Peer-reviewed version available at *Mathematical Inequalities & Applications* **2020**, *23*, 123-135; doi:10.7153/mia-2020-23-10

Some Inequalities of the Bell Polynomials

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Abstract. In the paper, the author

(1) presents an explicit formula and its inversion formula for higher order derivatives of generating functions of the Bell polynomials, with the help of the Faà di Bruno formula, properties of the Bell polynomials of the second kind, and the inversion theorem for the Stirling numbers of the first and second kinds;
(2) recovers an explicit formula and its inversion formula for the Bell polynomials in terms of the Stirling numbers of the first and second kinds, with the aid of the above explicit formula and its inversion formula for higher order derivatives of generating functions of the Bell polynomials;
(3) constructs some determinantal and product inequalities and deduces the logarithmic convexity of the Bell polynomials, with the assistance of the complete monotonicity of generating functions of the Bell polynomials. These inequalities are main results of the paper.

Contents

1. Introduction

In combinatorics, the Bell numbers, usually denoted by B_k for $k \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, count the number of ways a set with k elements can be partitioned into disjoint and nonempty subsets. These numbers

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2010 *Mathematics Subject Classification.* Primary 11B83; Secondary 11A25, 11B73, 11C08, 11C20, 15A15, 26A24, 26A48, 26C05, 26D05, 33B10, 34A05.

Key words and phrases. Bell polynomial; Bell number; Bell polynomial of the second kind; higher order derivative; generating function; Faà di Bruno formula; inversion theorem; Stirling number of the first kind; Stirling number of the second kind; explicit formula; inversion formula; absolute monotonicity; complete monotonicity; determinantal inequality; product inequality.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

 doi:10.20944/preprints201708.0079.v2

Peer-reviewed version available at *Mathematical Inequalities & Applications* **2020**, *23*, 123-135; doi:10.7153/mia-2020-23-10

have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930s. The Bell numbers B_k for $k\ge0$ can be generated by

$$e^{e^t-1}=\sum_{k=0}^\infty B_k\frac{t^k}{k!}=1+t+t^2+\frac{5}{6}t^3+\frac{5}{8}t^4+\frac{13}{30}t^5+\frac{203}{720}t^6+\frac{877}{5040}t^7+\cdots$$

and the first ten Bell numbers B_k for $0\le k\le 9$ are

$$\begin{aligned}
&B_0=1, && B_1=1, && B_2=2, && B_3=5, && B_4=15,\\
&B_5=52, && B_6=203, && B_7=877, && B_8=4140, && B_9=21147.
\end{aligned}$$

In [1, Section 3], Asai, Kubo, and Kuo mentioned applications of the Bell numbers B_n and its generalizations to white noise distribution theory. For more information on the Bell numbers B_k, please refer to [1, 2, 3, 4, 5, 15, 16, 20] and plenty of references therein.

As well-known generalizations of the Bell numbers B_k for $k\ge0$, the Bell polynomials $B_k(x)$ for $k\ge0$ can be generated by

$$\begin{aligned}
e^{x(e^t-1)}=\sum_{k=0}^\infty B_k(x)\frac{t^k}{k!}&=1+xt+\frac{1}{2}x(x+1)t^2+\frac{1}{6}x\bigl(x^2+3x+1\bigr)t^3\\
&+\frac{1}{24}x\bigl(x^3+6x^2+7x+1\bigr)t^4+\frac{1}{120}x\bigl(x^4+10x^3+25x^2+15x+1\bigr)t^5+\cdots
\end{aligned}$$

and the first seven Bell polynomials $B_k(x)$ for $0\le k\le 6$ are

$$\begin{gathered}
1,\quad x,\quad x(x+1),\quad x\bigl(x^2+3x+1\bigr),\quad x\bigl(x^3+6x^2+7x+1\bigr),\\
x\bigl(x^4+10x^3+25x^2+15x+1\bigr),\quad x\bigl(x^5+15x^4+65x^3+90x^2+31x+1\bigr).
\end{gathered}$$

In [18] it was pointed out that there have been studies on interesting applications of the Bell polynomials $B_k(x)$ in soliton theory, including links with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions. See, for example, [8, 9, 10]. Therefore, applications of the Bell polynomials $B_k(x)$ to integrable nonlinear equations are greatly expected and any amendment on multilinear forms of soliton equations, even on exact solutions, would be beneficial to interested audiences in the research community. For more information about the Bell polynomials $B_k(x)$, please refer to [6, 7, 18, 19] and closely related references therein.

