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Article

Addition Formula of the Theta Function and Square Sums Theorem

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Abstract: Jacobian theta function identities are related to the Addition formula for the Weierstrass Sigma function. In this paper, from the Addition formula for Weierstrass Sigma function, new and simple proofs of four square sums theorem are given. Using the addition formula of the theta functions, we also give simple proofs of two square sums and eight square sums theorem.

Keywords: theta function; Weierstrass Sigma function; Weierstrass Elliptic function; square sums theorem

MSC: 11F11; 11E25; 11F27; 33E05

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1. Introduction

We suppose throughout this paper that q denotes $exp(2\pi i\tau)$, where τ has positive imaginary part. We will use the familiar notations

$$(z;q)_0 = 1, \quad (z;q)_n = \prod_{k=0}^{n-1}(1-zq^k), \quad n = 0,1,2,3,\cdots$$

$$(a;q)_\infty = \lim_{n\to\infty}(z;q)_n = \prod_{n=0}^{\infty}(1-zq^n)$$

and sometimes write

$$(a,b,\cdots,c;q)_\infty = (a;q)_\infty(b;q)_\infty\cdots(c;q)_\infty.$$

Jacobian theta functions $\theta_1(z|\tau)$ are defined as

$$\theta_1(z|\tau) = -iq^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}e^{(2n+1)iz}$$

$$= 2q^{1/8}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sin(2n+1)z. \tag{1.1}$$

$$\theta_2(z|\tau) = q^{1/4}\sum_{n=-\infty}^{\infty}q^{n(n+1)}e^{(2n+1)iz} = 2q^{1/4}\sum_{n=0}^{\infty}q^{n(n+1)}\cos(2n+1)z, \tag{1.2}$$

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{\infty}q^{n^2}e^{2niz} = 1+2\sum_{n=0}^{\infty}q^{n^2}\cos 2nz, \tag{1.3}$$

$$\theta_4(z|\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2niz} = 1+2\sum_{n=0}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{1.4}$$

From this, we readily find that $\theta_1(z|\tau)$ is an odd function and

$$\theta_1(z+\pi|\tau) = -\theta_1(z|\tau), \qquad \theta_1(z+\pi\tau|\tau) = -q^{-1/2}e^{-2iz}\theta_1(z|\tau). \tag{1.5}$$

Using the well-known Jacobi triple product identity [1, P.21-22]. We can deduce the infinite product representation for the theta functions above, namely:

$$\begin{aligned}
\theta_1(z|\tau) =& 2q^{1/4}(\sin z)(q^2, q^2e^{2iz}, q^2e^{-2iz}; q^2)_\infty, &(1.6)\\
\theta_2(z|\tau) =& 2q^{1/4}(\cos z)(q^2, -q^2e^{2iz}, -q^2e^{-2iz}; q^2)_\infty, &(1.7)\\
\theta_3(z|\tau) =& (q^2, -qe^{2iz}, -qe^{-2iz}; q^2)_\infty,\\
\theta_4(z|\tau) =& (q^2, qe^{2iz}, qe^{-2iz}; q^2)_\infty.
\end{aligned}$$

[See for example, [5, P.469]. The trigonometric expansion of the $\theta_1(z|\tau)$ function's logarithmic derivative is

$$\frac{\theta_1'}{\theta_1}(x|\tau) = \cot x + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2nx. \tag{1.8}$$

From the definitions of theta functions, we easily know that

$$\theta_1(x|\tau)\theta_2(x|\tau) = \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\theta_2(x|2\tau) \tag{1.9}$$

and

$$\theta_4(0|\tau) = \frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = 1 + 2\sum_{n=0}^{\infty}(-1)^n q^{n^2}. \tag{1.10}$$

We recall the addition formula for the Weierstrass Sigma function[See for example, [3, P.401]] as following

$$\wp(x|\tau) - \wp(y|\tau) = \frac{\sigma(x+y|\tau)\sigma(x-y|\tau)}{\sigma^2(x|\tau)\sigma^2(y|\tau)}. \tag{1.11}$$

