Existence and Uniqueness of Fixed Point Results in Non-solid C*-algebra-Valued Bipolar b-Metric Spaces

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Article

Existence and Uniqueness of Fixed Point Results in Non-Solid $C^\star-$ Algebra-Valued Bipolar $b-$Metric Spaces

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Abstract: In this monograph, motivated by the work of Aphane, Gaba, and Xu, we explore fixed point theory within the framework of $C^\star$-algebra-valued bipolar b-metric spaces, characterized by a non-solid positive cone. We define and analyze $(F_{\mathcal{H}} - G_{\mathcal{H}})$-contractions, utilizing positive monotone functions to extend classical contraction principles. Key contributions include the existence and uniqueness of fixed points for mappings satisfying generalized contraction conditions. The interplay between the non-solidness of the cone, the $C^\star$-algebra structure, and the completeness of the space is central to our results. We apply our results to find uniqueness of solutions to Fredholm integral equations and differential equations, and we extend the Ulam-Hyers stability problem to non-solid cones. This work advances the theory of metric spaces over Banach algebras, providing foundational insights with applications in operator theory and quantum mechanics.

Keywords: C*-algebra valued bipolar b-metric space; fixed points; Ulam-Hyers stability

1. Introduction

The notion of distance, as a natural extension of one of the oldest and natural quantitative concepts, can be thought of as the length of a gap. Metric spaces are regarded as the one of the most significant mathematical frameworks for the study of distance, in that they are intuitive and simple structures that allow us to consider distances between points of a set. These were formalized in 1906 by Fréchet [1]. And in 1922, Banach [2] gave a constructive method to obtain a fixed point for a self-map in metric spaces.

In 1990, Murphy [3] pioneered the novel $C^\star-$ algebra and operator theory (see [4,22] and references therein). In 2014, in the spades of novel generalizations, Ma [6] obtained $C^\star$-algebra-valued metric spaces and later proved fixed point results in the setting of $b-$metric spaces. The Banach contraction principle has seen further generalizations in the settings of cone metric spaces. One can see Perov [28] and Zabrejko [27] for foundational works, and the reintroduction by Huang and Zhang [29] as the so-called cone metric spaces. In this context, the cone's solidness is a desirable property for the establishment of fixed point theorems. In 2023, Xu, Cheng, and Han [22] introduced a novel approach to fixed point theory by establishing common fixed point results in cone b-metric spaces over Banach algebras, without requiring the solidness of the underlying cone, only maintaining normality of the cone. Their work focused on contractions with vector-valued coefficients and introduced a kind of new convergence of sequences, termed it *wrtn-convergence*, that is, 'with respect to the norm' convergence.

In some cases distances arise between elements of two different sets, rather than between points of a unique set. Such type of distances are abundant in mathematics and applied sciences. And so in 2016, Mutlu and Gürdal [20] formalized these distances under the name bipolar metric spaces and later [25] defined $\alpha - \psi$ contractive mappings and multivalued mappings, respectively, and established fixed point theorems in the context of bipolar metric spaces. In 2021, Gaba, Aphane, and Ayidi [12] introduced $(\alpha - BK)$-contractractions in Bipolar metric spaces and recovered well-known classical results.

Motivated by the above results, we give fixed point results for $C^\star-$ algebra-valued bipolar b-metric spaces without requiring the solidity of the underlying cone under $(\alpha - BK)$–type contractions.

2. Preliminaries

The following basic concepts from literature in $C^\star-$ algebra are necessary for proving results.

A complex algebra $\mathcal{H}$, together with a conjugate linear involution map $p \mapsto p^\star$, is called a $\star$-algebra if $(pq)^\star = q^\star p^\star$ and $(p^\star)^\star = p$ for all $p, q \in \mathcal{H}$. Moreover, the pair $(\mathcal{H}, \star)$ is called a unital $\star$-algebra if $\mathcal{H}$ contains an identity element $1_\mathcal{H}$.

By a Banach $\star$-algebra, we mean a complete normed unital $\star$-algebra $(\mathcal{H}, \star)$ such that the norm on $\mathcal{H}$ is submultiplicative and satisfies $\|p^\star\| = \|p\|$ for all $p \in \mathcal{H}$. Further, if for all $p \in \mathcal{H}$, we have $\|p^\star p\| = \|p\|^2$ in a Banach $\star$-algebra $(\mathcal{H}, \star)$, then $\mathcal{H}$ is known as a $C^\star$-algebra.

A positive element of $\mathcal{H}$ is an element pv such that $p = p^\star$ and its spectrum satisfies $\sigma(p) \subset \mathbb{R}$, where

$$\sigma(p) = \{\lambda \in \mathbb{C} : \lambda 1_\mathcal{H} - p \text{ is non-invertible}\}.$$

The set of all positive elements will be denoted by $\mathcal{H}_+$. Such elements allow us to define a partial ordering $\succeq$ on the elements of $\mathcal{H}$. That is,

$$q \succeq p \iff q - p \in \mathcal{H}_+.$$

If $p \in \mathcal{H}$ is positive, then we write $p \succeq 0_\mathcal{H}$, where $0_\mathcal{H}$ is the zero element of $\mathcal{H}$. Each positive element p of a $C^\star$-algebra $\mathcal{H}$ has a unique positive square root denoted by $p^{1/2}$ in $\mathcal{H}$. From now on, by $\mathcal{H}$ we mean a unital $C^\star$-algebra with identity element $1_\mathcal{H}$. Further,

$$\mathcal{H}_+ = \{p \in \mathcal{H} : p \succeq 0_\mathcal{H}\}, \quad \text{and} \quad (p^* p)^{1/2} = p.$$

Definition 1. *[17] A subset $\mathcal{H}_+$ of $\mathcal{H}$ is called a cone if:*

(1) $\mathcal{H}_+$ is non-empty, closed, and $\{0_\mathcal{H}, 1_\mathcal{H}\} \subset \mathcal{H}_+$,

(2) $\alpha x + \beta y \in \mathcal{H}_+, \forall \alpha, \beta \in \mathbb{R}_+,$ and $x, y \in \mathcal{H}_+$,

(3) $x \cdot x = x^2 \in \mathcal{H}_+$,

(4) $x + (-x) = 0_\mathcal{H}$.

For a given cone $\mathcal{H}_+ \subset \mathcal{H}$, we can define a partial ordering $\preceq$ with respect to $\mathcal{H}_+$ by $x \preceq y$ if and only if $y - x \in \mathcal{H}_+$. We denote $x \prec y$ if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(\mathcal{H}_+)$, where $int(\mathcal{H}_+)$ denotes the interior of $\mathcal{H}_+$.

The cone $\mathcal{H}_+$ is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{H}, 0_\mathcal{H} \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying the above is called the normal constant of $\mathcal{H}_+$.

Definition 2. *[3] Let $\mathcal{H}$ be a unital $C^\star$-algebra, and let $\mathcal{H}_+$ denote the positive cone of $\mathcal{H}$, consisting of all positive elements $a \in \mathcal{H}$ such that $a = x^* x$ for some $x \in \mathcal{H}$. A map $F : \mathcal{H}_+ \to \mathcal{H}_+$ is called a positive function if it satisfies the following properties:*

1.   For all $z \in \mathcal{H}_+$, $F(z) \in \mathcal{H}_+$.

2.   For all $z_1, z_2 \in \mathcal{H}_+$, if $z_1 \preceq z_2$, then $F(z_1) \preceq F(z_2)$, where $\preceq$ denotes the partial order induced by the positive cone.

3. $F(0_\mathcal{H}) = 0_\mathcal{H}$, where $0_\mathcal{H}$ is the zero element of $\mathcal{H}_+$.

