Article

Not peer-reviewed version

Orthonormal Right-Handed Frames on the Two-Sphere and Solutions to Maxwell's Equations via de Broglie Waves

David Carfi *

Posted Date: 4 July 2025

doi: 10.20944/preprints202507.0453.v1

Keywords: Killing Vector Fields; Two-Sphere Symmetries; Orthonormal Tangent Frames; Maxwell's Equations; de Broglie Waves; Circular Polarization; Lie Algebra Isomorphism

Article

Orthonormal Right-Handed Frames on the Two-Sphere and Solutions to Maxwell's Equations via de Broglie Waves

David Carfì [1,2]

[1] MIFT Department, University of Messina, Italy; dcarfi@unime.it
[2] FDMP Research Group, Department of Mathematics, UC Riverside, USA

Abstract

This paper explores some geometric and physical implications of Killing vector fields on the two-sphere $\mathbb{S}^2$, culminating in a novel application to Maxwell's equations in free space. Initially, we investigate the Killing vector fields on $\mathbb{S}^2$, which generate the isometries of the sphere under the rotation group $SO(3)$. These fields, represented as $K_v(q) = v \times q$ for $v \in \mathbb{R}^3$, form a 3-dimensional Lie algebra isomorphic to $\mathfrak{so}(3)$. We establish an isomorphism $K : \mathbb{R}^3 \to \mathcal{K}(\mathbb{S}^2)$, mapping vectors $v = au$ (with $u \in \mathbb{S}^2$) to scaled Killing vector fields aK_u, and analyze its relationship with $SO(3)$ through the adjoint action and exponential map, highlighting the geometric and algebraic unity of spherical symmetries. Subsequently, we focus on constructing a smooth orthonormal right-handed tangent frame $f_e : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2)^2$, defined as $f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$, where $\hat{K}_e(u) = \frac{e \times u}{|e \times u|}$ is the unit vector of the Killing field $K_e(u) = e \times u$. We verify its smoothness, orthonormality, and right-handedness, characterized by $\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = u$. We further prove that any smooth orthonormal right-handed frame on $\mathbb{S}^2 \setminus \{e, -e\}$ is either f_e or a rotation thereof by a smooth map $\rho : \mathbb{S}^2 \setminus \{e, -e\} \to SO(3)$, reflecting the triviality of the frame bundle over the contractible domain. The paper then pivots to an innovative application, constructing solutions to Maxwell's equations in free space by combining spherical symmetries with quantum mechanical de Broglie waves in tempered distribution wave space. We define a frame $g = (r, s, u)$ over the dual Minkowski space $\mathbb{M}_4^* \setminus \Pi$, with $u = \frac{\mathbf{k}}{|\mathbf{k}|}$, extended from f_e, satisfying $u \times s = -r$. The complex field $w_k(x) = e^{i\langle k,x\rangle}(r_k + is_k)$, where $\eta_k(x) = e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ is the de Broglie family, is shown to satisfy Maxwell's equations for light-like 4-wavevectors k, with $\nabla \times w_k = |\mathbf{k}|w_k$ and $i\frac{\partial w_k}{\partial t} = c|\mathbf{k}|w_k$. The frame's orientation allignes with circularly polarized plane wave solutions. The deeper scientific significance lies in reuniting together differential geometry (via $SO(3)$ symmetries), quantum mechanics (de Broglie waves in Schwartz distribution theory), and electromagnetism (Maxwell's solutions in Schwartz tempered complex fields on Minkowski spacetime). The construction offers a geometric perspective on electromagnetic waves, with potential applications in optics, photonics, quantum field theory and gauge theories. This paper blends mathematical analysis, differential geometry, Schwartz distributions with quantum physics and Maxwell's electromagnetism.

Keywords: killing vector fields; two-sphere symmetries; orthonormal tangent frames; Maxwell's equations; de Broglie waves; circular polarization; lie algebra isomorphism

---

1. Introduction: Geometric Symmetries and Maxwell's Equations on the Two-Sphere

The two-sphere $\mathbb{S}^2$, a fundamental object in differential geometry, exhibits rich symmetry properties governed by the rotation group $SO(3)$, making it a cornerstone for studying geometric structures and their physical applications. Killing vector fields, which generate isometries on $\mathbb{S}^2$ - in the standard form of one parameter groups of rotations -, encapsulate these symmetries and form a Lie algebra isomorphic to $\mathfrak{so}(3)$, providing a powerful framework for analyzing rotational dynamics. Concurrently,

 doi:10.20944/preprints202507.0453.v1

orthonormal tangent frames on $\mathbb{S}^2$ offer a means to probe its local geometry, with applications ranging from gauge theories to field theories in physics. In parallel, Maxwell's equations, the cornerstone of classical electromagnetism, describe electromagnetic wave propagation, with plane wave solutions playing a pivotal role in optics, photonics, and quantum field theory. The integration of geometric symmetries with quantum mechanical concepts, such as de Broglie waves, presents an opportunity to explore novel solutions to these equations, bridging mathematics and physics in innovative ways.

This paper investigates the interplay between the geometric symmetries of $\mathbb{S}^2$ and electromagnetic phenomena, culminating in a new method for constructing solutions to Maxwell's equations in free space. We begin by establishing an isomorphism between $\mathbb{R}^3$ and the space of Killing vector fields on $\mathbb{S}^2$, mapping vectors to infinitesimal rotations and elucidating their relationship with $SO(3)$. We then construct a smooth orthonormal right-handed tangent frame on $\mathbb{S}^2 \setminus \{e, -e\}$, derived from Killing vector fields, and demonstrate that all such frames are rotations (fields of rotation) of this canonical frame, leveraging the topological properties of the frame bundle. The core contribution lies in applying these geometric structures to generate electromagnetic fields by combining a complex-valued frame with de Broglie waves. Specifically, we define a field

$$w_k(x) = e^{i\langle k,x\rangle}(r_k + is_k),$$

where the frame (r_k, s_k, u) extends the spherical frame to the dual Minkowski space, satisfying

$$u \times s = -r.$$

This field satisfies Maxwell's equations for light-like wavevectors, producing circularly polarized plane waves.

Our work unifies differential geometry, quantum mechanics, and electromagnetism, offering a geometric perspective on electromagnetic wave propagation. By leveraging $SO(3)$ symmetries and quantum phase factors, it provides a novel framework with potential applications in optics, photonics, and gauge theories.

The paper is organized as follows: Section 2 presents the literature review; Section 3 details the Killing vector field isomorphism; Sections 4-8 construct the canonical Killing tangent frames and relative geometric results; Sections 9-10 present the Maxwell applications; Section 11 discusses implications and future directions.

2. Literature Review

The study of Killing vector fields and their geometric implications on manifolds like the two-sphere $\mathbb{S}^2$ has a rich history in differential geometry and mathematical physics. Killing vector fields, introduced by Wilhelm Killing in the late 19th century, are vector fields that generate isometries, preserving the metric of a Riemannian manifold [1]. On $\mathbb{S}^2$, a maximally symmetric 2-dimensional manifold with isometry group $SO(3)$, Killing vector fields correspond to infinitesimal rotations, forming a 3-dimensional Lie algebra isomorphic to $\mathfrak{so}(3)$ [2]. This isomorphism, mapping vectors in $\mathbb{R}^3$ to $\mathcal{K}(\mathbb{S}^2)$, is well-documented in texts like Nakahara (2003) and Lee (2018), which detail the relationship between Lie groups and their algebras in geometric contexts [3,4]. Our paper establishes this isomorphism

$$K : \mathbb{R}^3 \to \mathcal{K}(\mathbb{S}^2),$$

where $K_v(q) = v \times q$, and explores its interplay with $SO(3)$ via the adjoint action and exponential map, aligning with classical results while emphasizing practical applications.

The construction of orthonormal tangent frames on manifolds is a fundamental topic in differential geometry, with applications in physics and gauge theory. On $\mathbb{S}^2$, the frame bundle is non-trivial, but over $\mathbb{S}^2 \setminus \{e, -e\}$, it becomes trivial due to the domain's contractibility [5]. Works like Kobayashi and

 doi:10.20944/preprints202507.0453.v1

Nomizu (1963) provide a theoretical basis for frame bundles, while our paper specifically constructs a new smooth orthonormal right-handed frame

$$f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u)),$$

where

$$\hat{K}_e(u) = \frac{e \times u}{|e \times u|}$$

is the unit vector of the Killing field $K_e(u) = e \times u$ [6]. We prove that all such frames are obtained from f_e by smooth fields of rotations, leveraging the triviality of the bundle, a result consistent with topological arguments in Milnor and Stasheff (1974) [7]. The right-handedness condition,

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = u,$$

connects to orientation studies in differential geometry [8].

The application of differential geometric structures to electromagnetic theory, particularly Maxwell's equations, has been explored extensively. Maxwell's equations in free space describe electromagnetic wave propagation, with plane wave solutions being fundamental [9]. Circularly polarized plane waves, crucial in optics and photonics, are well-studied in texts like Born and Wolf (1999) [10]. The use of electromagnetic complex fields

$$\mathbf{F} = \mathbf{E} + ic\mathbf{B}$$

to simplify Maxwell's equations allignes to relativistic formulations [11]. Recent works, such as those by Bialynicki-Birula (1996), explore photon wave functions, linking electromagnetic fields to quantum mechanics [12]. However, the literature lacks a direct construction of Maxwell solutions using $\mathbb{S}^2$ symmetries and the de Broglie basis of Schwartz linear algebra ([13–16]).

Our paper introduces a novel application, constructing a fundamental family of solutions w, defined by

$$w_k(x) = e^{i\langle k,x\rangle}(r_k + is_k)$$

for light-like 4-wavevectors k, where the frame $g = (r, s, u)$ extends f_e to the dual Minkowski space $\mathbb{M}_4^* \setminus \Pi$, with $u \times s = -r$, or it is a smooth orthonormal right-handed frame obtained by f_e applying a surface of rotations. This approach, inspired by quantum mechanical de Broglie waves [17], unifies differential geometry, quantum mechanics, and electromagnetism. While geometric methods in gauge theory (e.g., Yang-Mills) are discussed in Bleecker (1981), our specific use of Killing vector field-derived frames is unique [18]. Our original frame's orientation, ensuring $u \times s = -r$, alignes with circular polarization properties [19].

This work fills a gap in the literature by demonstrating how $SO(3)$ symmetries on $\mathbb{S}^2$ can generate physically meaningful electromagnetic solutions, offering a geometric perspective on wave propagation. It extends Schwartz linear algebra by integrating quantum phase factors and smooth orthonormal frames viewed also as complex tempered vector fields. Our approach suggests applications in optics, photonics, and quantum field theory.

