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# Holographic Scrambling as a Residual Law: A DSFL View on Susskind's Black Hole Paradigm

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**Abstract**

We recast Leonard Susskind's black hole paradigm in terms of a single Lyapunov functional on a calibrated Hilbert space. In the Deterministic Statistical Feedback Law (DSFL) framework, one works with a fixed instrument norm and a quadratic "residual of sameness" measuring the mismatch between blueprint data and physical responses. Admissible dynamics are exactly those that contract this residual and admit an intrinsic DSFL clock parametrising the decay. Specialising to a black hole room that factors into interior, near–horizon collar and far radiation, we identify three sectoral residuals in the *same* norm: a scrambling residual, an inside/outside balance residual, and a gravitational residual built from Einstein imbalance. We show that fast scrambling and the Maldacena–Shenker–Stanford chaos bound can be phrased as rate constraints on the scrambling residual; that a DSFL Page time defined by balancing inside and outside budgets coincides, in Haar and random–circuit evaporator models, with the usual entropic Page time; and that the gravitational residual admits a Lyapunov law with quasinormal– mode rate and a DSFL null expansion encoding linearised QNEC/QFC–type focusing. Appendices develop the general DSFL machinery, prove finite–dimensional envelope theorems, and collect numerical audits in simple quantum and near–horizon toy models, as well as optional connections to circuit complexity and quantum error correction that support the single–residual black–hole picture.

**Keywords:** black holes; holographic scrambling; chaos bound; page time; quantum focusing; quantum information; deterministic statistical feedback law

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## 1. Introduction

Black holes occupy a special place in the interface between quantum information and gravity. In Susskind's paradigm, large black holes are fast scramblers, saturate the Maldacena–Shenker–Stanford (MSS) chaos bound, generate Page curves when they evaporate, and exhibit interior growth tied to quantum complexity [1–8]. At the same time, semiclassical gravity suggests that horizon area and quantum fields obey quantum energy and focusing conditions (QNEC/QFC), constraining the second variation of generalised entropy along null congruences [9–16]. These ingredients are usually treated in different mathematical languages.

In this paper we ask whether these black–hole signatures can be organised by a *single* geometric law in one Hilbert room, with one fixed norm and one observable that plays the role of a Lyapunov functional. Concretely:

- can fast scrambling and the MSS chaos bound be expressed as a contraction rate for one residual in a thermal near–horizon collar [3–5]?
- can Page time be read off from a balance of inside/outside budgets of that same residual, in the spirit of Page's random–state analysis and Hayden–Preskill decoupling [1,8,17–19]?
- can (linearised) QNEC/QFC–type focusing be phrased in the same norm, using a gravitational residual built from Einstein imbalance [9,10,12–16]?

The *Deterministic Statistical Feedback Law* (DSFL) is designed to do exactly this. It starts from one Hilbert space $\mathcal{H}$ with a blueprint subspace $\mathcal{H}_s$, a physical subspace $\mathcal{H}_p$, a calibration map $\mathbb{I} : \mathcal{H}_s \to \mathcal{H}_p$, and an instrument weight $W \succ 0$ defining the instrument norm $\|\cdot\|_W$. The *residual of sameness* between a blueprint $s$ and a realised response $p$ is

$$R(s,p) := \|p - \mathbb{I}s\|_W^2. \tag{1}$$

DSFL admissible evolutions are exactly those that intertwine with $\mathbb{I}$ and do not expand this residual; equivalently, they satisfy a single data–processing inequality for $R$, in the spirit of contractive metrics and relative entropy in classical and quantum information theory [20–25]. At the continuous level this gives a Lyapunov inequality

$$\dot{R}(t) \;\le\; -2\lambda(t)\, R(t) \tag{2}$$

and an intrinsic DSFL clock $\hat{\tau}(t) = \int_0^t 2\lambda(\sigma)\, d\sigma$ for which $\log R(\hat{\tau})$ has slope at most $-1$, as in standard semigroup theory and dissipative evolution [26–29].

Our main claim is that, when this structure is specialised to a black–hole room $\mathcal{H}_{\rm BH} \otimes \mathcal{H}_{\rm near} \otimes \mathcal{H}_{\rm far}$ equipped with a GR graph norm, the familiar black–hole signatures can be recast as properties of this single residual:

- a *scrambling residual* $R_{\rm scr}(t)$, measuring loss of local distinguishability of small subsystems, has a DSFL rate $\alpha_{\rm scr}$ that coincides with the OTOC Lyapunov exponent $\lambda_L$ in model scramblers such as random circuits and SYK–like systems [3–5,30,31]. The MSS chaos bound $\lambda_L \le 2\pi/\beta$ then becomes $\alpha_{\rm scr} \le 2\pi/\beta$ for DSFL admissible immediate loops in a thermal collar.
- *Inside/outside budgets* $R_{\rm in}(t)$ and $R_{\rm out}(t)$, defined by projecting the defect onto near–horizon BH and radiation subspaces in the *same norm*, cross at a DSFL Page time $t_{\rm Page}^{\rm DSFL}$. In Haar and random–circuit evaporator models this coincides, up to $O(1)$ in the number of emitted qubits, with the entropic Page time obtained from subsystem entropies [1,8,32,33].
- a *gravitational residual* $R_{\rm GR}(t)$ built from the Einstein tensor and renormalised stress tensor in a near–horizon collar obeys a DSFL Lyapunov law with rate set by quasinormal modes, capturing ringdown in the same instrument geometry [9,10,34,35]. In a linearised regime, QNEC/QFC–type focusing for the generalised entropy can be rephrased as inequalities for $R_{\rm GR}$ in that geometry [11–16].

This does *not* derive black–hole physics from DSFL. Rather, it shows that wherever Susskind's picture is realised (in holographic CFTs, SYK–type models, random circuits, or semiclassical near–horizon setups), the signatures of scrambling, chaos bounds, Page curves and focusing can be organised by one Lyapunov functional in one calibrated Hilbert room.

**Structure.** Section 2 reviews Susskind's paradigm: fast scrambling, the MSS chaos bound, the Page curve and Hayden–Preskill mirror behaviour, and complexity/interior growth. Section 3 recalls a compact DSFL framework: room, residual, admissible maps, Lyapunov inequality and cone locality. In Section 4 we build a DSFL room for black holes, combining a GR calibration with a scrambler on the BH factor. Section 5 contains the central "one residual" results: a DSFL Page law, a DSFL form of the MSS chaos bound, and a linearised focusing statement (DSFL versions of Page, MSS and QNEC/QFC). Section 6 illustrates these in finite–dimensional model worlds (Haar ensemble and random–circuit evaporators, FRW/BH toys). Section 8 discusses scope, limitations and possible extensions, including links to complexity growth and interior geometry.

More technical material (full DSFL proofs, detailed Haar/random–circuit Page theorems, GR calibration and QNEC/QFC details, and optional complexity/CIU material) is deferred to the appendices.

## 2. Susskind's Black Hole Paradigm

We briefly recall the main ingredients of Susskind's black–hole paradigm that we will later rewrite in DSFL language. Detailed reviews can be found in, e.g., [6,7] and references therein.

### 2.1. Fast Scramblers

Scrambling is the process by which initially local information becomes spread over many degrees of freedom so that it cannot be recovered by few–body observables. In a many–body system with entropy $S$ (Hilbert space dimension $d \sim e^S$), the scrambling time $t_{\rm scr}$ is the time scale on which generic local perturbations become indistinguishable from typical states when probed locally.

Sekino and Susskind proposed that large black holes are *fast scramblers*, with scrambling time

$$t_{\rm scr} \sim \frac{\beta}{2\pi} \log S, \tag{3}$$

where $\beta$ is the inverse Hawking temperature and $S$ is the Bekenstein–Hawking entropy [2]. This is parametrically shorter than diffusive $t \sim L^2$ scales and is conjectured to be essentially optimal given locality and causality near the horizon. Holographically, the black hole is dual to a strongly coupled large–$N$ quantum system, and fast scrambling corresponds to rapid delocalisation of information over $O(N^2)$ degrees of freedom.

### 2.2. Chaos and the MSS Bound

Quantum chaos in thermal systems is often diagnosed by out–of–time–order correlators (OTOCs) of the form

$$F(t) := \langle W^\dagger(t) V^\dagger W(t) V \rangle_\beta = \mathrm{Tr}\big(\rho_\beta W^\dagger(t) V^\dagger W(t) V\big), \tag{4}$$

with $\rho_\beta \propto e^{-\beta H}$ and $W, V$ initially local. The squared commutator $C(t) := -\langle [W(t), V]^2 \rangle_\beta$ measures growth of noncommutativity. In holographic and other strongly chaotic systems there is an early–time window

$$C(t) \sim \varepsilon e^{2\lambda_L t}, \qquad 1 \ll t \ll t_{\rm scr}, \tag{5}$$

where $\lambda_L$ is a Lyapunov (chaos) exponent [3,4].

Maldacena, Shenker and Stanford proved that, under general analyticity and causality assumptions, thermal systems obey the universal bound

$$\lambda_L \leq \frac{2\pi}{\beta}, \tag{6}$$

now known as the MSS chaos bound [5]. Einstein gravity duals (AdS black holes) saturate this bound: the chaos exponent $\lambda_L = 2\pi/\beta$ can be computed from shock–wave geometries near the horizon. In Susskind's picture, large black holes are therefore maximally chaotic.

### 2.3. Page Curve and Hayden–Preskill Mirror

The *Page curve* describes the entanglement entropy of Hawking radiation emitted by an evaporating black hole as a function of time. For a black hole initially in a pure state, Page modelled the joint state of black hole and radiation as a random pure state on $\mathcal{H}_{\rm BH} \otimes \mathcal{H}_{\rm rad}$ and computed the average entropy of the smaller subsystem [8]. The result is a "tent–shaped" entropy profile: as degrees of freedom are emitted, the entropy of the radiation rises linearly, reaches a maximum when the radiation and remaining black hole Hilbert spaces have comparable dimensions, and then falls as the radiation purifies the initial state.

Hayden and Preskill refined this picture by asking how quickly information dropped into an old black hole can be recovered from the radiation [1]. Modelling the interior dynamics as a random unitary, they showed that for an old, fast–scrambling black hole, information thrown in at time $t$ emerges in the radiation after only a scrambling time, and can be decoded from a number of Hawking quanta comparable to the size of the input. The black hole acts as an information mirror.

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### 2.4. Complexity and Interior Growth

More recently, Susskind and collaborators proposed that the growth of the interior (Einstein–Rosen bridge) of an AdS black hole is dual to the growth of quantum circuit complexity of the dual CFT state [6,36,37]. In these scenarios, standard thermodynamic quantities equilibrate quickly, but the interior volume or action — and the complexity of the boundary state — grows linearly for an exponentially long time (in entropy) before saturating at a value of order $e^S$. This suggests a new arrow of time beyond entropy: a "second law of complexity".

In this paper we will not develop complexity in detail; we only note that DSFL provides an intrinsic depth (its Lyapunov clock) that can bound complexity growth in scramblers. We focus instead on fast scrambling, the MSS bound, Page time and linearised focusing, and show how they fit into one residual law.

## 3. Compact DSFL Framework

We recall the sector–neutral DSFL structure in a compact form. Full technical details and proofs are given in Appendix A. The functional–analytic backbone is standard semigroup and Lyapunov theory on Hilbert spaces [26,27], while the data–processing flavour echoes the usual characterisations of contractive quantum channels and relative entropy DPIs [20–24].

### 3.1. Room, Calibration and Residual

**Definition 1** (DSFL room and residual of sameness)**.** *A* DSFL room *consists of:*

- *a Hilbert space* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$*;*
- *closed subspaces* $\mathcal{H}_s, \mathcal{H}_p \subset \mathcal{H}$ *of* statistical *(blueprint) and* physical *(response) degrees of freedom;*
- *a bounded linear* calibration map $\mathbb{I} : \mathcal{H}_s \to \mathcal{H}_p$*;*
- *a bounded, selfadjoint, strictly positive* instrument weight $W : \mathcal{H}_p \to \mathcal{H}_p$*, inducing*

$$\langle u, v\rangle_W := \langle Wu, v\rangle, \qquad \|u\|_W^2 := \langle u, u\rangle_W. \tag{7}$$

*Given* $s \in \mathcal{H}_s$ *and* $p \in \mathcal{H}_p$*, the* DSFL defect *and* residual of sameness *are*

$$e(s, p) := p - \mathbb{I}s \in \mathcal{H}_p, \qquad R(s, p) := \|e(s, p)\|_W^2. \tag{8}$$

The zero set of $R$ is the calibration graph $\{(s, p) : p = \mathbb{I}s\}$, and $R \geq 0$ by strict positivity of $W$. Quadratic functionals of this kind are standard Lyapunov candidates in operator and semigroup theory [26,27].

### 3.2. Admissible Maps and a Single DPI

**Definition 2** (DSFL–admissible pair)**.** *Let* $(\mathcal{H}, \mathcal{H}_s, \mathcal{H}_p, \mathbb{I}, W)$ *be a DSFL room. A pair of bounded maps* $\widetilde{\Phi} : \mathcal{H}_s \to \mathcal{H}_s$ *and* $\Phi : \mathcal{H}_p \to \mathcal{H}_p$ *is* DSFL–admissible *if*

(a) *(*calibration preservation*)* $\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi}$*;*
(b) *(*nonexpansiveness*)* $\|\Phi u\|_W \leq \|u\|_W \quad \forall u \in \mathcal{H}_p$.

The nonexpansiveness condition is the same operator–norm constraint $\|\Phi\|_{W\to W} \leq 1$ familiar from contractive channel and semigroup theory [24,38].

**Theorem 1** (Admissibility $\Longleftrightarrow$ DPI for $R$)**.** *In a DSFL room, the pair* $(\widetilde{\Phi}, \Phi)$ *is DSFL–admissible if and only if for all* $(s, p) \in \mathcal{H}_s \times \mathcal{H}_p$ *one has the data–processing inequality*

$$R(\widetilde{\Phi}s, \Phi p) \leq R(s, p). \tag{9}$$

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This parallels the way standard quantum DPIs characterise completely–positive trace–preserving maps that contract relative entropy or other convex functionals [22–24]. Here, the single scalar observable $R$ plays that role.

### 3.3. Lyapunov Law and DSFL Clock

Let $t \mapsto (s(t), p(t))$ be a trajectory in a DSFL room and write $e(t) = p(t) - \mathbb{I}s(t)$ and $R(t) = \|e(t)\|_W^2$. A *DSFL Lyapunov law* is an inequality

$$\dot{R}(t) \le -2\lambda(t)R(t), \tag{10}$$

for some rate $\lambda(t) \ge 0$. Grönwall's lemma then gives

$$R(t) \le R(0)\exp\Big(-\int_0^t 2\lambda(\sigma)\,d\sigma\Big), \tag{11}$$

which is a standard Lyapunov estimate for dissipative evolutions on Hilbert spaces [26,27]. Defining the *DSFL clock*

$$\widehat{\tau}(t) := \int_0^t 2\lambda(\sigma)\,d\sigma, \tag{12}$$

this becomes

$$\log R(\widehat{\tau}) \le \log R(0) - \widehat{\tau}. \tag{13}$$

In DSFL time $\widehat{\tau}$, the semi–log plot $\widehat{\tau} \mapsto \log R(\widehat{\tau})$ has slope at most $-1$.

In discrete time, for iterates $e_{k+1} = \Phi_k e_k$ and $R_k = \|e_k\|_W^2$, one defines the depth

$$\widehat{\tau}_0 := 0, \qquad \widehat{\tau}_{k+1} := \widehat{\tau}_k - \log\frac{R_{k+1}}{R_k}, \tag{14}$$

so that $\log R_k = \log R_0 - \widehat{\tau}_k$ exactly. This is the discrete analogue of the Lyapunov parametrisation used in contractive Markov and quantum semigroups [25,28].

### 3.4. Cone Locality

To encode finite–speed propagation, we split $\mathcal{H}_p$ as functions or fields on a metric space $(X, 1.7em)$ and let $\Pi_O$ be multiplication by the indicator of a region $O \subset X$.

**Definition 3** (Cone locality in a DSFL room). *A family of maps $\Psi_t : \mathcal{H}_p \to \mathcal{H}_p$ satisfies a* cone bound *with parameters $(C, v, \xi)$ if for all measurable $O, O' \subset X$ and all $t \ge 0$,*

$$\|\Pi_{O'}\Psi_t\Pi_O\|_{W\to W} \le C\,\exp\Big(-\frac{1.7em(O,O') - vt}{\xi}\Big), \tag{15}$$

*where $\|\cdot\|_{W\to W}$ denotes the operator norm induced by the instrument norm $\|\cdot\|_W$ on $\mathcal{H}_p$.*

Here $v$ plays the role of an emergent "light–cone" speed in the instrument geometry and $\xi$ a decay length setting how sharply the tails fall off outside the cone. This is the DSFL analogue of a Lieb–Robinson bound on quasi–local propagation in lattice systems [39–41]. In the BH room we will use such bounds for the relay (QFT propagation) part of the evolution in the same instrument norm $\|\cdot\|_W$.

## 4. A DSFL Room for Black–Hole Scramblers

We now specialise the DSFL framework to a black–hole spacetime and identify three sectoral residuals in the same norm: a global BH residual, a scrambling residual, and a gravitational residual.

### 4.1. Hilbert Factorisation and Instrument Norm

We model "black hole + near zone + far radiation" by a tensor product

$$\mathcal{H}_{\text{room}} \cong \mathcal{H}_{\text{BH}} \otimes \mathcal{H}_{\text{near}} \otimes \mathcal{H}_{\text{far}}, \tag{16}$$

in line with standard information–theoretic treatments of black–hole evaporation and scrambling [1,2,8]. The blueprint subspace $\mathcal{H}_s \subset \mathcal{H}_{\text{room}}$ holds statistical data (e.g. interior microstates, boundary conditions and couplings), and the calibration map $\mathbb{I} : \mathcal{H}_s \to \mathcal{H}_p$ realises these as ideal responses.

On a near–horizon collar we choose a GR graph–norm weight $W_{\text{GR}}[g]$ built from the linearised Einstein–matter system on a background black–hole metric $g$ [9,10,34,35]. This induces an instrument inner product and norm

$$\|u\|_W^2 := \langle u, W_{\text{GR}}[g]u\rangle \tag{17}$$

on a physical space $\mathcal{H}_p \subset \mathcal{H}_{\text{room}}$ that controls energy and constraint norms for perturbations of $(g, \langle T\rangle)$ [10,35,42]. We use this norm throughout, so that Lyapunov envelopes and cone locality (Definition 3) are tested in a single, common instrument geometry.

The *BH defect* and *global BH residual* are

$$e(t) := p(t) - \mathbb{I}s(t), \qquad R(t) := \|e(t)\|_W^2, \tag{18}$$

and will be our main Lyapunov observable.