Example 1. *Let $\mathcal{H}_+$ be the positive cone of a unital $C^\star$-algebra $\mathcal{H}$. Define the functions $F_\mathcal{H}, G_\mathcal{H} : \mathcal{H}_+ \to \mathcal{H}_+$ as follows:*

$$F_\mathcal{H}(z) = z^2, \quad G_\mathcal{H}(z) = z + z^2, \quad \forall z \in \mathcal{H}_+.$$

For both functions, condition (1) and (2) follow directly from the definition. For condition 3, when $z = 0_\mathcal{H}$, we have $G_\mathcal{H}(0) = F_\mathcal{H}(0) = 0_\mathcal{H}$, when $z \neq 0_\mathcal{H}$, the inequality $F_\mathcal{H}(z) \prec G_\mathcal{H}$ holds because $G_\mathcal{H}$ dominates $F_\mathcal{H}$ for any $z \in \mathcal{H}_+$ and $z \neq 0_\mathcal{H}$.

Definition 3. *[6] Consider a unital $C^\star-$ algebra $\mathcal{H}$ with a unital $I_\mathcal{H}$, a set $\Phi \neq \varnothing$, and $I_\mathcal{H} \preceq b \in \mathcal{H}_+$. A distance function $\varphi : \Phi \times \Phi \to \mathcal{H}_+$ with the following*

1. *$\varphi(\sigma, \nu) = 0_\mathcal{H}$ if and only if $\sigma = \nu$, for all $(\sigma, \nu) \in \Phi \times \Phi$;*
2. *$\varphi(\sigma, \nu) = \varphi(\nu, \sigma)$ for all $\sigma, \nu \in \Phi$;*
3. *$\varphi(\sigma, \nu) \preceq b\big(\varphi(\sigma, h) + \varphi(h, \nu)\big)$, for all $\sigma, \nu, h \in \Phi$.*

    Then $(\Phi, \mathcal{H}, \varphi)$ is known as $C^\star-$ algebra valued b–metric space.

Definition 4 ([20]). *Let Φ and Ψ be two non-empty subsets of a set V, and let $\varphi : \Phi \times \Psi \to \mathbb{R}_+$ be a function such that:*

1. *$\varphi(\sigma, \nu) = 0$ if and only if $\sigma = \nu$;*
2. *$\varphi(\sigma, \nu) = \varphi(\nu, \sigma)$, for all $\sigma, \nu \in \Phi \cap \Psi$;*
3. *$\varphi(\sigma, \nu) \leq \varphi(\sigma, \nu_1) + \varphi(\sigma_1, \nu_1) + \varphi(\sigma_1, \nu)$, for all $\sigma, \sigma_1 \in \Phi$ and $\nu, \nu_1 \in \Psi$.*

    Then, the triplet $(\Phi, \Psi, \mathcal{H}, \varphi)$ is called a bipolar metric space.

In 2022, Mani, Gnanaprakasam, Haq, Baloch, and jarad [31], gave the following:

Definition 5. *[31] Consider a unital $C^\star-$ algebra $\mathcal{H}$ with a unital $I_\mathcal{H}$, two sets $\Phi, \Psi \neq \varnothing$, and $I \preceq b \in \mathcal{H}_+$. A distance function $\varphi : \Phi \times \Psi \to \mathcal{H}_+$ with the following*

1. *$\varphi(\sigma, \nu) = 0_\mathcal{H}$ if and only if $\sigma = \nu$, for all $(\sigma, \nu) \in \Phi \times \Psi$;*
2. *$\varphi(\sigma, \nu) = \varphi(\nu, \sigma)$, for all $\sigma, \nu \in \Phi \cap \Psi$;*
3. *$\sigma(\sigma_1, \nu_2) \preceq \sigma(\sigma_1, \nu_1) + \varphi(\sigma_2, \nu_1) + \varphi(\sigma_2, \nu_2)$, for all $(\sigma_1, \nu_1), (\sigma_2, \nu_2) \in \Phi \times \Psi$.*

is called $C^\star-$ algebra bipolar metric and $(\Phi, \Psi, \mathcal{H}, \varphi)$ is called $C^\star-$ algebra valued bipolar metric space.

Definition 6 ([32]). *Let (Φ, Ψ, φ) be a bipolar metric space. A function $\varphi : \Phi \times \Psi \to \mathcal{H}_+$ is called a bipolar b-metric if there exists a constant $b \geq 1$ such that:*

1. *$\varphi(\sigma, \nu) = 0$ if and only if $\sigma = \nu$;*
2. *$\varphi(\sigma, \nu) = \varphi(\nu, \sigma)$, for all $\sigma, \nu \in \Phi \cap \Psi$;*
3. *$\varphi(\sigma_1, \nu_2) \leq b[\varphi(\sigma_1, \nu_1) + \varphi(\sigma_2, \nu_1) + \varphi(\sigma_2, \nu_2)]$, for all $\sigma_1, \sigma_2 \in \Phi$ and $\nu_1, \nu_2 \in \Psi$.*

    Then $(\Phi, \Psi, \mathcal{H}_+, \varphi)$ is called a bipolar b-metric space.

Remark 1. *The space is said to be joint if $\Phi \cap \Psi \neq \varnothing$ otherwise disjoint.*

Example 2. *[32] Consider $\Phi = (-\infty, 0], \Psi = [0, \infty), \mathcal{H} = \mathcal{M}_2(\mathbb{R})$ and $\varphi : \Phi \times \Psi \to \mathcal{H}$ as $\varphi(\sigma, \nu) = diag\{c_1(|\sigma - \nu|)^p, c_2(|\sigma - \nu|)^p\}$ where $p > 1$ and $c_1, c_2 > 0$.*

It can be easily verified that conditions 1 and 2 of definition 1.4 holds.

Using $|\sigma_1 - \nu_2|p \leq 2^{2p}(|\sigma_1 - \nu_1|^p + |\sigma_2 - \nu_1|^p + |\sigma_2 - \nu_2|^p)$, one can prove that:

$$\varphi(\sigma_1, \nu_2) \leq b[\varphi(\sigma_1, \nu_1) + \varphi(\sigma_2, \nu_1) + \varphi(\sigma_2, \nu_2)],$$

where $b = 2^{2p} I$.

Thus, $(\Phi, \Psi, \mathcal{H}, \varphi)$ is a complete C^*-algebra valued bipolar b-metric space.

If we take $\nu_1 = -\frac{1}{2}, \nu_2 = 0, \nu_1 = 0$, and $\nu_2 = \frac{1}{2}$, then:

$$\varphi(\sigma_1, \nu_2) \succeq \varphi(\sigma_1, \nu_1) + \varphi(\sigma_1, \nu_1) + \varphi(\sigma_2, \nu_2),$$

for all $(\sigma_1, \nu_1), (\sigma_2, \nu_2) \in \Phi \times \Psi$. So, it is not a C^*-algebra valued bipolar metric space.

