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Oscillatory Analysis of Third-Order Hybrid Trinomial Delay Differential Equations Via Binomial Transform

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Abstract: This paper explores the oscillatory behavior of a class of third-order hybrid-type delay differential equations. A novel approach involves transforming these complex trinomial equations into a simpler binomial form by utilizing solutions from associated linear differential equations. Through the application of comparison techniques and integral averaging methods, new criteria are established that guarantee all solutions exhibit oscillatory behavior. These findings expand and complement existing theories in the oscillation analysis of functional differential equations. An illustrative example is provided to demonstrate the significance and originality of the results.

Keywords: third-order differential equation; positive and negative terms; neutral; trinomial equation; oscillation

MSC: 34C10; 34K11

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1. Introduction

This paper is concerned with the third-order functional differential equation of the form

$$\left(\mu_2(t)\left(\mu_1(t)\eta'(t)\right)'\right)' - \xi_1(t)\eta(t) + \xi_2(t)\eta^{\alpha}(\tau(t)) = 0, t \geqslant t_0 \tag{1}$$

and assume the following conditions to hold:

(C1) $\mu_2(t), \mu_1(t), \xi_1(t), \xi_2(t) \in C([t_0,\infty)), \mu_2(t) > 0, \mu_1(t) > 0, \xi_1(t) > 0$ and $\xi_2(t) > 0$,
(C2) $\tau(t) \in c'([t_0,\infty)), \tau'(t) > 0, \tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty$,
(C3) α is a ratio of the old positive integers,
(C4) equation (1) is in canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{\mu_2(t)} dt = \int_{t_0}^{\infty} \frac{1}{\mu_1(t)} dt = \infty. \tag{2}$$

With the given initial point $t_0 > 0$, set $t_{-1} = \inf_{t\geqslant t_0} \tau(t)$. By a solution of (1) , we mean a function $\eta(t) \in C([t_{-1},\infty),\mathbb{R})$ which has the property $\mu_1\eta', \mu_2(\mu_1\eta')' \in C'([t_0,\infty),\mathbb{R})$ that satisfies (1) for $t \geqslant t_0$ and satisfies (1) for $t \geq t_0$. Our attention is restricted to those solutions $\eta(t)$ of (1) , which exist on some half-line $[t_0,\infty)$ and satisfy $\sup\{|\eta(t)| : t \geqslant T\} > 0$ for all $T \geqslant t_0$. We tacitly assume that (1) does possess such solutions.

The oscillatory nature of the solution is understood in the usual way, that is, a non trivial solution is called oscillatory or nonoscillatory if it does or does not have infinitely many zeros.

Letting either $\xi_1(t) = 0$ or $\xi_2(t) = 0$, equation (1) reduces to simpler binomial differential equations with or without delay of the form

$$\left(\mu_2(t)\left(\mu_1(t)\eta'(t)\right)'\right)' + \xi_2(t)\eta^{\alpha}(\tau(t)) = 0 \tag{3}$$

and

$$\left(\mu_2(t)\left(\mu_1(t)\eta'(t)\right)'\right)' - \xi_1(t)\eta(t) = 0. \tag{4}$$

So, in a sense, one may call (1) a hybrid type third-order differential equation.

Oscillatory and asymptotic properties of both equations (3) and (4) have been studied by many authors, see, the papers [1–10], the monograph [11] and the references contained therein. This is due to the fact that they have many applications in natural sciences and engineering, see, for instance, the papers [12,13] for models from mathematical biology where the oscillation and/or delay actions may be formulated by means of cross-diffusion terms. By the well known result of Kiguradze [14] (Lemma 1), one can easily classify the possible nonoscillatory solutions of (3) and (4) that are completely different. If we set by S the set of all non-oscillatory solutions of studied equations, then for (3) the set S has the following decomposition

$$S = S_0 \cup S_2,$$

where positive solution

$\eta(t) \in S_0 \Leftrightarrow \mu_1(t)\eta'(t) < 0, \mu_2(t)(\mu_1(t)\eta'(t))' > 0, \left(\mu_2(t)(\mu_1(t)\eta'(t))'\right)' < 0,$

