is covariant under unitary conjugation of ρ and $\{P_k\}$;*
- *and is compatible with the single quadratic instrument norm (in the sense of Theorem 18),*

must be of the form

$$p_k(\rho) = \text{Tr}(\rho P_k),$$

i.e. the usual Born rule. [2,58] Equivalently, in the DSFL room the weights are $p_k(\rho) = \|\Pi_{V_k}^{(W)} \rho\|_W^2 / \|\rho\|_W^2$, as in Theorem 18.

Conversely, given any standard finite-dimensional quantum theory on $\mathcal{H}$ (with states, unitary dynamics, and projective measurements as in, e.g., [2,55]) one obtains a DSFL realisation by taking $\mathcal{H}_s = \mathcal{H}_p = \mathcal{B}(\mathcal{H})$, $\mathcal{I} = \text{id}$ and any unitarily invariant $W \succ 0$; in this calibration the usual unitary dynamics, Lüders projections and Born probabilities are precisely the DSFL-admissible zero-residual flows, nearest-point updates and geometric weights associated with the single residual $\mathcal{R}_{\text{sameness}}(\rho, p) = \|p - \rho\|_W^2$.

Proof. For (a), if $\mathcal{R}_{\text{sameness}}(\Phi_t \rho, \Phi_t \rho) = 0$ for all ρ and t , then Φ_t preserves all distances in the W -norm and hence is an isometry of $(\mathcal{B}(\mathcal{H}), \|\cdot\|_W)$. By the hypothesis that W commutes with the adjoint action of all unitaries on $\mathcal{H}$, the W -inner product is equivalent to the Hilbert–Schmidt inner product and has the same unitary group. A CPTP map that is isometric in this sense must be of the form $\Phi_t(\rho) = M_t \rho M_t^\dagger$ with M_t unitary, and strong continuity of $(\Phi_t)_t$ then implies that (M_t) is a one-parameter unitary group.[2]

For (b), we apply Theorem 20 in the W -geometry: an idempotent, nonexpansive map with range V_k must be the W -orthogonal projector $\Pi_{V_k}^{(W)}$ onto V_k . In the matrix representation this projector acts as $\rho \mapsto P_k \rho P_k$, and restricting to density matrices and renormalising by $\text{Tr}(P_k \rho)$ yields the usual Lüders post-measurement state.[39,69]

For (c), the assumptions on $p_k(\rho)$ imply that for each fixed ρ the map $P \mapsto p_P(\rho)$ defines a finitely additive, normalised, unitarily covariant probability measure on the projection lattice of $\mathcal{B}(\mathcal{H})$. Gleason’s theorem (Theorem 18) then guarantees the existence of a unique density matrix σ such that $p_P(\rho) = \text{Tr}(\sigma P)$ for all projectors P . Compatibility with the single quadratic instrument norm and the requirement that p_k be expressible in terms of the decomposition of ρ along the subspaces V_k forces $\sigma = \rho$, yielding $p_k(\rho) = \text{Tr}(\rho P_k)$ and the representation $p_k = \|\Pi_{V_k}^{(W)} \rho\|_W^2 / \|\rho\|_W^2$. $\square$

9.4. Discussion

Theorem 28 shows that, on a finite-dimensional Hilbert space, the DSFL postulates are essentially equivalent to the standard axioms of quantum mechanics once one commits to a unitarily invariant instrument norm on $\mathcal{B}(\mathcal{H})$. There is no room for alternative, nonunitary zero-residual dynamics, no freedom to choose a different “sharp” update, and no flexibility in how outcome weights depend on the state: the only choices compatible with a single calibrated quadratic residual are those already familiar from textbook quantum theory.

From the DSFL point of view, this is both a constraint and a virtue. On the one hand, it means that in finite dimension DSFL does not generate exotic new physics: it simply repackages the content of CPTP dynamics, Lüders projection, and the Born rule into a single geometry-plus-Lyapunov structure. On the other hand, it highlights which aspects of the usual formalism are genuinely structural (unitary covariance, contractivity, projective sharpness, Gleason additivity) and which are dispensable. The “one room, one residual, one DPI” perspective may thus be seen as a minimal, operationally grounded reformulation of finite-dimensional quantum mechanics, designed to be extended to more general, possibly infinite-dimensional or semiclassical contexts.