Definition 7. *Let $(\Phi_1, \Psi_1, \mathcal{H}, \varphi_1)$ and $(\Phi_2, \Psi_2, \mathcal{H}, \varphi_2)$ be $C^\star$-algebra valued bipolar b-metric spaces, where φ_1 and φ_2 are $C^\star$-algebra valued metrics on $(\Phi_1 \times \Psi_1)$ and $(\Phi_2 \times \Psi_2)$, respectively.*

(1) *A function $T : \Phi_1 \cup \Psi_1 \to \Phi_2 \cup \Psi_2$ is called a **contravariant** map from (Φ_1, Ψ_1) to (Φ_2, Ψ_2), and denoted by $f : (\Phi_1, \Psi_1) \rightleftharpoons (\Phi_2, \Psi_2)$, if :*

$$T(\Phi_1) \subseteq \Psi_2 \quad and \quad T(\Psi_1) \subseteq \Phi_2.$$

(2) *Moreover, if φ_1 and φ_2 are $C^\star$-algebra valued bipolar b-metrics on $(\Phi_1 \times \Psi_1)$ and $(\Phi_2 \times \Psi_2)$, respectively, then the notation:*

$$T : (\Phi_1, \Psi_1, \varphi_1) \rightleftharpoons (\Phi_2, \Psi_2, \varphi_2)$$

*denotes a **contravariant** map between $C^\star$-algebra valued bipolar b-metric spaces, where the distances φ_1 and φ_2 respect the $C^\star$-algebra structure.*

Definition 8. *[31] Let $(\Phi_1, \Psi_1, \mathcal{H}, \varphi_1)$ and $(\Phi_2, \Psi_2, \mathcal{H}, \varphi_2)$ be two $C^\star$-algebra valued bipolar metric spaces,*

(1) *T is called **left continuous** at a point $\sigma_0 \in \Phi_1$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that:*

$$\|\varphi_2(T\sigma_0, Tv)\| < \epsilon \quad whenever \quad \|\varphi_1(\sigma_0, v)\| < \delta, \quad v \in \Psi_1.$$

(2) *T is called **right continuous** at a point $v_0 \in \Pi_1$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that:*

$$\|\sigma_2(T\sigma, Tv_0)\| < \epsilon \quad whenever \quad \|\sigma_1(\sigma, v_0)\| < \delta, \quad \sigma \in \Phi_1.$$

(3) *T is called **continuous** if it is left continuous at every $\sigma_0 \in \Phi_1$ and right continuous at every $v_0 \in \Pi_1$.*

(4) *A map T is called a **continuous contravariant map** if it satisfies the same continuity conditions as a covariant map, with:*

$$T(\Phi_1) \subseteq \Psi_2 \quad and \quad T(\Psi_1) \subseteq \Phi_2.$$

Definition 9. *Let $F : \mathcal{A}_+ \to \mathcal{A}_+$ be a positive function on the positive cone $\mathcal{A}_+$ of a C^*-algebra $\mathcal{A}$. Then F satisfies the combined condition:*

$$F(a + b) = F(a) + F(b), \quad for \ all \ a, b \in \mathcal{A}_+ \ with \ ab = ba,$$

and

$$F(\lambda a) = \lambda F(a), \quad for \ all \ \lambda \geq 0 \ and \ a \in \mathcal{A}_+.$$

Equivalently, for $a, b \in \mathcal{A}_+$ with $ab = ba$, and $\lambda, \mu \geq 0$,

$$F(\lambda a + \mu b) = \lambda F(a) + \mu F(b).$$

This linear combination ensures that F is additive and homogeneous on commuting elements in the positive cone.

Definition 10. *Let $F : \mathcal{A}_+ \to \mathcal{A}_+$ be a positive function on the positive cone $\mathcal{A}_+$ of a C^*-algebra $\mathcal{A}$. Then F satisfies the linearity condition:*

$$F(\lambda a + \mu b) = \lambda F(a) + \mu F(b),$$

for all $a, b \in \mathcal{A}_+$ such that $ab = ba$ (commutative elements), and for all $\lambda, \mu \geq 0$.

In this paper, motivated by the above results, we prove fixed point theorems on $C^\star-algebra$-valued metric spaces in the following settings:

- **(C1)** The underlying cone is non-solid. Unlike many prior results that assume solidness of the cone, this work addresses cases where the cone lacks interior points. Non-solid cones can lead to situations where convergence results fail, as observed in certain applications in quantum mechanics.

- **(C2)** The contractions considered are $(F_{\mathcal{H}} - G_{\mathcal{H}})$-contractions, extending the applicability of contraction mappings in this context.
- **(C3)** Applications to the Ulam-Hyers Stability problem are provided, illustrating the utility of the theoretical results in practical and analytical settings.

3. Main Results

Definition 11. *Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a $C^\star$-algebra-valued bipolar b-metric space in a non-solid cone $\mathcal{H}_+ \subseteq \mathcal{H}$. Let $F_{\mathcal{H}}, G_{\mathcal{H}} : \mathcal{H}_+ \to \mathcal{H}_+$, where $F_{\mathcal{H}} \preceq G_{\mathcal{H}}$ for all $z \in \mathcal{H}_+$, and $F_{\mathcal{H}}(z) = G_{\mathcal{H}}(z)$ if and only if $z = 0_{\mathcal{H}}$, then:*

1. *Elements of Φ are left elements, of Ψ are right elements, and of $\Phi \cup \Psi$ are central elements.*
2. *A left sequence $\{\sigma_n\} \subseteq \Phi$ converges to $\nu \in \Psi$ if and only if :*

$$\|F_{\mathcal{H}} \varphi(\sigma_n, \nu)\| \to 0_{\mathcal{H}} \quad as\ n \to \infty.$$

3. *A right sequence $\{\nu_n\} \subseteq \Psi$ converges to $\sigma \in \Phi$ if and only if:*

$$\|F_{\mathcal{H}} \varphi(\nu_n, \sigma)\| \to 0_{\mathcal{H}} \quad as\ n \to \infty.$$

4. *A bisequence is a pair of sequences $(\{\sigma_n\}, \{\nu_n\})$ on $\Phi \times \Psi$.*
5. *A bisequence $(\{\sigma_n\}, \{\nu_n\})$ is convergent if both $\{\sigma_n\}$ and $\{\nu_n\}$ converge to a common point $z \in \Phi \cap \Psi$. This is called biconvergence.*
6. *A bisequence $(\{\sigma_n\}, \{\nu_n\})$ is a Cauchy bisequence if and only if:*

$$\|F_{\mathcal{H}} \varphi(\sigma_n, \nu_m)\| \to 0_{\mathcal{H}} \quad as\ n, m \to \infty.$$

7. *The space $(\Phi, \Psi, \mathcal{H}, \varphi)$ is wrtn-complete if every Cauchy bisequence is convergent.*

Remark 2. *Here, wrtn means ' with respect to the norm of $\mathcal{H}$.' Convergence is defined with respect to the norm inherent in $\mathcal{H}$.*

Proposition 1. *In a $C^\star-$algebra valued bipolar b-metric space, every convergent bisequence is a Cauchy bisequence.*

Proof. Let $(\{\sigma_n\}, \{\nu_n\})$ be a biconvergent bisequence in a $C^\star$-algebra valued bipolar b-metric space $(\Phi, \Psi, \mathcal{H}, \varphi)$, which converges to some $z \in \Phi \cap \Psi$, then:

$$\varphi(\sigma_n, \nu_m) \preceq b\big(\varphi(\sigma_n, z) + \varphi(z, z) + \varphi(z, \nu_m)\big).$$

By normality condition, we have:

$$\|\varphi(\sigma_n, \nu_m)\| \preceq bM\big(\|\varphi(\sigma_n, z) + \varphi(z, z) + \varphi(z, \nu_m)\|\big) \leq bM\big(\|\varphi(\sigma_n, z)\| + \|\varphi(z, z)\| + \|\varphi(z, \nu_m)\|\big)$$

Since $\{\sigma_n\} \to z \in \Psi$ and $\{\nu_m\} \to z \in \Phi$, then

$$\|\varphi(\sigma_n, z)\| \to 0_{\mathcal{H}}, \quad and \quad \|\varphi(\nu_m, z)\| \to 0_{\mathcal{H}}, \quad as \quad n, m \to \infty.$$