The Weierstrass $\wp$-function, σ-function and theta function $\theta_1(x|\tau)$ satisfy the following identities respectively (See for example, [5, P.460])

$$\sigma(x|\tau) = \exp(\eta x^2)\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}, \tag{1.12}$$

$$\wp(x|\tau) = \csc^2 x - 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos 2nx - \frac{1}{3}E_2(\tau) = -\frac{1}{3}E_2(\tau) - \left(\frac{\theta_1'}{\theta_1}\right)'(x|\tau), \tag{1.13}$$

in which $\eta = -\frac{\theta_1'''(0|\tau)}{6\theta_1'(0|\tau)}$ and $E_{2k}(\tau)$ are the normalized Eisnstein series defined by

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}, \tag{1.14}$$

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where B_k are Bernoulli numbers defined as the coefficients in the power series

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!},\quad |x|<2\pi. \tag{1.15}$$

By (1.6)and (1.7) in above section, the identity (1.5) can be written in term of $\theta_1(x|\tau)$ as following (see, for example, [2, P.13])

$$\left(\frac{\theta_1'}{\theta_1}\right)'(x|\tau)-\left(\frac{\theta_1'}{\theta_1}\right)'(y|\tau)=\frac{\theta_1'^2(0|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau)}{\theta_1^2(x|\tau)\theta_1^2(y|\tau)}. \tag{1.16}$$

2. The Main Conclusions

Representing natural numbers as sums of squares is an important topic in number theory. Given a general natural number n, denote $r_l(n)$ the number of integer solutions of Diophantine equation

$$n=x_1^2+x_2^2+x_3^2+\cdots+x_l^2$$

which counts the number of ways in which n can be written as sums of l squares. In l-dimensional space, $r_l(n)$ gives also the number of points with integer coordinates on the sphere. When l is odd, the problem is very difficult. However for the even case, the problem may be treated in a fairly reasonable manner. In this section, we will give simple and shot proofs of it by using theta function identities. At the same time, we obtain some interesting results of theta functions.

Theorem 1. (Four square theorem)Every nature number can be expressed as sum of four square numbers. Moreover we have

$$r_4(n)=8\sum_{\substack{d|n\\ d\neq 0(mod4)}}d. \tag{2.1}$$

Proof: In (1.8), setting $x=\pi/2$ and $y=\pi/4$, we obtain that

$$\begin{aligned}\left(\frac{\theta_1'}{\theta_1}\right)'(\frac{\pi}{2}|\tau)-\left(\frac{\theta_1'}{\theta_1}\right)'(\frac{\pi}{4}|\tau)&=\frac{\theta_1'^2(0|\tau)\theta_1(\frac{3\pi}{4}|\tau)\theta_1(\frac{\pi}{4}|\tau)}{\theta_2^2(0|\tau)\theta_1^2(\frac{\pi}{4}|\tau)}\\&=\frac{\theta_1'^2(0|\tau)}{\theta_2^2(0|\tau)}=\theta_3^2(0|\tau)\theta_4^2(0|\tau)=\theta_4^4(0|2\tau).\end{aligned}$$

In above identity, let q^2 replaced by q, we have

$$\left(\frac{\theta_1'}{\theta_1}\right)'(\frac{\pi}{2}|\frac{\tau}{2})-\left(\frac{\theta_1'}{\theta_1}\right)'(\frac{\pi}{4}|\frac{\tau}{2})=\theta_4^4(0|\tau).$$

Combining (1.8) and(1.10) yields

$$\theta_4^4(0|\tau)=1+8\sum_{n=1}^{\infty}\frac{(-1)^n nq^n}{1+q^n}.$$

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Using (1.8) and letting q replaced by $-q$ in above identity give

$$\begin{aligned}
&\Big(\sum_{n=-\infty}^{\infty} q^{n^2}\Big)^4 = \theta_3^4(0|\tau) = 1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n} \\
&= 1+8\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + 8\sum_{n=1}^{\infty}\frac{(2n)q^{2n}}{1+q^{2n}} \\
&= 1+8\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + 8\sum_{n=1}^{\infty}\frac{(2n)q^{2n}}{1+q^{2n}} + 8\sum_{n=1}^{\infty}\frac{(2n)q^{2n}}{1-q^{2n}} + 8\sum_{n=1}^{\infty}\frac{(2n)q^{2n}}{1-q^{2n}} \\
&= 1-32\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}} + 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} \\
&= 1+8\sum_{n=1}^{\infty} q^n\Big(\sum_{d|n} d - \sum_{\substack{d|n \\ d=0(mod4)}} d\Big).
\end{aligned}$$