In this paper, continuing the article [16], we present an explicit formula and its inversion formula for higher order derivatives with respect to t of generating functions $e^{xe^{\pm t}}$ for the Bell polynomials $B_k(x)$ with the help of the Faà di Bruno formula, properties of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, and the inversion theorem for the Stirling numbers $s(n,k)$ and $S(n,k)$, recover an explicit formula and its inversion formula for the Bell polynomials $B_k(x)$ in terms of the Stirling numbers $s(n,k)$ and $S(n,k)$, construct some determinantal and product inequalities for the Bell polynomials $B_k(x)$, and deduce the logarithmic convexity of the Bell polynomials $B_k(x)$.

2. Higher order derivatives for generating functions

In this section, by the Faà di Bruno formula, properties of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$, and the inversion theorem for the Stirling numbers $s(n,k)$

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and $S(n,k)$, we present an explicit formula and its inversion formula for higher order derivatives with respect to t of the generating function $e^{xe^{\pm t}}$.

Theorem 2.1. *For $n\ge 0$, the nth derivative of the generating function $e^{xe^{\pm t}}$ with respect to t can be computed by*

$$\frac{\partial^n e^{xe^{\pm t}}}{\partial t^n}=(\pm1)^n e^{xe^{\pm t}}\sum_{k=0}^{n}S(n,k)\bigl(xe^{\pm t}\bigr)^k \tag{2.1}$$

and the generating function $e^{xe^{\pm t}}$ satisfies a family of nonlinear ordinary differential equations

$$\sum_{k=0}^{n}(\pm1)^k s(n,k)f^{(k)}(t)=f(t)[\ln f(t)]^n, \tag{2.2}$$

where $x\in\mathbb{C}$, $S(n,k)$ for $n\ge k\ge 0$, which can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},$$

represent the Stirling numbers of the second kind, and $s(n,k)$ for $n\ge k\ge 0$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1,$$

stand for the Stirling numbers of the first kind.

Proof. In combinatorics, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{\ell_1,\dots,\ell_n\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\Bigl(\frac{x_i}{i!}\Bigr)^{\ell_i}$$

for $n\ge k\ge 0$, see [3, p. 134, Theorem A], and satisfy identities

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{2.3}$$

and

$$\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k), \tag{2.4}$$

see [3, p. 135], where a and b are any complex numbers. The Faà di Bruno formula for computing higher order derivatives of composite functions can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ g(x)=\sum_{k=0}^{n}f^{(k)}(g(x))\mathrm{B}_{n,k}\bigl(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\bigr), \tag{2.5}$$

see [3, p. 139, Theorem C]. Applying $f(u)=e^{xu}$ and $u=g(t)=e^t$ to (2.5) and making use of identities (2.3) and (2.4) yield

$$\begin{aligned}\frac{\partial^n e^{xe^t}}{\partial t^n}&=\sum_{k=0}^{n}\frac{\partial^k e^{xu}}{\partial u^k}\mathrm{B}_{n,k}\bigl(e^t,e^t,\dots,e^t\bigr)\\&=\sum_{k=0}^{n}x^ke^{xu}e^{kt}\mathrm{B}_{n,k}(1,1,\dots,1)=e^{xe^t}\sum_{k=0}^{n}x^ke^{kt}S(n,k).\end{aligned}$$

The explicit formula (2.1) for the plus sign case is thus proved.