### 4.2. Immediate Loop: Scrambler + Einstein Tightening

A single DSFL loop is split into an *immediate* tightening step and a *relay*. In the BH room, the immediate part combines:

- a *scrambler* on $\mathcal{H}_{\text{BH}}$ and $\mathcal{H}_{\text{near}}$: DSFL–admissible channels (or local circuits) with a nonzero DSFL gap, inducing a scrambling residual $R_{\text{scr}}(t)$ that decays at rate $\alpha_{\text{scr}}$. This includes, as model subclasses, $k$–local Hamiltonians on large–$N$ systems, SYK–type models and random local circuits used in the fast scrambler and OTOC literature [2–4,30,31,43–45].
- an *Einstein tightening* on the collar: nearest–point projection (in $\|\cdot\|_W$) onto an Einstein frame, followed by a GR two–loop DSFL evolution with residual

$$R_{\text{GR}}(t) = \left\|\mathcal{E}_{\kappa,\Lambda}(g(t); T(t))\right\|^2_{W_{\text{GR}}[g(t)]}, \tag{19}$$

  where $\mathcal{E}_{\kappa,\Lambda} = G[g] + \Lambda g - 8\pi G\langle T_{\mu\nu}\rangle$ is the Einstein imbalance. In linearised near–horizon models this residual decays at a rate $\lambda_{\text{GR}}(t)$ governed by the dominant quasinormal modes [34,35,46].

Each component is DSFL–admissible in $\|\cdot\|_W$ in the sense of Section 3. Their composition defines an immediate BH map with effective rate

$$\lambda_{\text{BH}}(t) \;\simeq\; \alpha_{\text{scr}}(t) + \lambda_{\text{GR}}(t), \tag{20}$$

which enters the global DSFL clock in the BH room.

### 4.3. Relay: QFT Propagation and Cone

Between immediate steps, the defect $e(t)$ is transported by a relay $\Psi_{\Delta t}$ induced by QFT propagation on $(\mathcal{M}, g)$ [10,47]. We assume:

- $\Psi_{\Delta t}$ is DSFL–admissible in $\|\cdot\|_W$, i.e. nonexpansive in the instrument geometry and intertwined with the calibration;
- $\Psi_{\Delta t}$ satisfies a cone bound as in Definition 3, with front speed $v$ and decay length $\xi$ set by the near–horizon geometry and field content. Here $v$ plays the role of an instrument–light–cone speed and $\xi$ the scale on which tails of $e(t)$ are exponentially suppressed outside the cone. This

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is the DSFL version of a Lieb–Robinson–type bound [39–41], realised in curved backgrounds by finite–speed propagation and quasinormal decay [10,34,35].

Altogether, the BH DSFL evolution consists of repeated application of a scrambler, an Einstein tightening, and a relay, each admissible in the same instrument norm and constrained by the same Lyapunov and cone structure.

### 4.4. Scrambling and Inside/Outside Residuals

Within this BH room we single out three residuals.

Scrambling residual.

For a small subsystem $A$ (e.g. a few exterior modes or boundary sites), let $\rho_A(t)$ be its reduced state and $\rho_A^{\mathrm{eq}}$ a thermal or Haar–typical equilibrium state dictated by the blueprint [1,8,32]. We define

$$R_{\mathrm{scr}}(t) := \|\rho_A(t) - \rho_A^{\mathrm{eq}}\|_{W_A}^2, \tag{21}$$

where $W_A$ is the restriction of $W_{\mathrm{GR}}[g]$ to the sector containing $A$. This is a DSFL version of "how far $A$ is from scrambled" and can be viewed as a quadratic proxy for relative entropy or mutual information in the small–subsystem limit [25,28,48,49]. In many chaotic models its decay rate matches the OTOC chaos exponent; see Section 5.

Inside/outside budgets.

Using the tensor split $\mathcal{H}_{\mathrm{room}} \cong \mathcal{H}_{\mathrm{BH}} \otimes \mathcal{H}_{\mathrm{out}}$, where $\mathcal{H}_{\mathrm{out}}$ aggregates $\mathcal{H}_{\mathrm{near}}$ and $\mathcal{H}_{\mathrm{far}}$, we define $\|\cdot\|_W$–orthogonal projections $\Pi_{\mathrm{in}}$ and $\Pi_{\mathrm{out}}$ onto BH and outside sectors and set

$$R_{\mathrm{in}}(t) := \|\Pi_{\mathrm{in}} e(t)\|_W^2, \qquad R_{\mathrm{out}}(t) := \|\Pi_{\mathrm{out}} e(t)\|_W^2. \tag{22}$$

Their balance will define a DSFL Page time, in analogy with the entropic Page crossing $S_{\mathrm{BH}} \approx S_{\mathrm{rad}}$ [1,8].

Gravitational residual.

On the GR side we take

$$R_{\mathrm{GR}}(t) := \big\|\mathcal{E}_{\kappa,\Lambda}(g(t); T(t))\big\|_{W_{\mathrm{GR}}[g(t)]}^2, \tag{23}$$

where $\mathcal{E}_{\kappa,\Lambda}(g;T) = G_{\mu\nu}[g] + \Lambda g_{\mu\nu} - 8\pi G\langle T_{\mu\nu}\rangle$ is the Einstein imbalance. In linearised near–horizon collars this residual obeys a DSFL Lyapunov law with rate set by QNM decay, and its null restriction can be used to express QNEC/QFC–type focusing inequalities in the same instrument geometry [10,12–16,34,35].

## 5. Page Curve, Chaos Bound and Focusing from One Residual

We now show how Page behaviour, the MSS chaos bound and linearised focusing/QNEC–QFC can all be phrased as inequalities for the three residuals introduced in Section 4, *all measured in the same instrument norm* $\|\cdot\|_W$ induced by $W_{\mathrm{GR}}[g]$. The Page part is inspired by Page's random–state analysis and the Hayden–Preskill mirror picture [1,8], the chaos part by the OTOC formalism and MSS bound [3–5], and the focusing part by the QNEC/QFC literature on generalised entropy [11–15].

### 5.1. DSFL Page Curve

We first model Page behaviour via a simple residual–entropy relation, playing the role of a quadratic proxy for entropies in the spirit of Pinsker- and log–Sobolev–type bounds [24,25,28]. The key point is that the interior entropy is taken to be a monotone function of the *interior residual* $R_{\mathrm{in}}(t)$ *in the DSFL norm*.

**Assumption 1** (Residual–entropy relation). *There exist $S_0 > 0$ and $\gamma > 0$ such that along the BH DSFL evolution*

$$S_{\mathrm{BH}}(t) = S_0 \, R_{\mathrm{in}}(t)^{\gamma}, \qquad S_{\mathrm{rad}}(t) = S_{\mathrm{tot}} - S_{\mathrm{BH}}(t), \tag{24}$$

*where $S_{\mathrm{BH}}$ and $S_{\mathrm{rad}}$ are coarse–grained entropies of interior and radiation, and $S_{\mathrm{tot}} = S_0$ is the total coarse–grained entropy. Here $R_{\mathrm{in}}(t)$ is the inside–budget of the global BH defect $e(t)$ measured in $\|\cdot\|_W$, as in Section 4.4.*

This is meant as a surrogate for the way the trace–norm (or Hilbert–Schmidt) distance to an equilibrium state controls relative entropy and hence von Neumann entropy [19,24,50].

**Assumption 2** (Lyapunov envelope for interior residual). *The interior residual obeys a Lyapunov law in the instrument norm,*

$$R_{\mathrm{in}}(t) = \exp(-2\alpha_{\mathrm{BH}} t), \tag{25}$$

*for some constant $\alpha_{\mathrm{BH}} > 0$ (after suitable rescaling of $t$). This mirrors the standard semigroup setting where exponential decay of a quadratic Lyapunov functional is controlled by a spectral gap or log–Sobolev constant [25–28]. In DSFL language, this is the statement that $R_{\mathrm{in}}(t)$ sits on a rate–bearing Lyapunov envelope with rate $\alpha_{\mathrm{BH}}$.*

**Theorem 2** (DSFL Page curve). *Under Assumptions 1–2, there is a unique time $t_{\mathrm{Page}}^{\mathrm{DSFL}}$ such that*

$$S_{\mathrm{BH}}(t_{\mathrm{Page}}^{\mathrm{DSFL}}) = S_{\mathrm{rad}}(t_{\mathrm{Page}}^{\mathrm{DSFL}}) = \frac{S_0}{2}, \tag{26}$$

*given by*

$$t_{\mathrm{Page}}^{\mathrm{DSFL}} = \frac{\log 2}{2\alpha_{\mathrm{BH}}\gamma}, \qquad R_{\mathrm{in}}(t_{\mathrm{Page}}^{\mathrm{DSFL}}) = 2^{-1/\gamma}. \tag{27}$$

*In this sense, Page turnover is a property of the single interior residual $R_{\mathrm{in}}$ in the DSFL instrument norm, paralleling the usual picture in which Page time is the balance point of entropies of black hole and radiation [1,8].*

**Proof.** Setting $S_{\mathrm{BH}} = S_{\mathrm{rad}} = S_0/2$ in Assumption 1 gives $R_{\mathrm{in}}(t_{\mathrm{Page}})^{\gamma} = 1/2$. Using $R_{\mathrm{in}}(t) = \mathrm{e}^{-2\alpha_{\mathrm{BH}} t}$ yields $-2\alpha_{\mathrm{BH}}\gamma t_{\mathrm{Page}} = \log(1/2)$ and hence the stated formulae. □

In finite–dimensional models we can define a *DSFL Page time* by balancing inside/outside residual budgets in expectation and check that it coincides with the usual entropic Page time. This is done in Section 6, where we compare to Page's original Haar ensemble [8] and to random–circuit evaporators in the spirit of [1,17,18].

### 5.2. MSS Chaos Bound as a DSFL Rate Constraint

We next connect the DSFL scrambling rate to the OTOC Lyapunov exponent and the MSS bound. We follow the OTOC diagnostics of [3,4,30] and the chaos bound of [5].

Let $\lambda_L$ be the chaos exponent extracted from OTOCs in the thermal BH state at inverse temperature $\beta$:

$$C(t) := -\langle [W(t), V]^2 \rangle_{\beta} \sim C_0 e^{2\lambda_L t}, \qquad 0 < t \ll t_{\mathrm{scr}}, \tag{28}$$

for suitable local $W, V$.

**Assumption 3** (OTOC–residual comparability). *There exist constants $c_1, c_2 > 0$ and a time window $[0, T_*]$ such that, for a DSFL scrambling residual $R_{\mathrm{scr}}(t)$ measured in the same instrument norm $\|\cdot\|_W$,*

$$c_1 \, \|[W(t), V]\|_2^2 \le R_{\mathrm{scr}}(t) \le c_2 \, \|[W(t), V]\|_2^2, \qquad 0 \le t \le T_*, \tag{29}$$

*and $R_{\mathrm{scr}}$ satisfies a DSFL Lyapunov inequality $\dot{R}_{\mathrm{scr}} \le -2\alpha_{\mathrm{scr}} R_{\mathrm{scr}}$ with some $\alpha_{\mathrm{scr}} > 0$.*

Such a comparability is natural in random–circuit, SYK and holographic models where OTOC growth, operator spreading and decay of local distinguishability are tightly correlated [3,4,43,49].

**Proposition 1** (DSFL rate vs. chaos exponent). *Under **??** 3, in the early–time regime* $0 < t \ll T_*$ *one has*

$$\frac{d}{dt} \log R_{\text{scr}}(t) \approx 2\lambda_L, \tag{30}$$

*so* $\alpha_{\text{scr}}$ *is proportional to* $\lambda_L$ *up to subleading corrections. In model scramblers one finds* $\alpha_{\text{scr}} \approx \lambda_L$ *numerically [30,43].*

Combining this with the MSS bound [5] yields:

**Theorem 3** (MSS chaos bound as a DSFL inequality). *Under **??** 3, if* $C(t)$ *satisfies the MSS assumptions (KMS condition, analyticity, boundedness of thermal OTOCs) then the DSFL scrambling rate obeys*

$$\alpha_{\text{scr}} \leq \frac{2\pi}{\beta}. \tag{31}$$

*In particular, no DSFL–admissible immediate loop in a thermal near–horizon collar can contract the scrambling residual* $R_{\text{scr}}$ *in the instrument norm* $\|\cdot\|_W$ *faster than* $2\pi/\beta$*, and Einstein–gravity black holes that saturate the MSS bound are precisely those that saturate this DSFL rate inequality.*

The proof is a direct combination of Proposition 1 with the MSS chaos bound and a rescaling of the DSFL clock (see Appendix A for details).

### *5.3. Linearised Focusing and QNEC/QFC*

Finally, we sketch how linearised QNEC/QFC–type focusing can be phrased in terms of the gravitational residual $R_{\text{GR}}$ in the same instrument norm, following the generalised–entropy framework of [11–15].

Let $\sigma$ be a spacelike $(d-2)$–surface in the BH exterior and $k^\mu$ a null normal generating a congruence of null hypersurfaces $\mathcal{N}(\lambda)$, with affine parameter $\lambda$. The *generalised entropy* is

$$S_{\text{gen}}(\lambda) := \frac{A(\lambda)}{4G} + S_{\text{out}}(\lambda), \tag{32}$$

where $A$ is area and $S_{\text{out}}$ the von Neumann entropy of fields outside $\sigma(\lambda)$.

In linearised QNEC/QFC, second variations of $S_{\text{gen}}$ along $k^\mu$ are bounded from above by local stress–energy data and are closely related to relative entropy inequalities on null cuts [15,16,51]. DSFL provides a way to package these into a single norm.

**Assumption 4** (Linearised control of $S_{\text{gen}}$ by $R_{\text{GR}}$). *In a linearised regime about a BH background, variations of* $S_{\text{gen}}$ *along* $\lambda$ *obey*

$$\frac{d^2}{d\lambda^2} S_{\text{gen}}(\lambda) \leq -c_{\text{foc}}\, R_{\text{GR}}(\lambda), \tag{33}$$

*for some* $c_{\text{foc}} > 0$ *and gravitational residual* $R_{\text{GR}}(\lambda)$ *measured in* $\|\cdot\|_{W_{\text{GR}}[g]}$ *built from the Einstein imbalance [9,10,34,35].*

**Theorem 4** (DSFL focusing). *Under **??** 4 and the GR DSFL Lyapunov law for* $R_{\text{GR}}$ *in the instrument norm* $\|\cdot\|_{W_{\text{GR}}[g]}$*, both* $R_{\text{GR}}(\lambda)$ *and an associated DSFL expansion*

$$\Theta_{\text{DSFL}}(\lambda) := -\frac{1}{2\pi} \frac{d}{d\lambda} \log R_{\text{GR}}(\lambda) \tag{34}$$

*obey inequalities of the same form as linearised QNEC and QFC:*

$$\frac{d^2}{d\lambda^2} S_{\text{gen}}(\lambda) \le 0, \qquad \frac{d}{d\lambda}\Theta_{\text{DSFL}}(\lambda) \le 0, \tag{35}$$

*with both driven by the same residual $R_{\text{GR}}$ in the DSFL norm.*

The details depend on the choice of $W_{\text{GR}}[g]$ and renormalisation scheme; we refer to Appendix D for a more technical discussion. For our purposes, the key point is conceptual: the same Lyapunov functional that governs quasinormal ringdown and decay in the collar [34,35] also controls linearised focusing inequalities in one geometry.

### 5.4. Summary: One Residual, Three Inequalities

In the black–hole DSFL room, the results above can be summarised as follows, emphasising the role of the *single instrument norm* $\|\cdot\|_W$:

- **Page law.** A DSFL Page time is the balance point of inside/outside residual budgets $R_{\text{in}}$ and $R_{\text{out}}$; in model worlds (Haar ensembles and random–circuit evaporators) it coincides with the entropic Page time [1,8].
- **Chaos bound.** The DSFL scrambling rate for $R_{\text{scr}}$ coincides with the OTOC Lyapunov exponent and is bounded by $2\pi/\beta$, in line with the MSS chaos bound [5]. No DSFL–admissible immediate loop in the BH room can contract $R_{\text{scr}}$ in the norm $\|\cdot\|_W$ faster than this bound.
- **Focusing.** The gravitational residual $R_{\text{GR}}$ obeys a Lyapunov law and drives linearised QNEC/QFC–type focusing in the same norm [11,15], via the DSFL null expansion $\Theta_{\text{DSFL}}$.

All three are inequalities for a single residual observable in one instrumented Hilbert room equipped with a single, fixed instrument norm $\|\cdot\|_W$.

## 6. Model Worlds and Numerical Support

We briefly illustrate the DSFL Page and chaos statements in finite models where everything can be computed explicitly. Full details and additional tests are in Appendices C and F.

### 6.1. Haar Ensemble: Static DSFL Page Point

Consider an $n$–qubit system $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes n}$ and a bipartition into "black hole" and "radiation" subsystems of dimensions $d_{\text{BH}} = 2^{n-k}$ and $d_{\text{rad}} = 2^k$, with $k$ emitted qubits. Let $\rho$ be a Haar–random pure state on $\mathcal{H}$. Define

$$R_{\text{BH}}(k) := \|\rho_{\text{BH}} - d_{\text{BH}}^{-1} I\|_2^2, \qquad R_{\text{rad}}(k) := \|\rho_{\text{rad}} - d_{\text{rad}}^{-1} I\|_2^2, \tag{36}$$

with $\rho_{\text{BH}}$ and $\rho_{\text{rad}}$ the reduced states.

Using Page's purity formula, one finds [8]:

**Theorem 5** (Static DSFL Page point)**.** *In the Haar ensemble,*

$$\mathbb{E}[R_{\text{BH}}(k)] \sim \frac{d_{\text{BH}}}{d_{\text{rad}}^2}, \qquad \mathbb{E}[R_{\text{rad}}(k)] \sim \frac{d_{\text{rad}}}{d_{\text{BH}}^2}, \qquad d_{\text{BH}}, d_{\text{rad}} \gg 1, \tag{37}$$

*and the index*

$$k_{\text{Page}}^{\text{DSFL}} := \arg\min_k \big|\mathbb{E}[R_{\text{BH}}(k)] - \mathbb{E}[R_{\text{rad}}(k)]\big| \tag{38}$$

*satisfies $k_{\text{Page}}^{\text{DSFL}} = n/2 + O(1)$ as $n \to \infty$. The usual entropic Page time (maximising $\mathbb{E}[S(\rho_{\text{rad}})]$) has the same asymptotics. Thus the DSFL Page point (balance of residuals) coincides asymptotically with Page's entropic crossing.*

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We confirmed this numerically for $n = 8, 10, 12$ by sampling Haar states and computing $\mathbb{E}[R_{\mathrm{BH}}(k)]$ and $\mathbb{E}[R_{\mathrm{rad}}(k)]$;

## 7. DSFL Page Audit

the curves cross at $k \approx n/2$, where the radiation entropy is maximal. See Section 7.

### 7.1. Random–Circuit Evaporation: Dynamical Page Time

To include dynamics, we consider a random–circuit evaporation model where one qubit is emitted at each step and between emissions we apply a finite–depth local scrambler. Under modest assumptions (each step implements a unitary 2–design on the interior), the reduced states at step $k$ are close to Haar–distributed on the corresponding bipartition [1,32].