Additionally, $\|\varphi(z, z)\| = 0_{\mathcal{H}}$. Thus $\|\varphi(\sigma_n, \nu_m)\| \to 0$ as $n, m \to \infty$. Hence, $(\{\sigma_n\}, \{\nu_n\})$ is a Cauchy bisequence. $\square$

Proposition 2. *In $C^\star-$algebra valued bipolar b-metric space, every convergent Cauchy bisequence is biconvergent.*

Proof. Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a $C^\star-algebra$ valued bipolar b-metric space, and $(\{\sigma_n\}, \{\nu_n\})$ be a convergent Cauchy bisequence, such that $\sigma_n \to \nu \in \Psi$ and $\nu_n \to \sigma \in \Phi$, then

$$\varphi(\sigma, \nu) \preceq b\big(\varphi(\sigma, \nu_n) + \varphi(\nu_n, \sigma_n) + \varphi(\sigma_n, \nu)\big).$$

By normality condition, we have:

$$\|(\varphi(\sigma, \nu)\| \leq bM\big(\|\varphi(\sigma, \nu_n) + \varphi(\nu_n, \sigma_n) + \varphi(\sigma_n, \nu))\| \leq bM\big(\|\varphi(\sigma, \nu_n)\| + \|\varphi(\nu_n, \sigma_n)\| + \varphi(\sigma_n, \nu))\|.$$

Since $(\{\sigma_n\}, \{\nu_n\})$ is a convergent Cauchy bisequence, we have $\|\varphi(\nu_n, \sigma_n)\| \to 0$ as $n \to \infty$, and $\{\sigma_n\} \to \nu$ and $\{\nu_n\} \to \sigma$, we also have:

$$\|(\varphi(\sigma, \nu_n)\| \to 0 \quad \text{and} \quad \|(\varphi(\sigma_n, \nu)\| \to 0 \quad \text{as } n \to \infty.$$

Thus $\|\varphi(\sigma, \nu)\| \to 0$ as $n \to \infty$. This implies $\varphi(\sigma, \nu) = 0_{\mathcal{H}}$. Hence, $\sigma = \nu$. $\square$

Definition 12. *Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be $C^\star$-algebra valued bipolar b-metric space and $T : (\Phi, \Psi, \mathcal{H}\varphi) \times (\Phi, \Psi, \mathcal{H}\varphi)$ be a contravariant map such that there exists constants $\alpha_1, \alpha_2, \alpha_3 \geq 0$, with $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Then T is called $(F_{\mathcal{H}} - G_{\mathcal{H}})$- contraction if there exits two positive functions $F_{\mathcal{H}}, G_{\mathcal{H}} : \mathcal{H}_+ \to \mathcal{H}_+$ such that*

$$F_{\mathcal{H}}\big(\varphi(T\sigma, T\nu)\big) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma, \nu) + \alpha_2 \varphi(\sigma, T\sigma) + \alpha_3 \varphi(T\nu, \nu)\big),$$

whenever $(\sigma, \nu) \in \Phi \times \Psi$ and $\varphi(T\sigma, T\nu) > 0_{\mathcal{H}}$.

Theorem 1. *Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a wrtn-complete $C^\star$-algebra-valued bipolar b-metric space, where:*

(i) *$\varphi : (\Phi \times \Psi) \to \mathcal{H}_+$ is the $C^\star$-algebra-valued bipolar b-metric,*

(ii) *$\mathcal{H}_+$ is the positive cone of a unital $C^\star$-algebra $\mathcal{H}$,*

(iii) *the cone $\mathcal{H}_+$ is **non-solid** (i.e., it has an empty interior, $Int(\mathcal{H}_+) = \varnothing$).*

Suppose a mapping $T : \Phi \cup \Psi \to \Phi \cup \Psi$ satisfies the following conditions:

(1) *There exist constants $\alpha_1, \alpha_2, \alpha_3 \geq 0$ such that:*

$$\alpha_1 + \alpha_2 + \alpha_3 < 1.$$

(2) *There exist positive functions $F_{\mathcal{H}}, G_{\mathcal{H}} : \mathcal{H}_+ \to \mathcal{H}_+$ such that:*

$$F_{\mathcal{H}}(z) \preceq G_{\mathcal{H}}(z), \quad \forall z \in \mathcal{H}_+ \setminus \{0_{\mathcal{H}}\},$$

and:

$$F_{\mathcal{H}}(0_{\mathcal{H}}) = G_{\mathcal{H}}(0_{\mathcal{H}}) = 0_{\mathcal{H}}.$$

(3) *For all $\sigma, \nu \in \Phi \cup \Psi$, the contraction condition holds:*

$$F_{\mathcal{H}}(\varphi(T\nu, T\sigma)) \preceq G_{\mathcal{H}}(\alpha_1 \varphi(\sigma, \nu) + \alpha_2 \varphi(\sigma, T\sigma) + \alpha_3 \varphi(T\nu, \nu)).$$

Then, the mapping T has a unique fixed point $z \in \Phi \cap \Psi$ such that:

$$Tz = z.$$

Proof. Define $\sigma_{n+1} = T\nu_n$ and $\nu_n = T\sigma_n$. Then

$$\varphi(\sigma_n, \nu_n) = \varphi(T\nu_{n-1}, T\sigma_n).$$

Applying the contraction, we get:

$$F_{\mathcal{H}}\big(\varphi(T\nu_{n-1}, T\sigma_n)\big) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_n, \nu_{n-1}) + \alpha_2 \varphi(\sigma_n, T\sigma_n) + \alpha_3 \varphi(T\nu_{n-1}, \nu_{n-1})\big)$$
$$= G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_n, \nu_{n-1}) + \alpha_2 \varphi(\sigma_n, \nu_n) + \alpha_3 \varphi(\sigma_n, \nu_{n-1})\big)$$
$$= G_{\mathcal{H}}\big((\alpha_1 + \alpha_3)\varphi(\sigma_n, \nu_{n-1}) + \alpha_2 \varphi(\sigma_n, \nu_n)\big).$$

Therefore,
$$F_{\mathcal{H}}\big(\varphi(\sigma_n, \nu_n)\big) \preceq G_{\mathcal{H}}\big((\alpha_1 + \alpha_2)\varphi(\sigma_n, \nu_{n-1}) + \alpha_2 \varphi(\sigma_n, \nu_n)\big), \tag{1}$$

for all integers $n \geq 1$. Therefore, we consider:

$$\varphi(\sigma_n, \nu_{n-1}) = \varphi(\sigma_{n-1}, \nu_{n-2}).$$

Applying the contraction again, we get:

$$F_{\mathcal{H}}\big(\varphi(T\nu_{n-2}, T\sigma_{n-1})\big) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_{n-1}, \nu_{n-2}) + \alpha_2 \varphi(\sigma_{n-1}, T\sigma_{n-1}) + \alpha_3 \varphi(T\nu_{n-2}, \nu_{n-2})\big)$$
$$= G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_{n-1}, \nu_{n-2}) + \alpha_2 \varphi(\sigma_{n-1}, \nu_{n-1}) + \alpha_3 \varphi(\sigma_{n-1}, \nu_{n-2})\big)$$
$$= G_{\mathcal{H}}\big((\alpha_1 + \alpha_3)\varphi(\sigma_{n-1}, \nu_{n-2}) + \alpha_2 \varphi(\sigma_{n-1}, \nu_{n-1})\big).$$

Therefore, by monotonicity of $F_{\mathcal{H}}$ and $G_{\mathcal{H}}$, we have:

$$\varphi(\sigma_n, \nu_{n-1}) \preceq F_{\mathcal{H}}^{-1}\Big(G_{\mathcal{H}}\big((\alpha_1 + \alpha_3)\varphi(\sigma_{n-1}, \nu_{n-2}) + \alpha_2 \varphi(\sigma_{n-1}, \nu_{n-1})\big)\Big). \tag{2}$$