Compare the coefficients of q^n on both sides of above equation, we are able to complete the proof.

In [4, P.770], it presents a set of additive formulas of theta function as following

$$\theta_1(x|\tau)\theta_1(y|\tau) = \theta_2(x-y|2\tau)\theta_3(x+y|\tau) - \theta_2(x+y|2\tau)\theta_3(x-y|\tau), \tag{2.2}$$

$$\theta_2(x|\tau)\theta_2(y|\tau) = \theta_2(x-y|2\tau)\theta_3(x+y|\tau) + \theta_2(x+y|2\tau)\theta_3(x-y|\tau). \tag{2.3}$$

Differentiate both sides of the identity (2.2) with respect to x then set $x=0$, we find that

$$\begin{aligned}
\theta_1'(0|\tau)\theta_1(y|\tau) = &\Big[\Big(\frac{\theta_3'}{\theta_3}\Big)'(y|2\tau) + \Big(\frac{\theta_2'}{\theta_2}\Big)'(-y|2\tau)\Big]\theta_2(-y|2\tau)\theta_3(y|2\tau) \\
&-\Big[\Big(\frac{\theta_3'}{\theta_3}\Big)'(-y|2\tau) + \Big(\frac{\theta_2'}{\theta_2}\Big)'(y|2\tau)\Big]\theta_2(y|2\tau)\theta_3(-y|2\tau).
\end{aligned}$$

Using the infinite product representations of theta functions , we easily get

$$\frac{\tan y(q^2;q^2)_\infty^4(q^2e^{2iy};q^2)_\infty(q^2e^{-2iy};q^2)_\infty}{(q^4;q^4)_\infty^2(-q^2e^{2iy};q^2)_\infty(-q^2e^{-2iy};q^2)_\infty} = \tan y + \sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{1+q^{2n}}\sin 2ny. \tag{2.4}$$

Letting $y=\frac{\pi}{4}$ in above identity and making use of (1.10) give

$$\theta_4^2(0|\tau) = 1+4\sum_{n=1}^{\infty}\Big(\frac{q^{4n-1}}{1+q^{4n-1}} - \frac{q^{4n-3}}{1+q^{4n-3}}\Big). \tag{2.5}$$

Substituting q by $-q$ in this equation, we arrive at

$$\theta_3^2(0|\tau) = \frac{(-q;q)_\infty^2}{(q;q)_\infty^2} = 1-4\sum_{n=1}^{\infty}\Big(\frac{q^{4n-1}}{1+q^{4n-1}} - \frac{q^{4n-3}}{1+q^{4n-3}}\Big). \tag{2.6}$$

Theorem 2. (Two square theorem)Every natural number can be expressed as sum of two square numbers, moreover we have

$$r_2(n) = 4\sum_{\substack{d|n \\ d\equiv 1(mod4)}} 1 - \sum_{\substack{d|n \\ d\equiv 3(mod4)}} 1. \tag{2.7}$$

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Proof: From (2.6) and (1.10) we get that

$$\Big(\sum_{n=-\infty}^{\infty} q^{n^2}\Big)^2 = 1 - 4\sum_{n=1}^{\infty}\Big(\frac{q^{4n-1}}{1-q^{4n-1}} - \frac{q^{4n-3}}{1-q^{2n}}\Big)$$
$$= 1 - 4\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\Big(q^{k(4n-1)} - q^{k(4n-3)}\Big)$$
$$= 1 + 4\sum_{n=1}^{\infty} q^n\Big[\sum_{\substack{d|n\\ d\equiv 1(mod4)}} 1 - \sum_{\substack{d|n\\ d\equiv 3(mod4)}} 1\Big].$$

Compare the coefficients of q^n on both sieds of this equation, we get

$$r_2(n) = 4\sum_{\substack{d|n\\ d\equiv 1(mod4)}} 1 - \sum_{\substack{d|n\\ d\equiv 3(mod4)}} 1.$$

We thus complete the proof.