In this setting one can show:

**Theorem 6** (Dynamical DSFL Page time). *In a random–circuit evaporation model with strong scrambling at each step, the step $k_{\mathrm{Page}}^{\mathrm{DSFL}}$ that minimises $\mathbb{E}|R_{\mathrm{BH}}(k) - R_{\mathrm{rad}}(k)|$ satisfies*

$$\frac{k_{\mathrm{Page}}^{\mathrm{DSFL}}}{n} \to \frac{1}{2}, \tag{39}$$

*and coincides, up to $O(1)$ fluctuations, with the step $k_{\mathrm{Page}}^{S}$ that maximises the expected radiation entropy $\mathbb{E}[S(\rho_{\mathrm{rad}}(k))]$.*

For $n = 8$ qubits and depth–4 brickwork scramblers we find numerically that both the entropic Page step and the DSFL Page step occur at $k = 4$, in line with Theorem 6.

### 7.2. Near–Horizon Ringdown Toy

As a GR illustration, consider a 1+1–dimensional wave equation with a potential mimicking a BH barrier (e.g. a Pöschl–Teller or Schwarzschild Regge–Wheeler potential). Let $e(t)$ be a perturbation restricted to a near–horizon collar and define

$$R_{\mathrm{GR}}(t) := \|e(t)\|^2_{W_{\mathrm{GR}}[g]} \tag{40}$$

with a suitable graph–norm weight. Numerically one finds that $\log R_{\mathrm{GR}}(t)$ has a linear tail with slope $-2\lambda_{\mathrm{QNM}}$, where $\lambda_{\mathrm{QNM}}$ is the imaginary part of the dominant quasinormal frequency, and that spatially resolved residuals obey a cone bound in the same norm. This fits the GR DSFL Lyapunov and cone structure assumed in Section 4.

DSFL residuals in standard quantum channels.

Before turning to gravitational and near–horizon systems, it is instructive to observe that the DSFL residual law is already realised — exactly and explicitly — in standard one–qubit channels.

In particular, for canonical CPTP maps such as amplitude damping, depolarisation, and dephasing, the DSFL residual

$$R_k = \|\mathcal{E}^k(\rho_0) - \rho_*\|_2^2 \tag{41}$$

decays at a rate set by the square of the second–largest Liouville eigenvalue of the channel. This matches the DSFL spectral envelope

$$R_k \sim C\,|\lambda_2|^{2k}, \tag{42}$$

with state–dependent saturation depending on alignment with fast or slow modes.

For example:

- In amplitude damping, $\rho_0 = |1\rangle\langle 1|$ decays at rate $(1-\gamma)^{2k}$, while generic states decay as $(1-\gamma)^k$.

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- In depolarising channels, all states decay uniformly at rate $(1-p)^{2k}$ — a maximally contracting DSFL scrambler.
- In dephasing, no global DSFL contraction occurs in $\|\cdot\|_2$; only coherences decay — a gapless DSFL case.

These explicit finite–dimensional behaviours confirm that the DSFL Lyapunov structure is not an artifact of black hole dynamics: it already governs residual contraction in standard quantum noise models. Details are given in Appendix B.

### 7.3. Summary of Supporting DSFL Results

For clarity we briefly summarise the additional material developed in the appendices and how it underpins the main claims of this article. All precise theorems and proofs are deferred to Appendices A and C–F.

Sector–neutral DSFL backbone (Appendix A).

The abstract DSFL framework is made completely precise: a calibrated room $(\mathcal{H}, \mathcal{H}_s, \mathcal{H}_p, \mathbb{I}, W)$, the residual of sameness $R(s,p) = \|p - \mathbb{I}s\|_W^2$, and DSFL–admissible pairs $(\widetilde{\Phi}, \Phi)$ are defined so that a *single* data–processing inequality for $R$ is equivalent to calibration preservation and nonexpansiveness in the instrument norm, in close analogy with standard quantum DPI results for relative entropy and contractive channels [20–24]. A Lyapunov inequality $\dot{R} \le -2\lambda(t)R$ and the associated DSFL clock $\widehat{\tau}(t) = \int_0^t 2\lambda(\sigma)\, d\sigma$ are proved in a standard semigroup setting [26,27], together with a discrete analogue for iterated maps. A cone locality result shows how Lieb–Robinson–type bounds on a local generator imply the DSFL cone bound

$$A_{O' \leftarrow O}(t) := \|\Pi_{O'} \Psi_t \Pi_O\|_{W \to W}\, A(0) \;\le\; C \exp\!\Big(-\frac{1.7em(O,O') - v|t|}{\xi}\Big)\, A(0), \tag{43}$$

(43) for the relay in the *same* norm [39–41]. These results justify the structural assumptions used throughout Sections 3 and 4.

Haar and random–circuit DSFL Page laws (Appendix C).

In the static Haar ensemble on $(\mathbb{C}^2)^{\otimes n}$, DSFL residuals $R_{\rm BH}(k)$ and $R_{\rm rad}(k)$ to maximally mixed states are computed explicitly using Page's purity formula [8]. The *DSFL Page index*, defined by balancing these residual budgets, converges to $k = n/2$ as $n \to \infty$, i.e. the DSFL Page point coincides asymptotically with the entropic Page time. A dynamical random–circuit evaporation model is then analysed: when each emission step is followed by a sufficiently strong local scrambler (a unitary 2–design on the interior) [1,32], the *dynamical* DSFL Page time, defined by minimizing $\mathbb{E}|R_{\rm BH}(k) - R_{\rm rad}(k)|$, coincides (up to $O(1)$ steps) with the step that maximises the expected radiation entropy. Numerical audits for $n = 8, 10, 12$ qubits confirm that, in both Haar and random–circuit evaporators, the DSFL Page balance and the entropic Page maximum occur at $k \approx n/2$.

GR calibration and $\|\cdot\|_{W_{\rm GR}[g]}$ QNEC/QFC (Appendix D).

On the GR side, we construct a DSFL room for perturbations of a fixed black–hole background $(\mathcal{M}, g)$ by using a graph or energy norm for the linearised Einstein–matter system to define the instrument weight $W_{\rm GR}[g]$ [10,35,42]. The Einstein imbalance $\mathcal{E}_{\kappa,\Lambda}(g;T) = G[g] + \Lambda g - 8\pi G\langle T_{\mu\nu}\rangle$ then induces a gravitational residual $R_{\rm GR}(t) = \|\mathcal{E}_{\kappa,\Lambda}(g(t);T(t))\|^2_{W_{\rm GR}[g(t)]}$. In a near–equilibrium collar, one shows that $R_{\rm GR}$ obeys a two–loop DSFL Lyapunov law with rate governed by the dominant quasinormal modes [34,35], and that a null restriction of $R_{\rm GR}$ can be related, to leading order, to relative entropy along lightsheets [15,16,51]. This yields a $\|\cdot\|_{W_{\rm GR}}$ version of QNEC/QFC in a linearised regime: convexity of the null residual encodes QNEC [12,15], and a DSFL expansion $\Theta_{\rm DSFL} = -\frac{1}{2\pi}\partial_\lambda \log R_{\rm GR}$ furnishes a QFC–type focusing inequality in the same norm [11,13,14]. These results justify the focusing statements in Section 5.3.

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DSFL depth and complexity bounds (Appendix E).

Viewing a DSFL loop as a depth–1 $k$–local circuit layer plus relay, we define the discrete DSFL depth $\widehat{\tau}_K$ from per–step residual losses $R_{k+1}/R_k$ and adapt standard Markov/dissipative techniques [25,28]. Under mild locality and "no strong backtracking" assumptions on the loops, we prove linear upper and lower bounds on any reasonable circuit complexity $\mathcal{C}(K)$ in terms of $\widehat{\tau}_K$: there exist constants $C_\pm > 0$ such that, in a scrambling window where $\widehat{\tau}_K \approx \alpha^* K$, one has $\mathcal{C}(K) \approx C_\pm \widehat{\tau}_K$ up to additive $O(1)$ terms. In particular, in models where $\alpha^*$ coincides with the OTOC Lyapunov exponent [3–5], these bounds support the heuristic that the linear–in–time complexity growth of holographic scramblers is anchored in the same Lyapunov envelope that contracts the DSFL residual [6,36,37,52].

Numerical envelope, cone and CIU audits (Appendix F).

Finally, we specify finite–sample envelope and cone audits for finite–dimensional and lattice scramblers, and describe how CIU (Contextual Importance/Utility) diagnostics [53,54] can be used to identify "high–headroom" handles in a DSFL calibration. The envelope audit fits $\log R_k$ versus $\widehat{\tau}_k$ and verifies unit–slope behaviour in DSFL time; the cone audit fits front speeds and tail decay rates in $\|\cdot\|_W$ and checks consistency with Lieb–Robinson cones [39–41]. Page–time audits in Haar and random–circuit models confirm the analytic DSFL Page laws [1,8,32], while CIU headroom maps on small chains and near–horizon toys illustrate how DSFL+CIU can locate control channels with the largest potential impact on the Lyapunov rate without violating cone locality. Together, these numerical tests show that the single–residual DSFL picture is not only formally consistent, but also empirically tight in the model worlds where explicit computation is possible.

## 8. Discussion and Outlook

We have argued that a large part of Susskind's black–hole paradigm can be reorganised as statements about a *single* residual of sameness in one instrumented Hilbert room.[1] In this representation, fast scrambling and the MSS chaos bound [2–5] appear as constraints on the DSFL Lyapunov rate $\alpha_{\mathrm{scr}}$ for a scrambling residual $R_{\mathrm{scr}}$, whose decay rate coincides with the OTOC Lyapunov exponent $\lambda_L$ in model scramblers. Page time and Hayden–Preskill mirror behaviour [1,8] emerge as a balance of inside/outside residual budgets $R_{\mathrm{in}}$ and $R_{\mathrm{out}}$ in the *same* norm, and the resulting DSFL Page time agrees, in Haar and random–circuit evaporator models, with the usual entropic Page time up to $O(1)$ corrections. Linearised QNEC/QFC–type focusing [11–16] can likewise be phrased as Lyapunov/focusing inequalities for a gravitational residual $R_{\mathrm{GR}}$ built from Einstein imbalance, measured in the same GR instrument geometry $\|\cdot\|_{W_{\mathrm{GR}}[g]}$, with rates governed by quasinormal modes and decay estimates in near–horizon collars [34,35,46].

Conceptually, this suggests that many of the "separate" equilibrium and arrow–of–time statements we usually make in quantum mechanics, statistical mechanics and general relativity — contractivity of quantum channels and relative entropies [19,25,50,55,56], mixing and entropy production in classical/quantum Markov dynamics [25,28,29], and Einstein balance and focusing in semiclassical GR [9–12,35] — may be different faces of a *single* contraction law for a single quadratic observable $R$ in a suitably chosen room. Technically, the DSFL framework is modest: it requires only a calibrated Hilbert space, a single weight $W$, and an admissibility condition equivalent to a DPI for $R$. Within that structure, fast scrambling, chaos bounds, Page turnover and linearised focusing become inequalities for the time profile of $R$ and its DSFL clock $\widehat{\tau}$.

The limitations of the present work are equally clear. Our GR analysis is linearised and semiclassical, relying on known quasinormal–mode and decay estimates on fixed backgrounds [9,10,34,35,46]; the OTOC–residual comparison is proved only in simple finite–dimensional models and random–circuit scramblers [3,4,30,43]; and DSFL itself is taken as a sector–neutral structural ansatz rather than derived from a UV completion such as string theory, SYK–type models or a concrete holographic dual [5,7,57].

[1] In the sense of a calibrated room $(\mathcal{H}, \mathcal{H}_s, \mathcal{H}_p, \mathbb{I}, W)$ with residual $R(s,p) = \|p - \mathbb{I}s\|_W^2$ and admissible maps characterised by a single DPI for $R$; see Section 3 and [20–24,26,27].

Nevertheless, the fact that black–hole signatures fit coherently into this "one room, one residual" picture provides both an organising principle and a set of falsifiable benchmarks: any purported scrambler or BH surrogate can be audited via its DSFL envelope, cone, Page balance and chaos rate in a single norm.

Several directions for further work are natural. One is to move beyond linearised near–horizon collars and extend the GR calibration and Lyapunov analysis to fully dynamical horizons, strong backreaction and nonperturbative regimes, testing whether an appropriately renormalised gravitational residual still satisfies a robust DSFL envelope and cone law. A second is to make the $\|\cdot\|_{W_{\mathrm{GR}}[g]}$ implementation of QNEC and QFC fully rigorous for interacting fields, building on existing relative–entropy and modular–Hamiltonian techniques [11,15,47,51]. A third is to explore DSFL depth $\hat{\tau}$ as a quantitative proxy for complexity growth in holographic and SYK–type scramblers, and to relate it more systematically to "complexity = volume/action" proposals and the second law of complexity [4,6,36,37,52,58]. Finally, it would be interesting to incorporate FRW/de Sitter expansion more fully, using Hubble–scaled DSFL clocks in BH+cosmology models and testing, in explicit toy worlds, the idea that the effective DSFL rate in a de Sitter patch is universally limited by $H(t)$ [5,59–61].

These questions are left for future work and, in some cases, for companion papers focused on cosmology, complexity and sector–neutral DSFL applications beyond the black–hole setting.

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## Notation (Symbols Only)

| Symbol | Type / Domain | Meaning / Assumptions |
|---|---|---|
| **Ambient Hilbert geometry ("one room")** | | |
| $\mathcal{H}$ | Hilbert space | Common comparison space with inner product $\langle x, y\rangle$ and norm $\|x\| = \sqrt{\langle x, x\rangle}$ |
| $\mathcal{H}_s, \mathcal{H}_p \subset \mathcal{H}$ | Closed subspaces | Statistical (blueprint, sDoF) and physical (response, pDoF) arenas |
| $\mathcal{S}, \mathcal{P} \subset \mathcal{H}$ | Closed subspaces | Alternative notation for $\mathcal{H}_s, \mathcal{H}_p$ (sector–neutral definitions) |
| $P_{\mathcal{S}}, P_{\mathcal{P}}$ | Projections | Orthogonal projectors onto $\mathcal{S}, \mathcal{P}$ in the ambient inner product |
| $\mathbb{I} : \mathcal{H}_s \to \mathcal{H}_p$ | Linear map | Calibration / interchangeability map (blueprint $\to$ ideal response) |
| $U := \overline{\operatorname{ran} \mathbb{I}}$ | Closed subspace | Coherent image of $\mathcal{H}_s$ on the physical side (after closure) |
| $P_U$ | Projection | Orthogonal projector onto $U$ |
| $W : \mathcal{H}_p \to \mathcal{H}_p$ | SPD operator | Instrument weight; induces instrument inner product and norm $\|x\|_W^2 = \langle x, Wx\rangle$ |
| $\|\cdot\|_W$ | Norm | Instrument norm on $\mathcal{H}_p$, associated with $W$ |

| Symbol | Type / Domain | Meaning / Assumptions |
|---|---|---|
| **Black–hole room and sectors** | | |
| $\mathcal{H}_{\mathrm{room}}$ | Hilbert space | "BH + near + far" DSFL room; typically $\mathcal{H}_{\mathrm{BH}} \otimes \mathcal{H}_{\mathrm{near}} \otimes \mathcal{H}_{\mathrm{far}}$ |
| $\mathcal{H}_{\mathrm{BH}}, \mathcal{H}_{\mathrm{near}}, \mathcal{H}_{\mathrm{far}}$ | Hilbert spaces | Interior, near–horizon collar, and far radiation factors |
| $\Pi_{\mathrm{BH}}, \Pi_{\mathrm{near}}, \Pi_{\mathrm{far}}$ | Projections | $W_{\mathrm{GR}}[g]$–orthogonal projectors onto the corresponding factors in the GR calibration |
| $R_{\mathrm{BH}}(t)$ | Scalar | Interior residual budget $\|\Pi_{\mathrm{BH}} e(t)\|^2_{W[g(t)]}$ |
| $R_{\mathrm{rad}}(t)$ | Scalar | Exterior (radiation) residual budget $\|\Pi_{\mathrm{far}} e(t)\|^2_{W[g(t)]}$ |
| $R_{\mathrm{in}}(t), R_{\mathrm{out}}(t)$ | Scalars | Generic notation for inside/outside residual budgets (BH vs. outside) |
| $R_{\mathrm{corr}}(t)$ | Scalar | Correlation residual; shared part of the defect between inside and outside |
| $R_{\mathrm{scr}}(t)$ | Scalar | Scrambling residual (loss of local distinguishability in a small subsystem) |
| **States, channels, duals** | | |
| $\mathcal{H}$ | Hilbert space | Finite–dimensional quantum system, typically $\mathcal{H} \cong \mathbb{C}^d$ |
| $\mathcal{B}(\mathcal{H})$ | Operator space | Matrices / bounded operators on $\mathcal{H}$ |
| $\mathcal{S}(\mathcal{H})$ | State space | Density matrices $\rho \geq 0$ with $\operatorname{Tr} \rho = 1$ |
| $\rho, \sigma$ | Operators | Quantum states in $\mathcal{S}(\mathcal{H})$ |
| $\Phi$ | Linear map | Quantum channel (CPTP) or general linear update on $\mathcal{H}_p$ |
| $\widetilde{\Phi}$ | Linear map | Blueprint update on $\mathcal{H}_s$; DSFL pair is $(\widetilde{\Phi}, \Phi)$ |
| $\Phi^*$ | Linear map | Hilbert–Schmidt adjoint of $\Phi$ |
| $\|\Phi\|_{W \to W}$ | Operator norm | Induced norm $\sup_{\|x\|_W = 1} \|\Phi x\|_W$; DSFL admissibility requires $\|\Phi\|_{W \to W} \leq 1$ |
| $K_i$ | Operators | Kraus operators of a CPTP channel, $\Phi(X) = \sum_i K_i X K_i^\dagger$ |
| $C_W(\Phi)$ | Operator on $\mathcal{H}_p$ | Gram operator $W^{-1/2} \Phi^* W \Phi W^{-1/2}$; controls $\|\Phi\|^2_{W \to W}$ via its largest eigenvalue |
| $\lambda_{\max}(C_W(\Phi))$ | Scalar | Largest eigenvalue of $C_W(\Phi)$; equals $\|\Phi\|^2_{W \to W}$ in finite dimension |
| $\gamma(\Phi; W)$ | Scalar | DSFL gap $1 - \|\Phi\|_{W \to W} = 1 - \sqrt{\lambda_{\max}(C_W(\Phi))}$ |
| **Residual of sameness and budgets** | | |
| $s \in \mathcal{H}_s$ | Vector | Statistical (blueprint, sDoF) state |
| $p \in \mathcal{H}_p$ | Vector | Physical (response, pDoF) state |
| $e := p - \mathbb{I}s$ | Vector in $\mathcal{H}_p$ | Calibrated mismatch (defect, residual direction) |
| $R(s,p)$, $\mathcal{R}_{\mathrm{sameness}}(s,p)$ | Scalar | Residual of sameness: $R(s,p) = \|p - \mathbb{I}s\|^2_W = \|e\|^2_W$ |
| $\mathcal{R}$ | Scalar | Generic notation for a DSFL residual (when no sector is specified) |
| $\varepsilon := \|e\|_W$ | Nonnegative scalar | Residual magnitude, e.g. half–width in DSFL band tests |
| $R_{\mathrm{tot}}$ | Scalar | Total residual in a block–diagonal or multi–sector room |
| $R_{\mathrm{QM}}, R_{\mathrm{TD}}, R_{\mathrm{GR}}$ | Scalars | Sectoral residuals in QM, TD and GR calibrations (when used) |
| **Frames and nearest–point updates** | | |
| $V \subset \mathcal{H}_p$ | Closed subspace | Measurement / reconstruction / constraint frame |
| $\Pi_V : \mathcal{H}_p \to V$ | Projection | Orthogonal projector onto $V$ (ambient or $W$–inner product) |
| $T$ | Linear map | Sharp update; DSFL–admissible sharp updates are nearest–point projectors onto $V$ |
| **Admissibility, DPI, and operators** | | |
| $\widetilde{\Phi} : \mathcal{H}_s \to \mathcal{H}_s$ | Linear map | Statistical update (blueprint side) |
| $\Phi : \mathcal{H}_p \to \mathcal{H}_p$ | Linear map | Physical update (response side) |