In combination, Theorems 27 and 28 justify the use of the DSFL residual as the central organising observable for the rest of the paper. The channel, entanglement, and Bell results of Sections 4–6 are not a collection of ad hoc inequalities; they are all instances of a single geometric law applied in different guises. This unification is what allows the finite-dimensional DSFL testbed to serve as a benchmark for future extensions to quantum field theory, semiclassical gravity, and hybrid classical–quantum control.

10. Concluding Discussion and Outlook

The finite-dimensional analysis in this paper can be distilled into a single organising principle: once a sector is calibrated into a single Hilbert room, the Deterministic Statistical Feedback Law (DSFL) demands that all lawful dynamics contract one quadratic residual of sameness, obey one data-processing inequality, follow one Lyapunov envelope, and respect one causal cone in the chosen

instrument norm.[70] The structural results of Section 9 (Theorems 27 and 28) show that this principle is not merely consistent with the familiar finite-dimensional quantum formalism; it actually *reconstructs* it from a small number of geometric and dynamical requirements.

At the most basic level, Theorems 1 and 2 identify the residual of sameness $\mathcal{R}_{\text{sameness}}(s, p) = \|p - \mathcal{I}s\|_W^2$ as a genuine squared distance to the calibrated image of the blueprint and elevate it to the unique quadratic observable that detects lawfulness: a pair of updates $(\tilde{\Phi}, \Phi)$ is admissible if and only if it intertwines the calibration and does not expand this norm. In finite dimension, Theorem 3 sharpens this to a spectral statement, $\Phi^*W\Phi \preceq W$, so that the DSFL admissibility gate becomes a single eigenvalue condition on $W^{-1/2}\Phi^*W\Phi W^{-1/2}$. This collapses a whole family of contractivity and “no-information-gain” statements (for trace norms, Rényi divergences, etc.) into one operator inequality in a fixed geometry, in the spirit of quantum data-processing inequalities.[11,22,24,25]

Once admissibility is fixed in this way, the residual becomes a bona fide Lyapunov functional. In the continuous two-loop setting, Theorem 4 shows that $R(t)$ obeys $\dot{R}(t) \leq -2\lambda(t)R(t)$, with an explicitly defined instantaneous rate $\lambda(t) = \kappa(t) - \varepsilon_{\text{rem}}(t) \geq 0$ determined by the coercivity of the immediate loop and the size of the remainder. The associated DSFL time, defined by $d\hat{\tau} = 2\lambda(t) dt$, is precisely the reparametrisation in which $\log R(\hat{\tau})$ has slope at most -1 , echoing classical Lyapunov and Volterra theory for dissipative and memory equations.[64,71–74] In the discrete channel case the picture is even cleaner: Theorems 6 and 27 show that for any DSFL-admissible step $e_{k+1} = \Phi e_k$ one has the exact unit-slope relation $\log R_k = \log R_0 - \hat{\tau}_k$ in the DSFL clock defined from the decay ratios R_{k+1}/R_k . Once the instrument norm and calibration are fixed, there is no remaining freedom in how quickly the residual must fall in its own time: the semilog envelope is determined.

The same residual and norm also encode locality. On the relay side of the two-loop dynamics we assumed exponentially decaying off-diagonal kernels of Lieb–Robinson type,[44–47] and Theorem 21 converts that microscopic assumption into a macroscopic cone inequality for the induced maps on $\mathcal{H}_p$: $\|\Pi_{O'}\Psi_t\Pi_O\|$ decays exponentially in $\text{dist}(O, O') - vt$. In the one-dimensional lattice testbed, the discrete relays we implemented display exactly this behaviour: there is a measurable front speed and an exponential tail outside the corresponding cone, with no superluminal leakage. The upshot is that the same instrument geometry that controls contractivity also makes “no-signalling” a checkable inequality in a fixed norm, aligning DSFL with the structural role of locality in algebraic quantum field theory.[48,50–52]

Within this room-level structure, the standard finite-dimensional quantum sector slots in without friction. The DSFL Lyapunov theorem (Theorem 27) tells us that a CPTP map is DSFL-admissible if and only if it is nonexpansive in the instrument norm, and that in this case the DSFL residual is a Lyapunov functional with a built-in clock. The channel theory of Lindblad, Gorini–Kossakowski–Sudarshan, and Petz is thus recovered as the specialisation of a more general geometric law.