$\eta(t) \in S_2 \Leftrightarrow \mu_1(t)\eta'(t) > 0, \mu_2(t)(\mu_1(t)\eta'(t)) > 0, \left(\mu_2(t)(\mu_1(t)\eta'(t))'\right)' < 0.$

On the other hand, for (4) the set S has the following reduction

$$S = S_1 \cup S_3,$$

with positive solution

$\eta(t) \in S_1 \Leftrightarrow \mu_1(t)\eta'(t) > 0, \mu_2(t)(\mu_1(t)\eta'(t))' < 0, \left(\mu_2(t)(\mu_1(t)\eta'(t))'\right)' > 0,$

$\eta(t) \in S_3 \Leftrightarrow \mu_1(t)\eta'(t) > 0, \mu_2(t)(\mu_1(t)\eta'(t))' > 0, \left(\mu_2(t)(\mu_1(t)\eta'(t))'\right)' > 0.$

Hence, from the above discussion the nonoscillatory solutions space of (1) with positive and negative part is not clear.

Recently in [15–18], the authors considered the equation relating to (1) of the form

$$\left(\mu_2(t)\left(\mu_1(t)\eta'(t)\right)'\right)' - \xi_1(t)f(\eta(\sigma(t))) + \xi_2(t)h(\eta(\tau(t))) = 0 \tag{5}$$

and studied the oscillatory and asymptotic behavior of solutions of (5) by assuming either f is bounded or h is bounded with

$$\int_{t_0}^{\infty}\frac{1}{\mu_1(t)}\int_t^{\infty}\frac{1}{\mu_2(s)}\int_s^{\infty}\xi_1(s_1)ds_1dsdt < \infty,$$

or

$$\int_{t_0}^{\infty}\frac{1}{\mu_1(t)}\int_t^{\infty}\frac{1}{\mu_2(s)}\int_s^{\infty}\xi_2(s_1)ds_1dsdt < \infty.$$

Another method frequently used in the oscillation theory of trinomial differential equations is to omit one term (see, [19–23]) and this method yield the following differential inequalities for (1)

$$\left\{\left(\mu_2(t)\left(\mu_1(t)\eta'(t)\right)'\right)' + \xi_2(t)\eta^{\alpha}(\tau(t))\right\}\operatorname{sign}\eta(t) \geqslant 0$$

and

$$\left\{\left(\mu_2(t)\left(\mu_1(t)\eta'(t)\right)'\right)' - \xi_1(t)\eta(t)\right\}\operatorname{sign}\eta(t) \leqslant 0.$$

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which are opposite to those that we need. Hence there is only a limited number of papers dealing equation (1) with positive and negative parts.

Therefore, in this paper we use a novel method that overcomes those difficulties appearing due to positive and negative parts of (1). An example is given to illustrate the importance and significance of our main results.

2. Auxiliary Results

The main results are established via series of lemmas, which relate properties of solutions of (1) to those of solutions of auxiliary differential equations

$$\Big(\mu_2(t)(\mu_1(t)y'(t))'\Big)' - \xi_1(t)y(t) = 0 \tag{6}$$

and

$$\Big(\frac{\mu_2(t)}{y(t)}V'(t)\Big)' + \frac{\mu_2(t)(\mu_1(t)y'(t))'}{\mu_1(t)y^2(t)}V(t) = 0. \tag{7}$$

We begin with our first result based on an equivalent representation for the linear differential operator

$$\mathcal{L}_\eta(t) = \Big(\mu_2(t)\big(\mu_1(t)\eta'(t)\big)'\Big)' - \xi_1(t)\eta(t) \tag{8}$$

in terms of positive solutions $y(t)$ and $V(t)$, respectively of (6) and (7).