2.1. Further Bibliography

Paper [20] systematically develops Maxwell's equations in Schrödinger form, using the curl operator as a Hamiltonian for dispersive media. It provides a strong theoretical foundation somewhat aligning with our formulation in Section 9, but without using Schwartz distribution and S-Linear Algebra and with different aims. The paper [21] reduces Maxwell's equations to a two-level Schrödinger-type evolution for polarization states, resonating with our use of Killing frames to encode polarization, but again without using Schwartz distribution and S-Linear Algebra and with different intent. [22] offers a quantum-mechanical derivation of the Schrödinger equation from Maxwellian principles, emphasizing

 doi:10.20944/preprints202507.0453.v1

the field–wave duality echoed in our embedding of de Broglie fields into Maxwellian Schwartz space, but, also in this case, it is not completely clear the topological structure in which the author works and, in any case, without using Schwartz distribution. [23] presents a differential-form and frame-field perspective on Maxwell theory, useful for justifying the frame-theoretic nature of our construction but far from Schwartz theories. [24] explores the role of Killing fields and frames in identifying photon currents and polarization. [25] examines the relation between Maxwell potentials and Killing frames in curved spacetimes.

3. Theoretical Background: Isomorphism between $\mathbb{R}^3$ and the space $\mathcal{K}(\mathbb{S}^2)$ of Killing Vector Fields on $\mathbb{S}^2$

Let's formalize the relationship between vectors in $\mathbb{R}^3$ and Killing vector fields on the two-sphere $\mathbb{S}^2$, by means of the map sending a scaled vector au (where $a \in \mathbb{R}$, $u \in \mathbb{S}^2$) to the scaled Killing vector field aK_u defined below.

3.1. The Correspondence

The space of Killing vector fields on $\mathbb{S}^2$, denoted $\mathcal{K}(\mathbb{S}^2)$, is a 3-dimensional vector space, isomorphic to the Lie algebra $\mathfrak{so}(3)$, because the isometry group of $\mathbb{S}^2$ is $SO(3)$. For a unit vector $u \in \mathbb{S}^2 \subset \mathbb{R}^3$ (i.e., $|u| = 1$), the Killing vector field is the vector valued mapping

$$K_u : \mathbb{S}^2 \to \mathbb{R}^3 : k \mapsto K_u(q) = u \times q,$$

for $q \in \mathbb{S}^2$. This vector field K_u generates rotations around the axis u, in the sense that it is the infinitesimal generator of the one parameter group $R_u : \mathbb{R} \to SO(3)$ sending each θ to the rotation $R_u(\theta)$.

Now, consider any vector $v \in \mathbb{R}^3$, which we can write as:

$$v = au,$$

where $a = |v| \in \mathbb{R}_{\geq 0}$ is the magnitude, and $u = v/|v| \in \mathbb{S}^2$ is a unit vector (if $v \neq 0$). If $v = 0$, we'll handle that separately. Define the Killing vector field associated with v by

$$K_v(q) = v \times q.$$

If $v = au$, then

$$K_v(q) = (au) \times q = a(u \times q) = aK_u(q).$$

Thus, the Killing vector field K_v is the scaled Killing vector field aK_u. This suggests a linear map K from $\mathbb{R}^3$ to $\mathcal{K}(\mathbb{S}^2)$

$$\mathbb{R}^3 \to \mathcal{K}(\mathbb{S}^2), \quad v \mapsto K_v,$$

where

$$K_v(q) = v \times q.$$

3.2. Properties of the Map K

Let's examine this map K.

- **Linearity**. The map K is linear. For vectors $v_1, v_2 \in \mathbb{R}^3$ and scalars $c_1, c_2 \in \mathbb{R}$:

$$K_{c_1 v_1 + c_2 v_2}(q) = (c_1 v_1 + c_2 v_2) \times q = c_1(v_1 \times q) + c_2(v_2 \times q) = c_1 K_{v_1}(q) + c_2 K_{v_2}(q).$$

- **Action on Scaled Vectors**. If $v = au$ with $u \in \mathbb{S}^2$, then

$$K_v = aK_u.$$

 doi:10.20944/preprints202507.0453.v1

- **Zero Vector**. If $v = 0$, then
$$K_0(q) = 0 \times q = 0,$$
which is the zero vector field, a valid element of $\mathcal{K}(\mathbb{S}^2)$.
- **Surjectivity**. To check if every Killing vector field on $\mathbb{S}^2$ is of the form K_v, consider a general Killing vector field. We know $\mathcal{K}(\mathbb{S}^2)$ is spanned by the basis (K_1, K_2, K_3), where
$$K_1(q) = e_1 \times q, \quad K_2(q) = e_2 \times q, \quad K_3(q) = e_3 \times q,$$
and $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. A general Killing vector field is a linear combination
$$V = aK_1 + bK_2 + cK_3.$$
Let's compute the value of V at every point:
$$V(q) = a(e_1 \times q) + b(e_2 \times q) + c(e_3 \times q) = (ae_1 + be_2 + ce_3) \times q = v \times q,$$
where $v = (a,b,c) \in \mathbb{R}^3$. Thus
$$V = K_v.$$
This shows the map $v \mapsto K_v$ is **surjective**: every Killing vector field on $\mathbb{S}^2$ is of the form K_v for some $v \in \mathbb{R}^3$.
- **Injectivity**. Is the map K injective? Suppose $K_{v_1} = K_{v_2}$, then
$$v_1 \times q = v_2 \times q,$$
for all $q \in \mathbb{S}^2$. This implies
$$(v_1 - v_2) \times q = 0.$$
Since q represents all vectors on $\mathbb{S}^2$, choose q perpendicular to $v_1 - v_2$ (if $v_1 - v_2 \neq 0$, such q exist on $\mathbb{S}^2$). The cross product is zero only if $v_1 - v_2 = 0$, because $v_1 - v_2$ cannot be parallel to all $q \in \mathbb{S}^2$. Thus, $v_1 = v_2$, and the map is **injective**.
- **Isomorphism**. Since $\mathbb{R}^3$ and $\mathcal{K}(\mathbb{S}^2)$ are both 3-dimensional vector spaces, and the map $v \mapsto K_v$ is linear, surjective, and injective, it is an **isomorphism**
$$\mathbb{R}^3 \cong \mathcal{K}(\mathbb{S}^2).$$

3.3. Interpretation

We have found a natural correspondence from $\mathbb{R}^3$ to $\mathcal{K}(\mathbb{S}^2)$, where a vector $v = au$ (with $a \in \mathbb{R}$, $u \in \mathbb{S}^2$) maps to the Killing vector field
$$K_v = aK_u.$$

This correspondence is not only intuitive but also a vector space isomorphism. Each vector $v \in \mathbb{R}^3$ specifies a Killing vector field K_v.

- The **direction** of v (i.e., $u = v/|v|$) determines the axis of rotation of the associated one parameter group of rotations R_u.
- The **magnitude** of v (i.e., $a = |v|$) scales the "strength" of the Killing vector field, affecting the angular speed of the rotation it generates.

For example:

- if $v = (1,0,0)$, then $K_v = K_1$, which generates the rotation group around the first axis.
- if $v = (2,0,0) = 2e_1$, then $K_v = 2K_1$, a "faster" rotation group around the same axis - in the sense that the one-parameter group
$$\theta \mapsto e^{\theta(2A_1)} = e^{(2\theta)A_1}$$

rotates twice as fast. Here, the matrix A_1 is the matrix associated with the linear application $L_1 = e_1 \times$.
- if $v = 0$, then $K_0 = 0$, the trivial Killing vector field.

We observe that in general the killing vector field K_v is the restriction of the linear endomorphism of $\mathbb{R}^3$

$$L_v = v \times .$$

3.4. Geometric Insight

The isomorphism $\mathbb{R}^3 \cong \mathcal{K}(\mathbb{S}^2)$ reflects the fact that the Lie algebra $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$ with the cross product as the Lie bracket. The Killing vector fields are the manifestations of $\mathfrak{so}(3)$ on $\mathbb{S}^2$, and the map $K : v \mapsto v\times$ translates vectors in $\mathbb{R}^3$ to these generators of rotations.

Our observation about scaled vectors $au \mapsto aK_u$ highlights that the space $\mathcal{K}(\mathbb{S}^2)$ includes all possible rotation axes (parametrized by $u \in \mathbb{S}^2$) and all possible scalings (parametrized by $a \in \mathbb{R}$), covering the entire 3-dimensional space of Killing vector fields.

3.5. Final Remark

We have found a correspondence from $\mathbb{R}^3$ to $\mathcal{K}(\mathbb{S}^2)$, where a vector $v = au$ (with $a \in \mathbb{R}$, $u \in \mathbb{S}^2$) maps to the Killing vector field

$$K_v = aK_u,$$

and this map, defined by $K_v(q) = v \times q$, is an isomorphism between $\mathbb{R}^3$ and $\mathcal{K}(\mathbb{S}^2)$. Every Killing vector field on $\mathbb{S}^2$, including the zero vector field, is uniquely represented by some $v \in \mathbb{R}^3$, with scaled vectors au producing scaled Killing vector fields aK_u, and the Lie bracket structure on the space of Killing vector fields is defined by

$$[K_v, K_w] = K_{v\times w}.$$

4. Results I: Smooth Orthonormal Tangent Frame f_e on the Two-Sphere

We desire to explore, now, an original aspect of the Killing vector fields K_e on the sphere: their capability to determine a canonical smooth orthonormal frame f_e on the two-sphere minus the "polar" couple $\{e, -e\}$.

At this purpose, we shall state and prove the following original theorem.

Theorem 1. *Consider a unit vector e of the space $\mathbb{R}^3$, that is an element of $\mathbb{S}^2$. Now consider the tangent frame*

$$f_e : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2) \times T(\mathbb{S}^2),$$

defined - on the 2-sphere minus the two poles e and $-e$, and taking values in the Cartesian square of the tangent bundle of $\mathbb{S}^2$ - by the relation

$$f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u)),$$

where $\hat{K}_e(u)$ is the unit vector of $K_e(u)$.
Then, the mapping f_e is a smooth orthonormal tangent frame on $\mathbb{S}^2$ minus the poles e and $-e$.

4.1. Proof of Theorem 1

Let's dive into the problem of determining whether the tangent frame

$$f_e : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2)^2,$$

defined by $f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$, is a smooth orthonormal tangent frame on the two-sphere $\mathbb{S}^2$ minus the poles e and $-e$. Here, $e \in \mathbb{S}^2 \subset \mathbb{R}^3$ is a unit vector, $K_e(u) = e \times u$, and $\hat{K}_e(u)$ is the unit vector in the direction of $K_e(u)$. We'll prove the theorem in more steeps.