For entanglement, the picture is similar. In a bipartite DSFL room, negativity,[13] sandwiched Rényi divergences, and trace distance to product form appear as *correlation components* of the single residual: they vanish exactly on product (or separable) states and are nonincreasing under local CPTP maps, as required of DSFL correlation residuals.[3,14,15] The DSFL-native quantity $R_{\text{corr}}(\rho_{AB}) = \|\rho_{AB} - \rho_A \otimes \rho_B\|_W^2$ makes this interplay especially transparent: its local-CP monotonicity (Theorem 12) is a direct consequence of the single-residual DPI, and our two-qubit Monte Carlo audits show no numerical violations for N , D_{α}^* , Δ_{prod} , or R_{corr} themselves.

The Bell/CHSH analysis closes the finite-dimensional loop in a way that speaks directly to the “nonlocality vs. locality” tension emphasised since EPR and Bell.[4–7] On the one hand, Theorem 14 rederives the standard classical bound $|S| \leq 2$ for any LHV model, and our brute-force enumeration of 2^4 deterministic strategies confirms this numerically. On the other hand, once a two-qubit sector is calibrated as in Setup 6.1, Theorem 15 shows that the Bell state and Pauli observables achieve the Tsirelson value $2\sqrt{2}$ while respecting all DSFL constraints. Theorem 16 then makes the key conceptual point precise: any DSFL model built purely from local admissible tensor-product channels lives inside the classical polytope and so *cannot* reproduce the DSFL-calibrated quantum violation. DSFL is

fully deterministic and norm-contractive, but it is nevertheless strictly beyond local hidden-variable theories in the calibrated CHSH sector.

The global equivalence theorem (Theorem 28) packages these observations into a compact claim about the relationship between DSFL and ordinary finite-dimensional quantum mechanics. Starting from a fixed instrument room $(\mathcal{H}, \mathcal{S}, \mathcal{P}, \mathcal{I}, W)$ with $\mathcal{H}_s = \mathcal{H}_p = \mathcal{B}(\mathcal{H})$ and W unitarily invariant, impose three conditions: (i) dynamical admissibility is exactly nonexpansiveness plus intertwining; (ii) admissible flows satisfy a Lyapunov/DSFL-time inequality; and (iii) sharp measurements are required to be idempotent, range-sharp, and nonexpansive in $\|\cdot\|_W$. Then zero-residual flows are forced to be unitary evolutions; sharp updates are uniquely Lüders projections; and the only frame-covariant way to read probabilities from the single quadratic form is the Born rule, by Gleason's theorem.[2,39,58] Conversely, any standard finite-dimensional quantum theory can be embedded into such a DSFL room so that its usual dynamics and measurements are precisely the DSFL-admissible maps. In this sense, DSFL on a finite-dimensional room does not replace quantum mechanics; it provides a minimal geometric superstructure in which the familiar postulates become consequences of a single residual law.

The analysis above is deliberately modest in scope: all Hilbert spaces are finite-dimensional, all channels are bounded matrices, and “locality” is modelled on simple lattice relays. Nevertheless, the logic of DSFL extends naturally to more ambitious settings, and the finite-dimensional theorems can be read as a design specification for such extensions.

In quantum field theory on a fixed background, the natural candidates for the physical room $\mathcal{H}_p$ are GNS Hilbert spaces of states on a local net, or suitable Sobolev/graph-norm completions of the space of smeared fields. In this context the instrument weight W should be constructed from Hadamard two-point functions or elliptic operators associated with the background geometry, so that $\|e(t)\|_W^2$ controls a physically meaningful combination of energy and curvature fluctuations.[75,76] If one can show that the relevant dynamical semigroups (e.g. those generated by Lindblad-type operators, or by interacting Euclidean path integrals through Osterwalder-Schrader reconstruction[52]) satisfy a spectral inequality of the form $\Phi_t^* W \Phi_t \preceq W$, then the continuous DSFL Lyapunov inequality extends verbatim: the “Einstein-plus-matter” residual decays (or at least does not grow) in DSFL time, modulo a controlled remainder $r(t)$. The cone condition of Theorem 21 becomes a quantitative version of microcausality: the relay kernel must be supported, up to exponentially small tails, within the light cone associated with the background metric.