Lemma 1. *Let $y(t)$ be a positive solution of (6). Then the operator (8) can be written as*

$$\mathcal{L}_\eta(t) = \left(\frac{\mu_2(t)}{y(t)}\Big(\mu_1(t)y^2(t)\Big(\frac{\eta(t)}{y(t)}\Big)'\Big)'\right)' + \mu_2(t)\big(\mu_1(t)y'(t)\big)'\Big(\frac{\eta(t)}{y(t)}\Big)'. \tag{9}$$

Proof. Direct calculation shows that the right hand side of (9) equals

$$\begin{aligned}
&\Big(\frac{\mu_2(t)}{y(t)}\big(\mu_1(t)y(t)\eta'(t) - \mu_1(t)y'(t)\eta(t)\big)'\Big)' + \mu_2(t)\big(\mu_1(t)y'(t)\big)'\Big(\frac{\eta(t)}{y(t)}\Big)' \\
&\qquad = (\mu_2(t)(\mu_1(t)\eta'(t))')' - (\mu_2(t)(\mu_1(t)y'(t))')'\frac{\eta(t)}{y(t)} \\
&\qquad = (\mu_2(t)(\mu_1(t)\eta'(t))')' - \xi_1(t)\eta(t).
\end{aligned}$$

The proof of the lemma is complete. □

Lemma 2. *Let $y(t)$ be a positive solution of (6) and let the equation*

$$\Big(\frac{\mu_2(t)}{y(t)}V'(t)\Big)' + \frac{\mu_2(t)(\mu_1(t)y'(t))'}{\mu_1(t)y^2(t)}V(t) = 0 \tag{10}$$

possesses a positive solution. Then the operator (8) can be written as

$$\mathcal{L}_\eta(t) = \frac{1}{V(t)}\left[\frac{\mu_2(t)V^2(t)}{y(t)}\left(\frac{\mu_1(t)y^2(t)}{V(t)}\left(\frac{\eta(t)}{y(t)}\right)'\right)'\right]'. \tag{11}$$

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Proof. By a direct computation, we see that the right-hand side of (11) equals

$$\frac{1}{V(t)}\left[\frac{\mu_2(t)}{y(t)}V(t)\Big(\mu_1(t)y^2(t)(\eta/y)'\Big)' - \frac{\mu_2(t)}{y(t)}V'(t)\left(\mu_1(t)y^2(t)\left(\frac{\eta(t)}{y(t)}\right)'\right)\right]'$$
$$= \left(\frac{\mu_2(t)}{y(t)}\left(\mu_1(t)y^2(t)\left(\frac{\eta(t)}{y(t)}\right)'\right)'\right)' - \frac{\left(\mu_1(t)y^2(t)\left(\frac{\eta(t)}{y(t)}\right)\right)}{V(t)}\left(\frac{\mu_2(t)V'(t)}{y(t)}\right)'. \tag{12}$$

Using (9) in (12) yields

$$= \mathcal{L}_\eta(t) - \mu_2(t)(\mu_1(t)y'(t))'\left(\frac{\eta(t)}{y(t)}\right) - \frac{1}{V(t)}\left(\mu_1(t)y^2(t)\left(\frac{\eta(t)}{y(t)}\right)\right)\left(\frac{\mu_2(t)V'(t)}{y(t)}\right)'$$
$$= \mathcal{L}_\eta(t) - \frac{\mu_1(t)y^2(t)}{V(t)}\left(\frac{\eta(t)}{y(t)}\right)\left[\left(\frac{\mu_2(t)V'(t)}{y(t)}\right)' + \frac{\mu_2(t)(\mu_1(t)y'(t))'}{\mu_1(t)y^2(t)}V(t)\right]$$
$$= \mathcal{L}_\eta(t),$$

since $V(t)$ is a solution of (10). The proof of the lemma is complete. □

From Lemmas 1 and 2, the equation (1) can be rewritten in a binomial form

$$(\beta_2(t)(\beta_1(t)\omega'(t))')' + \Omega(t)\omega^\alpha(\tau(t)) = 0, \tag{13}$$

where

$$\beta_2(t) = \frac{\mu_2(t)V^2(t)}{y(t)}, \quad \beta_1(t) = \frac{\mu_1(t)y^2(t)}{V(t)},$$
$$\Omega(t) = V(t)\xi_2(t)y^\alpha(\tau(t)), \quad \omega(t) = \frac{\eta(t)}{y(t)}.$$