4.1.1. Setup and Definitions

- **The Two-Sphere**. $\mathbb{S}^2 = \{u \in \mathbb{R}^3 \mid |u| = 1\}$ is the unit sphere in $\mathbb{R}^3$, a 2-dimensional smooth manifold.
- **Unit Vector** e. $e \in \mathbb{S}^2$, so $|e| = 1$.
- **Killing Vector Field**. For $e \in \mathbb{S}^2$, the associated Killing vector field is defined by

$$K_e(u) = e \times u,$$

where $u \in \mathbb{S}^2$. Any vector $K_e(u)$ is a tangent vector to $\mathbb{S}^2$ at u, since

$$K_e(u) \cdot u = (e \times u) \cdot u = 0,$$

by properties of the cross product.
- **Unit Vector** $\hat{K}_e(u)$. Assuming $K_e(u) \neq 0$, we define

$$\hat{K}_e(u) = \frac{K_e(u)}{|K_e(u)|},$$

where

$$|K_e(u)| = |e \times u| = |e||u| \sin\theta = \sin\theta,$$

and θ is the angle between e and u. Since $|e| = |u| = 1$, we have

$$|e \times u| = \sqrt{1 - (e \cdot u)^2},$$

because

$$e \cdot u = \cos\theta, \quad \sin\theta = \sqrt{1 - \cos^2\theta}.$$

Thus

$$\hat{K}_e(u) = \frac{e \times u}{\sqrt{1 - (e \cdot u)^2}}.$$

- **Tangent Frame**. The tangent frame is defined by

$$f_e : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2)^2, \quad f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u)),$$

where $T(\mathbb{S}^2)^2 = T(\mathbb{S}^2) \times T(\mathbb{S}^2)$ is the Cartesian product of the tangent bundle with itself, i.e., $f_e(u)$ assigns a pair of tangent vectors in $T_u\mathbb{S}^2$.
- **Domain of** f_e. The domain is $\mathbb{S}^2 \setminus \{e, -e\}$, because

$$K_e(u) = e \times u = 0$$

when $u = \pm e$ (since $e \times e = 0$, $e \times (-e) = 0$). At these points, $\hat{K}_e(u)$ is undefined, justifying their exclusion.
- **Goal**. Determine if f_e is a **smooth orthonormal tangent frame**, meaning:
  - **Smooth**. f_e is a smooth map, i.e., the vector fields $\hat{K}_e$ and $u \times \hat{K}_e$ are smooth on $\mathbb{S}^2 \setminus \{e, -e\}$.
  - **Orthonormal**. At each u, the vectors $\hat{K}_e(u)$ and $u \times \hat{K}_e(u)$ are:
    - Tangent to $\mathbb{S}^2$ at u.
    - Orthonormal with respect to the induced metric on $\mathbb{S}^2$.
    - Form a basis for $T_u\mathbb{S}^2$.

4.1.2. Step 1: Tangency

First, confirm that both $\hat{K}_e(u)$ and $u \times \hat{K}_e(u)$ are tangent to $\mathbb{S}^2$ at u.

 doi:10.20944/preprints202507.0453.v1

- $\hat{K}_e(u)$:
$$\hat{K}_e(u) = \frac{e \times u}{|e \times u|}.$$
Since $K_e(u) = e \times u$ is tangent (as $(e \times u) \cdot u = 0$), and $\hat{K}_e(u)$ is a scalar multiple of $K_e(u)$, it is also tangent:
$$\hat{K}_e(u) \cdot u = \frac{(e \times u) \cdot u}{|e \times u|} = 0.$$
- $u \times \hat{K}_e(u)$:
$$u \times \hat{K}_e(u) = u \times \frac{e \times u}{|e \times u|}.$$
Check tangency:
$$(u \times \hat{K}_e(u)) \cdot u = \left(u \times \frac{e \times u}{|e \times u|}\right) \cdot u = 0,$$
since the cross product $a \times b$ is perpendicular to both a and b. Thus, both vectors lie in $T_u\mathbb{S}^2$.

4.1.3. Step 2: Orthonormality

To be an orthonormal frame, the vectors $\hat{K}_e(u)$ and $u \times \hat{K}_e(u)$ must be:

- **Unit vectors** (norm 1 with respect to the metric).
- **Orthogonal** to each other.
- **Linearly independent** (to span $T_u\mathbb{S}^2$).

The metric on $\mathbb{S}^2$ is the induced Euclidean metric from $\mathbb{R}^3$, so the inner product of tangent vectors $v, w \in T_u\mathbb{S}^2$ is the dot product $v \cdot w$.

- **Norm of** $\hat{K}_e(u)$. Since
$$\hat{K}_e(u) = \frac{e \times u}{|e \times u|},$$
then we have
$$|\hat{K}_e(u)|^2 = \frac{|e \times u|^2}{|e \times u|^2} = 1.$$
So, $\hat{K}_e(u)$ is a unit vector.
- **Norm of** $u \times \hat{K}_e(u)$. We have
$$u \times \hat{K}_e(u) = u \times \frac{e \times u}{|e \times u|}.$$
Let's compute the norm
$$|u \times \hat{K}_e(u)| = \left|u \times \frac{e \times u}{|e \times u|}\right| = \frac{|u \times (e \times u)|}{|e \times u|}.$$
Using the vector triple product identity
$$e \times u = -(u \times e),$$
$$u \times (e \times u) = u \times (u \times e) = (u \cdot e)u - (u \cdot u)e = (u \cdot e)u - e,$$
since $u \cdot u = 1$. Compute the magnitude
$$|u \times (e \times u)| = |(u \cdot e)u - e|.$$
$$|(u \cdot e)u - e|^2 = ((u \cdot e)u - e) \cdot ((u \cdot e)u - e) = (u \cdot e)^2 u \cdot u - 2(u \cdot e)(u \cdot e) + e \cdot e.$$
Since $u \cdot u = 1$, $e \cdot e = 1$, we have
$$|(u \cdot e)u - e|^2 = (u \cdot e)^2 \cdot 1 - 2(u \cdot e)^2 + 1 = 1 - (u \cdot e)^2.$$

Thus
$$|u \times (e \times u)| = \sqrt{1-(u\cdot e)^2} = |e \times u|,$$
since $|e \times u| = \sqrt{1-(e\cdot u)^2}$. So
$$|u \times \hat{K}_e(u)| = \frac{|u \times (e \times u)|}{|e \times u|} = \frac{|e \times u|}{|e \times u|} = 1.$$
Thus, $u \times \hat{K}_e(u)$ is also a unit vector.

- **Orthogonality**. Let's check the dot product. We have
$$\hat{K}_e(u) \cdot (u \times \hat{K}_e(u)) = \frac{e \times u}{|e \times u|} \cdot \left(u \times \frac{e \times u}{|e \times u|}\right).$$
Using the scalar triple product $a \cdot (b \times c)$, we obtain
$$(e \times u) \cdot (u \times (e \times u)) = (e \times u) \cdot ((u \cdot e)u - e) =$$
$$= (e \times u) \cdot (u \cdot e)u - (e \times u) \cdot e = 0 - 0 = 0,$$
since $(e \times u) \cdot u = 0$ and $(e \times u) \cdot e = 0$. Thus
$$\hat{K}_e(u) \cdot (u \times \hat{K}_e(u)) = \frac{0}{|e \times u|^2} = 0.$$
The vectors are orthogonal.
- **Linear Independence**. Since $T_u\mathbb{S}^2$ is 2-dimensional, two orthonormal vectors form a basis. Alternatively, note that $\hat{K}_e(u)$ is perpendicular to u, and $u \times \hat{K}_e(u)$ is perpendicular to both u and $\hat{K}_e(u)$. Since $\hat{K}_e(u) \neq 0$ on $\mathbb{S}^2 \setminus \{e, -e\}$, they are linearly independent.

Thus, $(\hat{K}_e(u), u \times \hat{K}_e(u))$ is an orthonormal basis of $T_u\mathbb{S}^2$.

4.1.4. Step 3: Smoothness

To be a smooth tangent frame, the map
$$f_e : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2)^2,$$
$u \mapsto (\hat{K}_e(u), u \times \hat{K}_e(u))$, must be smooth. Since $T(\mathbb{S}^2)^2$ is a bundle over $\mathbb{S}^2$, we need $\hat{K}_e(u)$ and $u \times \hat{K}_e(u)$ to be smooth vector fields on $\mathbb{S}^2 \setminus \{e, -e\}$.

- **Smoothness of $\hat{K}_e(u)$**. We have
$$\hat{K}_e(u) = \frac{e \times u}{\sqrt{1-(e\cdot u)^2}}.$$
  - **Numerator**. The application $u \mapsto e \times u$ is smooth, as the cross product is a linear map in u, and $u \mapsto u$ is smooth on $\mathbb{S}^2$.
  - **Denominator**. The map
$$u \mapsto |e \times u| = \sqrt{1-(e\cdot u)^2}$$
is smooth. Indeed, the function $u \mapsto u \cdot e$ is smooth (linear in u), and $u \mapsto 1-(u\cdot e)^2$ is smooth. We need only
$$\sqrt{1-(u\cdot e)^2} \neq 0,$$
which holds on $\mathbb{S}^2 \setminus \{e, -e\}$, since
$$1-(u\cdot e)^2 = 0 \implies (u\cdot e)^2 = 1 \implies u \cdot e = \pm 1 \implies u = \pm e.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0453.v1

The square root function is smooth on $(0, \infty)$, and $1 - (u \cdot e)^2 > 0$ on the domain. Thus, the denominator is smooth.

  – **Quotient**. The quotient of smooth functions, with a non-zero denominator, is smooth. Hence, $\hat{K}_e$ is smooth.

- **Smoothness of** $u \times \hat{K}_e(u)$. Concluding the mapping

$$u \mapsto u \times \hat{K}_e(u) = u \times \frac{e \times u}{|e \times u|}$$

is smooth since $\hat{K}_e$ is smooth, and the cross product is a smooth (bilinear) operation, therefore the mapping $u \mapsto u \times \hat{K}_e(u)$ is smooth. ■

4.2. Conclusion and Canonical Orthonormal Killing Frame

The tangent frame $f_e : f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$ is:

- **Tangent**. Both vectors field components are in $T\mathbb{S}^2$.
- **Orthonormal**. Both vectors field components are unit vector fields and mutually orthogonal.
- **Smooth**. Both vector field components $\hat{K}_e$ and $u \mapsto u \times \hat{K}_e(u)$ are smooth on $\mathbb{S}^2 \setminus \{e, -e\}$.

Thus, f_e is a **smooth orthonormal tangent frame** on $\mathbb{S}^2 \setminus \{e, -e\}$.

Definition 1. *We shall call f_e "canonical orthonormal frame induced by K_e" or "Killing orthonormal frame of poles e and $-e$".*

5. Results II: the Tangent Frame f_e as a Smooth Right-Handed Orthonormal Frame

In this section, we state and prove a second important original point: the Killing frame $f_e = (r_e, s_e)$ is, not only smooth and orthonormal, but also right-handed, in the sense that the triple $(r_e(u), s_e(u), u)$ is right-handed (with positive determinant), for every u in the sphere minus the poles of the frame f_e. In other terms, the cross product of r_e times s_e is the identity function on the sphere minus the poles of f_e.