In semiclassical gravity and black-hole physics, DSFL offers a slightly different perspective. There the natural “blueprint” lives on a kinematical Hilbert space of metric plus matter configurations, while the “response” encodes coarse-grained observables (areas, fluxes, near-horizon stress-tensors). The residual of sameness can then be taken to be a graph-norm square of the Einstein tensor minus the renormalised stress-tensor, $\mathcal{R}(g, T) = \|\mathcal{E}_{\kappa, \Lambda}(g; T)\|_W^2$, built from a renormalised, covariant Green operator.[75,76] The DSFL demand that this quantity be a Lyapunov functional in an appropriately defined time variable $\hat{t}$ leads to an inequality of the schematic form $\dot{\mathcal{R}}(t) \leq -2\lambda(t)\mathcal{R}(t)$, with the “rate” $\lambda(t)$ determined by near-horizon redshifts and quasi-normal decay. In model spacetimes (e.g. CGHS, Vaidya, or near-extremal Reissner-Nordström) one can then ask whether the resulting DSFL-time evolution of $\log \mathcal{R}$ exhibits a Page-curve-like turnover, and whether the same instrument geometry that controls local energy fluxes also controls the “correlation residuals” built from algebraic entanglement measures across the horizon. If so, the DSFL framework would provide a new, explicitly norm-based route to the information-theoretic statements usually derived from replica trick or modular-flow arguments.[4,46]

What the present finite-dimensional analysis ultimately shows is that, for small quantum systems, one can gather channel contractivity, entanglement, and Bell nonlocality under a single geometric law.[2–4] In that sense, “one room, one ruler, one residual, one DPI, one envelope, one cone” is not merely a slogan but a concrete and surprisingly efficient way of packaging a large swathe of known quantum structure.

More broadly, this suggests taking “one room, one ruler, one residual, one DPI, one envelope, one cone” as a design principle for any putative fundamental theory of quantum spacetime. In such a theory, every admissible evolution would be contractive in a single, physically meaningful Hilbert geometry; every allowed coupling between “system” and “environment” would appear as an intertwining, norm–nonexpansive map on the common room; and every claim of locality or causality would reduce to an inequality in that same norm. The finite–dimensional results here show that this is not an empty mantra: in the simplest nontrivial quantum sectors, enforcing a single residual law already recovers and organises the full content of standard finite–dimensional quantum mechanics, including its most counterintuitive feature—Bell nonlocality. The open challenge, and the natural next step for the DSFL programme, is to see how far this single–room, single–ruler construction can be pushed into genuinely field–theoretic and gravitational regimes, where the deepest conceptual tensions of modern physics still lie.

11. Author’s Note

This paper applies a sector–neutral Lyapunov–residual framework (DSFL) to finite–dimensional quantum information. The aim is not to alter the Hilbert–space or operator–algebraic formalism of quantum mechanics, but to isolate a minimal calibrated quadratic residual in a common instrument geometry and make explicit what that alone can constrain: a single data–processing inequality and a Lyapunov envelope for channels, together with locality and correlation constraints under clear hypothesis gates (calibration, admissibility, coercivity, cone bounds).

Concretely, we take standard finite–dimensional tools—CPTP maps and quantum DPI [11,22,25], contractive Hilbert–space operators and projection geometry [63], entanglement monotones [3,13,14], and Bell/CHSH correlation bounds [4,7,18]—and organise them around a single observable, $R(s, p) = \|p - \mathcal{I}s\|_{\mathcal{W}}^2$. In that language, admissible dynamics are exactly the nonexpansive, calibration–preserving maps, entanglement measures become correlation parts of the same residual, and Bell nonlocality appears as the statement that no locally admissible factorisation can reproduce the calibrated two–qubit DSFL correlations. The contribution is therefore not a new axiomatics, but a re–packaging: one residual, one DPI, one Lyapunov envelope, one cone, applied systematically to the finite–dimensional quantum–information toolkit [2,3].

12. Declaration of Generative AI and AI-Assisted Technologies in the Writing Process

During preparation of this manuscript, the author used ChatGPT (OpenAI) in a limited, assistive capacity to: (i) convert draft formulas and definitions into $\text{\LaTeX}$, (ii) suggest editorial refinements to headings, boxes, and figure captions, and (iii) refactor small, non-critical plotting/data-wrangling snippets between R and Python. All outputs were reviewed, edited, and independently verified by the author. The author is solely responsible for all scientific content, mathematical claims, proofs, and conclusions. No generative system was used to fabricate, analyze, or select scientific results, and no proprietary or unpublished data were provided to any AI system.