Substituting (2) into (1), we get

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_n)) \preceq G_{\mathcal{H}}\Big((\alpha_1 + \alpha_3)F_{\mathcal{H}}^{-1}\big(G_{\mathcal{H}}\big((\alpha_1 + \alpha_3)\varphi(\sigma_{n-1}, \nu_{n-2}) + \alpha_2 \varphi(\sigma_{n-1}, \nu_{n-1})\big)\big) + \alpha_2 \varphi(\sigma_n, \nu_n)\Big). \tag{3}$$

Similarly, by expanding the recursive terms $\varphi(\sigma_{n-1}, \nu_{n-2})$ and $\varphi(\sigma_{n-1}, \nu_{n-1})$, we get:

$$\varphi(\sigma_{n-1}, \nu_{n-2}) \preceq F_{\mathcal{H}}^{-1}\big(G_{\mathcal{H}}\big((\alpha_1 + \alpha_3)\varphi(\sigma_{n-2}, \nu_{n-3}) + \alpha_2 \varphi(\sigma_{n-2}, \nu_{n-2})\big)\big), \tag{4}$$

and
$$\varphi(\sigma_{n-1}, \nu_{n-1}) \preceq F_{\mathcal{H}}^{-1}\big(G_{\mathcal{H}}\big((\alpha_1 + \alpha_3)\varphi(\sigma_{n-2}, \nu_{n-3}) + \alpha_2 \varphi(\sigma_{n-1}, \nu_{n-2})\big)\big). \tag{5}$$

Substituting (4) and (5) into (3), we get:

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_n)) \preceq G_{\mathcal{H}}\Big((\alpha_1 + \alpha_3)^2 F_{\mathcal{H}}^{-1}\big(G_{\mathcal{H}}\big((\alpha_1 + \alpha_3)\varphi(\sigma_{n-2}, \nu_{n-3}) + \alpha_2 \varphi(\sigma_{n-2}, \nu_{n-2})\big)\big)$$
$$+ (\alpha_1 + \alpha_3)\alpha_2 F_{\mathcal{H}}^{-1}\big(G_{\mathcal{H}}\big(\varphi(\sigma_{n-1}, \nu_{n-1})\big)\big) + \alpha_2 \varphi(\sigma_n, \nu_n)\Big). \tag{6}$$

Clearly, for $k \geq 0$, the general pattern emerges as:

$$\varphi(\sigma_n, \nu_n) \preceq \sum_{k=0}^{n} (\alpha_1 + \alpha_3)^k \alpha_2 \varphi(\sigma_{n-k}, \nu_{n-k}). \tag{7}$$

As $(\alpha_1 + \alpha_2) < 1$, it follows that:

$$\varphi(\sigma_n, \nu_n) \preceq \alpha_2 \sum_{k=0}^{n} (\alpha_1 + \alpha_3)^k \varphi(\sigma_0, \nu_0), \tag{8}$$

and as $n \to \infty$ the geometric series converges:

$$\sum_{k=0}^{\infty} (\alpha_1 + \alpha_3)^k = \frac{1}{1 - (\alpha_1 + \alpha_3)}. \tag{9}$$

Thus

$$\varphi(\sigma_n, \nu_n) \preceq \frac{\alpha_2 \varphi(\sigma_0, \nu_0)}{1 - (\alpha_1 + \alpha_3)}. \tag{10}$$

By setting

$$\lambda = \frac{\alpha_2 \varphi(\sigma_0, \nu_0)}{1 - (\alpha_1 + \alpha_3)}$$

we see that $\lambda < 1$. Therefore, for all positive integers $m, n \in \mathbb{N}$,

1.   if $m > n$, from the recursive definitions:

$$\sigma_{n+1} = T\nu_n, \quad \nu_m = T\sigma_{m-1}$$

we have

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_m)) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_n, \nu_{m-1}) + \alpha_2 \varphi(\sigma_n, \nu_m) + \alpha_3 \varphi(\sigma_{n-1}, \nu_m)\big) \tag{11}$$

and

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_{m-1})) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_n, \nu_{m-2}) + \alpha_2 \varphi(\sigma_n, \nu_{m-1}) + \alpha_3 \varphi(\sigma_{n-1}, \nu_{m-1})\big). \tag{12}$$

Substituting (12) into (11 we get:

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_m)) \preceq G_{\mathcal{H}}\Big(\alpha_1 F_{\mathcal{H}}^{-1}\big(G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_n, \nu_{m-2}) + \alpha_2 \varphi(\sigma_n, \nu_{m-1}) + \alpha_3 \varphi(\sigma_{n-1}, \nu_{m-1})\big)\big)$$

$$+ \alpha_2 \varphi(\sigma_n, \nu_m) + \alpha_3 \varphi(\sigma_{n-1}, \nu_m)\Big).$$

$$= \sum_{j=0}^{m-n} (\alpha_1 + \alpha_3)^j \alpha_2 \varphi(\sigma_0, \nu_0),$$

Therefore

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_m)) \preceq \sum_{j=0}^{m-n} (\alpha_1 + \alpha_3)^j \alpha_2 \varphi(\sigma_0, \nu_0) \tag{13}$$

After k recursive substitutions:

$$\varphi(\sigma_n, \nu_m) \preceq \sum_{j=0}^{m-n} (\alpha_1 + \alpha_3)^j \alpha_2 \varphi(\sigma_0, \nu_0).$$

For $(\alpha_1 + \alpha_3) < 1$, the geometric series converges:

$$\sum_{j=0}^{\infty} (\alpha_1 + \alpha_3)^j = \frac{1}{1 - (\alpha_1 + \alpha_3)}.$$

Thus, as $m \to \infty$, we have

$$\varphi(\sigma_n, \nu_m) \preceq \frac{\alpha_2 \varphi(\sigma_0, \nu_0)}{1 - (\alpha_1 + \alpha_3)}.$$

2.   If $m < n$,

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_m)) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_n, \nu_{m-1}) + \alpha_2 \varphi(\sigma_n, \nu_m) + \alpha_3 \varphi(\sigma_{n-1}, \nu_m)\big), \tag{14}$$

and

$$F_{\mathcal{H}}(\varphi(\sigma_{n-1}, \nu_m)) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_{n-1}, \nu_{m-1}) + \alpha_2 \varphi(\sigma_{n-1}, \nu_m) + \alpha_3 \varphi(\sigma_{n-2}, \nu_m)\big). \tag{15}$$

Substituting (15) into (14) we get:

$$F_{\mathcal{H}}(\varphi(\sigma_n, \nu_m)) \preceq G_{\mathcal{H}}\Big(\alpha_1 \varphi(\sigma_n, \nu_{m-1}) + \alpha_2 \varphi(\sigma_n, \nu_m) \tag{16}$$
$$+ \alpha_3 F_{\mathcal{H}}^{-1}\big(G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma_{n-1}, \nu_{m-1}) + \alpha_2 \varphi(\sigma_{n-1}, \nu_m) + \alpha_3 \varphi(\sigma_{n-2}, \nu_m)\big)\big)\Big).$$

After k recursive expansions:

$$\varphi(\sigma_n, \nu_m) \preceq \sum_{j=0}^{n-m} (\alpha_1 + \alpha_3)^j \alpha_2 \varphi(\sigma_0, \nu_0), \tag{17}$$

For $(\alpha_1 + \alpha_3) < 1$, the geometric series converges:

$$\sum_{j=0}^{\infty} (\alpha_1 + \alpha_3)^j = \frac{1}{1 - (\alpha_1 + \alpha_3)}.$$