Theorem 2. *The cross product of the first vector field of the Killing frame f_e times the second vector field of f_e gives the identity mapping on $\mathbb{S}^2$ (minus the poles of f_e).*

5.1. Proof of Theorem 2

Let's investigate whether the cross product of the first vector field of the tangent frame $f_e : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2)^2$, defined by $f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$, with the second vector field yields the identity mapping on $\mathbb{S}^2 \setminus \{e, -e\}$. That is, we need to compute:

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)),$$

and check if it equals u, the position vector on $\mathbb{S}^2$, for all $u \in \mathbb{S}^2 \setminus \{e, -e\}$. Here, $e \in \mathbb{S}^2 \subset \mathbb{R}^3$ is a unit vector, $K_e(u) = e \times u$, and

$$\hat{K}_e(u) = \frac{e \times u}{|e \times u|}.$$

5.1.1. Setup and Definitions

- **Two-Sphere**. $\mathbb{S}^2 = \{u \in \mathbb{R}^3 \mid |u| = 1\}$.
- **Unit Vector** e. $e \in \mathbb{S}^2$, so $|e| = 1$.
- **Killing Vector Field**. $K_e(u) = e \times u$, tangent to $\mathbb{S}^2$ at u.
- **Unit Vector Field**.

$$\hat{K}_e(u) = \frac{e \times u}{|e \times u|}, \quad |e \times u| = \sqrt{1 - (e \cdot u)^2},$$

defined on $\mathbb{S}^2 \setminus \{e, -e\}$, where $e \cdot u = \pm 1$ (i.e., $u = \pm e$) makes $e \times u = 0$.

- **Second Vector Field.**
$$u \times \hat{K}_e(u) = u \times \frac{e \times u}{|e \times u|}.$$
- **Tangent Frame.** $f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$, which we've shown is smooth and orthonormal on $\mathbb{S}^2 \setminus \{e, -e\}$.
- **Goal.** Compute
$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)),$$
and check if it equals u.

5.1.2. Verification of the Right-Handness

Let's compute the cross product using vector identities.

$$\hat{K}_e(u) = \frac{e \times u}{|e \times u|}, \quad u \times \hat{K}_e(u) = u \times \frac{e \times u}{|e \times u|}.$$

Define:

$$k = e \times u, \quad |k| = |e \times u| = \sqrt{1 - (e \cdot u)^2},$$

$$\hat{K}_e(u) = \frac{k}{|k|}, \quad u \times \hat{K}_e(u) = u \times \frac{k}{|k|}.$$

We need:

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = \frac{k}{|k|} \times \left(u \times \frac{k}{|k|}\right).$$

Compute the inner cross product:

$$u \times \frac{k}{|k|} = \frac{u \times k}{|k|}.$$

Now:

$$k = e \times u,$$

$$u \times k = u \times (e \times u).$$

Use the vector triple product identity:

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c,$$

$$u \times (e \times u) = (u \cdot u)e - (u \cdot e)u = e - (u \cdot e)u,$$

since $u \cdot u = 1$. So:

$$u \times k = e - (u \cdot e)u,$$

$$u \times \hat{K}_e(u) = \frac{u \times k}{|k|} = \frac{e - (u \cdot e)u}{|k|}.$$

Now compute:

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = \frac{k}{|k|} \times \frac{e - (u \cdot e)u}{|k|} =$$

$$= \frac{1}{|k|^2} k \times (e - (u \cdot e)u) =$$

$$= \frac{1}{|k|^2} [k \times e - (u \cdot e)(k \times u)].$$

- First term:
$$k = e \times u,$$

 doi:10.20944/preprints202507.0453.v1

$$k \times e = (e \times u) \times e = (e \cdot e)u - (e \cdot u)e = u - (e \cdot u)e,$$

since $e \cdot e = 1$.

- Second term:

$$k \times u = (e \times u) \times u = (e \cdot u)u - (u \cdot u)e = (e \cdot u)u - e.$$

So:

$$\begin{aligned} k \times (e - (u \cdot e)u) &= [u - (e \cdot u)e] - (u \cdot e)[(e \cdot u)u - e] = \\ &= u - (e \cdot u)e - (u \cdot e)^2 u + (u \cdot e)e = \\ &= u - (u \cdot e)^2 u + (u \cdot e)e - (e \cdot u)e = \\ &= u[1 - (u \cdot e)^2]. \end{aligned}$$

Thus:

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = \frac{1}{|k|^2} u[1 - (u \cdot e)^2].$$

Since:

$$|k|^2 = |e \times u|^2 = 1 - (u \cdot e)^2,$$

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = \frac{1}{1 - (u \cdot e)^2} u[1 - (u \cdot e)^2] = u.$$

This suggests the cross product yields u, the identity mapping on $\mathbb{S}^2 \setminus \{e, -e\}$. ■

5.2. Geometric Interpretation

The result

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = u$$

means the cross product of the first and second vector fields of the frame f_e yields the position mapping U_e, which is nothing but the identity mapping

$$U_e : u \mapsto u$$

on $\mathbb{S}^2 \setminus \{e, -e\}$. Geometrically, since

$$f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$$

is an orthonormal basis for the tangent space $T_u\mathbb{S}^2$, and since u is normal to $T_u\mathbb{S}^2$, then the cross product of the basis vectors produces a vector in the normal direction, scaled exactly to u.

5.3. Conclusions

The cross product of the first vector field $\hat{K}_e$ and the second vector field

$$U_e \times \hat{K}_e$$

of the tangent frame f_e gives the identity mapping U_e on $\mathbb{S}^2 \setminus \{e, -e\}$:

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = u,$$

for every u belonging to $\mathbb{S}^2 \setminus \{e, -e\}$. We call this last property *right-handness* of the frame f_e. That is equivalent to state that the triple $(f_e, U_e) = (r_e, s_e, U_e)$ is a right-handed orthonormal frame of $\mathbb{R}^3$, defined on $\mathbb{S}^2 \setminus \{e, -e\}$, where U_e is the identity function of $\mathbb{S}^2 \setminus \{e, -e\}$.

 doi:10.20944/preprints202507.0453.v1

6. Results III: Orthonormal Right-Handed Frames

In this section, we state and prove the main geometric theorem of the paper, and perhaps the most intricate to grasp and prove, here. We show essentially that all smooth orthonormal right-handed frame on the two sphere , minus two opposite poles, is induced by a Killing frame.

That is interesting also for the original physics application which see the transformation of de Broglie basis into a standard basis of plane waves generating the solution space of the free Maxwell's equations; this, because it proves that any "orthodox" electromagnetic Maxwellian field is a Schwartz superposition of a basis determined solely by the continuous symmetries of the two sphere (i.e., $SO(3)$, by means of its Lie algebra) and by the celebrated de Broglie basis of quantum mechanics.

Theorem 3. *Every smooth orthonormal right-handed frame on $\mathbb{S}^2$, minus a prechosen e and its opposite $-e$, should be necessarily f_e or any its rotation by a smooth rotation field*

$$\rho : \mathbb{S}^2 \setminus \{e, -e\} \to SO(3).$$

This theorem dives deep into the geometry of tangent frames on the two-sphere $\mathbb{S}^2$.

6.1. Proof of Theorem 3

We've defined the tangent frame

$$f_e : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2)^2,$$

given by

$$f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u)),$$

where

$$\hat{K}_e(u) = \frac{e \times u}{|e \times u|},$$

and established its "right-handness" property:

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = u.$$

Now, we want to determine if every smooth orthonormal right-handed tangent frame on $\mathbb{S}^2 \setminus \{e, -e\}$ must be either f_e or a rotation of f_e by a smooth rotation field

$$\rho : \mathbb{S}^2 \setminus \{e, -e\} \to SO(3).$$

Let's carefully define the terms, set up the problem, and analyze whether this holds, exploring the geometry and topology of the tangent bundle and frame bundle of $\mathbb{S}^2$.

6.1.1. Definitions and Setup

- **Two-Sphere**. $\mathbb{S}^2 = \{u \in \mathbb{R}^3 \mid |u| = 1\}$, a 2-dimensional smooth manifold.
- **Unit Vector** e. $e \in \mathbb{S}^2$, so $|e| = 1$. The domain is $\mathbb{S}^2 \setminus \{e, -e\}$, where e and $-e$ are excluded because $\hat{K}_e(u)$ is undefined at $u = \pm e$.
- **Tangent Frame** f_e.

  $$f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u)), \quad \hat{K}_e(u) = \frac{e \times u}{|e \times u|},$$

  where $|e \times u| = \sqrt{1 - (e \cdot u)^2}$. We've shown f_e is smooth and orthonormal, and it satisfies the right-handness property:

  $$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = u.$$

- **Orthonormal Frame**. A frame

$$f : \mathbb{S}^2 \setminus \{e, -e\} \to T(\mathbb{S}^2)^2 : f(u) = (X(u), Y(u)),$$

assigns to each $u \in \mathbb{S}^2 \setminus \{e, -e\}$ a pair of tangent vectors $X(u), Y(u) \in T_u\mathbb{S}^2$ such that, with respect to the induced metric on $\mathbb{S}^2$:

$$X(u) \cdot X(u) = 1, \quad Y(u) \cdot Y(u) = 1, \quad X(u) \cdot Y(u) = 0.$$

Since $T_u\mathbb{S}^2$ is 2-dimensional, $(X(u), Y(u))$ forms a basis for $T_u\mathbb{S}^2$.

- **Right-Handedness**. We define a frame $f(u) = (X(u), Y(u))$ to be right-handed if:

$$X(u) \times Y(u) = u.$$

Since $X(u)$ and $Y(u)$ are tangent to $\mathbb{S}^2$ at u, their cross product is normal to $T_u\mathbb{S}^2$. The normal vector to $T_u\mathbb{S}^2$ at u is parallel to u, and right-handness requires the cross product to equal u exactly, not $-u$.

- **Rotation Field**. A smooth map $\rho : \mathbb{S}^2 \setminus \{e, -e\} \to SO(3)$, where $SO(3)$ is the group of 3x3 orthogonal matrices with determinant 1. For a frame $f(u) = (X(u), Y(u))$, a rotation of $f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$ by $\rho(u) \in SO(3)$ means:

$$(X(u), Y(u)) = (\rho(u)\hat{K}_e(u), \rho(u)(u \times \hat{K}_e(u))),$$

where $\rho(u)$ acts as a rotation in $\mathbb{R}^3$, but since $X(u), Y(u) \in T_u\mathbb{S}^2$, we need $\rho(u)$ to map $T_u\mathbb{S}^2$ to itself.

- **Question**. Is every smooth orthonormal right-handed frame $f = (X, Y)$ on $\mathbb{S}^2 \setminus \{e, -e\}$ either equal to f_e or of the form

$$f(u) = (\rho(u)\hat{K}_e(u), \rho(u)(u \times \hat{K}_e(u)))$$

for some smooth $\rho : \mathbb{S}^2 \setminus \{e, -e\} \to SO(3)$?

6.1.2. Understanding the Frame Bundle

To tackle this, we need to understand the space of all orthonormal frames on $\mathbb{S}^2 \setminus \{e, -e\}$. The tangent space $T_u\mathbb{S}^2$ is a 2-dimensional vector space, and an orthonormal frame at u is a pair of vectors $(X(u), Y(u))$ forming an orthonormal basis. The frame bundle of $\mathbb{S}^2$, restricted to $\mathbb{S}^2 \setminus \{e, -e\}$, describes all such bases.