In [21, p. 171, Theorem 12.1], it is stated that, if b_α and a_k are a collection of constants independent of n, then

$$a_n=\sum_{\alpha=0}^{n}S(n,\alpha)b_\alpha \quad \text{if and only if} \quad b_n=\sum_{k=0}^{n}s(n,k)a_k. \tag{2.6}$$

Combining this inversion theorem for the Stirling numbers with (2.1) arrives at equations

$$\sum_{k=0}^{n}s(n,k)\frac{\partial^k e^{xe^t}}{\partial t^k}=e^{xe^t}\bigl(xe^t\bigr)^n$$

which is equivalent to that the generating function e^{xe^t} satisfies the family of nonlinear ordinary differential equations in (2.2) for the plus sign case.

The equation (2.12) in the second proof of [16, Theorem 2.2] reads that

$$\mathrm{B}_{n,n}(\alpha)=\alpha^n \quad \text{and} \quad \mathrm{B}_{n+k+1,k}(\alpha,0,\dots,0)=0, \tag{2.7}$$

where $\alpha\in\mathbb{C}$ and $k,n\in\{0\}\cup\mathbb{N}$. Applying $f(u)=e^{xe^u}$ and $u=g(t)=-t$ in (2.5), taking $\alpha=-1$ in (2.7), and utilizing (2.1) lead to

$$\begin{aligned}\frac{\partial^n e^{xe^{-t}}}{\partial t^n}&=\sum_{k=0}^{n}\frac{\partial^k e^{xe^u}}{\partial u^k}\mathrm{B}_{n,k}(-1,0,\dots,0)=\frac{\partial^n e^{xe^u}}{\partial u^n}\mathrm{B}_{n,n}(-1)\\&=(-1)^n e^{xe^u}\sum_{k=0}^{n}S(n,k)\bigl(xe^u\bigr)^k=(-1)^n e^{xe^{-t}}\sum_{k=0}^{n}S(n,k)\bigl(xe^{-t}\bigr)^k.\end{aligned}$$

The formula (2.1) for the minus sign case follows immediately.

Employing the inversion theorem for Stirling numbers, demonstrated in (2.6), to consider the formula (2.1) for the minus sign case give

$$\sum_{k=0}^{n}s(n,k)(-1)^k\frac{\partial^k e^{xe^{-t}}}{\partial t^k}=e^{xe^{-t}}\bigl(xe^{-t}\bigr)^n$$

which implies that the generating function $e^{xe^{-t}}$ satisfies the family of nonlinear ordinary differential equations in (2.2) for the minus sign case. The proof of Theorem 2.1 is thus complete. □

3. An explicit formula and its inversion formula

In this section, with the aid of Theorem 2.1, we recover an explicit formula and its inversion formula for the Bell polynomials $B_n(x)$ in terms of the Stirling numbers $s(n,k)$ and $S(n,k)$.

Theorem 3.1. *For $n\ge0$, the Bell polynomials $B_n(x)$ can be computed by*

$$B_n(x)=\sum_{k=0}^{n}S(n,k)x^k \tag{3.1}$$

and satisfy

$$\sum_{k=0}^{n}s(n,k)B_k(x)=x^n. \tag{3.2}$$

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Proof. In light of the theory of series, it is easy to see that

$$B_n(x)=(\pm1)^n\lim_{t\to0}\frac{\partial^n e^{x(e^{\pm t}-1)}}{\partial t^n}=(\pm1)^n e^{-x}\lim_{t\to0}\frac{\partial^n e^{xe^{\pm t}}}{\partial t^n}.$$

Combining this with (2.1) gives

$$B_n(x)=(\pm1)^n e^{-x}\lim_{t\to0}(\pm1)^n e^{xe^{\pm t}}\sum_{k=0}^{n}S(n,k)\bigl(xe^{\pm t}\bigr)^k=\sum_{k=0}^{n}S(n,k)x^k.$$

The formula (3.1) follows.