| Symbol | Type / Domain | Meaning / Assumptions |
|---|---|---|
| $\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi}$ | Identity | Intertwining (coherent blueprint → coherent response) |
| $\Phi^* W \Phi \preccurlyeq W$ | Operator inequality | Spectral DPI / nonexpansiveness in $\|\cdot\|_W$ |
| *DSFL–admissible* | Property | $\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi}$ and $\|\Phi\|_{W\to W} \le 1$ (equivalently DPI for $R$) |
| $R(\widetilde{\Phi}s, \Phi p) \le R(s,p)$ | Inequality | Data–processing inequality for the single observable $R$ |
| **Two–loop dynamics and DSFL time** | | |
| $e(t)$ | Trajectory in $\mathcal{H}_p$ | Time–dependent residual $e(t) = p(t) - \mathbb{I}s(t)$ |
| $K_{\rm imm}(t)$ | Operator on $\mathcal{H}_p$ | Immediate loop generator; selfadjoint, $K_{\rm imm}(t) \succeq 0$ |
| $M(\sigma)$ | Operator kernel | Retarded, Loewner–positive memory kernel for $\sigma \ge 0$ (relay loop) |
| $r(t)$ | Vector in $\mathcal{H}_p$ | Remainder; $\lvert\langle r(t), x\rangle_W\rvert \le \varepsilon_{\rm rem}(t)\|x\|_W^2$ |
| $\kappa(t)$ | Nonnegative scalar | Coercivity bound: $\langle K_{\rm imm}(t)x, x\rangle_W \ge \kappa(t)\|x\|_W^2$ |
| $\varepsilon_{\rm rem}(t)$ | Nonnegative scalar | Remainder bound in the Lyapunov inequality |
| $\lambda(t) := \kappa(t) - \varepsilon_{\rm rem}(t)$ | Nonnegative scalar | Instantaneous decay rate (envelope rate) |
| $R(t) := \|e(t)\|_W^2$ | Nonnegative scalar | Residual energy in time |
| $\widehat{\tau}$ | Scalar | DSFL time via $d\widehat{\tau} = 2\lambda(t)\,dt$; unit-slope Lyapunov clock |
| $\lambda_{\rm BH}, \lambda_{\rm scr}$ | Nonnegative scalars | BH/near–horizon and scrambling Lyapunov rates in the BH calibration |
| $\lambda_{\rm QM}, \lambda_{\rm TD}, \lambda_{\rm GR}$ | Nonnegative scalars | Sectoral Lyapunov rates in other DSFL calibrations (when used) |
| **Causality and cone parameters (relay loop)** | | |
| $c$ | Speed constant | Instrument light–cone speed (cone front speed) |
| $\Psi_t$ | Semigroup on $\mathcal{H}_p$ | Relay evolution; cone bound with margin $\xi$ |
| $\Pi_O$ | Projection/localiser | Projection onto defects supported in region $O$ |
| $\xi$ | Length/time scale | Cone sharpness in bounds of the form $\|\Pi_{O'}\Psi_t\Pi_O\| \le C\,\exp(-(\mathrm{dist}(O,O') - ct)/\xi)$ |
| $\Lambda_{\rm UV}$ | Cutoff | Ultraviolet regulator; typically $\xi \sim \mathrm{const}/\Lambda_{\rm UV}$ |
| **GR calibration and focusing (BH sector)** | | |
| $\mathcal{E}_{\mu\nu}[g;T]$ | Tensor | Einstein imbalance $G_{\mu\nu}[g] + \Lambda g_{\mu\nu} - 8\pi G\,\langle T_{\mu\nu}\rangle$ in a collar |
| $W_{\rm GR}[g]$ | SPD operator | GR instrument weight; graph/energy norm for perturbations of $(g, \langle T\rangle)$ |
| $R_{\rm GR}(t)$ | Scalar | GR residual $\|\mathcal{E}_{\mu\nu}[g(t);T(t)]\|^2_{W_{\rm GR}[g(t)]}$ |
| $\lambda_{\rm GR}(t)$ | Scalar | GR Lyapunov rate in the near–horizon calibration, tied to QNM damping |
| $k^\mu$ | Null vector | Generator of a null congruence along a horizon or lightsheet |
| $\lambda$ | Affine parameter | Affine parameter along $k^\mu$ |
| $T_{kk}$ | Scalar | Null–null component $T_{\mu\nu}k^\mu k^\nu$ of the stress tensor |
| $S_{\rm out}(\lambda)$ | Scalar | Von Neumann entropy of quantum fields outside a deformed cut |
| $S_{\rm gen}(\lambda)$ | Scalar | Generalised entropy $A/4G + S_{\rm out}$ along a null congruence |
| **Scrambling, Page time, and complexity** | | |
| $t^{\rm DSFL}_{\rm scr}(\varepsilon)$ | Scalar | DSFL scrambling time: minimal time/steps to reduce a residual by factor $\varepsilon$ |
| $t^{\rm DSFL}_{\rm Page}$ | Scalar | DSFL Page–like time (inside/outside DSFL residuals equal, correlation residual peaked) |
| $S_{\rm BH}(t), S_{\rm rad}(t)$ | Scalars | Coarse–grained BH and radiation entropies (or DSFL proxies thereof) |
| $\alpha, \alpha^*$ | Scalars | DSFL Lyapunov rates in simple toy models, $R(t) \approx R(0)e^{-2\alpha t}$; extracted from audits |

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| Symbol | Type / Domain | Meaning / Assumptions |
|---|---|---|
| $\gamma$ | Scalar | Entropy–response exponent, e.g. $S_{BH}(t) = S_0 R(t)^\gamma$ in the toy Page law |
| $\mathcal{C}_{DSFL}$ | Scalar | DSFL depth / complexity (Lyapunov depth or cumulative DSFL time) |
| $\mathcal{C}(t)$ | Scalar | Computational complexity (when used in conditional Susskind–type bounds) |
| $\lambda_L$ | Scalar | Chaos / Lyapunov exponent in MSS OTOC bound |
| $\beta$ | Scalar | Inverse temperature, $\beta = 1/(k_B T)$ |
| **Cosmology and Hubble sector (when used)** | | |
| $a(t)$ | Scalar | Scale factor in FRW backgrounds |
| $H(t) := \dot{a}(t)/a(t)$ | Scalar | Hubble parameter; geometric expansion rate |
| $N(t)$ | Scalar | Number of e–folds, $N(t) = \log(a(t)/a(0)) = \int_0^t H(\sigma)\,d\sigma$ |
| $T_{dS}$ | Scalar | de Sitter temperature $T_{dS} = H/(2\pi)$ in a static patch |
| $T_{BH}$ | Scalar | Hawking temperature of a black hole (e.g. $1/(8\pi M)$ in Schwarzschild) |
| **CIU handles and scores (Contextual Importance & Utility)** | | |
| $u(\mathcal{R})$ | Scalar in $[0,1]$ | Utility / score derived from a residual, e.g. $u(\mathcal{R}) = 1 - e^{-k\mathcal{R}}$ |
| $[u_{min}^{ref}, u_{max}^{ref}]$ | Interval | Fixed reference band for $u$ (Främling CIU normalisation) |
| $u_{min}, u_{max}$ | Scalars | Sweep band for $u$ reached by varying a handle family $\{T_\theta\}$ |
| $u_{now}$ | Scalar | Current score at a nominal handle setting $\theta_0$ |
| CI | Scalar in $[0,1]$ | Contextual Importance (headroom), as in CIU; normalised bandwidth |
| CU | Scalar in $[0,1]$ | Contextual Utility (efficacy); placement of $u_{now}$ within the sweep band |
| $u_{best}$ | Scalar | CIU–predicted best reachable score within the current handle family |
| $\mathcal{R}_{best}$ | Scalar | CIU–predicted best reachable residual via $u^{-1}(u_{best})$ |
| **Miscellaneous symbols** | | |
| $\|\cdot\|$ | Norm | Ambient $\mathcal{H}$ norm (context may indicate operator, Hilbert–Schmidt, or trace norm) |
| $\langle\cdot,\cdot\rangle$ | Inner product | Ambient $\mathcal{H}$ inner product |
| $A \succeq 0$ | Loewner order | Positive semidefinite operator: $\langle Ax, x\rangle \geq 0$ for all $x$ |
| $\mathrm{dist}(x, V)$ | Nonnegative scalar | Distance of $x$ to subspace $V$ in $\mathcal{H}$ |

## Master Definitions — Compact Longtable

| Entry | Definition / Formula | Role / Notes |
|---|---|---|
| **Ambient Hilbert geometry ("one room")** | | |
| Instrument room | $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, $\|x\|^2 = \langle x, x\rangle$ | Single calibrated comparison space for all DSFL sectors (QM, TD, GR, model worlds) |
| Blueprint / response | $\mathcal{H}_s, \mathcal{H}_p \subset \mathcal{H}$ (closed) | Statistical blueprints (sDoF) and physical responses (pDoF) |
| Calibration | $\mathbb{I} : \mathcal{H}_s \to \mathcal{H}_p$ (bounded, linear) | Interprets blueprints as ideal responses; coherent image $U := \overline{\mathrm{ran}\,\mathbb{I}}$ |
| Instrument weight | $W : \mathcal{H}_p \to \mathcal{H}_p$, $W \succ 0$ | Defines instrument inner product and norm $\|x\|_W^2 = \langle x, Wx\rangle$ |
| Frames | $V \subset \mathcal{H}_p$, projector $\Pi_V$ | Measurement / reconstruction / constraint subspaces (frames) |

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| Entry | Definition / Formula | Role / Notes |
|---|---|---|
| **Single observable ("one residual")** | | |
| Residual of sameness | $R(s,p) := \|p - \mathbb{I}s\|_W^2$ | Only scalar used for DPI, Lyapunov rates, cones, Page time and audits |
| Residual direction | $e := p - \mathbb{I}s \in \mathcal{H}_p$ | Carries all mismatch / budget; $R(s,p) = \|e\|_W^2$ |
| DPI / admissibility | $R(\widetilde{\Phi}s, \Phi p) \le R(s,p)$ | Holds iff $\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi}$ and $\|\Phi\|_{W\to W} \le 1$ (single–residual DPI) |
| Spectral test | $\Phi^* W \Phi \preccurlyeq W$ | Equivalent operator inequality for nonexpansiveness in $\|\cdot\|_W$ |
| Channel Gram operator | $C_W(\Phi) := W^{-1/2}\Phi^* W \Phi W^{-1/2}$ | $\|\Phi\|_{W\to W}^2 = \lambda_{\max}(C_W(\Phi))$; largest eigenvalue = slowest sameness mode |
| **Nearest point (collapse / tightening mechanism)** | | |
| Orthogonal projector | $\Pi_V : \mathcal{H}_p \to V$ | Idempotent, nonexpansive, range $V$; canonical DSFL sharp update |
| Lüders–type uniqueness | $T^2 = T$, $\operatorname{ran} T \subseteq V$, $\|T\|_{W\to W} \le 1 \Rightarrow T = \Pi_V$ | Sharp update = unique DSFL–admissible nearest point onto $V$ in the instrument norm |
| **Two–loop law and DSFL time ("one ruler")** | | |
| Immediate loop | $K_{\mathrm{imm}}(t) \succeq 0$, $\langle K_{\mathrm{imm}}(t)x, x\rangle_W \ge \kappa(t)\|x\|_W^2$ | Time–local contraction of the defect in the instrument norm |
| Relay (memory) | Retarded $M(\sigma) \succeq 0$ for $\sigma \ge 0$ | Finite–speed transport of mismatch across the room (Volterra loop) |
| Envelope rate | $\lambda(t) := \kappa(t) - \varepsilon_{\mathrm{rem}}(t) \ge 0$ | Printed decay rate in $\dot{R}(t) \le -2\lambda(t)R(t)$ |
| DSFL clock | $d\widehat{\tau} = 2\lambda(t)\,dt$ | Time parametrisation with unit–slope Lyapunov envelope |
| Unit slope | $\frac{d}{d\widehat{\tau}} \log R(\widehat{\tau}) \le -1$ | In DSFL time, $\log R$ vs. $\widehat{\tau}$ is a line of slope $-1$ (or steeper) |
| DSFL depth / complexity | $\mathcal{C}_{\mathrm{DSFL}} \sim \widehat{\tau}$ | Intrinsic Lyapunov depth; appears in scrambling and complexity comparisons |
| **Locality from relay (cone bounds)** | | |
| Cone speed | $c > 0$ | Emergent signal / light–cone speed in the instrument norm |
| Cone margin | $\xi \sim \mathrm{const}/\Lambda_{\mathrm{UV}}$ | Smearing scale; sharper cone as cutoff $\Lambda_{\mathrm{UV}}$ increases |
| Instrument cone | $\|\Pi_{O'}\Psi_t\Pi_O\|_{W\to W} \le C\,\exp(-(1.7em(O,O') - ct)/\xi)$ | Causality constraint for relay dynamics in the DSFL room |
| **Black–hole room and residual budgets** | | |
| BH room | $\mathcal{H}_{\mathrm{room}} \cong \mathcal{H}_{\mathrm{BH}} \otimes \mathcal{H}_{\mathrm{near}} \otimes \mathcal{H}_{\mathrm{far}}$ | Calibrated DSFL room for interior, near–zone collar, and far radiation |
| BH residual | $R(t) = \|p(t) - \mathbb{I}s(t)\|_{W[g(t)]}^2$ | Single BH DSFL residual in the GR/near–horizon weight $W[g]$ |
| Inside / outside budgets | $R_{\mathrm{in}}(t) = \|\Pi_{\mathrm{BH}}e(t)\|_{W[g(t)]}^2$, $R_{\mathrm{out}}(t) = \|\Pi_{\mathrm{far}}e(t)\|_{W[g(t)]}^2$ | Interior vs. radiation residual budgets in the same norm |
| Correlation residual | $R_{\mathrm{corr}}(t) = R_{\mathrm{tot}}(t) - R_{\mathrm{in}}(t) - R_{\mathrm{out}}(t)$ | Part of the defect genuinely shared between BH and radiation |
| Scrambling residual | $R_{\mathrm{scr}}(t)$ | DSFL proxy for loss of local distinguishability (e.g. distance of a small subsystem to equilibrium) |
| **Finite–dimensional DSFL channels (toys)** | | |

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| Entry | Definition / Formula | Role / Notes |
|---|---|---|
| Single system | $\mathcal{H} \cong \mathbb{C}^d$, $\mathcal{B}(\mathcal{H})$ | Finite–dimensional Hilbert space and operator algebra |
| State space | $\mathcal{S}(\mathcal{H})$ | Density matrices (positive, trace one) |
| Channel | $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ | Completely positive trace–preserving (CPTP) map |
| Admissible channel | $\|\Phi\|^2_{W\to W} = \lambda_{\max}(C_W(\Phi)) \leq 1$ | DSFL–lawful evolution (single–residual DPI holds) |
| **GR calibration and focusing (near horizon)** | | |
| Einstein imbalance | $\mathcal{E}_{\mu\nu}[g;T] = G_{\mu\nu}[g] + \Lambda g_{\mu\nu} - 8\pi G \langle T_{\mu\nu}\rangle$ | Curvature–matter mismatch; vanishes on semiclassical solutions |
| GR weight | $W_{\rm GR}[g]$ | Graph/energy norm controlling linearised Einstein–matter perturbations in a collar |
| GR residual | $R_{\rm GR}(t) = \|\mathcal{E}_{\mu\nu}[g(t);T(t)]\|^2_{W_{\rm GR}[g(t)]}$ | DSFL Lyapunov functional in near–horizon GR calibration |
| QNM rate | $\lambda_{\rm QNM} > 0$ | Dominant quasinormal–mode decay rate controlling $R_{\rm GR}(t)$ |
| Generalised entropy | $S_{\rm gen}(\lambda) = A(\lambda)/(4G) + S_{\rm out}(\lambda)$ | Area plus outside entropy along null generators; subject to QNEC/QFC–type focusing |
| **Scrambling, MSS chaos bound, and Page time** | | |
| OTOC exponent | $\lambda_L$, with MSS bound $\lambda_L \leq 2\pi/\beta$ | Lyapunov exponent extracted from out–of–time–order correlators in a thermal state |
| DSFL scrambling rate | $\alpha_{\rm scr}$ | Lyapunov rate extracted from $R_{\rm scr}(t) \sim e^{-2\alpha_{\rm scr}t}$; identified with $\lambda_L$ in suitable models |
| DSFL scrambling time | $t^{\rm DSFL}_{\rm scr}(\varepsilon)$ | Minimal depth/time to reduce a residual by factor $\varepsilon$ |
| Toy residual | $R(t) = e^{-2\alpha t}$ | Simple DSFL Lyapunov model used in finite–dimensional Page law |
| Toy entropies | $S_{\rm BH}(t) = S_0 R(t)^\gamma$, $S_{\rm rad}(t) = S_0(1 - R(t)^\gamma)$ | DSFL Page–curve toy; $t_{\rm Page} = \frac{\log 2}{2\alpha\gamma}$ |
| DSFL Page time | $t^{\rm DSFL}_{\rm Page}$ | Time when $R_{\rm in}(t) \approx R_{\rm out}(t)$ and $R_{\rm corr}(t)$ peaks (Page–like balance) |
| **Cosmology and Hubble scaling (when used)** | | |
| Hubble parameter | $H(t) = \dot{a}(t)/a(t)$ | Geometric expansion rate in FRW backgrounds |
| e–fold time | $N(t) = \log(a(t)/a(0)) = \int_0^t H(\sigma)\,d\sigma$ | Natural coarse–grained time variable for cosmological DSFL laws |
| de Sitter temperature | $T_{\rm dS} = H/(2\pi)$ | Gibbons–Hawking temperature of a de Sitter static patch |
| BH temperature | $T_{\rm BH}$ | Hawking temperature of a black hole (e.g. $1/(8\pi M)$ in Schwarzschild) |
| Minimal rate | $\lambda_{\min}(t) = \min\{\lambda_{\rm BH}(t), H(t), \ldots\}$ | Effective DSFL Lyapunov rate when BH and cosmological sectors compete |
| **Complexity and DSFL depth** | | |
| DSFL depth | $\hat{\tau}(t) = \int_0^t 2\lambda(\sigma)\,d\sigma$ | Accumulated Lyapunov depth; unit–slope coordinate for $\log R$ |
| Circuit complexity | $\mathcal{C}(t)$ | Minimal gate count to prepare a state from a reference (in a given gate set) |

| Entry | Definition / Formula | Role / Notes |
|---|---|---|
| DSFL–complexity bounds | $A_- + B_-\widehat{\tau}(t) \leq \mathcal{C}(t) \leq A_+ + B_+\widehat{\tau}(t)$ | Coarse linear upper/lower bounds linking complexity growth to DSFL depth in scrambling window |
| **CIU handles and utilities (Contextual Importance & Utility)** | | |
| Utility | $u(\mathcal{R}) \in [0,1]$ | Monotone score derived from a residual (e.g. $u = 1 - e^{-k\mathcal{R}}$) |
| Reference band | $[u^{\text{ref}}_{\text{min}}, u^{\text{ref}}_{\text{max}}]$ | Global normalisation interval for CIU scores |
| Sweep band | $u_{\text{min}}, u_{\text{max}}$ from a handle family $\{T_\theta\}$ | Min/max utility achieved by varying a specific handle in context |
| Contextual Importance | $\text{CI} = \dfrac{u_{\text{max}} - u_{\text{min}}}{u^{\text{ref}}_{\text{max}} - u^{\text{ref}}_{\text{min}}}$ | Headroom of a handle (or region) in the current context |
| Contextual Utility | $\text{CU} = \dfrac{u_{\text{now}} - u_{\text{min}}}{u_{\text{max}} - u_{\text{min}}}$ | How good the current setting is, relative to its sweep band |
| CIU forecast | $u_{\text{best}} = u_{\text{now}} + (1 - \text{CU})\,\text{CI}(u^{\text{ref}}_{\text{max}} - u^{\text{ref}}_{\text{min}})$ | Predicted best reachable score; inverted to a best residual $\mathcal{R}_{\text{best}}$ |
| **Falsification / stress tests (device level)** | | |
| Unit–slope test | Upper convex hull of $(\widehat{\tau}, \log R)$ has slope $\leq -1$ | Fails if non–admissible steps or miscalibrated DSFL clock |
| Cone test | No response for $t < 1.7em(O,O')/c$, tails $\sim \exp(-(1.7em(O,O') - ct)/\xi)$ | Violations indicate superluminal or nonlocal relay evolution |
| Band test | One– / two–frame bands tighten under DPI | Dimension–free audit of calibration, frames, and nonexpansiveness |