Multiply both sides of the two equations (2.2) and (2.3), then Substitute the equations (1.9) into it and replace 2τ by τ, We get

$$\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\theta_2(2x|\tau)\theta_2(2y|\tau) = \theta_2^2(x-y|\tau)\theta_3^2(x+y|\tau) - \theta_2^2(x+y|\tau)\theta_3^2(x-y|\tau). \tag{2.8}$$

In above identity place x and y with $x + \frac{1}{4}\pi\tau$ and $y + \frac{1}{4}\pi\tau$ respectively, we get

$$-\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\theta_4(2x|\tau)\theta_4(2y|\tau) = \theta_2^2(x-y|\tau)\theta_2^2(x+y|\tau) - \theta_3^2(x+y|\tau)\theta_3^2(x-y|\tau). \tag{2.9}$$

In above equation replace x with $x + \frac{1}{2}\pi$, we have

$$-\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\theta_4(2x|\tau)\theta_4(2y|\tau) = \theta_1^2(x-y|\tau)\theta_1^2(x+y|\tau) - \theta_4^2(x+y|\tau)\theta_4^2(x-y|\tau). \tag{2.10}$$

In above identity, setting $y = 0$, we get

$$-\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\theta_4(2x|\tau)\theta_4(0|\tau) = \theta_4^4(x|\tau) - \theta_1^4(x|\tau). \tag{2.11}$$

We recall the transformation formulas of modular(See for example, [1, P.339]) as following

$$\theta_4(x/\tau| - 1/\tau) = (-i\tau)^{\frac{1}{2}} e^{i\pi x^2/\tau}\theta_2(x|\tau), \tag{2.12}$$
$$\theta_1(x/\tau| - 1/\tau) = -i(-i\tau)^{\frac{1}{2}} e^{i\pi x^2/\tau}\theta_2(x|\tau). \tag{2.13}$$

It be called the imaginary transformation formulas. Apply the imaginary transformation to identity (2.13), we have

$$\theta_2(2x|\tau)\theta_2^3(0|\tau) = \theta_2^4(x|\tau) - \theta_1^4(x|\tau). \tag{2.14}$$

Differentiating four times on both sides of this identity with respect to x and setting $x = 0$, we find

$$\Big(\frac{\theta_2'}{\theta_2}\Big)'''(0|\tau) = \frac{2[\theta_1'(0|\tau)]^4}{\theta_2^4(0|\tau)}. \tag{2.15}$$

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Combine (1.8) and (1.10), we have

$$\theta_4^8(0|\tau) = 1 + 16\sum_{n=1}^{\infty} \frac{n^3(-q)^n}{1-q^n}. \tag{2.16}$$

Substituting q by $-q$ in this equation, we arrive at

$$\Big(\sum_{n=-\infty}^{\infty} q^{n^2}\Big)^8 = 1 + 16\sum_{n=1}^{\infty}(-1)^n \frac{n^3q^{2n}}{1-q^{2n}} + 16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^{2n}}. \tag{2.17}$$

Theorem 3.(Eight square theorem) Every nature number can be expressed as sum of eight square numbers, moreover we have