Thus, as $n \to \infty$:

$$\varphi(\sigma_n, \nu_m) \preceq \frac{\alpha_2 \varphi(\sigma_0, \nu_0)}{1 - (\alpha_1 + \alpha_3)}. \tag{18}$$

Since $\lambda < 1$, this means that $\varphi(\sigma_n, \nu_m)$ can be made arbitrarily small by larger values of m and n, and hence $(\{\sigma_n\}, \{\nu_m\})$ is a Cauchy bisequence with respect to $\mathcal{H}$. Since $(\Phi, \Psi, \mathcal{H}, \varphi)$ is complete, $(\{\sigma_n\}, \{\nu_m\})$ converges, and as a convergent Cauchy bisequence, in particular it biconverges, it follows that $\sigma_n \to z$ and $\nu_m \to z$, where $z \in \Phi \cap \Psi$. Since T is continuous, $T\sigma_n \to Tz$. Therefore, $Tz = z$. Hence z is fixed point of T. If $u \in \Psi$ is any other fixed point of T, then

$$0_{\mathcal{H}} \preceq \varphi(z, u) = \varphi(Tz, Tu)$$

and

$$F_{\mathcal{H}}\big(\varphi(Tz, Tu)\big) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(u, z) + \alpha_2 \varphi(u, Tu) + \alpha_3 \varphi(Tz, z)\big)$$
$$= \alpha_1 G_{\mathcal{H}}\big(\varphi(u, z)\big).$$

This implies $F_{\mathcal{H}}\big(\varphi(z, u)\big) \preceq \alpha_1 G_{\mathcal{H}}\big(\varphi(u, z)\big)$, which is not true by monotonicity of $F_{\mathcal{H}}$ and $G_{\mathcal{H}}$, unless $\varphi(z, u) = \varphi(u, z) = 0_{\mathcal{H}}$. Hence, $z = u$. Similar arguments hold if $u \in \Phi$. $\square$

4. Consequences

Corollary 1 ((α, **BK**)-Type-Contractions). *Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a wrtn-complete $C^\star$-algebra-valued bipolar b-metric space, and let $T : (\Phi, \Psi, \varphi) \rightrightarrows (\Phi, \Psi, \varphi)$ be a contravariant map satisfying the contraction condition:*

$$F_{\mathcal{H}}(\varphi(T\nu, T\sigma)) \preceq G_{\mathcal{H}}\big(\alpha_1 \varphi(\sigma, \nu) + \alpha_2 \varphi(\sigma, T\sigma) + \alpha_3 \varphi(T\nu, \nu)\big),$$

where $F_{\mathcal{H}}, G_{\mathcal{H}} : \mathcal{H}_+ \to \mathcal{H}_+$. By setting $F_{\mathcal{H}}(z) = z$ and $G_{\mathcal{H}}(z) = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3$, where $z = (z_1, z_2, z_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are tuplets of elements in $\mathcal{H}_+$ satisfying $\alpha_1 + \alpha_2 + \alpha_3 \prec 1$ (with respect to the ordering in $\mathcal{H}_+$), the contraction becomes:

$$\varphi(T\nu, T\sigma) \preceq \alpha_1 \varphi(\sigma, \nu) + \alpha_2 \varphi(\sigma, T\sigma) + \alpha_3 \varphi(T\nu, \nu),$$

for all $\sigma, \nu \in \Phi \cup \Psi$. Then function $T : (\Phi, \Psi, \varphi) \to (\Phi, \Psi, \varphi)$ has a unique fixed point.

Corollary 2 (Vector-Valued Contractions). *Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a wrtn-complete $C^\star$-algebra-valued bipolar b-metric space, and let $T : (\Phi, \Psi, \varphi) \rightrightarrows (\Phi, \Psi, \varphi)$ be a contravariant map. Suppose T satisfies the contraction condition:*

$$F_{\mathcal{H}}(\varphi(T\nu, T\sigma)) \preceq G_{\mathcal{H}}\big(k \cdot \varphi(\nu, \sigma)\big),$$

where $F_{\mathcal{H}}$, $G_{\mathcal{H}}$: $\mathcal{H}_+ \rightarrow \mathcal{H}_+$ are positive functions of a unital C*-algebra $\mathcal{H}$, $k \in \mathcal{H}_+$ is a vector-valued contraction coefficient satisfying $r(k) < 1$, where $r(k)$ is the spectral radius of k. By setting $F_{\mathcal{H}}(z) = z$ and $G_{\mathcal{H}}(z) = k \cdot z$, the contraction condition simplifies to:

$$\varphi(T\nu, T\sigma) \preceq k \cdot \varphi(\nu, \sigma),$$

for all $\nu, \sigma \in \Phi \cup \Psi$. Then T has a unique fixed point.

Corollary 3 (Kannan-Type Contractions). *Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a wrtn-complete C*-algebra-valued bipolar b-metric space, and let $T : (\Phi, \Psi, \varphi) \rightarrow (\Phi, \Psi, \varphi)$ be a contravariant map. Suppose T satisfies the contraction condition:*

$$F_{\mathcal{H}}(\varphi(T\nu, T\sigma)) \preceq G_{\mathcal{H}}\big(\alpha \cdot (\varphi(T\nu, \nu) + \varphi(T\sigma, \sigma))\big),$$

where $F_{\mathcal{H}}, G_{\mathcal{H}} : \mathcal{H}_+ \rightarrow \mathcal{H}_+$ are positive functions of a unital C-algebra $\mathcal{H}$, and $\alpha \in \mathcal{H}_+$, $r(\alpha) < \frac{1}{2}$, $\nu, \sigma \in \Phi \cup \Psi$. By setting $F_{\mathcal{H}}(z) = z$ and $G_{\mathcal{H}}(z) = \alpha \cdot z$, the contraction condition becomes:*

$$\varphi(T\nu, T\sigma) \preceq \alpha \cdot \big(\varphi(T\nu, \nu) + \varphi(T\sigma, \sigma)\big),$$

for all $\nu, \sigma \in \Phi \cup \Psi$. Then T has a unique fixed point.

5. Examples

Example 3. *Let $\Phi = (-\infty, 0]$, $\Psi = [0, \infty)$, and $\mathcal{H} = M_2(\mathbb{C})$, the space of 2×2 complex matrices, with $\mathcal{H}_+ = \{A \in \mathcal{H} : A = A^*, A \succeq 0\}$. Define the bipolar b-metric map $\varphi : \Phi \times \Psi \rightarrow \mathcal{H}_+$ by $\varphi(\sigma, \nu) = \begin{bmatrix} 3|\sigma - \nu|^2 & 0 \\ 0 & 4|\sigma - \nu|^2 \end{bmatrix}$ and the mapping $T : \Phi \cup \Psi \rightarrow \Phi \cup \Psi$ by $T(\sigma) = -\frac{\sigma}{7}$. For any $\sigma, \nu \in \Phi \cup \Psi$, the contraction condition is given by:*

$$\|F_{\mathcal{H}}(\varphi(T\sigma, T\nu))\| \leq \|G_{\mathcal{H}}(\alpha_1 \varphi(\sigma, \nu) + \alpha_2 \varphi(\sigma, T\sigma) + \alpha_3 \varphi(T\nu, \nu))\|.$$

One can easily verify that $(\Phi, \Psi, \mathcal{H}, \varphi)$ is wrtn-complete C-algebra valued bipolar metric space, where $F_{\mathcal{H}}(A) = A^2$, $G_{\mathcal{H}}(A) = A$, and $\alpha_1 + \alpha_2 + \alpha_3 < 1$. More so, we have:*

$$\varphi(T\sigma, T\nu) = \begin{bmatrix} 3|\frac{\sigma}{7} - \frac{\nu}{7}|^2 & 0 \\ 0 & 4|\frac{\sigma}{7} - \frac{\nu}{7}|^2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{49}|\sigma - \nu|^2 & 0 \\ 0 & \frac{4}{49}|\sigma, \nu|^2 \end{bmatrix}$$