Substituting $f(t)=e^{xe^{\pm t}}$ into (2.2) and taking $t\to0$ result in

$$\sum_{k=0}^{n}(\pm1)^k s(n,k)(\pm1)^k e^x B_k(x)=e^x x^n$$

which can be simplified as (3.2). The proof of Theorem 3.1 is complete. □

4. Inequalities for the Bell polynomials

In light of complete monotonicity of generating functions $e^{xe^{-t}}$ and with the assistance of properties of completely monotonic functions, we now start out to construct some determinantal and product inequalities for the Bell polynomials $B_n(x)$. From these inequalities, we can derive the logarithmic convexity of the sequence of the Bell polynomials $B_n(x)$. These inequalities are our main results in this paper.

Theorem 4.1. *Let $m\ge1$ be a positive integer and let $|a_{ij}|_m$ denote a determinant of order m with elements a_{ij}.*

(1) *If a_i for $1\le i\le m$ are non-negative integers, then*

$$\bigl|B_{a_i+a_j}(x)\bigr|_m\ge0,\quad x>0 \tag{4.1}$$

and

$$\bigl|(-1)^{a_i+a_j}B_{a_i+a_j}(x)\bigr|_m\ge0,\quad x>0. \tag{4.2}$$

(2) *If $a=(a_1,a_2,\dots,a_n)$ and $b=(b_1,b_2,\dots,b_n)$ are non-increasing n-tuples of non-negative integers such that $\sum_{i=1}^k a_i\ge\sum_{i=1}^k b_i$ for $1\le k\le n-1$ and $\sum_{i=1}^n a_i=\sum_{i=1}^n b_i$, then*

$$\prod_{i=1}^{n}B_{a_i}(x)\ge\prod_{i=1}^{n}B_{b_i}(x),\quad x>0. \tag{4.3}$$

Proof. Recall from [13, 25] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t)\ge0$ for $t\in I$ and $k\in\mathbb{N}$. Recall from [13, Chapter XIII], [22, Chapter 1], and [25, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^k f^{(k)}(x)\ge0$ on I for all $k\ge0$. Theorem 2b in [25, p. 145] reads that, if $f_1(x)$ is absolutely monotonic and $f_2(x)$ is completely monotonic on their defined intervals, then their composite function $f_1(f_2(x))$ is completely monotonic on its defined interval. Consequently, the function $e^{xe^{-t}}$ for $x>0$ is completely monotonic with respect to $t\in(0,\infty)$.

In [12] and [13, p. 367], it was obtained that if f is completely monotonic on $[0,\infty)$, then

$$\bigl|f^{(a_i+a_j)}(t)\bigr|_m\ge0 \tag{4.4}$$

and

$$\left|(-1)^{a_i+a_j}f^{(a_i+a_j)}(t)\right|_m \ge 0. \tag{4.5}$$

Applying $f(t)$ to the function $e^{xe^{-t}}$ in (4.4) and (4.5) and taking the limit $t\to0^+$ give

$$\lim_{t\to0^+}\left|\left(e^{xe^{-t}}\right)^{(a_i+a_j)}\right|_m = \left|(-1)^{a_i+a_j}e^xB_{a_i+a_j}(x)\right|_m \ge 0$$

and

$$\lim_{t\to0^+}\left|(-1)^{a_i+a_j}\left(e^{xe^{-t}}\right)^{(a_i+a_j)}\right|_m = \left|(-1)^{a_i+a_j}(-1)^{a_i+a_j}e^xB_{a_i+a_j}(x)\right|_m \ge 0.$$

The determinant inequalities (4.1) and (4.2) follow.