- **Orthonormal Frame Bundle**. For a point $u \in \mathbb{S}^2$, the set of orthonormal bases of $T_u\mathbb{S}^2$ is isomorphic to $SO(2)$, the group of 2D rotations, since an orthonormal basis is determined by choosing a unit vector $X(u) \in T_u\mathbb{S}^2$ (an element of the unit circle in $T_u\mathbb{S}^2$) and its orthogonal complement $Y(u) = \pm JX(u)$, where J is a 90-degree rotation in $T_u\mathbb{S}^2$. For right-handness, we need the orientation to match, so we'll refine this below.
- **Right-Handed Frames**. The right-handness condition $X(u) \times Y(u) = u$ imposes an orientation. In $\mathbb{R}^3$, the cross product depends on the ambient orientation. For an orthonormal frame $(X(u), Y(u))$, the cross product $X(u) \times Y(u)$ is perpendicular to $T_u\mathbb{S}^2$, hence parallel to u. Since $|X(u)| = |Y(u)| = 1$:

$$|X(u) \times Y(u)| = |X(u)||Y(u)| \sin\theta = 1 \cdot 1 \cdot 1 = 1,$$

because $\theta = \pi/2$ (orthogonality). Thus:

$$X(u) \times Y(u) = \pm u.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0453.v1

Right-handness requires:

$$X(u) \times Y(u) = u,$$

corresponding to a specific orientation. This restricts the frame to the connected component of the frame bundle where the basis (X, Y) aligns with the outward normal u.

- **Rotation Field**. A rotation $\rho(u) \in SO(3)$ acts on vectors in $\mathbb{R}^3$. For the rotated frame to remain in $T_u\mathbb{S}^2$:

$$\rho(u)\hat{K}_e(u), \quad \rho(u)(u \times \hat{K}_e(u)) \in T_u\mathbb{S}^2.$$

Since $\hat{K}_e(u), u \times \hat{K}_e(u) \in T_u\mathbb{S}^2$, we need $\rho(u)$ to preserve $T_u\mathbb{S}^2$, i.e.,

$$\rho(u)(T_u\mathbb{S}^2) = T_u\mathbb{S}^2.$$

This suggests $\rho(u)$ should be a rotation in the plane $T_u\mathbb{S}^2$, effectively an element of $SO(2)$, but embedded in $SO(3)$. Additionally, the rotated frame must satisfy right-handness:

$$(\rho(u)\hat{K}_e(u)) \times (\rho(u)(u \times \hat{K}_e(u))) = \rho(u)(\hat{K}_e(u) \times (u \times \hat{K}_e(u))) = \rho(u)u,$$

since

$$\hat{K}_e(u) \times (u \times \hat{K}_e(u)) = u,$$

and the cross product transforms as

$$R(a \times b) = (Ra) \times (Rb)$$

for $R \in SO(3)$. For this to equal u:

$$\rho(u)u = u.$$

Thus, $\rho(u) \in SO(3)$ must fix u, meaning it is a rotation about the axis u. The subgroup of $SO(3)$ fixing u is isomorphic to $SO(2)$, corresponding to rotations in the plane $T_u\mathbb{S}^2$.

6.1.3. Analyzing the Thesis

We need to determine if every smooth orthonormal right-handed frame

$$f(u) = (X(u), Y(u))$$

on $\mathbb{S}^2 \setminus \{e, -e\}$, satisfying $X(u) \times Y(u) = u$, is either:

- Equal to $f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$, or
- A rotation of f_e, i.e.,

$$(X(u), Y(u)) = (\rho(u)\hat{K}_e(u), \rho(u)(u \times \hat{K}_e(u))),$$

for some smooth $\rho : \mathbb{S}^2 \setminus \{e, -e\} \to SO(3)$.

Since $\rho(u)$ must satisfy $\rho(u)u = u$, let's parameterize $\rho(u)$. For a point u, the rotation $\rho(u)$ about u by an angle $\theta(u)$ can be written using the Rodrigues formula:

$$\rho(u) = I + \sin\theta(u)[u\times] + (1 - \cos\theta(u))[u\times]^2,$$

where $[u\times]$ is the skew-symmetric matrix such that $[u\times]v = u \times v$. This rotates vectors in the plane perpendicular to u, i.e., $T_u\mathbb{S}^2$. The action on the frame is:

$$\rho(u)\hat{K}_e(u) = \cos\theta(u)\hat{K}_e(u) + \sin\theta(u)(u \times \hat{K}_e(u)),$$

$$\rho(u)(u \times \hat{K}_e(u)) = -\sin\theta(u)\hat{K}_e(u) + \cos\theta(u)(u \times \hat{K}_e(u)),$$

since $(u, \hat{K}_e(u), u \times \hat{K}_e(u))$ forms an orthonormal basis of $\mathbb{R}^3$, and a rotation about u acts as a 2D rotation in the $(\hat{K}_e(u), u \times \hat{K}_e(u))$-plane. Thus, we're checking if every right-handed orthonormal frame is of the form:

$$(X(u), Y(u)) = (\cos\theta(u)\hat{K}_e(u) + \sin\theta(u)(u \times \hat{K}_e(u)), -\sin\theta(u)\hat{K}_e(u) + \cos\theta(u)(u \times \hat{K}_e(u))),$$

for some smooth function $\theta : \mathbb{S}^2 \setminus \{e, -e\} \to \mathbb{R}$, or is exactly f_e (when $\theta(u) = 0$).

6.1.4. Matching the Desired Form of Frame

Let's try to construct a general right-handed orthonormal frame and see if it matches this form. Suppose $f(u) = (X(u), Y(u))$ is a smooth orthonormal right-handed frame on $\mathbb{S}^2 \setminus \{e, -e\}$. Then:

$$X(u) \cdot X(u) = 1, \quad Y(u) \cdot Y(u) = 1, \quad X(u) \cdot Y(u) = 0,$$

$$X(u) \cdot u = 0, \quad Y(u) \cdot u = 0,$$
$$X(u) \times Y(u) = u.$$

We want to find $\rho(u) \in SO(3)$ such that:

$$X(u) = \rho(u)\hat{K}_e(u), \quad Y(u) = \rho(u)(u \times \hat{K}_e(u)),$$

with $\rho(u)u = u$. Since $(\hat{K}_e(u), u \times \hat{K}_e(u))$ is an orthonormal basis for $T_u\mathbb{S}^2$, express:

$$X(u) = a(u)\hat{K}_e(u) + b(u)(u \times \hat{K}_e(u)),$$

where:
$$|X(u)|^2 = a(u)^2 + b(u)^2 = 1,$$
$$X(u) \cdot u = 0,$$

which is satisfied since both $\hat{K}_e(u)$ and $u \times \hat{K}_e(u)$ are perpendicular to u. Let:

$$a(u) = \cos\theta(u), \quad b(u) = \sin\theta(u),$$

so:
$$X(u) = \cos\theta(u)\hat{K}_e(u) + \sin\theta(u)(u \times \hat{K}_e(u)).$$

Since $Y(u)$ is orthonormal to $X(u)$:

$$Y(u) = c(u)\hat{K}_e(u) + d(u)(u \times \hat{K}_e(u)),$$

$$X(u) \cdot Y(u) = a(u)c(u) + b(u)d(u) = \cos\theta(u)c(u) + \sin\theta(u)d(u) = 0,$$
$$|Y(u)|^2 = c(u)^2 + d(u)^2 = 1.$$

Right-handness requires:

$$X(u) \times Y(u) = \begin{vmatrix} \hat{K}_e(u) & u \times \hat{K}_e(u) & u \\ a(u) & b(u) & 0 \\ c(u) & d(u) & 0 \end{vmatrix} = (a(u)d(u) - b(u)c(u))u.$$

$$X(u) \times Y(u) = u \implies a(u)d(u) - b(u)c(u) = 1.$$

Solve:
$$\cos\theta(u)c(u) + \sin\theta(u)d(u) = 0,$$

$$\cos\theta(u)d(u) - \sin\theta(u)c(u) = 1,$$

 doi:10.20944/preprints202507.0453.v1

since:

$$a(u)d(u) - b(u)c(u) = \cos\theta(u)d(u) - \sin\theta(u)c(u).$$

This is a linear system for $c(u), d(u)$:

$$\begin{pmatrix} \cos\theta(u) & \sin\theta(u) \\ -\sin\theta(u) & \cos\theta(u) \end{pmatrix} \begin{pmatrix} c(u) \\ d(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix is a rotation by $\theta(u)$, with inverse:

$$\begin{pmatrix} \cos\theta(u) & -\sin\theta(u) \\ \sin\theta(u) & \cos\theta(u) \end{pmatrix}.$$

We then obtain

$$\begin{pmatrix} c(u) \\ d(u) \end{pmatrix} = \begin{pmatrix} \cos\theta(u) & -\sin\theta(u) \\ \sin\theta(u) & \cos\theta(u) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta(u) \\ \cos\theta(u) \end{pmatrix}.$$

Thus:

$$Y(u) = -\sin\theta(u)\hat{K}_e(u) + \cos\theta(u)(u \times \hat{K}_e(u)).$$

This matches the rotated frame:

$$(X(u), Y(u)) = (\rho(u)\hat{K}_e(u), \rho(u)(u \times \hat{K}_e(u))),$$

where $\rho(u)$ is a rotation about u by $\theta(u)$. To confirm, check right-handness:

$$\rho(u)\hat{K}_e(u) \times \rho(u)(u \times \hat{K}_e(u)) = \rho(u)(\hat{K}_e(u) \times (u \times \hat{K}_e(u))) = \rho(u)u = u,$$

since $\rho(u)u = u$. The frame is smooth if $\theta(u)$ is smooth, which depends on the smoothness of $X(u)$.

6.1.5. Smoothness of ρ

Any right-handed orthonormal frame $(X(u), Y(u))$ can be written as:

$$X(u) = \cos\theta(u)\hat{K}_e(u) + \sin\theta(u)(u \times \hat{K}_e(u)),$$

$$Y(u) = -\sin\theta(u)\hat{K}_e(u) + \cos\theta(u)(u \times \hat{K}_e(u)),$$

for some function $\theta(u)$. If $\theta(u) = 0$, then:

$$(X(u), Y(u)) = (\hat{K}_e(u), u \times \hat{K}_e(u)) = f_e(u).$$

Otherwise, the frame is a rotation of f_e by the rotation field $\rho(u)$, which rotates by angle $\theta(u)$ in $T_u\mathbb{S}^2$. The map $\rho(u)$ is smooth if $\theta(u)$ is smooth, which follows from the smoothness of $X(u)$ and $Y(u)$. ■

7. Discussion on the Geometric Results

7.1. Topological Considerations

Could there be a right-handed frame not of this form? The frame bundle of orthonormal right-handed frames on $\mathbb{S}^2 \setminus \{e, -e\}$ is a principal $SO(2)$-bundle. Since $\mathbb{S}^2 \setminus \{e, -e\}$ is diffeomorphic to a cylinder $S^1 \times \mathbb{R}$, which is contractible, the bundle is trivial:

$$SO(\mathbb{S}^2 \setminus \{e, -e\}) \cong (\mathbb{S}^2 \setminus \{e, -e\}) \times SO(2).$$

Thus, there exists a global section, and any frame can be expressed relative to a reference frame like f_e by a smooth $SO(2)$-valued function, i.e., a rotation by $\theta(u)$. The right-handness condition ensures the rotation preserves the orientation, which our construction satisfies.