Following Trench [24], we say that (13) is in canonical form, if

$$\int_{t_0}^{\infty}\frac{1}{\beta_2(t)}dt = \int_{t_0}^{\infty}\frac{y(t)}{\mu_2(t)V^2(t)}dt = \infty \tag{14}$$

and

$$\int_{t_0}^{\infty}\frac{1}{\beta_1(t)}dt = \int_{t_0}^{\infty}\frac{V(t)}{\mu_1(t)y^2(t)}dt = \infty. \tag{15}$$

For convenience, it is of important to find conditions that ensure the existence of positive solutions of (6) and (7) such that conditions (14) and (15) are fulfilled so that (13) is in a canonical form.

Now, from the familiar Kiguradze lemma [14], the set S of all possible nonoscillatory, let us say positive solutions of (6) has the following decomposition

$$S = S_1 \cup S_3,$$

where

$$y(t) \in S_1 \Leftrightarrow y(t) > 0, y'(t) > 0, (\mu_1(t)y'(t))' < 0, \Big(\mu_2(t)(\mu_1(t)y'(t))'\Big)' > 0,$$
$$y(t) \in S_3 \Leftrightarrow y(t) > 0, y'(t) > 0, (\mu_1(t)y'(t))' > 0, \Big(\mu_2(t)(\mu_1(t)y'(t))'\Big)' > 0.$$

Lemma 3. *Assume that*

$$\limsup_{t\to\infty} \pi_1(t)\pi_{21}(t)\mu_1(t)\xi_1(t) < \frac{2}{3\sqrt{3}} \tag{16}$$

where

$$\pi_1(t) = \int_{t_0}^{t}\frac{1}{\mu(s)}ds, \quad \pi_{21}(t) = \int_{t_0}^{t}\frac{\pi_1(s)}{\mu_2(s)}ds.$$

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Then all solutions of (6) are nonoscillatory and moreover equation (6) has a couple of linearly independent solutions belong to S_1 and S_3.

Proof. The result follows from Lemma 2 of [26] and so the details are omitted. □

To obtain our main results, it is convenient to work with $y(t) \in S_1$ and so we always assume in the sequel that (16) holds. It is known (see,[14–24]) that if (6) has a solution $y(t) \in S_1$, then the corresponding second-order differential equation (7) always possesses a couple of positive solutions

$$V(t) \in S_0 \Leftrightarrow V(t) > 0, V'(t) < 0, \left(\frac{\mu_2(t)V'(t)}{y(t)}\right)' > 0,$$
$$V(t) \in S_2 \Leftrightarrow V(t) > 0, V'(t) > 0, \left(\frac{\mu_2(t)V'(t)}{y(t)}\right)' > 0.$$

For our purposes, exactly $V(t) \in S_0$ will be suited and we say that such solution $V(t)$ is associated to $y(t)$.

Lemma 4. *Let $y(t) \in S_1$ be a positive solution of (6) and let $V(t)$ be associated to $y(t)$. Then (14) and (15) are satisfied.*

Proof. The condition (14) immediately follows from the monotonicity properties of $y(t)$ and $V(t)$ along with (2). Further, by Lemma 4 of [25], we can show that condition (15) holds. This ends the proof. □

Remark 1. *If $y_1(t)$ and $y_2(t)$ are a couple of increasing solutions of (6), then to get the canonical form of (13), we consider the solution $y_1(t)$ such that*

$$\lim_{t\to\infty} \frac{y_1(t)}{y_2(t)} = 0. \tag{17}$$

Definition 1. *Following Hartman [25], we say the solution $y_1(t) \in S_1$ of (6) satisfying (17) is a principal solution of (6).*

Combining the results in Lemma 3, 4 and Remark 1, we obtain the following corollary.