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Notation (Symbols Only)

Symbol	Type / Domain	Meaning / Assumptions
Ambient Hilbert geometry (“one room”)		
$\mathcal{H}$	Hilbert space	Common comparison space with inner product $\langle x, y \rangle$ and norm $\ x\ = \sqrt{\langle x, x \rangle}$
$\mathcal{S}, \mathcal{P} \subset \mathcal{H}$	Closed subspaces	Statistical (blueprint, sDoF) and physical (response, pDoF) arenas
$P_{\mathcal{S}}, P_{\mathcal{P}}$	Projections	Orthogonal projectors onto $\mathcal{S}, \mathcal{P}$
$\mathcal{I} : \mathcal{S} \rightarrow \mathcal{P}$	Linear map	Calibration / interchangeability map (blueprint $\rightarrow$ ideal response)
$U := \overline{\text{ran } \mathcal{I}}$	Closed subspace	Coherent image of $\mathcal{S}$ on the physical side (after closure)
P_U	Projection	Orthogonal projector onto U
$W : \mathcal{P} \rightarrow \mathcal{P}$	SPD operator	Instrument weight; induces instrument inner product and norm $\ x\ _W^2 = \langle x, Wx \rangle$
$\ \cdot\ _W$	Norm	Instrument norm on $\mathcal{P}$, associated with W
States, channels, duals		
$\mathcal{H}$	Hilbert space	Finite-dimensional quantum system, typically $\mathcal{H} \cong \mathbb{C}^d$
$\mathcal{B}(\mathcal{H})$	Operator space	Bounded operators / matrices on $\mathcal{H}$
$\mathcal{S}(\mathcal{H})$	State space	Density matrices $\rho \geq 0$ with $\text{Tr } \rho = 1$
ρ, σ	Operators	Quantum states in $\mathcal{S}(\mathcal{H})$
Φ	Linear map	Quantum channel (CPTP) or general linear update on $\mathcal{P}$
Φ^*	Linear map	Hilbert–Schmidt adjoint of Φ
$\ \Phi\ _{W \rightarrow W}$	Operator norm	Induced norm $\sup_{\ x\ _W=1} \ \Phi x\ _W$; admissibility requires $\ \Phi\ _{W \rightarrow W} \leq 1$
K_i	Operators	Kraus operators of a CPTP channel, $\Phi(X) = \sum_i K_i X K_i^\dagger$
Residual of sameness and defects		
$s \in \mathcal{S}$	Vector	Statistical (blueprint, sDoF) state
$p \in \mathcal{P}$	Vector	Physical (response, pDoF) state
$e := p - \mathcal{I}s$	Vector in $\mathcal{P}$	Calibrated mismatch (defect, residual direction)
$R(s, p)$ or $\mathcal{R}_{\text{sameness}}(s, p)$	Scalar	Residual of sameness: $R(s, p) = \ p - \mathcal{I}s\ _W^2 = \ e\ _W^2$
$\varepsilon := \ e\ _W$	Nonnegative scalar	Residual magnitude; e.g. half-width of band tests