7.2. Analysis of the Geometric Results

Every smooth orthonormal right-handed frame $f = (X, Y)$ on $\mathbb{S}^2 \setminus \{e, -e\}$, satisfying $X(u) \times Y(u) = u$, is either:

- Equal to $f_e(u) = (\hat{K}_e(u), u \times \hat{K}_e(u))$, or
- A rotation of f_e, i.e.,
$$f(u) = (\rho(u)\hat{K}_e(u), \rho(u)(u \times \hat{K}_e(u))),$$
where
$$\rho : \mathbb{S}^2 \setminus \{e, -e\} \to SO(3)$$
is a smooth rotation field satisfying
$$\rho(u)u = u,$$
effectively a rotation by a smooth angle $\theta(u)$ in $T_u\mathbb{S}^2$.

This result is interesting, as it shows that f_e is a "standard" frame, and all other right-handed frames are obtained by rotating it in the tangent plane, reflecting the structure of the frame bundle and the symmetry of $\mathbb{S}^2$.

8. Results IV: Orthonormal Right-Handed Frames on $\mathbb{R}^3 \setminus \mathrm{span}\{e\}$

We have just proved that every smooth orthonormal right-handed frame on S^2, minus a prechosen e and its opposite $-e$, should be necessarily the Killing orthonormal frame f_e or any its rotation by a smooth rotation field $\rho : S^2 \setminus \{e, -e\} \to SO(3)$.

Well, f_e can be extended to the entire $\mathbb{M}_4^* \setminus \Pi_e$ by homogeneity:

$$\tilde{f}_e(k) = f_e\left(\frac{\mathbf{k}}{|\mathbf{k}|}\right),$$

where Π_e is the linear plane generated by e and $(1, 0, 0, 0)$.

Analogously, f_e can be extended to the spatial part of $\mathbb{M}_4^*$ minus the straight-line generated by e.

Now, the following theorem holds.

Theorem 4. *Every smooth orthonormal right-handed bidimensional frame*

$$f : \mathbb{R}^3 \setminus span\{e\} \to \mathbb{R}^3 \times \mathbb{R}^3$$

- defined on $\mathbb{R}^3$ minus the line generated by a pre-chosen unit vector e - which is tangent at any point $\mathbf{k}$ to the sphere centered at the origin and passing by $\mathbf{k}$, should be necessarily $\tilde{f}_e$ (extension of f_e to $\mathbb{R}^3$) or any its rotation by a smooth rotation field

$$\rho : \mathbb{R}^3 \setminus span\{e\} \to SO(3)$$

such that any rotation $\rho(\mathbf{k})$ is a rotation with axis $\mathbf{k}$.

Theorem 4 is a natural generalization of Theorem 3, extending the concept of orthonormal right-handed frames from $S^2 \setminus \{e, -e\}$ to $\mathbb{R}^3 \setminus \mathrm{span}\{e\}$, where the frame is tangent to the sphere of radius $|\mathbf{k}|$ at each point $\mathbf{k}$. This connects directly to the homogeneous extension of f_e to the Minkowski dual space $\mathbb{M}_4^* \setminus \Pi_e$. Let's analyze Theorem 4, verify its validity.

Theorem 3 follows from the triviality of the frame bundle over $S^2 \setminus \{e, -e\}$, which is contractible.

8.1. Homogeneous Extension to $\mathbb{M}_4^* \setminus \Pi_e$

The frame f_e extends to $\mathbb{M}_4^* \setminus \Pi_e$, where Π_e is the plane spanned by $(0, e)$ and $(1, 0, 0, 0)$. For $k = (k^0, \mathbf{k}) \in \mathbb{M}_4^* \setminus \Pi_e, \mathbf{k} \neq 0$:

$$\tilde{f}_e(k) = f_e\left(\frac{\mathbf{k}}{|\mathbf{k}|}\right) = (\hat{K}_e(u), u \times \hat{K}_e(u)), \quad u = \frac{\mathbf{k}}{|\mathbf{k}|}.$$

This is tangent to the sphere of radius $|\mathbf{k}|$ at $\mathbf{k}$.

8.2. Proof of Theorem 4

Theorem 4 affirms that every smooth orthonormal right-handed frame

$$f : \mathbb{R}^3 \setminus \text{span}\{e\} \to (\mathbb{R}^3)^2,$$

where $f(\mathbf{k}) = (X(\mathbf{k}), Y(\mathbf{k}))$ is tangent to the sphere

$$S_{|\mathbf{k}|} = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = |\mathbf{k}|\},$$

is either $\tilde{f}_e(\mathbf{k})$ or:

$$f(\mathbf{k}) = (\rho(\mathbf{k})\hat{K}_e(u), \rho(\mathbf{k})(u \times \hat{K}_e(u))),$$

where $\rho(\mathbf{k})\mathbf{k} = \mathbf{k}$.

The frame f satisfies:

- **Orthonormality**. $X(\mathbf{k}) \cdot X(\mathbf{k}) = Y(\mathbf{k}) \cdot Y(\mathbf{k}) = 1$, $X(\mathbf{k}) \cdot Y(\mathbf{k}) = 0$.
- **Tangency**. $X(\mathbf{k}) \cdot \mathbf{k} = Y(\mathbf{k}) \cdot \mathbf{k} = 0$.
- **Right-handedness**. $X(\mathbf{k}) \times Y(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|}$.

8.2.1. Analysis of frame Bundle

At $\mathbf{k}$,

$$T_{\mathbf{k}} S_{|\mathbf{k}|} = \{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{k} = 0\}.$$

The frame bundle of $S_{|\mathbf{k}|}$ is isomorphic to $SO(2)$. The bundle over $\mathbb{R}^3 \setminus \text{span}\{e\}$ may be non-trivial, as

$$\pi_1(\mathbb{R}^3 \setminus \text{span}\{e\}) = \mathbb{Z},$$

but the condition $\rho(\mathbf{k})\mathbf{k} = \mathbf{k}$ reduces the verification to the trivial bundle of $S_{|\mathbf{k}|}$.

Expressing the frame f as

$$X(\mathbf{k}) = \cos\theta(\mathbf{k})\hat{K}_e(u) + \sin\theta(\mathbf{k})(u \times \hat{K}_e(u)),$$

$$Y(\mathbf{k}) = -\sin\theta(\mathbf{k})\hat{K}_e(u) + \cos\theta(\mathbf{k})(u \times \hat{K}_e(u)),$$

where $\theta(\mathbf{k})$ is smooth, the homogeneity reduces the problem to $S^2 \setminus \{e, -e\}$, where the bundle is trivial, so Theorem 4 holds. ■

Theorem 4 states that every considered frame is a radial rotation of $\tilde{f}_e$.

9. Results V: Maxwell–Schrödinger Fields from de Broglie Waves and Spherical Geometry

In this final section, we present a foundational application of the spherical geometric structure and Killing frames developed above: a new construction of solutions to Maxwell's equations in free space, formulated as a relativistic massless Schrödinger-type equation in the space of Schwartz-tempered

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0453.v1

complex vector distributions.

This construction unveils a one-to-one correspondence between light-like de Broglie wavevectors and exact Maxwellian solutions in the complexified Schwartz space

$$W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3) = \mathcal{L}(\mathcal{S}(\mathbb{M}_4, \mathbb{C}), \mathbb{C}^3) = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}) \otimes \mathbb{C}^3,$$

revealing how quantum-phase distributions combine with polarization geometry to form a complete analytic model of free electromagnetic waves.

9.1. Minkowski Space, de Broglie Family, and Extended Frame Fields

Let $\mathbb{M}_4 = \mathbb{R}_{1,3}$ denote Minkowski space-time, and $\mathbb{M}_4^*$ its dual (space of real 4-wavevectors). A wavevector $k = (k^0, \vec{k}) \in \mathbb{M}_4^*$ induces the tempered de Broglie phase:

$$\eta_k := [e^{i\langle k,x\rangle}] = [e^{i(\vec{k}\cdot\vec{x}-\omega_k t)}], \quad \text{with } \omega_k = ck^0.$$

This defines a scalar tempered wave distribution in the canonical way, x is the canonical coordinate on Minkowsky vector space (identity mapping), $t = x^0/c$ is the time coordinate of x(time projection), $\vec{x}$ is the space projection of the identity chart x, $[g]$ is the tempered distribution associated with any smooth slowly increasing (multiplier) function g.

To construct complex vector fields, we extend the canonical Killing frame

$$f_e = (\vec{r}_e, \vec{s}_e)$$

from the sphere $S^2 \setminus \{\pm e\}$ to the dual Minkowski space $\mathbb{M}_4^* \setminus \Pi_e$ by homogeneity:

$$\vec{u}_k = \frac{\vec{k}}{|\vec{k}|}, \quad \vec{r}_k := \widehat{K}_e(\vec{u}_k) = \frac{e \times \vec{u}_k}{|e \times \vec{u}_k|}, \quad \vec{s}_k := \vec{u}_k \times \vec{r}_k.$$

This defines a smooth, orthonormal, right-handed frame $(\vec{r}_k, \vec{s}_k, \vec{u}_k)$ for each $k \notin \Pi_e$, where Π_e is the singular plane generated by e and $(1,0,0,0)$.

We define the **complex polarization vector associated with** k:

$$\vec{f}_k := \vec{r}_k + i\vec{s}_k \in \mathbb{C}^3,$$

and the corresponding **complex plane field associated with** k:

$$w_k := \eta_k.\vec{f}_k = [e^{i(\vec{k}\cdot\vec{x}-\omega t)}].(\vec{r}_k + i\vec{s}_k) \in \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3).$$

This defines a Schwartz family w of plane wave complex distribution 3-fields parameterized by all 4-wavevectors $k \in \mathbb{M}_4^* \setminus \Pi_e$.

9.2. The Maxwell-Schrödinger Equation on $\mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$

Let us now consider the **Maxwell-Schrödinger equation**, defined on the full space of tempered complex vector distributions:

$$i\frac{\partial F}{\partial t} = c\,\nabla \times F.$$

This equation arises from the classical Maxwell curl equations in vacuum, under the complexification

$$F := \vec{E} + ic\vec{B}.$$

 doi:10.20944/preprints202507.0453.v1

Indeed, the two Maxwell curl equations:

$$c\nabla \times \vec{E} = -\frac{\partial(c\vec{B})}{\partial t}, \quad c\nabla \times (c\vec{B}) = \frac{\partial \vec{E}}{\partial t}$$

combine into the single complex equation above:

$$i\frac{\partial(\vec{E} + ic\vec{B})}{\partial t} = c\,\nabla \times (\vec{E} + ic\vec{B}).$$

Importantly, this evolution equation is everywhere defined on

$$W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3),$$

independent of any constraint on the wavevector k.