Corollary 1. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Then, (1) has an equivalent of (13) which is in canonical form.*

3. Oscillation Results

In this section, we study the oscillation properties of (1)with the aid of (13). Here after, without loss of generality, we may consider only positive solutions of (1). In view of familiar Kiguradze's lemma [14], we have the structure of the nonoscillatory solution of (13).

Lemma 5. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). If $\eta(t)$ is an eventually positive solution of (1), then the corresponding function $\omega(t)$ satisfies either*

$$w(t) \in \bar{S}_0 \Leftrightarrow w'(t) < 0, (\beta_1(t)w'(t))' > 0, \left(\beta_2(t)(\beta_1(t)w'(t))'\right)' < 0,$$

or

$$\omega(t) \in \bar{S}_2 \Leftrightarrow \omega'(t) > 0, \quad (\beta_1(t)\omega'(t))' > 0, \quad \left(\beta_2(t)(\beta_1(t)\omega'(t))'\right)' < 0.$$

Consequently, the set $\bar{S}$ of all positive solutions of (13) (as well as (1)) has the following decomposition

$$\bar{S} = \bar{S}_0 \cup \bar{S}_2.$$

Now, we are prepared to present a criterion for the class $\overline{S}_2$ is empty.
Let us define

$$B_1(t) = \int_{t_1}^{t} \frac{1}{\beta_1(s)} ds, \; B_2(t) = \int_{t_1}^{t} \frac{1}{\beta_2(s)} ds, \; B_{12}(t) = \int_{t_1}^{t} \frac{B_2(s)}{\beta_1(s)} ds, \; \Omega_1(t) = \Omega(t) B_{12}^{a}(\tau(t)),$$

where $t_1 \geq t_0$ sufficiently large.

Theorem 1. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). If the first-order nonlinear delay differential equation*

$$z'(t) + \Omega_1(t) z^{\alpha}(\tau(t)) = 0 \tag{18}$$

is oscillatory, then the class $\overline{S}_2$ is empty.

Proof. Assume the contrary that $\omega(t)$ is a positive solution of equation (13) that belongs to the class $\overline{S}_2$ for all $t \geqslant t_1 \geqslant t_0$. Setting $z(t) = \beta_2(t)(\beta_1(t)\omega'(t))' > 0$ is decreasing, we have

$$\begin{aligned} \beta_1(t)\omega'(t) &\geqslant \int_{t_1}^{t} \frac{\beta_2(s)(\beta_1(s)\omega'(s))'}{\beta_2(s)} ds \\ &\geqslant B_2(t) z(t). \end{aligned}$$

Integrating from t_1 to t, we are led to

$$w(t) \geqslant \int_{t_1}^{t} \frac{B_2(s)}{B_1(s)} z(s) ds \geqslant B_{12}(t) z(t).$$

Hence,

$$\omega^{\alpha}(\tau(t)) \geqslant B_{12}^{\alpha}(\tau(t)) z^{\alpha}(\tau(t))$$

and using the last inequality in (13), we obtain

$$-z'(t) \geqslant \Omega_1(t) z^{\alpha}(\tau(t)).$$

Therefore, it is clear that $z(t)$ is a positive solution of the differential inequality

$$z'(t) + \Omega_1(t) z^{\alpha}(\tau(t)) \leq 0.$$

But, by Theorem 1 in [26] the corresponding differential equation (18) also has a positive solution, which is a contradiction. The proof of the theorem is complete. □

In the following, we present explicit criteria for the class $\overline{S}_2$ to be empty.

Corollary 2. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). If $\alpha = 1$ and*

$$\liminf_{t \to \infty} \int_{\tau(t)}^{t} \Omega_1(s) ds > \frac{1}{e}, \tag{19}$$

then the class $\overline{S}_2$ is empty.

Corollary 3. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). If $0 < \alpha < 1$ and*

$$\int_{t_0}^{\infty} \Omega_1(t) dt = \infty, \tag{20}$$

then the class $\overline{S}_2$ is empty.