## Appendix A. Technical DSFL Framework Details

In this appendix we collect the main proofs underlying Section 3: the DPI characterisation of admissible maps, the Lyapunov law and DSFL clock, and the cone bound. We only use standard Hilbert–space operator technology and well–known locality results for lattice systems [24,26,27,38–41].

Throughout, $(\mathcal{H}, \mathcal{H}_s, \mathcal{H}_p, \mathbb{I}, W)$ is a fixed DSFL room in the sense of Definition 1, with residual $R(s,p) = \|p - \mathbb{I}s\|^2_W$.

### *Appendix A.1. Proof of Theorem 1*

We restate the admissibility/DPI equivalence for convenience.

**Theorem A1** (Admissibility $\iff$ DPI for $R$)**.** *Let $(\mathcal{H}, \mathcal{H}_s, \mathcal{H}_p, \mathbb{I}, W)$ be a DSFL room and let $\widetilde{\Phi} : \mathcal{H}_s \to \mathcal{H}_s$, $\Phi : \mathcal{H}_p \to \mathcal{H}_p$ be bounded linear maps. The following are equivalent:*

(i) *$(\widetilde{\Phi}, \Phi)$ is DSFL–admissible, i.e.*

$$\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi} \quad \text{and} \quad \|\Phi u\|_W \leq \|u\|_W \;\; \forall u \in \mathcal{H}_p; \tag{A1}$$

(ii) *for all $(s,p) \in \mathcal{H}_s \times \mathcal{H}_p$ the residual data–processing inequality holds:*

$$R(\widetilde{\Phi}s, \Phi p) \;\leq\; R(s,p). \tag{A2}$$

**Proof.** We write $R(s,p) = \|p - \mathbb{I}s\|^2_W$ and use only linearity and strict positivity of $W$.

*(i) ⇒ (ii).* Assume $\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi}$ and $\|\Phi u\|_W \leq \|u\|_W$ for all $u \in \mathcal{H}_p$. Then for any $(s,p)$,

$$\begin{aligned} R(\widetilde{\Phi}s, \Phi p) &= \|\Phi p - \mathbb{I}\widetilde{\Phi}s\|_W^2 \\ &= \|\Phi p - \Phi\mathbb{I}s\|_W^2 = \|\Phi(p - \mathbb{I}s)\|_W^2 \\ &\leq \|p - \mathbb{I}s\|_W^2 = R(s,p), \end{aligned}$$

as claimed.

*(ii) ⇒ (i).* Assume now that $R(\widetilde{\Phi}s, \Phi p) \leq R(s,p)$ for all $(s,p)$.

*Step 1: calibration preservation.* Fix $s \in \mathcal{H}_s$ and set $p = \mathbb{I}s$. Then $R(s, \mathbb{I}s) = 0$, so

$$0 \leq R(\widetilde{\Phi}s, \Phi\mathbb{I}s) \leq R(s, \mathbb{I}s) = 0. \tag{A3}$$

Thus $R(\widetilde{\Phi}s, \Phi\mathbb{I}s) = 0$, which means $\Phi\mathbb{I}s - \mathbb{I}\widetilde{\Phi}s = 0$ by strict positivity of $W$. Hence $\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi}$ on all of $\mathcal{H}_s$.

*Step 2: nonexpansiveness.* Set $s = 0$ and keep $p \in \mathcal{H}_p$ arbitrary. Then the DPI gives

$$R(\widetilde{\Phi}0, \Phi p) \;\leq\; R(0,p), \tag{A4}$$

i.e.

$$\|\Phi p - \mathbb{I}\widetilde{\Phi}0\|_W^2 \leq \|p\|_W^2 \qquad \forall p \in \mathcal{H}_p. \tag{A5}$$

Define $y := \mathbb{I}\widetilde{\Phi}0 \in \mathcal{H}_p$. We first show that $y = \Phi 0$. Apply the DPI with $(s,p) = (0,0)$:

$$R(\widetilde{\Phi}0, \Phi 0) \leq R(0,0) = 0, \tag{A6}$$

so $\Phi 0 - \mathbb{I}\widetilde{\Phi}0 = 0$, hence $\Phi 0 = y$.

Substituting $y = \Phi 0$ into (A5), we obtain

$$\|\Phi p - \Phi 0\|_W^2 \leq \|p\|_W^2 \qquad \forall p \in \mathcal{H}_p. \tag{A7}$$

Let $x \in \mathcal{H}_p$ be arbitrary and apply (A7) with $p = x$ and $p = -x$:

$$\begin{gathered} \|\Phi x - \Phi 0\|_W^2 \leq \|x\|_W^2, \\ \|\Phi(-x) - \Phi 0\|_W^2 = \|-\Phi x - \Phi 0\|_W^2 \leq \|-x\|_W^2 = \|x\|_W^2. \end{gathered}$$

Using the parallelogram identity in the Hilbert norm induced by $W$ [38, Sec. 1.2],

$$\|\Phi x + \Phi 0\|_W^2 + \|\Phi x - \Phi 0\|_W^2 = 2\|\Phi x\|_W^2 + 2\|\Phi 0\|_W^2, \tag{A8}$$

and the two inequalities above, we find

$$2\|\Phi x\|_W^2 + 2\|\Phi 0\|_W^2 \;\leq\; 2\|x\|_W^2, \tag{A9}$$

hence

$$\|\Phi x\|_W^2 \leq \|x\|_W^2 - \|\Phi 0\|_W^2 \leq \|x\|_W^2. \tag{A10}$$

Taking square roots shows $\|\Phi x\|_W \leq \|x\|_W$ for all $x$, i.e. $\Phi$ is nonexpansive in $\|\cdot\|_W$.

We have shown both $\Phi\mathbb{I} = \mathbb{I}\widetilde{\Phi}$ and $\|\Phi\|_{W\to W} \leq 1$, so $(\widetilde{\Phi}, \Phi)$ is DSFL–admissible. □

This is the exact analogue, for the single quadratic residual $R$, of standard DPIs characterising contractive channels and relative entropy in quantum information theory [22–24].

*Appendix A.2. Lyapunov Inequality and DSFL Clock*

We now make precise the Lyapunov inequality and the associated DSFL clock for continuous time. This is a standard computation in the spirit of dissipative semigroups [26,27].

**Assumption 5** (Dissipativity in the instrument norm)**.** *Let $t \mapsto e(t) \in \mathcal{H}_p$ be differentiable and satisfy*

$$\frac{d}{dt}e(t) = A(t)e(t), \tag{A11}$$

*for a bounded operator–valued map $t \mapsto A(t)$. Assume there is a measurable function $\lambda(t) \geq 0$ such that for all $x \in \mathcal{H}_p$,*

$$\operatorname{Re}\langle x, WA(t)x\rangle \;\leq\; -\lambda(t)\,\langle x, Wx\rangle \qquad \textit{for a.e. } t \geq 0. \tag{A12}$$

**Theorem A2** (Lyapunov inequality and DSFL clock)**.** *Under* **??** *5, the residual $R(t) := \|e(t)\|_W^2$ satisfies*

$$\frac{d}{dt}R(t) \leq -2\lambda(t)R(t) \tag{A13}$$

*for almost every $t \geq 0$. Consequently,*

$$R(t) \leq R(0)\exp\Big(-\int_0^t 2\lambda(\sigma)\,d\sigma\Big). \tag{A14}$$

*Defining the DSFL clock*

$$\hat{\tau}(t) := \int_0^t 2\lambda(\sigma)\,d\sigma, \tag{A15}$$

*one has*

$$\log R(\hat{\tau}) \leq \log R(0) - \hat{\tau} \tag{A16}$$

*for all $t \geq 0$.*

**Proof.** Differentiating $R(t) = \langle e(t), We(t)\rangle$ gives

$$\begin{aligned}\frac{d}{dt}R(t) &= \langle \dot{e}(t), We(t)\rangle + \langle e(t), W\dot{e}(t)\rangle \\ &= \langle A(t)e(t), We(t)\rangle + \langle e(t), WA(t)e(t)\rangle \\ &= 2\,\operatorname{Re}\langle e(t), WA(t)e(t)\rangle.\end{aligned}$$

Applying (A12) with $x = e(t)$ yields

$$\frac{d}{dt}R(t) \leq -2\lambda(t)\langle e(t), We(t)\rangle = -2\lambda(t)R(t) \tag{A17}$$

for almost every $t \geq 0$. Grönwall's lemma [26, Thm. X.1.1] then gives

$$R(t) \leq R(0)\exp\Big(-\int_0^t 2\lambda(\sigma)\,d\sigma\Big). \tag{A18}$$

Defining $\hat{\tau}$ as above, we obtain $\log R(\hat{\tau}) \leq \log R(0) - \hat{\tau}$. □

In many applications one has $\lambda(t) \equiv \lambda_0 > 0$, in which case $\hat{\tau} = 2\lambda_0 t$ and $\log R$ decays as a straight line of slope $-2\lambda_0$ in physical time and slope $-1$ in DSFL time.

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Discrete time.

For iterates $e_{k+1} = \Phi_k e_k$, nonexpansiveness $\|\Phi_k\|_{W\to W} \le 1$ implies $R_{k+1} \le R_k$. Defining

$$\widehat{\tau}_0 := 0, \qquad \widehat{\tau}_{k+1} := \widehat{\tau}_k - \log \frac{R_{k+1}}{R_k}, \tag{A19}$$

one has $\log R_k = \log R_0 - \widehat{\tau}_k$ exactly. This discrete construction is the one used in our numerical audits and mirrors standard entropy/Dirichlet form parametrisations for Markov chains [25,28].

*Appendix A.3. Cone Locality from Lieb–Robinson Bounds*

We finally justify the cone bound in Definition 3 by quoting standard Lieb–Robinson results. We focus on a lattice setting; the continuum case is analogous.

Let $\Lambda$ be a countable discrete set equipped with a metric $d : \Lambda \times \Lambda \to [0, \infty)$ (e.g. $\mathbb{Z}^d$ with graph distance). Let $\mathcal{H} = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ with $\dim \mathcal{H}_x < \infty$, and denote by $\mathcal{A}_X$ the algebra of operators supported on a finite region $X \subset \Lambda$.

Consider a local generator on the observable algebra of the form

$$\mathcal{L} = \sum_{Z\subset\Lambda} \mathcal{L}_Z, \tag{A20}$$

where each $\mathcal{L}_Z$ acts nontrivially only on $\mathcal{A}_Z$ and satisfies an exponential decay bound

$$\sum_{Z\ni x,y} \|\mathcal{L}_Z\|\, e^{\mu\ \mathrm{diam}(Z)} \le J\, e^{-\mu d(x,y)}, \tag{A21}$$

for some $J, \mu > 0$ and all $x, y \in \Lambda$. This is the standard locality hypothesis in Lieb–Robinson theory [39,41].

Let $\Psi_t := \mathrm{e}^{t\mathcal{L}^*}$ be the induced Heisenberg evolution on observables. The Lieb–Robinson bound states that there exist constants $C, \mu', v > 0$ such that for any observables $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ with finite supports $X, Y \subset \Lambda$,

$$\|[\Psi_t(A), B]\| \le C\,\|A\|\,\|B\|\,\exp\big(-\mu'\,[d(X,Y) - v|t|]_+\big), \tag{A22}$$

where $[z]_+ := \max\{z, 0\}$ [39–41].

To translate (A22) into a DSFL cone bound, we think of $\mathcal{H}_p$ as a Hilbert space of "defects" (e.g. vectors in a Hilbert–Schmidt representation of observables or perturbations of a reference state) and of $\Psi_t$ as a linear map on $\mathcal{H}_p$ which is nonexpansive in the instrument norm $\|\cdot\|_W$ (for suitable $W$).

Let $O, O' \subset \Lambda$ be finite regions and let $\Pi_O$, $\Pi_{O'}$ denote the orthogonal projections onto the subspaces of $\mathcal{H}_p$ corresponding to defects supported in $O$ and $O'$. Then the Lieb–Robinson estimate implies a bound of the form

$$\|\Pi_{O'}\Psi_t\Pi_O\|_{W\to W} \le C'\,\exp\Big(-\frac{d(O,O') - vt}{\xi}\Big), \tag{A23}$$

for some $C' > 0$ and decay length $\xi \sim 1/\mu'$, after adjusting constants and taking into account the finite support structure of basis vectors in $\mathcal{H}_p$. The precise identification of $W$ and the projection structure depends on the model (spin systems, free fields, etc.), but the qualitative form (A23) is universal: information cannot propagate faster than a finite speed $v$ in the instrument norm.

**Theorem A3** (Cone locality from Lieb–Robinson)**.** *Let $\Psi_t$ be a nonexpansive DSFL relay arising from a local generator satisfying* (A21)*. Then there exist constants $(C, v, \xi)$ such that the cone bound of Definition 3 holds:*

$$\|\Pi_{O'}\Psi_t\Pi_O\|_{W\to W} \le C\,\exp\Big(-\frac{1.7em(O,O') - vt}{\xi}\Big) \tag{A24}$$

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*for all finite regions $O, O' \subset \Lambda$ and all $t \geq 0$.*

**Proof sketch.** Combine the Lieb–Robinson commutator bound (A22) with the representation of vectors in $\mathcal{H}_p$ as superpositions of localised basis elements (e.g. local operators or localised wave packets) and use standard arguments to convert bounds on commutators into decay of matrix elements between spatially separated sectors [40,41, Sec. 3]. Nonexpansiveness in $\|\cdot\|_W$ ensures that the instrument norm and the operator norm appearing in (A22) are compatible up to model–dependent constants. □

Thus, under standard locality assumptions, the DSFL relay part of the evolution inherits a Lieb–Robinson cone in the same instrument norm used for the Lyapunov envelope. This is the cone structure implicitly used in our BH room and numerical audits.

## Appendix B. Finite–Dimensional DSFL Diagnostics: Qubit Channels

This appendix records the basic DSFL spectral envelopes for three standard one–qubit channels: amplitude damping, depolarising noise and pure dephasing. Their behaviour illustrates, in a fully controlled setting, how the DSFL residual

$$R_k = \|\Phi^k(\rho_0) - \rho_*\|_W^2 \tag{A25}$$

decays at a rate fixed by the spectrum of the channel $\Phi$. Although elementary, these examples provide a useful benchmark: the same DSFL envelope structure reappears in black–hole scrambling, where the role of $|\lambda_2|$ is played by the QNM gap or the OTOC chaos exponent.

Throughout this appendix we take the instrument norm to be the Hilbert–Schmidt norm ($W = \mathbb{I}$).

### Appendix B.1. Amplitude–Damping Channel

The amplitude–damping channel $\mathcal{E}_\gamma$ with parameter $\gamma \in (0,1)$ has Kraus operators

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \tag{A26}$$

and unique fixed point $\rho_* = |0\rangle\langle 0|$.

**Lemma A1** (Spectrum)**.** *In Liouville representation, the superoperator $S_\gamma$ has eigenvalues*

$$\sigma(S_\gamma) = \{1, \sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma\}. \tag{A27}$$

Let $\lambda_2 = \sqrt{1-\gamma}$ be the second–largest modulus. Define $R_k = \|\mathcal{E}_\gamma^k(\rho_0) - \rho_*\|_2^2$.

**Proposition A1** (DSFL envelope)**.** *For generic initial states,*

$$R_k \sim C(1-\gamma)^k. \tag{A28}$$

*For special states aligned with the population mode (e.g. $\rho_0 = |1\rangle\langle 1|$),*

$$R_k = C_{\text{pop}}(1-\gamma)^{2k}. \tag{A29}$$

*Thus the DSFL Lyapunov rate is $-\log|\lambda_2|$, and state–dependent alignment determines whether the envelope is saturated or decays faster.*

These two regimes reflect exactly what happens in the BH room: generic states saturate the slowest Lyapunov mode, while specially aligned states decay at a higher rate.

*Appendix B.2. Depolarising Channel*

For the depolarising channel

$$\mathcal{D}_p(\rho) = (1-p)\rho + p\,\frac{I}{2}, \qquad p \in (0,1), \tag{A30}$$

the Liouville spectrum is

$$\{1,\, 1-p,\, 1-p,\, 1-p\}. \tag{A31}$$

**Proposition A2** (Uniform DSFL scrambler)**.** *For every initial state,*

$$R_k = (1-p)^{2k}\, R_0, \qquad \log R_k = 2\log(1-p)\,k + \log R_0. \tag{A32}$$

*Thus $\alpha_{\mathrm{eff}} = -\log(1-p)$ is a state–independent Lyapunov rate: the depolarising channel is a uniform DSFL scrambler.*

This behaviour mirrors high–temperature holographic scramblers: the DSFL envelope is always saturated by every state.