In [13, p. 367, Theorem 2], it was stated that if f is a completely monotonic function on $[0,\infty)$, then

$$\prod_{i=1}^{n}\left[(-1)^{a_i}f^{(a_i)}(t)\right] \ge \prod_{i=1}^{n}\left[(-1)^{b_i}f^{(b_i)}(t)\right]. \tag{4.6}$$

Applying $f(t)$ to the function $e^{xe^{-t}}$ in (4.6) and taking the limit $t\to0^+$ give

$$\begin{aligned}\lim_{t\to0^+}\prod_{i=1}^{n}\left[(-1)^{a_i}\left(e^{xe^{-t}}\right)^{(a_i)}\right] &= \prod_{i=1}^{n}[e^xB_{a_i}(x)]\\ &\ge \lim_{t\to0^+}\prod_{i=1}^{n}\left[(-1)^{b_i}\left(e^{xe^{-t}}\right)^{(b_i)}\right] = \prod_{i=1}^{n}[e^xB_{b_i}(x)].\end{aligned}$$

The product inequality (4.3) follows. The proof of Theorem 4.1 is complete. □

Corollary 4.1. *The sequence $\{B_n(x)\}_{n\ge0}$ for $x>0$ is logarithmically convex.*

Proof. In [13, p. 369] and [14, p. 429, Remark], it was stated that if $f(t)$ is a completely monotonic function such that $f^{(k)}(t)\ne0$ for $k\ge0$, then the sequence

$$\ln\left[(-1)^{k-1}f^{(k-1)}(t)\right], \quad k\ge1 \tag{4.7}$$

is convex. Applying this result to the function $e^{xe^{-t}}$ for $x>0$ figures out that the sequence

$$\ln\left[(-1)^{k-1}\left(e^{xe^{-t}}\right)^{(k-1)}\right] \to x+\ln B_{k-1}(x), \quad t\to0^+$$

for $k\ge1$ is convex. Hence, the sequence $\{B_n(x)\}_{n\ge0}$ is logarithmically convex.

Alternatively, letting

$$\ell\ge1, \quad n=2, \quad a_1=\ell+2, \quad a_2=\ell, \quad \text{and} \quad b_1=b_2=\ell+1$$

in the inequality (4.3) leads to $B_\ell(x)B_{\ell+2}(x) \ge B_{\ell+1}^2(x)$ which means that the sequence $\{B_k(x)\}_{k\in\mathbb{N}}$ is logarithmically convex. The proof of Corollary 4.1 is complete. □

Corollary 4.2. *If $\ell\ge0$ and $n\ge k\ge0$, then*

$$[B_{n+\ell}(x)]^k[B_\ell(x)]^{n-k} \ge [B_{k+\ell}(x)]^n, \quad x>0.$$

Proof. This follows from taking

$$a=(\overbrace{n+\ell,\dots,n+\ell}^{k},\overbrace{\ell,\dots,\ell}^{n-k}) \quad \text{and} \quad b=(k+\ell,k+\ell,\dots,k+\ell)$$

in the inequality (4.3). The proof of Corollary 4.2 is complete. □

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Peer-reviewed version available at Mathematical Inequalities & Applications 2020, 23, 123-135; doi:10.7153/mia-2020-23-10

Theorem 4.2. *If $\ell \ge 0$, $n \ge k \ge m$, $2k \ge n$, and $2m \ge n$, then*

$$B_{k+\ell}(x)B_{n-k+\ell}(x) \ge B_{m+\ell}(x)B_{n-m+\ell}(x), \quad x > 0. \tag{4.8}$$

Proof. In [23, p. 397, Theorem D], it was recovered that if $f(t)$ is completely monotonic on $(0,\infty)$ and if $n \ge k \ge m$, $k \ge n-k$, and $m \ge n-m$, then

$$(-1)^n f^{(k)}(t) f^{(n-k)}(t) \ge (-1)^n f^{(m)}(t) f^{(n-m)}(t).$$

Replacing $f(t)$ by the function $(-1)^\ell\bigl(e^{xe^{-t}}\bigr)^{(\ell)}$ in the above inequality leads to

$$(-1)^n\bigl(e^{xe^{-t}}\bigr)^{(k+\ell)}\bigl(e^{xe^{-t}}\bigr)^{(n-k+\ell)} \ge (-1)^n\bigl(e^{xe^{-t}}\bigr)^{(m+\ell)}\bigl(e^{xe^{-t}}\bigr)^{(n-m+\ell)}.$$

Further taking $t \to 0^+$ finds the inequality (4.8). The proof of Theorem 4.2 is complete. □