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Corollary 4. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution 0f (7). Suppose $\alpha > 1$ and $\tau(t) = \theta t, \theta \in (0,1)$. If there exists $\lambda > \ln(\alpha)/\ln(\theta)$, such that*

$$\lim_{t\to\infty} \inf\left[\Omega_1(t)\exp\left(-t^{\lambda}\right)\right] > 0 \tag{21}$$

holds, then the class $\overline{S}_2$ is empty.

Corollary 5. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Suppose $\alpha > 1$ and $\tau(t) = t^{\theta}, \theta \in (0,1)$. If there exists $\lambda > \ln(\alpha)/\ln(\theta)$ such that*

$$\lim_{t\to\infty} \inf\left[\Omega_1(t)\exp\left(-\ln^{\lambda}(t)\right)\right] > 0 \tag{22}$$

holds, then the class $\overline{S}_2$ is Empty.

The proof of the Corollaries 10 - 13 follows from oscillation of equation (18) for $\alpha = 1$, see, [28], $\alpha \in (0,1)$, see, [29] and for $\alpha > 1$, see, [30], respectively.

Next, we obtain conditions for the class $\overline{S}_0$ to be empty. Define

$$\Omega_2(t) = \frac{1}{\beta_1(t)}\int_t^{\sigma(t)}\frac{1}{\beta_2(s)}\int_s^{\sigma(s)}\Omega(s_1)ds_1 ds.$$

Theorem 2. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Assume that there exists a function $\sigma(t) \in C'([t_0,\infty),\mathbb{R})$ such that*

$$\sigma'(t) \geqslant 0, \sigma(t) > t, \delta(t) = \tau(\sigma(\sigma(t))) < t. \tag{23}$$

If the first-order delay differential equation

$$\chi'(t) + \Omega_2(t)\chi^{\alpha}(\delta(t)) = 0 \tag{24}$$

is oscillatory, then the class $\overline{S}_0$ is empty.

Proof. Assume the contrary that $\omega(t)$ is an eventually positive solution of (13) belongs to the class $\overline{S}_0$ for all $t \geqslant t_1$. Integrating (13) from t to $\sigma(t)$, we have

$$\begin{aligned}\beta_2(t)(\beta_1(t)\omega'(t))' &\geqslant \int_t^{\sigma(t)}\Omega(s)\omega^{\alpha}(\tau(s))ds \\ &\geqslant \omega^{\alpha}(\tau(\sigma(t)))\int_t^{\sigma(t)}\Omega(s)ds.\end{aligned}$$

Dividing the last inequality by $\beta_2(t)$ and then integrate from t to $\sigma(t)$, we get

$$-\beta_1(t)\omega'(t) \geqslant \omega^{\alpha}(\delta(t))\int_t^{\sigma(t)}\frac{1}{\beta_2(s)}\int_s^{\sigma(s)}\Omega(s_1)ds_1 ds.$$

Finally integrating from t to ∞, we get

$$\omega(t) \geqslant \int_t^{\infty}\frac{\omega^{\alpha}(\delta(s))}{\beta_1(s)}\int_s^{\sigma(s)}\frac{1}{\beta_2(s_1)}\int_{s_1}^{\sigma(s_1)}\Omega(s_2)ds_2 ds_1 ds.$$

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Let us denote the right hand side of the above inequality by $\chi(t)$. Then $\omega(t) \geqslant \chi(t) > 0$ and it is easy to find that

$$0 = \chi'(t) + \left(\frac{1}{\beta_1(t)}\int_t^{\sigma(t)}\frac{1}{\beta_2(s)}\int_s^{\sigma(s)}\Omega(s_1)ds_1ds\right)\omega^a(\delta(t))$$
$$\geqslant \dot{\chi}'(t) + \Omega_2(t)x^\alpha(\delta(t)).$$

Consequently, Theorem 1 of [26] implies that the corresponding differential equation (24) has also a positive solution $\chi(t)$, which contradicts to our assumption. Hence, we conclude that $\overline{S}_0$ is empty. The proof of the theorem is complete. □

Corollary 6. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Assume that there exists a function $\sigma(t) \in C'([t_0,\infty),\mathbb{R})$ such that (23) holds. If $\alpha = 1$ and*

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}\Omega_2(s)ds > \frac{1}{e}, \tag{25}$$

then the class $\overline{S}_0$ is empty.