*Appendix B.3. Pure Dephasing*

For the dephasing channel

$$\mathcal{F}_p(\rho) = (1-p)\rho + p\,Z\rho Z, \qquad p \in (0,1), \tag{A33}$$

$$\sigma(S_p) = \{1,\, 1,\, 1-2p,\, 1-2p\}. \tag{A34}$$

**Proposition A3** (Gapless DSFL channel)**.** *The induced HS–norm satisfies*

$$\|\mathcal{F}_p\|_{2\to 2} = 1. \tag{A35}$$

*Diagonal deviations do not decay, and only coherences contract at rate $|1-2p|$. Hence no global DSFL Lyapunov rate exists.*

This provides the canonical example of a *gapless* DSFL system. In the BH setting, this corresponds to cases where the gravitational residual $R_{\mathrm{GR}}$ fails to contract globally unless QNM damping is present.

*Appendix B.4. Why These Channels Matter for Black–Hole DSFL*

These one–qubit models demonstrate three phenomena that reappear in black–hole scrambling:

- The amplitude–damping channel shows *mode–dependent* decay rates, exactly as BH quasinormal spectra do.
- The depolarising channel shows *uniform saturation* of the DSFL envelope, analogous to near–maximal chaos where $\lambda_{\mathrm{scr}} = 2\pi/\beta$.
- The dephasing channel shows what happens when no spectral gap exists — a direct analogue of geometries without QNM damping.

In this sense, the DSFL framework already contains, in finite dimension, the same spectral logic that governs black–hole scrambling, Page curves and QNEC/QFC focusing in curved spacetime.

## Appendix C. Haar and Random–Circuit DSFL Page Theorems

In this appendix we give more detailed versions of Theorems 5 and 6 and briefly describe the numerical audits used in Section 6. Throughout we work with qubits for concreteness; the qudit case is analogous.

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*Appendix C.1. Static Haar Ensemble*

Let $\mathcal{H}_{\rm tot}^{(n)} \cong (\mathbb{C}^2)^{\otimes n}$ and fix a bipartition into "black hole" and "radiation" subsystems

$$\mathcal{H}_{\rm tot}^{(n)} = \mathcal{H}_{\rm BH}(k) \otimes \mathcal{H}_{\rm rad}(k), \qquad \dim \mathcal{H}_{\rm BH}(k) = 2^{n-k}, \ \dim \mathcal{H}_{\rm rad}(k) = 2^k, \tag{A36}$$

with $k = 0, \ldots, n$ interpreted as the number of emitted qubits. Let $\rho = |\Psi\rangle\langle\Psi|$ be Haar–random on $\mathcal{H}_{\rm tot}^{(n)}$ and denote by $\rho_{\rm BH}(k)$ and $\rho_{\rm rad}(k)$ the reduced density matrices.

In the Hilbert–Schmidt DSFL geometry ($W = \mathbb{I}$), the BH and radiation residuals to the maximally mixed state are

$$R_{\rm BH}^{(n)}(k) := \|\rho_{\rm BH}(k) - 2^{-(n-k)}\mathbb{I}\|_2^2, \qquad R_{\rm rad}^{(n)}(k) := \|\rho_{\rm rad}(k) - 2^{-k}\mathbb{I}\|_2^2. \tag{A37}$$

Page's purity formula for Haar–random pure states on $\mathcal{H}_A \otimes \mathcal{H}_B$ with dimensions $d_A \le d_B$ reads [8]

$$\mathbb{E}\big[\mathrm{Tr}(\rho_A^2)\big] = \frac{d_A + d_B}{d_A d_B + 1}. \tag{A38}$$

Since $R_A = \mathrm{Tr}(\rho_A^2) - 1/d_A$, one finds, for $d_A, d_B \gg 1$ with $d_A d_B$ fixed,

$$\mathbb{E}[R_A] = \frac{d_A^2 - 1}{d_A(d_A d_B + 1)} \sim \frac{d_A}{d_B^2}, \qquad \mathbb{E}[R_B] \sim \frac{d_B}{d_A^2}, \tag{A39}$$

which gives Theorem 5 as stated in the main text. The entropic Page analysis in [8] shows independently that the average subsystem entropy $\mathbb{E}[S(\rho_{\rm rad}(k))]$ is maximal when $d_{\rm rad} \approx d_{\rm BH}$, i.e. when $k \approx n/2$.

Defining the *DSFL Page index*

$$k_{\rm Page}^{\rm DSFL}(n) := \arg\min_{0 \le k \le n} \big|\mathbb{E}[R_{\rm BH}^{(n)}(k)] - \mathbb{E}[R_{\rm rad}^{(n)}(k)]\big| \tag{A40}$$

one checks directly from the asymptotics above that $k_{\rm Page}^{\rm DSFL}(n) = n/2 + O(1)$ and that $k_{\rm Page}^{\rm DSFL}(n)/n \to \frac{1}{2}$, agreeing with the entropic Page fraction in the large–$n$ limit [1,8].

*Appendix C.2. Random–Circuit Evaporation*

To model evaporation dynamically we follow the random–unitary picture of Hayden–Preskill [1] and subsequent decoupling work [17–19]. Let $\mathcal{H}^{(n)} \cong (\mathbb{C}^2)^{\otimes n}$ and, at step $k$, split

$$\mathcal{H}^{(n)} = \mathcal{H}_{\rm BH}(k) \otimes \mathcal{H}_{\rm rad}(k), \qquad \dim \mathcal{H}_{\rm BH}(k) = 2^{n-k}, \ \dim \mathcal{H}_{\rm rad}(k) = 2^k. \tag{A41}$$

We start from some initial pure state $|\Psi_0\rangle$ and define a discrete evolution:

(a) (*Emission*) Relabel one qubit from $\mathcal{H}_{\rm BH}(k)$ to $\mathcal{H}_{\rm rad}(k+1)$.
(b) (*Scrambling*) Apply a depth–$D$ nearest–neighbour brickwork circuit of random two–qubit gates on the full $n$–qubit system.

Assume that each scrambling layer forms, or closely approximates, a local unitary 2–design on the interior register [1,32]. Then, for fixed $k$, the reduced state $\rho_{\rm BH}(k)$ (and similarly $\rho_{\rm rad}(k)$) is close in low moments and trace distance to the Haar ensemble on the corresponding bipartition.

As a result, the expectations $\mathbb{E}[R_{\rm BH}(k)]$ and $\mathbb{E}[R_{\rm rad}(k)]$ at step $k$ are asymptotically equal to their static Haar values from Appendix C.1, up to errors controlled by the design quality. The same holds for the expected entropies $\mathbb{E}[S(\rho_{\rm BH}(k))]$ and $\mathbb{E}[S(\rho_{\rm rad}(k))]$ [1,19,32]. This yields Theorem 6: the *dynamical* DSFL Page step

$$k_{\rm Page}^{\rm DSFL} := \arg\min_k \mathbb{E}\big|R_{\rm BH}(k) - R_{\rm rad}(k)\big| \tag{A42}$$

satisfies $k_{\rm Page}^{\rm DSFL}/n \to 1/2$ and coincides, up to $O(1)$ in $k$, with the entropic Page step $k_{\rm Page}^{S} = \arg\max_k \mathbb{E}[S(\rho_{\rm rad}(k))]$.

*Appendix C.3. Numerical Audits*

For the static Haar case we generated $N = 40$ Haar–random pure states on $\mathcal{H}^{(n)}$ for $n = 8, 10, 12$ (using standard algorithms based on QR decomposition of random complex matrices) and evaluated $\overline{R}_{\mathrm{BH}}(k)$, $\overline{R}_{\mathrm{rad}}(k)$ and $\overline{S}_{\mathrm{rad}}(k)$ for all $k$. In all cases the DSFL residual curves cross at $k \approx n/2$, and the radiation entropy is maximal at the same $k$, in agreement with the theorems above.

For the random–circuit evaporator we took $n = 8$ qubits, depth–4 brickwork layers of Haar two–qubit gates between emissions, and $N = 32$ realisations. We again computed $R_{\mathrm{BH}}(k)$, $R_{\mathrm{rad}}(k)$ and $S_{\mathrm{rad}}(k)$ at each step. The measured DSFL Page step (minimal difference of residuals) and the entropic Page step (maximum of $\overline{S}_{\mathrm{rad}}$) both occurred at $k = 4$, where $d_{\mathrm{BH}} = d_{\mathrm{rad}} = 16$ (see Table A1), providing numerical support for Theorem 6 and for the DSFL Page prescription in a genuinely dynamical model.
For the random–circuit evaporator we took $n = 8$ qubits, depth–4 brickwork layers of Haar two–qubit gates between emissions, and $N = 32$ realisations. We again computed $R_{\mathrm{BH}}(k)$, $R_{\mathrm{rad}}(k)$ and $S_{\mathrm{rad}}(k)$ at each step. The measured DSFL Page step (minimal difference of residuals) and the entropic Page step (maximum of $\overline{S}_{\mathrm{rad}}$) both occurred at $k = 4$, where $d_{\mathrm{BH}} = d_{\mathrm{rad}} = 16$ (see Table A1), providing numerical support for Theorem 6 and for the DSFL Page prescription in a genuinely dynamical model.

**Table A1.** Random–circuit Page audit for $n = 8$ qubits ($N = 32$ realisations, depth–4 brickwork scrambler between emissions). Both the DSFL Page step $k^{\mathrm{DSFL}}_{\mathrm{Page}}$ (minimiser of $\overline{|R_{\mathrm{BH}}(k) - R_{\mathrm{rad}}(k)|}$) and the entropic Page step $k^{S}_{\mathrm{Page}}$ (maximiser of $\overline{S_{\mathrm{rad}}(k)}$) occur at $k = 4$, where the black–hole and radiation Hilbert spaces have equal dimension.

| Quantity | Symbol | Value |
|---|---|---|
| DSFL Page step | $k^{\mathrm{DSFL}}_{\mathrm{Page}}$ | 4 |
| Entropic Page step | $k^{S}_{\mathrm{Page}}$ | 4 |
| BH Hilbert space dimension at $k = 4$ | $d_{\mathrm{BH}}$ | 16 |
| Radiation Hilbert space dimension at $k = 4$ | $d_{\mathrm{rad}}$ | 16 |

## Appendix D. GR Calibration and $\|\cdot\|_{W[g]}$ QNEC/QFC

This appendix expands on the GR calibration used in Appendix D and on the interpretation of QNEC/QFC–type statements in the GR instrument norm $\|\cdot\|_{W_{\mathrm{GR}}[g]}$.

*Appendix D.1. GR DSFL Room and Lyapunov Law*

Let $(\mathcal{M}, g)$ be a fixed black–hole spacetime solving the semiclassical Einstein equation

$$G_{\mu\nu}[g] + \Lambda g_{\mu\nu} = 8\pi G\,\langle T_{\mu\nu}\rangle, \tag{A43}$$

with $\langle T_{\mu\nu}\rangle$ a renormalised stress tensor of a Hadamard QFT state [9,10,47]. Perturbations of $(g, \langle T\rangle)$ on an exterior or collar Cauchy slice $\Sigma$ can be modelled in a Hilbert space $\mathcal{H}^{\mathrm{GR}}_p$ completed in a suitable graph– or energy–type norm for the linearised Einstein–matter system [35,42].

Define the *Einstein imbalance*

$$\mathcal{E}_{\kappa,\Lambda}(g; T) := G_{\mu\nu}[g] + \Lambda g_{\mu\nu} - 8\pi G\,\langle T_{\mu\nu}\rangle, \tag{A44}$$

and choose a positive, selfadjoint "graph weight" $W_{\mathrm{GR}}[g] : \mathcal{H}^{\mathrm{GR}}_p \to \mathcal{H}^{\mathrm{GR}}_p$ so that

$$\|u\|^2_{W_{\mathrm{GR}}[g]} := \langle u, W_{\mathrm{GR}}[g]u\rangle \tag{A45}$$

controls both the energy norm and the linearised constraints on $\Sigma$ [10,35]. The GR residual used in the main text is

$$R_{\mathrm{GR}}(t) := \|\mathcal{E}_{\kappa,\Lambda}(g(t); T(t))\|^2_{W_{\mathrm{GR}}[g(t)]}. \tag{A46}$$

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Linearising the Einstein–matter system around $g$ and restricting to near–equilibrium collars, one finds that the evolution of the defect $e(t) := \mathcal{E}_{\kappa,\Lambda}(g(t); T(t))$ can be cast in a two–loop DSFL form,

$$\dot{e}(t) = -K_{\mathrm{imm}}[g(t)]e(t) - \int_0^t M_{g(t)}(t-\tau)e(\tau)\,\mathrm{d}\tau + r_g(t), \tag{A47}$$

with $K_{\mathrm{imm}}[g(t)] \succeq 0$, $M_{g(t)}(\sigma) \succeq 0$ for $\sigma \geq 0$ and $r_g(t)$ small in the quadratic–form sense; see [35] and [34,46] for detailed QNM/ringdown analyses. Under standard coercivity and boundedness assumptions, Theorem A2 then gives a GR Lyapunov inequality

$$\dot{R}_{\mathrm{GR}}(t) \;\leq\; -2\lambda_{\mathrm{GR}}(t)R_{\mathrm{GR}}(t), \tag{A48}$$

with $\lambda_{\mathrm{GR}}(t)$ controlled by the spectral gap of a $W_{\mathrm{GR}}[g(t)]$–graph operator associated with $K_{\mathrm{imm}}[g(t)]$, and hence by the dominant quasinormal modes of the background [34,46].

*Appendix D.2. QNEC in the GR Instrument Norm*

The quantum null energy condition (QNEC) relates local stress–energy to the second variation of outside entropy along a null generator. In $d$ spacetime dimensions it can be written schematically as

$$\langle T_{kk}(\lambda, y)\rangle_\psi \;\geq\; \frac{1}{2\pi}\,\partial_\lambda^2 S_{\mathrm{out}}(\lambda, y), \tag{A49}$$

where $k^\mu$ is a null vector, $T_{kk} = T_{\mu\nu}k^\mu k^\nu$ and $S_{\mathrm{out}}$ is an entanglement entropy density for a null half–space; see [12,15,16] for precise formulations and proofs in QFT.

In a null DSFL calibration along a lightsheet $\mathcal{N}$, one can choose $W_{\mathrm{GR}}[g;\lambda]$ so that the null residual $R_{\mathrm{null}}(\lambda)$ is proportional, to leading order, to a relative entropy functional measuring the deviation of the outside state from a reference state on the cut $S_\lambda$ [47,51]. In that case the convexity of $R_{\mathrm{null}}(\lambda)$ in $\lambda$ is equivalent to QNEC in a linearised regime, as stated in

**Theorem A4** (Weighted QNEC as convexity in the DSFL norm)**.** *Let $(\mathcal{M}, g)$ be a semiclassical spacetime and let $\sigma$ be a compact, spacelike codimension–two surface with a future–directed null normal $k^\mu$. Consider the family of deformed cuts $\{\sigma(\lambda)\}$ obtained by shifting $\sigma$ an affine parameter distance $\lambda$ along $k^\mu$, and let*

$$S_{\mathrm{gen}}(\lambda) := \frac{A(\lambda)}{4G} + S_{\mathrm{out}}(\lambda) \tag{A50}$$

*be the associated generalised entropy, where $S_{\mathrm{out}}$ is the von Neumann entropy of the quantum fields outside $\sigma(\lambda)$. Assume:*

(a) (GR DSFL calibration) *The near–horizon collar admits a GR DSFL room with weight $W_{\mathrm{GR}}[g]$ and gravitational residual $R_{\mathrm{GR}}(\lambda) = \|\mathcal{E}_{\kappa,\Lambda}[g(\lambda); T(\lambda)]\|^2_{W_{\mathrm{GR}}[g(\lambda)]}$, where $\mathcal{E}_{\kappa,\Lambda}$ is the Einstein imbalance built from $G_{\mu\nu}[g] + \Lambda g_{\mu\nu} - 8\pi G\langle T_{\mu\nu}\rangle$ [9,10,34,35].*

(b) (QNEC) *For each generator $k^\mu$, the quantum null energy condition holds in the form*

$$\frac{d^2}{d\lambda^2}S_{\mathrm{out}}(\lambda) \;\geq\; 2\pi \int_{\sigma(\lambda)} f^2(\theta)\,\langle T_{kk}(\lambda,\theta)\rangle\, d^{d-2}\theta, \tag{A51}$$

*for suitable smearing functions $f$ along the cut, as in [15,16,51].*

(c) (Weighted control) *The GR weight $W_{\mathrm{GR}}[g]$ is chosen so that along the null deformation there is a constant $c_Q > 0$ with*

$$\left|\int_{\sigma(\lambda)} f^2(\theta)\,\langle T_{kk}(\lambda,\theta)\rangle\, d^{d-2}\theta\right| \;\leq\; c_Q\, R_{\mathrm{GR}}(\lambda)^{1/2} \tag{A52}$$

*for all admissible $f$ in the linearised regime.*

*Then:*

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(i) *There exists a constant $C > 0$ such that*

$$\frac{d^2}{d\lambda^2} S_{\rm out}(\lambda) \;\geq\; -C\, R_{\rm GR}(\lambda). \tag{A53}$$

*In particular, along any segment where $R_{\rm GR}(\lambda)$ is small, the outside entropy $S_{\rm out}(\lambda)$ is convex up to $O(R_{\rm GR})$ corrections.*

(ii) *If, moreover, the area term $A(\lambda)$ is linear to first order in $\lambda$ in the chosen gauge (as in the linearised QFC setup [11,13,14]), then the generalised entropy satisfies*

$$\frac{d^2}{d\lambda^2} S_{\rm gen}(\lambda) \;\geq\; -C\, R_{\rm GR}(\lambda), \tag{A54}$$

*and in the near–solution regime $R_{\rm GR}(\lambda) \to 0$ one has $\frac{d^2}{d\lambda^2} S_{\rm gen}(\lambda) \geq 0$, i.e. $S_{\rm gen}(\lambda)$ is convex along the null deformation.*

More precisely, one has

$$R_{\rm null}(\lambda) \;\simeq\; 2\pi\, S_{\rm rel}(\rho(\lambda) \| \rho_{\rm ref}(\lambda)), \tag{A55}$$

and the known monotonicity/convexity properties of relative entropy along null deformations translate into $\partial_\lambda^2 R_{\rm null} \geq 0$ [15,47,51].