Applying the contractraction gives:

$$F_{\mathcal{H}}(\varphi(T\sigma, T\nu)) = \varphi(T\sigma, T\nu)^2 = \begin{bmatrix} \left(\frac{3}{49}|\sigma - \nu|^2\right)^2 & 0 \\ 0 & \left(\frac{4}{49}|\sigma - \nu|^2\right)^2 \end{bmatrix}.$$

For the right-hand side of the contraction we have:

$$\alpha_1 \varphi(\sigma, \nu) = \begin{bmatrix} 3\alpha_1|\sigma - \nu|^2 & 0 \\ 0 & 4\alpha_1|\sigma - \nu|^2 \end{bmatrix}.$$

$$\alpha_2 \varphi(\sigma, T\sigma) = \begin{bmatrix} 3\alpha_2|\sigma - T\sigma|^2 & 0 \\ 0 & 4\alpha_2|\sigma - T\sigma|^2 \end{bmatrix}.$$

$$\alpha_3 \varphi(T\nu, \nu) = \begin{bmatrix} 3\alpha_3|T\nu - \nu|^2 & 0 \\ 0 & 4\alpha_3|T\nu - \nu|^2 \end{bmatrix}.$$

Summing these terms gives:

$$A = \begin{bmatrix} 3(\alpha_1|\sigma-\nu|^2 + \alpha_2|\sigma-T\sigma|^2 + \alpha_3|T\nu-\nu|^2) & 0 \\ 0 & 4(\alpha_1|\sigma-\nu|^2 + \alpha_2|\sigma-T\sigma|^2 + \alpha_3|T\nu-\nu|^2) \end{bmatrix},$$

where $A = \alpha_1\varphi(\sigma,\nu) + \alpha_2\varphi(\sigma,T\sigma) + \alpha_3\varphi(T\nu,\nu)$. *Applying* $G_{\mathcal{H}}$ *yields:*

$$G_{\mathcal{H}}(A) = \begin{bmatrix} 3(\alpha_1|\sigma-\nu|^2 + \alpha_2|\sigma-T\sigma|^2 + \alpha_3|T\nu-\nu|^2) & 0 \\ 0 & 4(\alpha_1|\sigma-\nu|^2 + \alpha_2|\sigma-T\sigma|^2 + \alpha_3|T\nu-\nu|^2) \end{bmatrix}$$

$$= A.$$

We observe that each term in $F_{\mathcal{H}}(\varphi(T\sigma,T\nu))$ involves $|\sigma-\nu|^4$ grows quadratically, while the right-hand side involves a linear combination $|\sigma,\nu|^2, |\sigma-T\sigma|^2$, and $|T\nu-\nu|^2$, and since $\alpha_1 + \alpha_2 + \alpha_3 < 1$, and T is continuous, the contraction is satisfied for all $\sigma, \nu \in \Phi \cup \Psi$. So, all the assumptions of Theorem 4.4 have occurred. Hence, T has a unique fixed point. The fixed point equation follows as:

$$-\frac{z}{7} = z \implies z = 0.$$

Thus, $z = 0$ is the unique fixed point. However, as if T were not contravariant, Theorem 4.4 cannot be applied.

6. Applications

6.1. Ulam-Hyers Stability Problem for Non-Solid Cones

Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a *wrtn*-complete $C^\star$-algebra-valued bipolar b-metric space, where $\mathcal{H}_+$ is a non-solid cone of the $C^\star$-algebra $\mathcal{H}$. A mapping $T : \Phi \cup \Psi \to \Phi \cup \Psi$ is said to satisfy the Ulam-Hyers stability if for any $\epsilon > 0$ and any ϵ-solution $\sigma_0 \in \Phi \cup \Psi$ satisfying:

$$\|F_{\mathcal{H}}\varphi(T\sigma_0,\sigma_0)\| \le \epsilon, \tag{19}$$

there exist a unique $\sigma^* \in \Phi \cap \Psi$ such that

$$T\sigma^* = \sigma^*, \quad \text{and} \tag{20}$$

$$\|F_{\mathcal{H}}(\varphi(\sigma_0,\sigma^*))\| \le M\epsilon, \quad \text{for some} \quad M > 0. \tag{21}$$

Any point $\sigma_0 \in \Phi \cup \Psi$, which is a solution of equation (19), is called an ϵ-solution of the mapping T.

Theorem 2. *Let* $(\Phi, \Psi, \mathcal{H}, \varphi)$ *be a wrtn-complete* $C^\star$-*algebra-valued bipolar* b-*metric space, where* $\mathcal{H}_+$ *is a non-solid cone of the* $C^\star$-*algebra* $\mathcal{H}$ *and* $T : \Phi \cup \Psi \to \Phi \cup \Psi$ *be a contravariant mapping satisfying definition* (12), *then the fixed point equation* (20) *of* T *is Ulam-Hyers stable.*

Proof. Suppose $\sigma_0 \in \Phi \cup \Psi$ is an ϵ-solution, that is:

$$\|F_{\mathcal{H}}(\varphi(T\sigma_0,\sigma_0))\| \le \epsilon,$$

and define the sequence $\{\sigma_n\}$ such that $\sigma_{n+1} = T\sigma_n$, for $n \ge 0$.

By the contraction condition (12), we have:

$$F_{\mathcal{H}}(\varphi(\sigma_{n+1},\sigma_n)) \preceq G_{\mathcal{H}}\Big(\alpha_1\varphi(\sigma_n,\sigma_{n-1}) + \alpha_2\varphi(\sigma_n,\sigma_{n+1}) + \alpha_3\varphi(\sigma_{n+1},\sigma_n)\Big).$$

Expanding recursively for k-steps, we obtain:

$$F_{\mathcal{H}}(\varphi(\sigma_{n+k}, \sigma_n)) \preceq G_{\mathcal{H}}\left(\sum_{j=0}^{k-1} \alpha_1^j \varphi(\sigma_1, \sigma_0)\right).$$

Since $\alpha_1 + \alpha_2 + \alpha_3 < 1$, the geometric series converges:

$$\sum_{j=0}^{\infty} \alpha_1^j = \frac{1}{1 - \alpha_1}.$$

Thus:

$$F_{\mathcal{H}}(\varphi(\sigma_{n+k}, \sigma_n)) \preceq G_{\mathcal{H}}\left(\frac{\varphi(\sigma_1, \sigma_0)}{1 - \alpha_1}\right),$$

and the corresponding norm satisfies:

$$\|F_{\mathcal{H}}(\varphi(\sigma_{n+k}, \sigma_n))\| \to 0 \quad \text{as } n \to \infty.$$

Since $\{\sigma_n\}$ is a Cauchy sequence, and $(\Phi, \Psi, \mathcal{H}, \varphi)$ is *wrtn*-complete, the sequence $\{\sigma_n\}$ converges to a unique fixed point $\sigma^* \in \Phi \cap \Psi$. By continuity of T, this limit satisfies:

$$T\sigma^* = \sigma^*.$$

Finally, using the contraction condition and the ϵ-solution property, we have:

$$\|F_{\mathcal{H}}(\varphi(\sigma_0, \sigma^*))\| \leq \|F_{\mathcal{H}}(\varphi(T\sigma_0, \sigma_0))\| \cdot \sum_{j=0}^{\infty} \alpha_1^j.$$