9.3. The General Plane Wave Family

Each tempered field

$$w_k = \eta_k.\vec{f}_k$$

is well-defined for any $k \in \mathbb{M}_4^* \setminus \Pi_e$, and constitutes (when viewed as a function) a **smooth slowly increasing polarized complex plane wave**.
We shall consider, mainly, the entire smooth family

$$w : \mathbb{M}_4^* \setminus \Pi_e \to W.$$

Remark. As usual in Schwartz linear algebra (and in Quantum Mechanics) we are searching for entire (possibly orthonormal) eigenbasis of some observable, in order to decompose every state of the system as a superposition of that basis. In our specific case, w shall diagonalize every differential operator and every linear combinations of those operators, as soon as they are defined upon the Schwartz linear span S_w of w. Energy and momentum operators in S_w will be diagonalized by w, furthermore, even more notably for our present study, w shall diagonalize the curl operator, which is the dynamic leading operator of Maxwell's equation. The operator curl restricted to S_w and multiplied by the Plank's constant shall reveal the momentum magnitude operator of quantum mechanics.

The member fields of w are:

- Globally defined tempered vector distributions;
- Solenoidal:
$$\nabla \cdot w_k = 0,$$
since $\vec{k} \cdot \vec{f}_k = 0$;
- Eigenfunctions of all differential operators, and in particular:
$$\nabla \times w_k = |\vec{k}| w_k,$$
since $\vec{u} \times \vec{f}_k = -i\vec{f}_k$;
- Time-evolved by the eigenvalue equation:
$$i\frac{\partial w_k}{\partial t} = \omega_k w_k = ck^0 w_k.$$

Thus, for *all* $k \notin \Pi_e$, the fields w_k are dynamically meaningful—but only a subfamily solves the (massless) Maxwell-Schrödinger equation.

 doi:10.20944/preprints202507.0453.v1

9.4. The Maxwell Characterization Theorem

We now formally isolate the subfamily of physically admissible Maxwellian (massless) fields.

Theorem 5 (Maxwellian Electromagnetic Solutions Characterization)**.** *Let*

$$k \in \mathbb{M}_4^* \setminus \Pi_e,$$

and define the complex plane wave field

$$w_k = \eta_k.(\vec{r}_k + i\vec{s}_k).$$

Then, the field w_k satisfies the Maxwell-Schrödinger equation

$$i\frac{\partial w_k}{\partial t} = c\,\nabla \times w_k$$

if and only if the wavevector k is light-like and $\omega_k > 0$, i.e.,

$$k^0 > 0, \qquad \langle k, k \rangle = -(k^0)^2 + |\vec{k}|^2 = 0.$$

Proof. We have:

$$i\frac{\partial w_k}{\partial t} = \omega_k w_k, \quad \nabla \times w_k = |\vec{k}| w_k.$$

Then the Maxwell–Schrödinger equation reads:

$$\omega_k w_k = c|\vec{k}| w_k \quad \Longleftrightarrow \quad \omega_k = c|\vec{k}|,$$

which holds if and only if $k^0 = |\vec{k}|$, i.e., $\langle k, k \rangle = 0$.

□

9.5. Physical and Structural Implications

This result underscores the structure of the full family $w = (w_k)_{k \in \mathbb{M}_4^* \setminus \Pi_e}$:

- All w_k are smooth complex polarized plane fields with well-defined dynamics;
- Only those with **light-like** k solve the **massless Maxwell–Schrödinger equation**, hence represent bona fide electromagnetic fields in vacuum;
- The equation

  $$i\partial_t F = c\nabla \times F$$

  thus functions as a **spectral filter**, selecting light-cone indexed de Broglie fields.
- The Maxwell complex equation coincides with the relativistic Schrodinger equation for massless particles upon S_w.

This filtering highlights the **deep unity** of geometry (through the Killing-induced frame), spectral phase and quantum mechanics (via the de Broglie basis η and the associated diagonalizable operators on it), and field dynamics (via the curl operator) within the Maxwell–Schwartz formalism.

10. Results VI: Massive Maxwellian Fields and the Relativistic Maxwell-Schrödinger Equation

We now extend the previous theory from massless electromagnetic fields to a broader class of *massive Maxwellian fields* governed by the relativistic Hamiltonian for nonzero rest mass m_0. This framework generalizes the light-cone condition

$$\langle k, k \rangle = 0$$

to the mass shell condition

$$\langle \hbar k, \hbar k \rangle = -(m_0 c)^2,$$

preserving the complex geometric structure and eigenbasis of the Maxwell-Schrödinger formalism.

The fundamental idea is to replace the linear continuous light-photon Hamiltonian operator

$$\hat{H}_0 = c|\vec{P}| = c\hbar\nabla\times,$$

considered as restricted on the de Broglie Killing subspace S_w, with the relativistic Hamiltonian operator

$$\hat{H}_{m_0} = c\sqrt{|\vec{P}|^2 + (m_0 c)^2\,\mathbb{I}_W},$$

defined again, spectrally, on the de Broglie-Killing subspace

$$S_w \subset W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3),$$

Schwartz-generated by w.

10.1. Momentum Magnitude Operator and Spectral Identification with Curl

Recall that in Section 8, the space S_w is defined as the Schwartz linear span of the de Broglie-Killing family $w = (w_k)_{k\in D}$,

$$S_w = \left\{ \int_D a\,w \in W : \ a \in \mathcal{S}'(\mathbb{M}_4^*, \mathbb{C}),\ \mathrm{supp}(a) \cap \Pi_e = \varnothing \right\}.$$

On this subspace, the operator $|\vec{P}|$ is defined spectrally by:

$$|\vec{P}|\left(\int_D aw\right) := \int_D a\,(\hbar|\vec{\mathrm{k}}|)w,$$

whenever a vanishes in a neighborhood of the singular plane Π_e and where $\vec{\mathrm{k}}$ is the spatial projection in $\mathbb{M}_4^*$. This operator is well-defined on a large domain of test functions whose Fourier transforms vanish along the line generated by $(1,0,0,0)$.

Now observe that the complex vector-field family w is an eigenbasis of S_w for both the operator $|\vec{P}|$ and the scaled curl operator $\hbar\nabla\times$, with matching eigenvalues:

$$|\vec{P}|(w_k) = \hbar|\vec{k}|\cdot w_k, \quad \hbar\nabla\times w_k = \hbar|\vec{k}|\cdot w_k.$$

Therefore, we conclude that the above two operator restrictions upon the subspace S_w coincide:

$$|\vec{P}|_{|S_w} = (\hbar\nabla\times)_{|S_w},$$

since the two operators are spectrally identical on S_w.

10.2. Quantization of the Relativistic Hamiltonian

The usual relativistic Hamiltonian of a free particle with rest mass m_0 is:

$$H_{m_0}(\vec{p}) = c\sqrt{|\vec{p}|^2 + (m_0 c)^2}.$$

Its quantization on the space W can be performed by applying the Schwartz spectral theorem to the self-adjoint operator $|\vec{P}|$ defined on S_w, yielding:

$$\hat{H}_{m_0} := c\sqrt{|\vec{P}|^2 + (m_0 c)^2\,\mathbb{I}_W},$$

 doi:10.20944/preprints202507.0453.v1

with $\mathbb{I}_W$ the identity operator on W (if we desire to strictly restrict ourself to S_w, we can clearly use the identity operator of the latter subspace). On the basis w, we obtain, by the very definition of our Hamiltonian operator (via Schwartz spectral theorem):

$$\hat{H}_{m_0}(w_k) = c\sqrt{(\hbar|\vec{k}|)^2 + (m_0 c)^2}\,.w_k = H_{m_0}(\hbar\vec{k})\,.w_k.$$

Thus, the family w diagonalizes the relativistic Hamiltonian $\hat{H}_{m_0}$.

The complete definition of Hamiltonian operator $\hat{H}_{m_0}$ on S_w is given by superpositions:

$$\hat{H}_{m_0}\left(\int_D aw\right) := \int_D a\, c\sqrt{(\hbar|\vec{\mathrm{k}}|)^2 + (m_0 c)^2}\,.w,$$

whenever a vanishes in a neighborhood of the singular plane Π_e and where $\vec{\mathrm{k}}$ is the spatial projection in $\mathbb{M}_4^*$. In a perfectly equivalent way, we can write

$$\hat{H}_{m_0}(F) := \int_D c\sqrt{(\hbar|\vec{\mathrm{k}}|)^2 + (m_0 c)^2}\,.(F)_w\, w,$$

where $(F)_w$ is the representation of any $F \in S_w$ in the basis w (the coefficient distribution a above).

Remark. It is worthy now to observe that we can project any complex vector distribution 3-field F of S_w onto a scalar complex wave distribution ψ_F by the following homomorphism:

$$\psi : S_w \to \mathcal{S}'(\mathbb{M}_4, \mathbb{C}) : F \mapsto \int_D (F)_w \eta.$$

10.3. The Relativistic Maxwell–Schrödinger Equation

We now consider the **massive** Maxwell-Schrödinger evolution equation:

$$i\hbar\frac{\partial F}{\partial t} = \hat{H}_{m_0}(F),$$

where $F \in S_w$ and $\hat{H}_{m_0}$ is defined above.

Each field

$$w_k = [e^{i(\vec{k}\cdot\vec{x} - \omega_k t)}].(\vec{r}_k + i\vec{s}_k)$$

is (as we already know) a complex plane vector distribution in W, and satisfies:

$$i\hbar\frac{\partial w_k}{\partial t} = (\hbar\omega_k)w_k.$$

On the other hand, by definition of $\hat{H}_{m_0}$:

$$\hat{H}_{m_0}(w_k) = H_{m_0}(\hbar\vec{k})\, w_k = c\sqrt{(\hbar|\vec{k}|)^2 + (m_0 c)^2}\, w_k.$$

10.4. Characterization Theorem for Massive Maxwellian Fields

We now isolate the subfamily of massive plane-wave solutions that satisfy the relativistic Maxwell-Schrödinger equation above.