*Appendix D.3. $\|\cdot\|_{W[g]}$ QFC and DSFL Null Expansion*

The quantum focusing conjecture (QFC) asserts that the *quantum expansion* $\Theta(\lambda, y)$ of the generalised entropy

$$S_{\rm gen}[\sigma(\lambda)] = \frac{A[\sigma(\lambda)]}{4G} + S_{\rm out}[\sigma(\lambda)] \tag{A56}$$

along each null generator $k^\mu$ is nonincreasing: $\partial_\lambda \Theta \leq 0$ [11,13,14]. In the DSFL null calibration one can define a *DSFL null expansion*

$$\Theta_{\rm DSFL}(\lambda) := -\frac{1}{2\pi}\, \partial_\lambda \log R_{\rm null}(\lambda), \tag{A57}$$

where $R_{\rm null}$ is the null restriction of the GR residual measured in $\| \cdot \|_{W_{\rm GR}[g;\lambda]}$.

Under linearisation about a semiclassical background obeying the Einstein equation and assuming QNEC for the outside QFT state, one finds [see 13–15] that $\Theta_{\rm DSFL}(\lambda)$ differs from the usual quantum expansion only by higher–order corrections in the perturbation. Combining Raychaudhuri's equation, the semiclassical Einstein equation and QNEC then yields a DSFL focusing inequality $\partial_\lambda \Theta_{\rm DSFL}(\lambda) \leq 0$ in the linearised regime, as summarised in Theorem 4. From the DSFL standpoint, this shows that QNEC/QFC–type focusing can be viewed as a shape constraint on the null profile of a single residual $R_{\rm GR}$ in the same norm that controls Lyapunov decay and cone locality.

Taken together, these constructions justify the use of the GR instrument norm $\| \cdot \|_{W_{\rm GR}[g]}$ as a common metric for ringdown, QNEC and QFC in the near–horizon sector: quasinormal decay, null energy inequalities and quantum focusing all become inequalities for the same GR DSFL residual in one calibrated Hilbert room.

## Appendix E. DSFL Depth and Complexity (Optional)

Here we summarise how DSFL depth provides a natural "scrambling clock" that bounds circuit complexity growth in $k$–local scramblers, as a complement to Section 2.4. The discussion is conceptual rather than fully axiomatic, but it closely parallels the geometric–complexity and random–circuit literature [4,6,36,37,52,55,62].

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*Appendix E.1. Setup and Assumptions*

We work in a finite–dimensional DSFL room $(\mathcal{H}, \mathcal{H}_s, \mathcal{H}_p, \mathbb{I}, W)$ with $\mathcal{H} \cong (\mathbb{C}^d)^{\otimes n}$ for some local dimension $d$ and number of sites $n$. A single DSFL loop at time step $k$ is implemented by a pair of maps $(\widetilde{\Phi}_k, \Phi_k)$ as in Definition 2, with $\Phi_k$ realised by a depth–1 circuit layer of $k$–local gates on the physical system (e.g. nearest–neighbour or $k$–local Hamiltonian evolution) [55].

Let $\psi_0 \in \mathcal{H}$ be a reference state (e.g. a product state) and let $\psi_K$ be the state after $K$ DSFL loops. We define complexity relative to a fixed, finite gate set $\mathcal{G}$ as in the usual circuit–complexity literature [36,62].

**Definition A1** (Circuit complexity)**.** *Fix a universal gate set $\mathcal{G}$ and an accuracy threshold $\varepsilon > 0$. The circuit complexity $\mathcal{C}(K)$ of $\psi_K$ (relative to $\psi_0$ and $\mathcal{G}$) is the least number of gates from $\mathcal{G}$ needed to map $\psi_0$ to a state $\widetilde{\psi}_K$ with $\|\widetilde{\psi}_K - \psi_K\| \le \varepsilon$.*

We make the following mild assumptions on the DSFL loops:

(C1) ***k*–locality and bounded gate cost.** Each $\Phi_k$ can be realised by a circuit of depth 1 using at most $g_{\max}$ gates from $\mathcal{G}$ acting on at most $k$ sites (e.g. a brickwork layer of two–site gates) [45,55,63].

(C2) **Nontrivial tightening.** There exists $\eta_{\min} > 0$ such that, for a fraction $1 - \delta$ of the steps up to time $K$, the one–step DSFL loss satisfies $1 - \frac{R_{k+1}}{R_k} \ge \eta_{\min}$, where $R_k := \|e_k\|_W^2$ and $e_k$ is the defect after $k$ loops. This is a discrete "second law" condition, analogous to the "no macroscopic backtracking" assumptions in complexity arguments [36,52].

As in Appendix A.2, the discrete DSFL depth is defined by

$$\widehat{\tau}_0 := 0, \qquad \widehat{\tau}_{k+1} := \widehat{\tau}_k - \log \frac{R_{k+1}}{R_k}, \tag{A58}$$

so that $\log R_k = \log R_0 - \widehat{\tau}_k$ exactly. In models with approximately constant Lyapunov rate, one has $\widehat{\tau}_K \approx \alpha^* K$, with $\alpha^*$ determined by the one–step contractions.

*Appendix E.2. Linear Upper and Lower Bounds from DSFL Depth*

Under the assumptions above, DSFL depth controls complexity growth in both directions.

**Proposition A4** (Linear upper bound from DSFL depth)**.** *Under assumptions* (C1)*, the complexity $\mathcal{C}(K)$ after K DSFL loops obeys*

$$\mathcal{C}(K) \;\le\; \mathcal{C}(0) + g_{\max} K \;\le\; \mathcal{C}(0) + g_{\max}\, \widehat{\tau}_K. \tag{A59}$$

*If, in addition, $\widehat{\tau}_K \approx \alpha^* K$ in a scrambling window, then $\mathcal{C}(K)$ is at most linear in $K$ with slope of order $\alpha^*$, as expected from holographic and random–circuit scramblers [4,36,37].*

**Proof.** The first inequality is immediate from (C1): each loop adds at most $g_{\max}$ gates, so after $K$ loops $\mathcal{C}(K) \le \mathcal{C}(0) + g_{\max} K$. Since $\widehat{\tau}_K \le K$ (each step contributes $-\log(R_{k+1}/R_k) \ge 0$), the second inequality follows. □

**Proposition A5** (Coarse linear lower bound from DSFL depth)**.** *Under assumptions* (C1)–(C2)*, there exist constants $C_- > 0$ and $K_0 \in \mathbb{N}$ such that for all $K \ge K_0$,*

$$\mathcal{C}(K) \;\ge\; \mathcal{C}(0) + C_-\, \widehat{\tau}_K - O(1), \tag{A60}$$

*where the $O(1)$ term depends only on $\delta, \eta_{\min}$ and the accuracy threshold $\varepsilon$ in Definition A1.*

**Idea of proof.** Let $\eta_k := 1 - R_{k+1}/R_k$ be the one–step loss. Assumption (C2) says that at least a fraction $1 - \delta$ of the steps have $\eta_k \ge \eta_{\min}$, i.e. $R_{k+1} \le (1 - \eta_{\min}) R_k$ on those steps. Each such "good" step cannot be undone without using at least one gate in the opposite direction (up to the tolerance $\varepsilon$), so roughly speaking $\mathcal{C}(K)$ must grow at least like $(1-\delta)K$ up to $O(1)$ fluctuations; see also [36,52] for similar arguments. At the same time, the depth satisfies $\widehat{\tau}_K \geq (1-\delta)\theta(\eta_{\min})K$, for some $\theta(\eta_{\min}) > 0$, so $K \lesssim \widehat{\tau}_K$ and one obtains the stated lower bound with $C_- \sim (1-\delta)/\theta(\eta_{\min})$. □

Taken together, these propositions imply that in any window where $\widehat{\tau}_K \approx \alpha^* K$ the complexity $\mathcal{C}(K)$ is squeezed between two lines of slope $C_-\alpha^*$ and $C_+\alpha^*$, consistent with the essentially linear growth expected in holographic black holes and $k$–local random circuits [6,36,45,63].

**Corollary A1** (Complexity vs. DSFL depth in the scrambling window)**.** *Suppose that for $K$ in a scrambling window $[K_1, K_2]$ one has $\widehat{\tau}_K = \alpha^* K + O(1)$ with $\alpha^* > 0$. Then, under* (C1)–(C2)*, there exist $A_\pm, B_\pm$ such that*

$$A_- + B_- \widehat{\tau}_K \;\leq\; \mathcal{C}(K) \;\leq\; A_+ + B_+ \widehat{\tau}_K, \qquad K \in [K_1, K_2], \tag{A61}$$

*with $0 < C_- \leq B_- \leq B_+ \leq C_+$ proportional to $\alpha^*$.*

In this sense, DSFL depth is a norm–based "scrambling clock" whose increments provide both upper and lower bounds on complexity growth in $k$–local scramblers.

## Appendix F. Numerical Audits and CIU Diagnostics (Optional)

This appendix describes the finite–sample envelope and cone audits, the Haar and random–circuit Page audits, and optional CIU headroom maps used to analyse model scramblers. The philosophy is to treat DSFL as a set of falsifiable hypotheses about the residual $R$: if the envelope, cone and Page tests fail badly in a toy model, that model is not DSFL–lawful.

### *Appendix F.1. Envelope Audit*

The *envelope audit* checks whether the semi–log plot of $R$ in DSFL time has slope $-1$, as predicted by Theorem A2. Given a discrete sequence $R_k > 0$ (e.g. after each DSFL loop) and the corresponding depths $\widehat{\tau}_k$, we:

(E1) Select a tail window $\{k_0, \ldots, k_0 + N - 1\}$ after an initial transient.
(E2) Fit the linear model $\log R_k \approx a + b\,\widehat{\tau}_k$ by ordinary least squares (and optionally a robust method such as Theil–Sen).
(E3) Record the fitted slope $b$, its standard error, and the coefficient of determination $R^2$.

If $R^2$ is close to 1 and the confidence interval for $b$ contains $-1$, the DSFL Lyapunov envelope is supported by the data. In many of our small scramblers and ringdown toys we find $R^2 \gtrsim 0.999$ and $b = -1.00 \pm O(10^{-3})$, in line with the theory; see also [25,28] for related spectral–gap envelopes in Markov and Lindblad semigroups.

### *Appendix F.2. Cone Audit*

The *cone audit* tests the locality structure of a relay map $\Psi_t$ in the sense of Definition 3. On a lattice or chain we proceed as follows [39–41,45,63]:

(C1) Initialise a defect $e(0)$ supported near some region $O$.
(C2) Evolve to times $t_j$ under $\Psi_t$ alone (with any immediate tightening turned off) and compute regional residuals $R_{O'}(t_j) = \|\Pi_{O'} e(t_j)\|_W^2$ for regions $O'$ at various distances from $O$.
(C3) For each $t_j$, define a front radius $R_{\text{front}}(t_j)$ as the minimal distance beyond which $R_{O'}(t_j)$ falls below a fixed threshold (e.g. a fraction of the peak). Fit $R_{\text{front}}(t_j) \approx v t_j$ to extract an effective cone speed $v$.
(C4) In the tail region beyond the front, fit $\log R_{O'}(t_j)$ versus $1.7em(O, O') - v t_j$ to estimate an exponential decay length $\xi$.

If the DSFL relay obeys a cone bound of the form $\|\Pi_{O'} \Psi_t \Pi_O\|_{W \to W} \leq C \exp[-(1.7em(O, O') - vt)/\xi]$, then the fitted $(v, \xi)$ should remain stable across times and different initial supports. This is precisely

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the behaviour observed in lattice models with Lieb–Robinson bounds [39–41] and in random–circuit scramblers where operator fronts propagate ballistically [45,63].

*Appendix F.3. Haar and Random–Circuit Page Audits*

To test the DSFL Page–time picture we use two classes of toy models:

- **Static Haar ensemble:** As in Section 6.1, we sample Haar–random pure states on $(\mathbb{C}^2)^{\otimes n}$, compute reduced states on bipartitions with $k$ radiated qubits, and record both the entropies $S_{\mathrm{rad}}(k)$ and DSFL residuals $R_{\mathrm{BH}}(k)$, $R_{\mathrm{rad}}(k)$. We then compare the entropic Page point (where $S_{\mathrm{rad}}$ is maximal) with the DSFL Page point (where $R_{\mathrm{BH}} \approx R_{\mathrm{rad}}$).
- **Random–circuit evaporators:** Following [1,32,63], we build evaporation circuits that gradually move qubits from a BH register to a radiation register, interspersed with random local scramblers. We track $S_{\mathrm{rad}}(k)$, $R_{\mathrm{BH}}(k)$ and $R_{\mathrm{rad}}(k)$ and estimate the dynamical entropic and DSFL Page times. In all our small–$n$ tests they coincide up to $O(1)$ steps.

These audits numerically support the analytic Haar and random–circuit results summarised in Section 6.

*Appendix F.4. CIU Headroom Maps*

Finally, we use the Contextual Importance and Utility (CIU) framework [53,54] to build *headroom maps* for DSFL scramblers. In CIU, a *handle* (input, parameter, or control) is evaluated via:

- a utility function $u \in [0, 1]$ for the current context (here, typically $u(R)$ or a simple function of the fractional reduction in $R$ over one step); and
- two scores: contextual importance (CI), which measures the width of the utility sweep when the handle is varied, and contextual utility (CU), which measures how good the current setting is within that sweep.

In a DSFL calibration, a natural utility at time $t$ for a given handle $h$ is

$$u(h) := 1 - \frac{R^{(h)}(t + \Delta t)}{R(t)}, \tag{A62}$$

i.e. the fractional reduction of the residual achieved by changing $h$ and evolving one DSFL step. For each region or control channel we sweep $h$ over an admissible interval, record $u_{\mathrm{min}}$ and $u_{\mathrm{max}}$, and compute CI and CU as in [53]. The resulting CIU fields on a lattice or near–horizon collar identify where DSFL admissible changes can most effectively alter the Lyapunov rate (high CI, low CU). This gives a systematic way to locate "sensitive" regions in a scrambler, complementary to envelope and cone audits.

In our small models, high–CI handles (e.g. certain couplings or gate families) indeed have the largest impact on the fitted DSFL rate and Page time when perturbed, in line with CIU theory [53,54].

## Appendix G. DSFL Admissibility of QEC Cycles: Technical Details

In this appendix we make the informal arguments of Appendix G precise enough for finite–dimensional DSFL purposes. We work throughout in the Schrödinger picture, with $\mathcal{H}_p$ a Hilbert space of density operators on $\mathcal{H}_{\mathrm{room}}$, equipped with the instrument inner product $\langle \cdot, \cdot \rangle_W$ and norm $\| \cdot \|_W$.

*Appendix G.1. From Local Noise to an Effective Logical Channel*

We first recall, in compressed form, the standard fault–path expansion for a QEC gadget built from a distance–$d$ stabiliser code and local noise, following [64–69].

Let the ideal QEC gadget $\mathcal{G}$ be a finite sequence of gates, measurements, resets and classical processing, acting on the encoded data, ancillas and an environment sector.

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**Assumption 6** (Local noise model)**.** *The QEC gadget acts on a finite collection of physical locations (just before or after each single– or two–qubit gate, idle step, measurement, or reset). Let $L$ denote the total number of such locations. At location $j \in \{1, \ldots, L\}$ the ideal operation is followed by a noisy completely positive trace–preserving (CPTP) map*

$$\mathcal{N}_j \;=\; (1-p_j)\,\mathrm{Id}_j \;+\; p_j\,\mathcal{E}_j, \qquad 0 \le p_j \le p_{\max}, \tag{A63}$$

*where $\mathcal{E}_j$ is a fixed CPTP "error channel" acting only on the degrees of freedom of location $j$, and the parameters $p_j$ are independent and bounded by a common constant $p_{\max}$. The full noisy gadget is obtained by composing the ideal gadget $\mathcal{G}$ with the maps $\mathcal{N}_j$ at their respective locations, and faults at distinct locations are statistically independent conditioned on the pattern $\{j : \mathcal{E}_j \text{ acts}\}$.*

**Definition A2** (Fault patterns and weight)**.** *A* fault pattern *$\omega$ is a subset of $\{1, \ldots, L\}$, interpreted as the set of locations at which the error part $\mathcal{E}_j$ acts. The* weight *of $\omega$ is $|\omega|$. We write $\Omega_r$ for the set of fault patterns of weight $r$.*

Each pattern $\omega \subset \{1, \ldots, L\}$ defines a CPTP map $\mathcal{F}_\omega$ obtained by inserting $\mathcal{E}_j$ at locations $j \in \omega$ and the identity channel at all other locations, and composing with the ideal gadget $\mathcal{G}$. A standard combinatorial expansion then yields:

**Lemma A2** (Fault–path expansion of the noisy gadget)**.** *Let $\Phi_{\mathrm{ec}}(p)$ denote the CPTP map implemented by the noisy QEC gadget with local noise (A63). Then*

$$\Phi_{\mathrm{ec}}(p) = \sum_{\omega \subset \{1,\ldots,L\}} \left[ \prod_{j\in\omega} p_j \prod_{j\notin\omega} (1-p_j) \right] \mathcal{F}_\omega, \tag{A64}$$

*where each $\mathcal{F}_\omega$ is a CPTP map corresponding to the fault pattern $\omega$. Grouping by weight gives*

$$\Phi_{\mathrm{ec}}(p) = \sum_{r=0}^{L} \mathcal{F}_r(p), \qquad \mathcal{F}_r(p) := \sum_{\omega\in\Omega_r} \left[ \prod_{j\in\omega} p_j \prod_{j\notin\omega} (1-p_j) \right] \mathcal{F}_\omega. \tag{A65}$$

**Proof.** At each location $j$ we may write the noisy map as $\mathcal{N}_j = \mathcal{N}_j^{(0)} + \mathcal{N}_j^{(1)}$ with $\mathcal{N}_j^{(0)} = (1-p_j)\,\mathrm{Id}_j$ and $\mathcal{N}_j^{(1)} = p_j \mathcal{E}_j$. Expanding the product over locations in the Heisenberg or Schrödinger picture gives a sum over all choices of which locations use $\mathcal{N}_j^{(1)}$; these choices are indexed exactly by fault patterns $\omega \subset \{1, \ldots, L\}$. The coefficient for a given $\omega$ is the product of the corresponding $p_j$ and $(1-p_j)$, and the resulting channel is precisely the composition $\mathcal{F}_\omega$ of $\mathcal{E}_j$ at $j \in \omega$ and identity at $j \notin \omega$ with the ideal gadget. Grouping these terms by weight $r = |\omega|$ yields (A65). □

We now specialise to the encoded subspace. Let $V : \mathcal{H}_{\log} \to \mathcal{H}_{\mathrm{data}} \otimes \mathcal{H}_{\mathrm{anc}}$ be the encoding isometry, and let $\Pi_{\mathrm{enc}}$ denote the projector onto the encoded subspace (data+ancilla) tensored with some fixed environment reference state. As usual [66,67], the gadget $\mathcal{G}$ is designed so that any fault pattern with at most $t = \lfloor (d-1)/2 \rfloor$ faulty locations is correctable: at the logical level such patterns act as the identity channel $\Phi_{\log}$.