Theorem 4.3. *For $x, \ell \ge 0$ and $m, n \in \mathbb{N}$, let*

$$\begin{aligned}
\mathcal{G}_{\ell,m,n} &= B_{\ell+2m+n}(x)[B_\ell(x)]^2 - B_{\ell+m+n}(x)B_{\ell+m}(x)B_\ell(x)\\
&\quad - B_{\ell+n}(x)B_{\ell+2m}(x)B_\ell(x) + B_{\ell+n}(x)[B_{\ell+m}(x)]^2,\\
\mathcal{H}_{\ell,m,n} &= B_{\ell+2m+n}(x)[B_\ell(x)]^2 - 2B_{\ell+m+n}(x)B_{\ell+m}(x)B_\ell(x)\\
&\quad + B_{\ell+n}(x)[B_{\ell+m}(x)]^2,\\
\mathcal{I}_{\ell,m,n} &= B_{\ell+2m+n}(x)[B_\ell(x)]^2 - 2B_{\ell+n}(x)B_{\ell+2m}(x)B_\ell(x)\\
&\quad + B_{\ell+n}(x)[B_{\ell+m}(x)]^2.
\end{aligned}$$

Then

$$\begin{gathered}
\mathcal{G}_{\ell,m,n} \ge 0, \quad \mathcal{H}_{\ell,m,n} \ge 0,\\
\mathcal{H}_{\ell,m,n} \lesseqqgtr \mathcal{G}_{\ell,m,n} \quad \text{when } m \lesseqgtr n,\\
\mathcal{I}_{\ell,m,n} \ge \mathcal{G}_{\ell,m,n} \ge 0 \quad \text{when } n \ge m.
\end{gathered} \tag{4.9}$$

Proof. In [24, Theorem 1 and Remark 2], it was obtained that if f is completely monotonic on $(0,\infty)$ and

$$\begin{aligned}
G_{m,n} &= (-1)^n\bigl\{f^{(n+2m)}f^2 - f^{(n+m)}f^{(m)}f - f^{(n)}f^{(2m)}f + f^{(n)}\bigl[f^{(m)}\bigr]^2\bigr\},\\
H_{m,n} &= (-1)^n\bigl\{f^{(n+2m)}f^2 - 2f^{(n+m)}f^{(m)}f + f^{(n)}\bigl[f^{(m)}\bigr]^2\bigr\},\\
I_{m,n} &= (-1)^n\bigl\{f^{(n+2m)}f^2 - 2f^{(n)}f^{(2m)}f + f^{(n)}\bigl[f^{(m)}\bigr]^2\bigr\}
\end{aligned}$$

for $n, m \in \mathbb{N}$, then

$$\begin{gathered}
G_{m,n} \ge 0, \quad H_{m,n} \ge 0,\\
H_{m,n} \lesseqqgtr G_{m,n} \quad \text{when } m \lesseqgtr n,\\
I_{m,n} \ge G_{m,n} \ge 0 \quad \text{when } n \ge m.
\end{gathered} \tag{4.10}$$

Replacing $f(t)$ by $(-1)^\ell\bigl(e^{xe^{-t}}\bigr)^{(\ell)}$ in $G_{m,n}$, $H_{m,n}$, and $I_{m,n}$ and simplifying produce

$$\begin{aligned}
G_{m,n} &= (-1)^{\ell+n}\Bigl\{\bigl(e^{xe^{-t}}\bigr)^{(\ell+2m+n)}\Bigl[\bigl(e^{xe^{-t}}\bigr)^{(\ell)}\Bigr]^2\\
&\quad - \bigl(e^{xe^{-t}}\bigr)^{(\ell+m+n)}\bigl(e^{xe^{-t}}\bigr)^{(\ell+m)}\bigl(e^{xe^{-t}}\bigr)^{(\ell)}\\
&\quad - \bigl(e^{xe^{-t}}\bigr)^{(\ell+n)}\bigl(e^{xe^{-t}}\bigr)^{(\ell+2m)}\bigl(e^{xe^{-t}}\bigr)^{(\ell)} + \bigl(e^{xe^{-t}}\bigr)^{(\ell+n)}\Bigl[\bigl(e^{xe^{-t}}\bigr)^{(\ell+m)}\Bigr]^2\Bigr\},\\
H_{m,n} &= (-1)^{\ell+n}\Bigl\{\bigl(e^{xe^{-t}}\bigr)^{(\ell+2m+n)}\Bigl[\bigl(e^{xe^{-t}}\bigr)^{(\ell)}\Bigr]^2
\end{aligned}$$