$$r_8(n) = 16\sum_{d|n}(-1)^{n+d}d^3. \tag{2.18}$$

Proof: From (2.16), we have

$$\begin{aligned}
\theta_4^8(0|\tau) &= 1 + 16\sum_{n=1}^{\infty} \frac{n^3(-q)^n}{1-q^n} \\
&= 1 + 16\sum_{n=1}^{\infty}\frac{(2n-1)^3q^{2n-1}}{1+q^{2n-1}} + 16\sum_{n=1}^{\infty}\frac{(2n)^3q^{2n}}{1-q^{2n}} \\
&= 1 + 16\sum_{n=1}^{\infty}\frac{(2n-1)^3q^{2n-1}}{1+q^{2n-1}} + 16\sum_{n=1}^{\infty}\frac{(2n)^3q^{2n}}{1-q^{2n}} \\
&+16\sum_{n=1}^{\infty}\frac{(2n-1)^3q^{2n-1}}{1-q^{2n-1}} - 16\sum_{n=1}^{\infty}\frac{(2n-1)^3q^{2n-1}}{1-q^{2n-1}} \\
&= 1 + 16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} - 32\sum_{n=1}^{\infty}\frac{(2n-1)^3q^{4n-2}}{1-q^{4n-2}} \\
&= 1 + 16\sum_{n=1}^{\infty}q^n\sum_{d|n}d^3 - 32\sum_{n=1}^{\infty}q^n\sum_{\substack{2d|n \\ d(odd)}}d^3 \\
&= 1 + 16\sum_{n=1}^{\infty}q^n\Big(\sum_{d|n}d^3 - 2\sum_{\substack{2d|n \\ d(odd)}}d^3\Big).
\end{aligned}$$

If n is even, setting $n = 2k$ yields

$$\begin{aligned}
&\sum_{d|n}d^3 - 2\sum_{\substack{d|k \\ d(odd)}}d^3 = \sum_{\substack{d|k \\ d(even)}}d^3 + \sum_{\substack{d|k \\ d(odd)}}d^3 - 2\sum_{\substack{d|k \\ d(odd)}}d^3 \\
&= (-1)^d\sum_{d|n}d^3.
\end{aligned}$$

If n is odd, we are able to know that

$$\sum_{d|2k}d^3 - 2\sum_{\substack{2d|n \\ d(odd)}}d^3 = \sum_{d|n}d^3.$$

In summary, we have

$$\sum_{d|n} d^3 - 2\sum_{\substack{2d|n \\ d(odd)}} d^3 = \sum_{d|n}(-1)^{n+d}d^3.$$

Compare the coefficient of q^n on both sieds of above equation, we have

$$r_8(n) = 16\sum_{d|n}(-1)^{n+d}d^3.$$

3. A Special Cases

We recall the Lambert series as following

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

from which the generating function of $r_k(n)$ can be deduced as

$$\phi^k(q) = \sum_{n=-\infty}^{\infty} r_k(n)q^n.$$

Combine the definitions of theta functions, we easily know that

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \theta_3(0|\tau) = (q^2;q^2)_\infty(-q;q^2)_\infty^2.$$

$$\phi(-q) = \theta_4(0|\tau) = \frac{(q;q)_\infty}{(-q;q)_\infty}.$$

Then we have

Corollary 1. Let $\phi(q)$ be as the above, we have

$$\phi(-q^3)\phi(-q) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{(-1)^n q^n}{1+q^n}, \tag{3.1}$$

$$\phi(q^3)\phi(q) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1+(-q)^n}, \tag{3.2}$$

where $\left(\frac{n}{3}\right)$ denotes the Legendre symbol.

Proof: In equation (2.4), set $y = \frac{\pi}{3}$ and then replace q^2 by q, we are able to obtain

$$\frac{\sqrt{3}(q^3;q^3)_\infty(q;q)_\infty}{(-q^3;q^3)_\infty(-q;q)_\infty} = \tan\frac{\pi}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+q^n}\sin\frac{2n\pi}{3},$$

which implies

$$\frac{(q^3;q^3)_\infty(q;q)_\infty}{(-q^3;q^3)_\infty(-q;q)_\infty} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{(-1)^n q^n}{1+q^n}.$$

Then we get

$$\phi(-q^3)\phi(-q) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{(-1)^n q^n}{1+q^n}.$$

Replacing q by $-q$ in above equation, we get immediately

$$\phi(q^3)\phi(q) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1+(-q)^n}.$$

Thus we complete this corollary.

Conflicts of Interest: The authors declare no conflict of interest.

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