Symbol	Type / Domain	Meaning / Assumptions
Frames, projectors, angles		
$V \subset \mathcal{P}$	Closed subspace	Measurement / reconstruction / constraint frame
$\Pi_V : \mathcal{P} \rightarrow V$	Projection	Orthogonal projector onto V (in the ambient inner product)
V_A	Closed subspace	Spectral frame of observable A (after graph transport)
$\theta(U, V)$	Angle	Friedrichs angle; $\ P_U P_V\ _{\text{op}} = \cos \theta(U, V)$
c_{AB}	Scalar in $(0, 1]$	Overlap constant $c_{AB} = \ \Pi_{V_A} \Pi_{V_B}\ _{\text{op}}$
$V_A \vee V_B$	Closed subspace	Closed linear span of two frames V_A, V_B
Variance and graph transport (when used)		
$\Delta_\psi A$	Operator	Centered observable: $\Delta_\psi A := \hat{A} - \langle \hat{A} \rangle_\psi I$
J_A	Isomorphism	Graph-norm isomorphism $J_A = (I + (\Delta_\psi A)^2)^{1/2}$
$V_A := J_A(\ker \Delta_\psi A)$	Closed subspace	A -frame in $\mathcal{H}$ after graph transport
$\sigma_\psi(A)$	Nonnegative scalar	Standard deviation: $\sigma_\psi(A) = \ (I - \Pi_{V_A})J_A\psi\ $
Admissibility, DPI, and operators		
$\tilde{\Phi} : \mathcal{S} \rightarrow \mathcal{S}$	Linear map	Statistical update (blueprint side)
$\Phi : \mathcal{P} \rightarrow \mathcal{P}$	Linear map	Physical update (response side)
$\Phi \mathcal{I} = \mathcal{I} \tilde{\Phi}$	Identity	Intertwining (coherent blueprint $\rightarrow$ coherent response)
$\Phi^* W \Phi \preceq W$	Operator inequality	Spectral DPI / nonexpansiveness in $\ \cdot\ _W$
<i>admissible</i>	Property	$\Phi \mathcal{I} = \mathcal{I} \tilde{\Phi}$ and $\ \Phi\ _{W \rightarrow W} \leq 1$ (iff DPI for R)
$R(\tilde{\Phi}s, \Phi p) \leq R(s, p)$	Inequality	Data-processing for the single observable R
Two-loop dynamics and DSFL time		
$e(t)$	Trajectory in $\mathcal{P}$	Time-dependent residual $e(t) = p(t) - \mathcal{I}s(t)$
$K_{\text{imm}}(t)$	Operator on $\mathcal{P}$	Immediate loop generator; selfadjoint, $K_{\text{imm}}(t) \succeq 0$
$M(\sigma)$	Operator kernel	Retarded, Loewner-positive memory kernel for $\sigma \geq 0$
$r(t)$	Vector in $\mathcal{P}$	Remainder; $ \langle r(t), x \rangle \leq \varepsilon_{\text{rem}}(t) \ x\ _W^2$
$\kappa(t)$	Nonnegative scalar	Coercivity bound: $\langle K_{\text{imm}}(t)x, x \rangle \geq \kappa(t) \ x\ _W^2$
$\varepsilon_{\text{rem}}(t)$	Nonnegative scalar	Remainder bound in the Lyapunov inequality
$\lambda(t) := \kappa(t) - \varepsilon_{\text{rem}}(t)$	Nonnegative scalar	Instantaneous decay rate (envelope rate)
$R(t) := \ e(t)\ _W^2$	Nonnegative scalar	Residual energy in time
$\hat{\tau}$	Scalar (time)	DSFL time via $d\hat{\tau} = 2\lambda(t) dt$; unit-slope clock

Symbol	Type / Domain	Meaning / Assumptions
Causality & cone parameters (relay loop)		
c	Speed constant	Sharp causal speed (instrument light-cone speed)
Ψ_t	Semigroup on $\mathcal{P}$	Relay evolution; cone bound with margin ξ
Π_O	Projection/localizer	Projection onto observables / defects supported in region O
ξ	Length/time margin	Cone sharpness in $\ \Pi_O \Psi_t v - \Pi_O v\ \leq$ $C e^{-(\text{dist}(O, \text{supp } v) - ct)/\xi} \ v\ _W$
Λ_{UV}	Cutoff (bandwidth)	UV regulator; typically $\xi \sim \text{const}/\Lambda_{UV}$
Finite-dimensional quantum sector		
$\mathcal{H}_A, \mathcal{H}_B$	Hilbert spaces	Local qubit / qudit spaces, typically $\mathbb{C}^{d_A}, \mathbb{C}^{d_B}$
$\mathcal{H}_{AB}$	Hilbert space	Tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$
ρ_{AB}	Density matrix	Bipartite state on $\mathcal{H}_{AB}$
ρ_A, ρ_B	Density matrices	Reduced states: $\rho_A = \text{Tr}_B(\rho_{AB}),$ $\rho_B = \text{Tr}_A(\rho_{AB})$
ρ_Φ	Density matrix	Bell state $ \Phi^+\rangle\langle\Phi^+ $ in two-qubit sector
T_B	Partial transpose	Transposition on subsystem B in a fixed basis
Entanglement / correlation proxies		
$N(\rho_{AB})$	Scalar	Negativity: $\frac{1}{2}(\ \rho_{AB}^{T_B}\ _1 - 1)$
$D_\alpha^*(\rho_{AB} \ \rho_A \otimes \rho_B)$	Scalar	Sandwiched Rényi divergence to product form
$\Delta_{\text{prod}}(\rho_{AB})$	Scalar	Trace-distance to product: $\ \rho_{AB} - \rho_A \otimes \rho_B\ _1$
$\mathcal{R}_{\text{corr}}(\rho_{AB})$	Scalar	DSFL correlation residual $\ \rho_{AB} - \rho_A \otimes \rho_B\ _W^2$
Bell / CHSH notation		
A_0, A_1	Observables	Local dichotomic (± 1) observables on $\mathcal{H}_A$
B_0, B_1	Observables	Local dichotomic (± 1) observables on $\mathcal{H}_B$
C	Operator	CHSH operator $C = A_0 \otimes$ $(B_0 + B_1) + A_1 \otimes (B_0 - B_1)$
$S(\rho_{AB})$	Scalar	CHSH value $\text{Tr}(\rho_{AB} C)$
S_{LHV}	Scalar	Classical (local hidden-variable) CHSH value; $ S_{\text{LHV}} \leq 2$
S_{DSFL}	Scalar	DSFL / quantum CHSH value; $ S_{\text{DSFL}} \leq 2\sqrt{2}$ (Tsirelson bound)
Miscellaneous symbols		
$\ \cdot\ $	Norm	Ambient $\mathcal{H}$ norm (context may indicate operator / trace norm)
$\langle\cdot, \cdot\rangle$	Inner product	Ambient $\mathcal{H}$ inner product
$A \succeq 0$	Loewner order	Positive semidefinite operator: $\langle Ax, x \rangle \geq 0$ for all x
$\text{dist}(x, V)$	Nonnegative scalar	Distance of x to subspace V in $\mathcal{H}$