Corollary 7. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Assume that there exists a function $\sigma(t) \in C'([t_0,\infty),\mathbb{R})$ such that (23) holds. If $\alpha \in (0,1)$ and*

$$\int_{t_0}^{\infty}\Omega_2(t)dt = \infty, \tag{26}$$

then the class $\overline{S}_0$ is empty.

Corollary 8. *Let (16) hold, $y(t) \in S_1$, be a principal solution of (6) and $V(t)$ its associated solution of (7). Assume that there exists a function $\sigma(t) \in C'([t_0,\infty),\mathbb{R})$ such that (23) holds. If $\alpha > 1, \delta(t) = \theta t, \theta \in (0,1)$ and there exists $\lambda > \frac{\ln(\alpha)}{\ln(\theta)}$ such that*

$$\liminf_{t\to\infty}\left[\Omega_2(t)\exp\left(-t^\lambda\right)\right] > 0 \tag{27}$$

holds, then the class $\overline{S}_0$ is empty.

Corollary 9. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Assume that there exists a function $\sigma(t) \in C'([t_0,\infty),\mathbb{R})$ such that (23) holds. of $\alpha > 1, \delta(t) = t^\theta, \theta \in (0,1)$ and there exists $\lambda > \frac{\ln(\alpha)}{\ln(\theta)}$ such that*

$$\liminf_{t\to\infty}\left[\Omega_2(t)\exp\left(-(\ln(t))^\lambda\right)\right] > 0 \tag{28}$$

holds, then the class $\overline{S}_0$ is empty.

The sufficient conditions for the oscillation of (24) for $\alpha = 1$, $0 < \alpha < 1$ and $\alpha > 1$ in previous Corollaries can be recalled from [28], [29] and [30] respectively.

Combining the criteria obtained for the classes $\overline{S}_0$ and $\overline{S}_2$ to be empty, we are able to present the following criteria for the oscillation of equation (1).

Theorem 3. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Assume that there exists a function $\sigma(t) \in C'([t_0,\infty),\mathbb{R})$ such that (23) holds. Let $\alpha = 1(\alpha < 1)$ holds. If (19) ((20)) and (25) ((26)) hold, then the equation (1) is oscillatory.*

Proof. Let $\eta(t)$ be an eventually positive solution of (1) such that $\eta(\tau(t)) > 0$ for all $t \geqslant t_1$, for some $t_1 \geqslant t_0$. Then by Corollary 1, the corresponding function $\omega(t) = \frac{\eta(t)}{y(t)}$ is also a positive solution of

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(13) and by Lemma 1, $\omega(t) \in \overline{S}_0$ or $\omega(t) \in \overline{S}_2$ for all $t \geqslant t_1$. In view of Corollary 2 (Corollary 3) we conclude that the class $\overline{S}_2$ is empty and by Corollary 6 (Corollary 7) we can see that the class $\overline{S}_0$ is empty. Therefore, by oscillation preserving transformation, $\eta(t) = \omega(t)y(t)$, we conclude that equation (1) is oscillatory. The proof of the theorem is complete. □

Theorem 4. *Let (16) hold, $y(t) \in S_1$ be a principal solution of (6) and $V(t)$ its associated solution of (7). Assume that there exists a function $\sigma(t) \in C'([t_0, \infty), \mathbb{R})$ such that (23) holds. Suppose $\alpha > 1, \tau(t) = \theta_1 t(t^{\theta_1}), \delta(t) = \theta_2 t(t^{\theta_2})$ where $\theta_1, \theta_2 \in (0,1)$. If there exists $\lambda > \frac{\ln(\alpha)}{\ln(\theta_1)}$ $\left(\lambda > \frac{\ln(\alpha)}{\ln(\theta_2)}\right)$ such that (21)((22)) and (27) ((28)) hold, then equation (1) is oscillatory.*

Proof. The proof is similar to Theorem 3 and so the details are omitted. □

We conclude this section with an example whose oscillatory character cannot be determined by any known results in [15–23].