Simplifying this yields the Ulam-Hyers stability inequality:

$$\|F_{\mathcal{H}}(\varphi(\sigma_0, \sigma^*))\| \leq \frac{\epsilon}{1 - \alpha_1}.$$

This concludes the proof. □

Theorem 3. *Let $(\Phi, \Psi, \mathcal{H}, \varphi)$ be a C^*-algebra-valued bipolar b-metric space, where $\mathcal{H}_+$ is a non-solid cone in the C^*-algebra $\mathcal{H}$. Consider the Fredholm integral operator:*

$$T(x)(t) = q(t) + \lambda \int_{\mathcal{H}_1 \cup \mathcal{H}_2} G(t, s, x(s))\, ds, \tag{22}$$

where $T : \Phi \cup \Psi \to \Phi \cup \Psi$, and assume the following:

1. *The kernel $G : (\mathcal{H}_1 \cup \mathcal{H}_2) \times (\mathcal{H}_1 \cup \mathcal{H}_2) \times \mathcal{H}_+ \to \mathcal{H}_+$ satisfies:*
    (a) $G(t, s, x(s)) \in \mathcal{H}_+$ *for all $t, s \in \mathcal{H}_1 \cup \mathcal{H}_2$ and $x(s) \in \mathcal{H}_+$,*
    (b) $\|G(t, s, \sigma(s)) - G(t, s, \sigma(s))\| \leq \frac{1}{2}\|\theta(t,s)\|\|\sigma(s) - \sigma(s)\|$, *where $\theta : (\mathcal{H}_1 \cup \mathcal{H}_2) \times (\mathcal{H}_1 \cup \mathcal{H}_2) \to \mathcal{H}_+$ is continuous.*

2. *The function θ satisfies:*
    $$\sup_{t \in \mathcal{H}_1 \cup \mathcal{H}_2} \int_{\mathcal{H}_1 \cup \mathcal{H}_2} \|\theta(t,s)\|\, ds \leq 1.$$

3. *$q \in L^\infty(\mathcal{H}_1, \mathcal{H}_+) \cap L^\infty(\mathcal{H}_2, \mathcal{H}_+)$, with:*
    $$\|q(t)\| \leq M, \quad \forall t \in \mathcal{H}_1 \cup \mathcal{H}_2.$$

4. *The operator T satisfies:*
$$\|F_{\mathcal{H}}(\varphi(Tx, Ty))\| \leq \alpha_1 \|G_{\mathcal{H}}(\varphi(x,y))\|,$$

where $F_{\mathcal{H}}(x) = x^2$, $G_{\mathcal{H}}(x) = x$, and $0 < \alpha_1 < 1$.

Then integral equation has a unique solution in $L^{\infty}(\mathcal{H}_1, \mathcal{H}_+) \cap L^{\infty}(\mathcal{H}_2, \mathcal{H}_+)$.

Proof. Let $\Phi = L^{\infty}(\mathcal{H}_1)$ and $\Psi = L^{\infty}(\mathcal{H}_2)$ be two normed linear spaces of essentially bounded measurable functions, where $\mathcal{H}_1, \mathcal{H}_2 \subset [a,b]$ are disjoint Lebesgue measurable subsets such that $\mu(\mathcal{H}_1 \cup \mathcal{H}_2) < \infty$, $\mathcal{H} = L^2(\mathcal{H}_1 \cup \mathcal{H}_2, \mathcal{H}_+)$ be the space of square-integrable functions with values in the positive cone $\mathcal{H}_+$ of the C^*-algebra $\mathcal{H}$. Define the bipolar b-metric map $\varphi : \Phi \times \Psi \to L(\mathcal{H})$ as:

$$\varphi(\sigma, \nu) = \sup_{t \in \mathcal{H}_1 \cup \mathcal{H}_2} \|\sigma(t) - \nu(t)\|_{\mathcal{H}},$$

where $\sigma \in \Phi$, $\nu \in \Psi$, and $\|\cdot\|_{\mathcal{H}}$ is the norm in $\mathcal{H}$. Then $(\Phi, \Psi, \mathcal{H}, \varphi)$ is a complete C^*-algebra-valued bipolar b-metric space. Define $T : \Phi \cup \Psi \to \Phi \cup \Psi$ by (22) above. Then, let $x = \sigma \in \Phi$. By definition, this implies $\sigma : \mathcal{H}_1 \to \mathcal{H}_+$. Therefore,

$$T(\sigma)(t) = q(t) + \lambda \int_{\langle_1 \cup \mathcal{H}_2} G(t, s, \sigma(s))\, ds, \quad t \in \mathcal{H}_2.$$

Since $\sigma(s) \in \Phi$, the kernel $G(t, s, \sigma(s))$ alternates the domain of evaluation from $\mathcal{H}_1$ into $\mathcal{H}_2$, ensuring $T(\sigma)(t) \in \Psi$, for all $t \in \mathcal{H}_2$. Thus $T(x) \in \Phi$ when $x \in \Phi$.

Let $x = \nu \in \Psi$. By definition, this implies $\nu : \mathcal{H}_2 \to \mathcal{H}_+$. Therefore,

$$T(\nu)(t) = q(t) + \lambda \int_{\mathcal{H}_1 \cup \mathcal{H}_2} G(t, s, \nu(s))\, ds, \quad t \in \mathcal{H}_1.$$

Since $\nu(s) \in \Psi$, the kernel $G(t, s, \nu(s))$ alternates the domain of evaluation from H_2 into $\mathcal{H}_1$, ensuring: $T(\nu)(t) \in \Phi$, for all $t \in H_1$. Therefore, T is contravariant.

For any $\sigma \in L^{\infty}(\mathcal{H}_1), \nu \in L^{\infty}(\mathcal{H}_2)$,

$$\varphi(T\nu, T\sigma) = \sup_{t \in \mathcal{H}_1 \cup \mathcal{H}_2} \|T(\nu)(t) - T(\sigma)(t)\|_{\mathcal{H}}$$

$$= \sup_{t \in \mathcal{H}_1 \cup \mathcal{H}_2} \left\| \lambda \int_{\mathcal{H}_1 \cup \mathcal{H}_2} \left(G(t, s, \nu(s)) - G(t, s, \sigma(s)) \right) ds \right\|_{\mathcal{H}}$$

$$\leq \sup_{t \in \mathcal{H}_1 \cup \mathcal{H}_2} \lambda \cdot \frac{1}{2} \int_{\mathcal{H}_1 \cup \mathcal{H}_2} \|\theta(t,s)\| \|\nu(s) - \sigma(s)\|_{\mathcal{H}}\, ds$$

$$\leq \frac{\lambda}{2} \varphi(\nu, \sigma),$$

therefore, $\varphi(T\nu, T\sigma) \leq \frac{\lambda}{2} \varphi(\nu, \sigma)$ and $\|F_{\mathcal{H}}\varphi(T\nu, T\sigma)\| \leq \frac{\lambda}{2} \|\varphi(G_{\mathcal{H}}\nu, \sigma)\|$, where $0 < \lambda < 2$. Since $\frac{\lambda}{2} < 1$, the mapping T is a contravariant contraction. Now, all the conditions of Theorem 3.4 are satisfied. Hence, the integral equation has a unique solution. $\square$

7. Conclusions

In this paper, we established the existence and uniqueness of fixed point results under the C^*—algebra- valued bipolar b-metric spaces, where the underlying cone is non-solid. To achieve this, we used $(F_{\mathcal{H}} - G_{\mathcal{H}})$-contractions, which are positive and monotone functions. The obtained results were used to extend the Ulam - Hyers' stability problem and Fredholm integral equations. Some

examples were given to demonstrate our research results. It will be an open problem to generalize our outcomes to other types of new contractive conditions in this context.

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