Master Definitions — Compact Longtable

Entry	Definition / Formula	Role / Notes
Ambient Hilbert geometry (“one room”)		
Instrument room	$(\mathcal{H}, \langle \cdot, \cdot \rangle), \ x\ ^2 = \langle x, x \rangle$	Single calibrated space for all DSFL audits (channels, entanglement, Bell)
Blueprint / response	$\mathcal{S}, \mathcal{P} \subset \mathcal{H}$ (closed)	Statistical blueprints (sDoF) and physical readouts (pDoF)
Calibration	$\mathcal{I} : \mathcal{S} \rightarrow \mathcal{P}$ (bounded, linear)	Interprets blueprints as ideal responses; coherent image $U := \overline{\text{ran } \mathcal{I}}$
Instrument weight	$W : \mathcal{P} \rightarrow \mathcal{P}, W \succ 0$	Defines instrument inner product and norm $\ x\ _W^2 = \langle x, Wx \rangle$
Frames	$V \subset \mathcal{P}$, projector Π_V	Measurement / reconstruction / constraint subspaces
Angles	$\theta(U, V)$ via $\ P_U P_V\ _{\text{op}} = \cos \theta(U, V)$	Geometry of what a frame can “see” from U
Single observable (“one residual”)		
Residual of sameness	$R(s, p) := \ p - \mathcal{I}s\ _W^2$	Only scalar used for DPI, Lyapunov rates, and audits
Residual direction	$e := p - \mathcal{I}s \in \mathcal{P}$	Carries all mismatch / budget; $R = \ e\ _W^2$
DPI / admissibility	$R(\tilde{\Phi}s, \Phi p) \leq R(s, p)$	Iff $\Phi \mathcal{I} = \mathcal{I} \tilde{\Phi}$ and $\ \Phi\ _{W \rightarrow W} \leq 1$ (spectral DPI)
Spectral test	$\Phi^* W \Phi \preceq W$	Equivalent operator inequality for nonexpansiveness in $\ \cdot\ _W$
Nearest point (collapse as mechanism)		
Orthogonal projector	$\Pi_V : \mathcal{P} \rightarrow V$	Idempotent, nonexpansive, range V ; DSFL-lawful sharp update
Lüders uniqueness	$T^2 = T, \text{ran } T = V,$ $\ T\ _{W \rightarrow W} \leq 1 \Rightarrow T = \Pi_V$	Collapse = unique DSFL-admissible nearest point onto V (no extra postulate)
Two-loop law & DSFL time (“one ruler”)		
Immediate loop	$K_{\text{imm}}(t) \succeq 0,$ $\langle K_{\text{imm}}(t)x, x \rangle \geq \kappa(t)\ x\ _W^2$	Time-local contraction of the defect
Relay (memory)	Retarded $M(\sigma) \succeq 0$ for $\sigma \geq 0$	Finite-speed transport of mismatch across the room
Envelope rate	$\lambda(t) := \kappa(t) - \varepsilon_{\text{rem}}(t) \geq 0$	Printed decay rate in $\dot{R}(t) \leq -2\lambda(t)R(t)$
DSFL clock	$d\hat{\tau} = 2\lambda(t) dt$	Time parametrisation with unit-slope semilog envelope
Unit slope	$\frac{d}{d\hat{\tau}} \log R(\hat{\tau}) \leq -1$	In DSFL time, $\log R$ vs. $\hat{\tau}$ is a line of slope -1 (or steeper)
Locality from relay (cone bounds)		
Cone speed	$c = \sup_k \partial_k \omega(k)$	Emergent signal / light-cone speed in the instrument norm
Cone margin	$\xi \sim \text{const} / \Lambda_{\text{UV}}$	Smearing scale; sharper cone as cutoff $\Lambda_{\text{UV}} \uparrow$
Instrument cone	$\ \Pi_Y \Psi_t v - \Pi_Y v\ \leq C e^{-(d(X,Y)-ct)/\xi} \ v\ _W$	Causality constraint for relay dynamics in the DSFL room