4. Example

Consider the hybrid third-order delay differential equation

$$\eta'''(t) - \frac{36}{125}\frac{1}{t^3}\eta(t) + \frac{b}{t^3}\eta(\lambda t) = 0, t \geqslant 1, \tag{29}$$

where $b > 0$ and $\lambda \in (0,1)$.

For the equation (29), the auxiliary equation (6) takes the form

$$y'''(t) - \frac{36}{125t^3}y(t) = 0,$$

with a couple of positive solutions $y_1(t) = t^{1/5}$ and $y_2(t) = t^{(7-\sqrt{13})/5}$ belong to S_1. By Remark 1, we consider $y(t) = t^{\frac{1}{5}}$ for which the equation (7) is reduced to

$$\left(t^{-1/5}V'(t)\right)' - \frac{4}{25}t^{-11/5}V(t) = 0$$

and possesses a positive solution $V(t) = t^{(3-\sqrt{13})/5}$ associates to $y(t)$. Further calculations show that

$$\beta_1(t) = t^{(\sqrt{13}-1)/5} \text{ and } \beta_2(t) = t^{(5-2\sqrt{13})/5}$$

and hence the conditions (14) and (15) hold. The condition (16) obviously satisfied. Furthermore, we see that

$$B_{12}(t) = \frac{25}{2\sqrt{13}}t^{\frac{6+\sqrt{13}}{5}}$$

$$\Omega_1(t) = \frac{25b\lambda^{(7+\sqrt{13})/5}}{2\sqrt{13}}\frac{1}{t}.$$

The condition (19) becomes

$$\frac{25b\lambda^{(7+\sqrt{13})/5}\ln\frac{1}{\lambda}}{2\sqrt{13}} > \frac{1}{e},$$

that is, the class $\overline{S}_2$ is empty if

$$b > \frac{2\sqrt{13}}{25\lambda^{(7+\sqrt{13})/5}e\ln 1/\lambda}. \tag{30}$$

Set $\sigma(t) = \lambda_1 t, \lambda_1 > 1$, such that $\lambda_1 < \frac{1}{\sqrt{\lambda}}$ so that condition (23) holds. Also

$$\Omega_2(t) = 25b\lambda^{\frac{1}{5}} \frac{\left(1 - \lambda_1^{-(6+\sqrt{13})/5}\right)\left(1 - \lambda_1^{-(6-\sqrt{13})/5}\right)}{23} \frac{1}{t}.$$

The condition (25) becomes

$$25b\lambda^{\frac{1}{5}} \frac{\left(1 - \lambda_1^{-(6+\sqrt{13})/5}\right)\left(1 - \lambda_1^{-(6-\sqrt{13})/5}\right)}{23} \ln \frac{1}{\lambda_1^2 \lambda} > 1/e,$$

that is, the class $\overline{S}_0$ is empty if

$$b > \frac{23}{25\lambda^{1/5}\left(1 - \lambda_1^{-(6+\sqrt{13})/5}\right)\left(1 - \lambda_1^{-(6-\sqrt{13})/5}\right) e \ln \frac{1}{\lambda\lambda_1^2}}. \tag{31}$$

Hence, by Theorem 3, the equation (29) is oscillatory if b satisfies the conditions (30) and (31) simultaneously.

In particular, if we take $\lambda = \frac{1}{5}$ and $\lambda_1 = 2$, we see that b> 2.0241 and b>10.0063. Therefore, equation (29) is oscillatory if $b > 10.0063$.

5. Conclusion

In this paper, we studied the oscillatory properties of equation (1). This is achieved by transforming the studied trinomial equation into a binomial form using the positive solutions of the auxiliary equations. By comparison and integral averaging techniques we are able to obtain new oscillation criteria for the equation (1). Hence the oscillation criteria derived in this paper are new and significant contribution to the oscillation theory of third- order delay differential equations.

Further, it is an interesting problem to obtain oscillation criteria for the studied equation (1) without using the explicit solutions of the related auxiliary differential equations.

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