$$
\begin{aligned}
&- 2\bigl(e^{xe^{-t}}\bigr)^{(\ell+m+n)}\bigl(e^{xe^{-t}}\bigr)^{(\ell+m)}\bigl(e^{xe^{-t}}\bigr)^{(\ell)} + \bigl(e^{xe^{-t}}\bigr)^{(\ell+n)}\Bigl[\bigl(e^{xe^{-t}}\bigr)^{(\ell+m)}\Bigr]^2\Bigr\},\\
I_{m,n} &= (-1)^{\ell+n}\Bigl\{\bigl(e^{xe^{-t}}\bigr)^{(\ell+2m+n)}\Bigl[\bigl(e^{xe^{-t}}\bigr)^{(\ell)}\Bigr]^2\\
&- 2\bigl(e^{xe^{-t}}\bigr)^{(\ell+n)}\bigl(e^{xe^{-t}}\bigr)^{(\ell+2m)}\bigl(e^{xe^{-t}}\bigr)^{(\ell)} + \bigl(e^{xe^{-t}}\bigr)^{(\ell+n)}\Bigl[\bigl(e^{xe^{-t}}\bigr)^{(\ell+m)}\Bigr]^2\Bigr\}.
\end{aligned}
$$

Further taking $t \to 0^+$ reveals

$$
\lim_{t\to0^+} G_{m,n} = e^3\mathcal{G}_{\ell,m,n}, \quad \lim_{t\to0^+} H_{m,n} = e^3\mathcal{H}_{\ell,m,n}, \quad \text{and} \quad \lim_{t\to0^+} I_{m,n} = e^3\mathcal{I}_{\ell,m,n}.
$$

Substituting these quantities into (4.10) and simplifying bring about inequalities in (4.9). The proof of Theorem 4.3 is complete. □

Theorem 4.4. *For $x, k \ge 0$ and $n \in \mathbb{N}$, we have*

$$
\left[\prod_{\ell=0}^{n} B_{k+2\ell}(x)\right]^{1/(n+1)} \ge \left[\prod_{\ell=0}^{n-1} B_{k+2\ell+1}(x)\right]^{1/n}. \tag{4.11}
$$

Proof. If f is a completely monotonic function on $(0,\infty)$, then, by the convexity of the sequence (4.7) and Nanson's inequality listed in [11, p. 205, 3.2.27],

$$
\left[\prod_{\ell=0}^{n}(-1)^{k+2\ell+1}f^{(k+2\ell+1)}(t)\right]^{1/(n+1)} \ge \left[\prod_{\ell=1}^{n}(-1)^{k+2\ell}f^{(k+2\ell)}(t)\right]^{1/n}
$$

for $k \ge 0$. Replacing $f(t)$ by $e^{xe^{-t}}$ in the above inequality results in

$$
\left[\prod_{\ell=0}^{n}(-1)^{k+2\ell+1}\bigl(e^{xe^{-t}}\bigr)^{(k+2\ell+1)}\right]^{1/(n+1)} \ge \left[\prod_{\ell=1}^{n}(-1)^{k+2\ell}\bigl(e^{xe^{-t}}\bigr)^{(k+2\ell)}\right]^{1/n}
$$

for $k \ge 0$. Letting $t \to 0^+$ in the above inequality leads to (4.11). The proof of Theorem 4.4 is complete. □

Remark 4.1. This paper is a revised version of the preprint [17].

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