Entry	Definition / Formula	Role / Notes
Finite-dimensional quantum sector		
Single system	$\mathcal{H} \cong \mathbb{C}^d, \mathcal{B}(\mathcal{H})$	Finite-dimensional Hilbert space and operator algebra
State space	$\mathcal{S}(\mathcal{H})$	Density matrices (positive, trace one)
Channel	$\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$	Completely positive trace-preserving (CPTP) map
Admissible channel	$\ \Phi\ _{W \rightarrow W}^2 = \lambda_{\max}(W^{-1/2} \Phi^* W \Phi W^{-1/2}) \leq 1$	DSFL-lawful evolution (single-residual DPI holds)
Bipartite systems and entanglement		
Bipartite space	$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$	Two-party quantum system (qubits / qudits)
Reductions	$\rho_A = \text{Tr}_B(\rho_{AB}),$ $\rho_B = \text{Tr}_A(\rho_{AB})$	Local states of A and B
Separable state	$\rho_{AB} = \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)}$	Admits a classical mixture of product states
Entangled state	not separable	Cannot be written as convex combination of product states
Negativity	$N(\rho_{AB}) = \frac{1}{2}(\ \rho_{AB}^{T_B}\ _1 - 1)$	Entanglement measure; monotone under local CPTP maps
Rényi corr. residual	$D_\alpha^*(\rho_{AB} \parallel \rho_A \otimes \rho_B)$	Sandwiched Rényi divergence to product form
Trace corr. residual	$\Delta_{\text{prod}}(\rho_{AB}) = \ \rho_{AB} - \rho_A \otimes \rho_B\ _1$	Trace-distance to product state
DSFL corr. residual	$R_{\text{corr}}(\rho_{AB}) = \ \rho_{AB} - \rho_A \otimes \rho_B\ _W^2$	Correlation part of the DSFL mismatch (nonlocal budget)
Bell tests (CHSH sector)		
Local observables	A_0, A_1 on A , B_0, B_1 on B	Dichotomic (± 1) measurements per party
CHSH operator	$C = A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1)$	Defines $S(\rho_{AB}) = \text{Tr}(\rho_{AB} C)$
Classical bound	$ S_{\text{LHV}} \leq 2$	Local hidden-variable constraint (polytope inequality)
Tsirelson bound	$ S_{\text{DSFL}} \leq 2\sqrt{2}$	Maximal quantum / DSFL CHSH value in calibrated two-qubit sector
Bell state	$\rho_\Phi = \Phi^+\rangle\langle\Phi^+ $	Saturates Tsirelson bound with suitable (A_j, B_k)
Born weights and frequencies (geometric content)		
Born weights	$p_k = \frac{\ P_{V_k} p\ _W^2}{\ p\ _W^2}$	Unique DSFL-compatible outcome weights in a projective frame
Collapse map	$T(\rho) = \frac{P_k \rho P_k}{\text{Tr}(\rho P_k)}$	Sharp Lüders update = nearest point in the instrument norm
DSFL LLN (sketch)	$\ f_n - p\ _\infty \lesssim (\log n)^d / n$	Deterministic law of large numbers turning geometric weights into frequencies

Entry	Definition / Formula	Role / Notes
Falsification / stress tests (device level)		
Unit-slope test	Upper convex hull of $(\hat{\tau}, \log R)$ has slope ≤ -1	Fails if non-admissible steps or miscalibrated DSFL clock
Cone test	No response for $t < d/c$, tails $\sim e^{-(d-ct)/\xi}$	Violations indicate superluminal or nonlocal relay evolution
Band test	One-/two-frame bands tighten under DPI	Dimension-free audit of calibration, frames, and nonexpansiveness

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