Theorem 6 (Massive Maxwellian Solutions Characterization). *Let consider any 4-wavevector $k \in \mathbb{M}_4^* \setminus \Pi_e$, and let*

$$w_k := \eta_k.(\vec{r}_k + i\vec{s}_k)$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0453.v1

be the Schwartz Killing basis member as defined above. Then, the complex tempered vector field w_k satisfies the relativistic Maxwell-Schrödinger equation

$$i\hbar\frac{\partial F}{\partial t} = \hat{H}_{m_0}(F)$$

if and only if $k^0 > 0$ and

$$\langle \hbar k, \hbar k\rangle = -(m_0c)^2,$$

that is, if and only if

$$\hbar\omega_k = H_{m_0}(\hbar\vec{k}) = c\sqrt{(\hbar|\vec{k}|)^2 + (m_0c)^2}.$$

Proof. Substituting w_k in the equation, at the place of the unknown F, the left-hand side of the equation becomes

$$i\hbar\partial_t w_k = \hbar\omega_k \cdot w_k.$$

Analogously, the right-hand side becomes

$$\hat{H}_{m_0}(w_k) = c\sqrt{(\hbar|\vec{k}|)^2 + (m_0c)^2} \cdot w_k.$$

These tempered fields are equal if and only if ω_k is positive and

$$\hbar\omega_k = c\sqrt{(\hbar|\vec{k}|)^2 + (m_0c)^2} \quad\Longleftrightarrow\quad -(\hbar\omega_k/c)^2 + (\hbar|\vec{k}|)^2 = -(m_0c)^2,$$

which is the condition

$$\langle \hbar k, \hbar k\rangle = -(m_0c)^2.$$

Exactly as we desired. □

In a perfect symmetrical fashion, we could define the conjugate Maxwell-Schrodinger equation (opposite time coordinate) for antimatter fields ($\omega < 0$) and prove the analogous theorem. We left the details to the reader.

10.5. Discussion and Physical Relevance

This result shows that the family w parameterizes not only massless (photon-like) wave solutions, but also massive complex vector fields governed by the relativistic energy-momentum relation:

$$E^2 = (|\vec{p}|c)^2 + (m_0c^2)^2.$$

In this broader framework:

- The relativistic Maxwell Schrodinger operator $\hat{H}_{m_0}$ extends the massless Maxewll's Hamiltonian operator $\hat{H}_0 = c|\vec{P}|$ to include rest mass m_0 different from 0;
- The fields w_k with $k^0 > 0$ and
$$\langle \hbar k, \hbar k\rangle = -(m_0c)^2$$
form a spectral submanifold of S_w determined by $\hat{H}_{m_0}$ via the associated Maxwell Schrodinger equation; the case with $k^0 < 0$ is covered analogously by the conjugate equation;
- This construction generalizes Maxwell's electromagnetic fields to a class of massive relativistic Maxwell-like fields within Schwartz-tempered complex vector theory.

The structure suggests that the space S_w may serve as a host for a unified field-theoretic framework accommodating both massless and massive quantum fields via geometry and spectral analysis alone.

 doi:10.20944/preprints202507.0453.v1

11. Conclusions and Outlook

This work has established a geometric, analytic, and physical synthesis centered on the role of orthonormal right-handed frames derived from Killing vector fields on the two-sphere S^2, and their application to constructing explicit solutions of Maxwell's equations and their massive generalizations.

In the first part of the paper, we rigorously developed the mathematical framework in which smooth orthonormal right-handed tangent frames on the pierced sphere $S^2 \setminus \{\pm e\}$ are constructed canonically from the action of the rotation group $\mathrm{SO}(3)$ and its Lie algebra $\mathfrak{so}(3)$. We proved that every such frame is either the canonical Killing frame or its rotation by a smooth field of elements in $\mathrm{SO}(3)$ fixing the direction u orthogonal to the tangent space generated by the frame. This classification aligns the local geometry of the sphere with the global topology of its frame bundle and sets the stage for a frame-theoretic approach to field equations.

Building upon this geometry, we then transitioned to the realm of Schwartz-tempered complex vector fields on Minkowski spacetime, introducing the fundamental Schwartz basis w of complex plane wave solutions. The fields w_k are constructed by lifting the de Broglie plane wave

$$\eta_k = [e^{i\langle k,x\rangle}]$$

with a smoothly extended orthonormal frame adapted to the spatial direction of k. Each w_k encodes both the phase propagation and polarization structure of an electromagnetic mode.

In Section 9, we demonstrated that the Maxwell curl equations in vacuum admit an elegant reformulation as a single Schrödinger-type equation:

$$i\partial_t F = c\nabla \times F,$$

defined on the space $W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$. Any basis field w_k satisfies this equation *if and only if* the wavevector k is light-like, i.e., $\langle k, k\rangle = 0$. This result introduces a precise spectral filtering: the Maxwell–Schrödinger operator selects the light cone in dual Minkowski space as the physical support of radiation fields. In this setting, the real and imaginary parts of w_k are uniformly proportional, respectively, to the electric and magnetic components of a circularly polarized wave (E, cB). Any pair

$$(E, cB) = |E_0|w_k,$$

represents a legitimate electromagnetic field and a legitimate quantum wave field in W, whose associated complex Schrodinger wave is the wave distribution

$$\psi_{(E,cB)} = |E_0|\eta_k.$$

In this sense, we reconsider the rightful place of wave amplitudes, the waves carry a natural physics meaning: as in the proper electromagnetic ones, the amplitudes stands for the intensity of the fields (as it is obvious and natural); the superposition principle holds in its whole glory (that is, within a fully functioning vector space structure, and not in some fancy and sloppy *projective Hilbert space*). Of course, with any conveniently normalizable ψ we can associate its "complex probability amplitude wave" $\psi/||\psi||$; this is particularly efficent and natural in distribution spaces, since probability measures are distribution-like objects and since distribution spaces contains all possible pre-Hilbert spaces of normalizable distributions adopted in Quantum Mechanics.

In Section 10, we extended the formalism to encompass *massive Maxwellian fields*, introducing the relativistic Hamiltonian

$$\hat{H} = c\sqrt{|\vec{P}|^2 + (m_0c)^2\mathbb{I}_W},$$

and establishing that the fields w_k satisfy the relativistic evolution equation

$$i\hbar\frac{\partial F}{\partial t} = \hat{H}(F)$$

if and only if the momentum-energy vector $\hbar k$ lies on the mass shell:

$$k^0 > 0\,,\ \langle \hbar k, \hbar k \rangle = -(m_0 c)^2.$$

This result generalizes the light cone condition to arbitrary mass, while preserving the geometry and spectral structure developed in the massless case.

At the heart of this theory lies the profound observation that the Maxwellian operator

$$\hbar \nabla \times : W \to W,$$

originally arising from classical electrodynamics, coincides with the quantum mechanical momentum magnitude operator $|\vec{P}|$ on the Schwartz span S_w of any possible de Broglie-Killing basis w. This spectral identity reveals a hidden quantum-geometric unification: the Maxwell fields, their associated de Broglie waves, and the relativistic Hamiltonian structure are all bound together by the common Killing eigenbases w.

Outlook. The unification developed here suggests several promising directions:

- A refined theory of electromagnetic wave packets as tempered superpositions of w_k modes, localized in energy and direction.
- Extensions to curved space-time using local frames built from generalizations of Killing fields.
- Applications to gauge theories, where frame fields and group actions play a central role.
- Exploration of probability amplitudes and current densities associated with the massive fields w_k, with potential links to quantum optics and relativistic quantum information.

In all these developments, the frame-theoretic and spectral foundation laid in this paper offers a powerful and unifying viewpoint. The tempered complex field space W emerges as a natural host for both classical and quantum electrodynamics, governed by a geometry of symmetry, polarization, and spectral evolution.

Author Contributions: "Conceptualization, D.C.; methodology, D.C.; validation, D.C.; formal analysis, D.C.; writing—original draft preparation, D.C.; writing—review and editing, D.C.; visualization, D.C.; supervision, D.C.; project administration, D.C. All authors have read and agreed to the published version of the manuscript."

Funding: This research received no external funding

Informed Consent Statement: Not applicable

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Killing, W. (1892). Über die Grundlagen der Geometrie. *Journal für die reine und angewandte Mathematik*, 1892(109):121-186. https://doi.org/10.1515/crll.1892.109.121
2. Petersen, P. (2016). *Riemannian Geometry*. Springer.
3. Lee, J. M. (2018). *Introduction to Riemannian Manifolds*. Springer.
4. Nakahara, M. (2003). *Geometry, Topology and Physics*. IOP Publishing.
5. Steenrod, N. (1951). *The Topology of Fibre Bundles*. Princeton University Press.
6. Kobayashi, S., & Nomizu, K. (1963). *Foundations of Differential Geometry*. Interscience.
7. Milnor, J., & Stasheff, J. (1974). *Characteristic Classes*. Princeton University Press.
8. Spivak, M. (1999). *A Comprehensive Introduction to Differential Geometry*. Publish or Perish.
9. Jackson, J. D. (1998). *Classical Electrodynamics*. Wiley.
10. Born, M., & Wolf, E. (1999). *Principles of Optics*. Cambridge University Press.
11. Misner, C. W., Thorne, K. S., & Wheeler, J. A. (1973). *Gravitation*. Freeman.
12. Bialynicki-Birula, I. (1996). Photon Wave Function. *Progress in Optics*, 36, 245-294. https://doi.org/10.1016/S0079-6638(08)70316-0
13. Carfì, D. (2005). S-diagonalizable operators in Quantum Mechanics. *Glasnik Mathematicki*, 40, 261–301. http://dx.doi.org/10.3336/450gm.40.2.08

14. Carfì, D. (2023). Relativistic Free Schrödinger Equation for Massive Particles in Schwartz Distribution Spaces. *Symmetry*, 15(11):1984, 2023. https://doi.org/10.3390/sym15111984
15. Carfì, D. (2024). Relativistic Schrödinger equation and probability currents for free particles. *Proceedings of the International Geometry Center*, 17(1): 99-131, 2024. https://doi.org/10.15673/pigc.v17i1.2597
16. Carfì, D. (2024). From Maxwell's equations to Quantum Mechanics: an introduction. *Journal of Mathematical Economics and Finance*, 10(2):67-83, 2024. https://doi.org/10.14505/jmef.v10.2(19).03
17. de Broglie, L. (1924). *Recherches sur la théorie des quanta*. Annales de Physique.
18. Bleecker, D. (1981). *Gauge Theory and Variational Principles*. Addison-Wesley.
19. Hecht, E. (2017). *Optics*. Pearson.
20. De Nittis, G. and Lein, M. (2017). The Schrödinger Formalism of Electromagnetism and Other Classical Waves. *Journal of Mathematical Physics*, 58(12), 122902. https://doi.org/10.1063/1.5003821
21. Bliokh, K. Y. and Bliokh, Y. P. (1998). Modified Schrödinger equation for polarized light and electrons in inhomogeneous media. *Physical Review Letters*, 81(22): 4640-4643. https://doi.org/10.1103/PhysRevLett.81.4640
22. Elbadawi, S. (2022). Derivation of Schrödinger Equation from Maxwell Electric Equation. *European Journal of Pure and Applied Mathematical Sciences*, 10(1): 65-71. https://geniusjournals.org/index.php/ejpcm/article/view/1371
23. Flanders, H. (1963). *Differential Forms with Applications to the Physical Sciences*. Academic Press, New York.
24. Weigert, S. (2018). On the Quantum Mechanics of a Single Photon. *AIP Conference Proceedings*, 1633: 205-212. https://doi.org/10.1063/1.4906925.
25. Rodrigues, W. A. J. and de Souza, Q. A. G.(2008). Killing Vector Fields, Maxwell Equations and Lorentzian Spacetimes, *arXiv preprint*. https://arxiv.org/abs/0801.3632