**Assumption 7** (Ideal logical channel for correctable faults)**.** *There exists a CPTP map $\Phi_{\log}$ on $\mathcal{H}_{\log}$ such that for all $\omega \in \Omega_r$ with $r \le t$ and all logical states $\rho_{\log}$,*

$$\mathrm{Tr}_{\mathrm{anc,env}} \left[ \Pi_{\mathrm{enc}}\, \mathcal{F}_\omega (\mathbb{I}\rho_{\log})\, \Pi_{\mathrm{enc}} \right] = \Phi_{\log}(\rho_{\log}), \tag{A66}$$

*where $\mathbb{I}$ is the calibration map defined from $V$ and the fixed ancilla and environment reference.*

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The quantity of interest is the effective logical channel obtained by applying the noisy gadget to an encoded state, then tracing out ancilla and environment.

**Definition A3** (Effective logical channel). *For each noise vector $p$ define the effective logical channel $\Lambda_{\log}(p)$ on $\mathcal{H}_{\log}$ by*

$$\Lambda_{\log}(p)(\rho_{\log}) := \mathrm{Tr}_{\mathrm{anc,env}}\left[\Pi_{\mathrm{enc}}\,\Phi_{\mathrm{ec}}(p)\,\mathbb{I}\rho_{\log}\,\Pi_{\mathrm{enc}}\right]. \tag{A67}$$

Combining the fault–path expansion with **??** 7 yields the usual threshold–type bound.

**Theorem A5** (Threshold–type bound for the effective logical channel). *Assume **??** 6**??** 7. Let $t = \lfloor (d-1)/2 \rfloor$ be the number of correctable faults for the code and gadget. Then there exist constants $A > 0$, $\alpha > t$ and $p_* \in (0,1)$ (depending on the gadget and noise model, but not on the logical input) such that for all $0 \le p_{\max} < p_*$,*

$$\sup_{\rho_{\log}} \left\| \Lambda_{\log}(p)(\rho_{\log}) - \Phi_{\log}(\rho_{\log}) \right\|_1 \;\le\; A\, p_{\max}^{\alpha}. \tag{A68}$$

*In particular, one may take $\alpha = t+1$ under generic combinatorial assumptions on the gadget [64,69].*

**Theorem A6** (Explicit logical error bound from fault paths). *A5 Assume **??** 6 and **??**, and let $\Phi_{\mathrm{ec}}(p)$ be the CPTP map implemented by a single QEC gadget with $L$ physical locations subject to independent local noise of strength $p \in [0, p_{\max}]$ as in (**??**). Let $d$ be the minimum distance of the underlying code and $t := \lfloor (d-1)/2 \rfloor$ the number of correctable faults per block. Let $\Lambda_{\log}(p)$ be the induced logical channel as in*

**Definition A4** (Effective logical channel). *Let $\Phi_{\mathrm{ec}}(p)$ be the noisy QEC gadget on the physical Hilbert space and let $\mathbb{I}$ be the encoding/calibration map from the logical space $\mathcal{H}_{\log}$ to the encoded subspace. Denote by $\Pi_{\mathrm{enc}}$ the projector onto the encoded subspace (data + ancilla, tensored with a fixed environment reference state), and by $\mathrm{Tr}_{\mathrm{anc,env}}$ the partial trace over ancilla and environment degrees of freedom.*

*For each noise parameter $p$, the* effective logical channel *$\Lambda_{\log}(p)$ is the CPTP map on $\mathcal{H}_{\log}$ defined by*

$$\Lambda_{\log}(p)(\rho_{\log}) := \mathrm{Tr}_{\mathrm{anc,env}}\left[\Pi_{\mathrm{enc}}\,\Phi_{\mathrm{ec}}(p)\,\mathbb{I}\rho_{\log}\,\Pi_{\mathrm{enc}}\right], \tag{A69}$$

*for all logical input states $\rho_{\log}$. In words, $\Lambda_{\log}(p)$ is the logical channel induced by applying the noisy QEC gadget to an encoded state and discarding all nonlogical degrees of freedom.*

*Then there exist constants $A > 0$ and $p_* > 0$ (depending only on the gadget and the code) such that for all $0 \le p < p_*$,*

$$\sup_{\rho_{\log} \in \mathcal{S}(\mathcal{H}_{\log})} \left\| \Lambda_{\log}(p)(\rho_{\log}) - \Phi_{\log}(\rho_{\log}) \right\|_1 \;\le\; A\, p^{t+1}. \tag{??}$$

*In particular, (**??**) holds with any $A \ge \frac{2L^{t+1}}{1-Lp_*}$ and any $p_* \in (0, 1/L]$, and the leading exponent satisfies $\alpha = t+1 > 1$.*

**Proof.** By Lemma A2, we can write the noisy gadget as

$$\Phi_{\mathrm{ec}}(p) = \sum_{\omega \subset \{1,\dots,L\}} p^{|\omega|}(1-p)^{L-|\omega|}\,\mathcal{F}_\omega = \sum_{r=0}^{L} \mathcal{F}_r(p), \qquad \mathcal{F}_r(p) := \sum_{\omega \in \Omega_r} p^r (1-p)^{L-r}\,\mathcal{F}_\omega, \tag{A65}$$

where $\Omega_r$ is the set of fault patterns of weight $r$ and each $\mathcal{F}_\omega$ is a CPTP map corresponding to a specific choice of $r$ faulty locations.

By Definition A3, the logical channel is

$$\Lambda_{\log}(p)(\rho_{\log}) = \mathrm{Tr}_{\mathrm{anc,env}}\left[\Pi_{\mathrm{enc}}\,\Phi_{\mathrm{ec}}(p)\,\mathbb{I}\rho_{\log}\,\Pi_{\mathrm{enc}}\right]. \tag{A70}$$

Inserting the expansion (A65), we obtain

$$\Lambda_{\log}(p) = \sum_{r=0}^{L} \Lambda_r(p), \qquad \Lambda_r(p)(\rho_{\log}) := \mathrm{Tr}_{\mathrm{anc,env}}\big[\Pi_{\mathrm{enc}}\, \mathcal{F}_r(p)\, \mathbb{I}\rho_{\log}\, \Pi_{\mathrm{enc}}\big]. \tag{A71}$$

Each $\Lambda_r(p)$ is a CPTP map on $\mathcal{S}(\mathcal{H}_{\log})$, since it is a convex combination of CPTP maps and partial trace is CPTP.

By **??**, whenever $\omega$ has weight $r \le t$, the action of $\mathcal{F}_\omega$ on encoded inputs is logically equivalent to $\Phi_{\log}$ after decoding. In particular, for every $\rho_{\log}$,

$$\Lambda_r(p)(\rho_{\log}) = \mathrm{Tr}_{\mathrm{anc,env}}\big[\Pi_{\mathrm{enc}}\, \mathcal{F}_r(p)\, \mathbb{I}\rho_{\log}\, \Pi_{\mathrm{enc}}\big] = \Phi_{\log}(\rho_{\log}) \quad \text{for all } r \le t. \tag{A72}$$

Therefore the deviation of the logical channel from the ideal one is entirely due to the uncorrectable fault patterns:

$$\Lambda_{\log}(p) - \Phi_{\log} = \sum_{r=t+1}^{L} \Lambda_r(p) \quad \text{on } \mathcal{S}(\mathcal{H}_{\log}). \tag{A73}$$

Fix a logical input state $\rho_{\log} \in \mathcal{S}(\mathcal{H}_{\log})$. Using (A73) and the triangle inequality in trace norm,

$$\big\|\Lambda_{\log}(p)(\rho_{\log}) - \Phi_{\log}(\rho_{\log})\big\|_1 \le \sum_{r=t+1}^{L} \big\|\Lambda_r(p)(\rho_{\log})\big\|_1. \tag{A74}$$

Each $\Lambda_r(p)$ is a convex combination of CPTP maps, so it is itself CPTP; hence it is trace–norm contractive on states:

$$\|\Lambda_r(p)(\rho)\|_1 \le \|\rho\|_1 = 1 \quad \text{for all } \rho \in \mathcal{S}(\mathcal{H}_{\log}). \tag{A75}$$

Thus a trivial bound is $\|\Lambda_r(p)(\rho_{\log})\|_1 \le 1$. However, we can refine this using the explicit coefficients of the fault–path expansion. Recall from (A71) that

$$\Lambda_r(p)(\rho_{\log}) = \sum_{\omega\in\Omega_r} p^r(1-p)^{L-r}\, \mathrm{Tr}_{\mathrm{anc,env}}\big[\Pi_{\mathrm{enc}}\, \mathcal{F}_\omega\big(\mathbb{I}\rho_{\log}\big)\, \Pi_{\mathrm{enc}}\big] =: \sum_{\omega\in\Omega_r} c_\omega(p)\, \Lambda_\omega(\rho_{\log}), \tag{A76}$$

where $c_\omega(p) = p^r(1-p)^{L-r} \ge 0$ and each $\Lambda_\omega$ is CPTP on $\mathcal{S}(\mathcal{H}_{\log})$. Hence $\|\Lambda_\omega(\rho_{\log})\|_1 \le 1$ for each $\omega$, and

$$\|\Lambda_r(p)(\rho_{\log})\|_1 \le \sum_{\omega\in\Omega_r} c_\omega(p) = \binom{L}{r} p^r(1-p)^{L-r}. \tag{A77}$$

Therefore

$$\big\|\Lambda_{\log}(p)(\rho_{\log}) - \Phi_{\log}(\rho_{\log})\big\|_1 \le \sum_{r=t+1}^{L} \binom{L}{r} p^r(1-p)^{L-r}. \tag{A78}$$

The right–hand side is just the tail of a binomial distribution and is bounded by a geometric series in $Lp$. Using $\binom{L}{r} \le L^r/r!$ and $(1-p)^{L-r} \le 1$, we have

$$\sum_{r=t+1}^{L} \binom{L}{r} p^r(1-p)^{L-r} \le \sum_{r=t+1}^{\infty} L^r p^r = \sum_{r=t+1}^{\infty} (Lp)^r = \frac{(Lp)^{t+1}}{1-Lp}, \tag{A79}$$

which converges and is finite whenever $Lp < 1$. Thus, if we choose any $p_* \in (0, 1/L]$ and restrict to $0 \le p < p_*$, then

$$\sum_{r=t+1}^{L} \binom{L}{r} p^r(1-p)^{L-r} \le \frac{(Lp)^{t+1}}{1-Lp} \le \frac{(Lp)^{t+1}}{1-Lp_*} \le \frac{L^{t+1}}{1-Lp_*}\, p^{t+1}. \tag{A80}$$

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Combining this with (A78) and taking the supremum over $\rho_{\log} \in \mathcal{S}(\mathcal{H}_{\log})$ gives

$$\sup_{\rho_{\log}} \big\|\Lambda_{\log}(p)(\rho_{\log}) - \Phi_{\log}(\rho_{\log})\big\|_1 \le \frac{L^{t+1}}{1 - Lp_*}\, p^{t+1}. \tag{A81}$$

Thus (??) holds with any $A \ge \frac{L^{t+1}}{1-Lp_*}$ and any $p_* \in (0, 1/L]$. Since $t = \lfloor (d-1)/2 \rfloor$, the leading exponent is $\alpha = t + 1 > 1$, which is all that is required for the DSFL application. □

*Appendix G.2. Intertwining up to Higher–Order Defects*

We now formalise the "almost intertwining" relation between the physical QEC map $\Phi_{\mathrm{ec}}(p)$ and the ideal logical map $\Phi_{\log}$ inside the DSFL room.

Recall that $\mathbb{I}$ maps logical density operators to encoded physical states (data+ancilla+environment reference). Let $\Phi_{\log}^{\mathrm{phys}}$ denote the ideal, noise–free, physical realisation of $\Phi_{\log}$ at the gadget level.

**Lemma A3** (Intertwining up to higher–order faults)**.** *Under* **??** *6***??** *7, and with W satisfying the norm equivalence*

$$c_1 \|x\|_1 \le \|x\|_W \le c_2 \|x\|_1, \qquad \forall x, \tag{A82}$$

(A82)*, there exist constants* $A' > 0$*,* $\alpha > 1$ *and* $p_* \in (0, 1)$ *such that for all* $0 \le p_{\max} < p_*$ *and all logical inputs* $\rho_{\log}$*,*

$$\big\|\Phi_{\mathrm{ec}}(p)\, \mathbb{I}\rho_{\log} - \Phi_{\log}^{\mathrm{phys}} \mathbb{I}\rho_{\log}\big\|_W \le A'\, p_{\max}^{\alpha}. \tag{A83}$$

*Equivalently,*

$$\Phi_{\mathrm{ec}}(p)\, \mathbb{I} - \mathbb{I}\Phi_{\log} = \mathcal{O}_W(p_{\max}^{\alpha}) \quad \textit{on the calibration graph.} \tag{A84}$$

**Proof.** By definition,

$$\Phi_{\log}(\rho_{\log}) = \mathrm{Tr}_{\mathrm{anc,env}}\big[\Phi_{\log}^{\mathrm{phys}} \mathbb{I}\rho_{\log}\big]. \tag{A85}$$

Subtracting this from the definition of $\Lambda_{\log}(p)$ and using Theorem A5 yields

$$\sup_{\rho_{\log}} \big\|\mathrm{Tr}_{\mathrm{anc,env}}\big[(\Phi_{\mathrm{ec}}(p) - \Phi_{\log}^{\mathrm{phys}})\mathbb{I}\rho_{\log}\big]\big\|_1 \le A\, p_{\max}^{\alpha}. \tag{A86}$$

Since partial trace is a contraction for the trace norm, we have

$$\big\|(\Phi_{\mathrm{ec}}(p) - \Phi_{\log}^{\mathrm{phys}})\mathbb{I}\rho_{\log}\big\|_1 \ge \big\|\mathrm{Tr}_{\mathrm{anc,env}}\big[(\Phi_{\mathrm{ec}}(p) - \Phi_{\log}^{\mathrm{phys}})\mathbb{I}\rho_{\log}\big]\big\|_1, \tag{A87}$$

so the same bound holds with the partial trace removed. Finally, **??** 9 gives norm equivalence (A82) between $\|\cdot\|_1$ and $\|\cdot\|_W$ on encoded states, hence

$$\big\|(\Phi_{\mathrm{ec}}(p) - \Phi_{\log}^{\mathrm{phys}})\mathbb{I}\rho_{\log}\big\|_W \le c_2 \big\|(\Phi_{\mathrm{ec}}(p) - \Phi_{\log}^{\mathrm{phys}})\mathbb{I}\rho_{\log}\big\|_1 \le A'\, p_{\max}^{\alpha} \tag{A88}$$

for some $A' = c_2 A$. This is (A83) and thus (A84). □

Lemma A3 is the precise version of the heuristic intertwining condition

$$\big\|\Phi_{\mathrm{ec}}(p)\, \mathbb{I}\rho_{\log} - \mathbb{I}\,\Phi_{\log}(\rho_{\log})\big\|_W = \mathcal{O}(p^{\alpha}), \tag{A89}$$

uniformly for encoded logical states $\rho_{\log}$.

(A89) and (A88): any failure of perfect calibration preservation is a higher–order effect in the DSFL defect and does not affect the leading DSFL Lyapunov envelope.

*Appendix G.3. Finite–Dimensional DSFL Envelope and Norm Contraction*

We now relate the logical error bound to a contraction estimate for $\Phi_{\mathrm{ec}}(p)$ in the instrument norm.

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**Theorem A7** (Finite–dimensional DSFL envelope)**.** *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite–dimensional Hilbert space and let $W : V \to V$ be a bounded, selfadjoint, strictly positive operator. Equip $V$ with the instrument norm*

$$\|x\|_W^2 := \langle x, Wx\rangle, \qquad x \in V. \tag{A90}$$

*Let $\Phi : V \to V$ be a linear map and denote by $\rho$ the spectral radius of $\Phi$ restricted to the invariant subspace on which $\Phi$ has no fixed point, i.e.*

$$\rho := \max\{|\lambda| : \lambda \in \sigma(\Phi),\ \lambda \neq 1\}. \tag{A91}$$

*Assume $\rho < 1$ and define the DSFL residual*

$$R_k(x_0) := \|\Phi^k x_0\|_W^2, \qquad k \in \mathbb{N}_0. \tag{A92}$$

*Then:*

1. *There exists a constant $C \geq 1$, depending only on $\Phi$ and $W$, such that for all $x_0 \in V$ and all $k \in \mathbb{N}_0$,*

$$R_k(x_0) \;\leq\; C\,\rho^{2k}\,\|x_0\|_W^2. \tag{A93}$$

2. *For any $x_0$ whose component along at least one eigenvector with eigenvalue modulus $\rho$ is nonzero, one has the asymptotic behaviour*

$$\log R_k(x_0) \;=\; 2\log\rho\,k + O(1), \qquad k \to \infty. \tag{A94}$$

*In particular, $\rho$ is the DSFL spectral rate for $\Phi$ in the instrument norm: it controls both the worst–case Lyapunov envelope (A93) and the asymptotic slope of $\log R_k(x_0)$ for generic initial data.*

Recall the finite–dimensional DSFL envelope theorem (Theorem A7): for any bounded linear map $\Phi : \mathcal{H}_p \to \mathcal{H}_p$,

$$\|\Phi\|_{W\to W}^2 = \lambda_{\max}\big(C_W(\Phi)\big), \qquad C_W(\Phi) := W^{-1/2}\Phi^* W \Phi W^{-1/2}. \tag{A95}$$

In particular, $\Phi$ is DSFL–admissible (nonexpansive in $\|\cdot\|_W$) if and only if $\lambda_{\max}\big(C_W(\Phi)\big) \leq 1$.

We want to show that for a genuinely noisy QEC cycle, the inequality is strict and the difference from 1 is controlled by the logical error scale $p_{\max}^{\alpha}$. To avoid pathologies, we make a mild nondegeneracy assumption.

**Assumption 8** (Nontrivial noise on all directions)**.** *10 There exists a constant $\delta_0 > 0$ such that for every nonzero $X \in \mathcal{H}_p$ supported on the encoded subspace, the restriction of $\Phi_{\mathrm{ec}}(p)$ to the two–dimensional subspace $\mathrm{span}\{X, \mathbb{I}\}$ is not unitary whenever $p_{\max} > \delta_0$. Equivalently, for $p_{\max} > \delta_0$ the channel $\Phi_{\mathrm{ec}}(p)$ is not an isometry in any nontrivial direction corresponding to a change of encoded state.*

**Assumption 9** (Instrument compatibility)**.** *The instrument weight $W$ is chosen so that, on the encoded and logically relevant subspace of $\mathcal{H}_p$, the induced norm $\|\cdot\|_W$ is equivalent to the trace norm. More precisely, there exist constants $0 < c_1 \leq c_2 < \infty$ such that for every encoded state (or difference of encoded states) $X$ one has*

$$c_1\,\|X\|_1 \;\leq\; \|X\|_W \;\leq\; c_2\,\|X\|_1. \text{A82} \tag{A96}$$

*In particular, $\|\cdot\|_W$ can be regarded as a quadratic proxy for operational distinguishability (trace distance) on encoded logical states and effective logical channels.*

This is natural: once the noise rates are nonzero on all physical locations, no encoded direction should be perfectly preserved by the full noisy QEC gadget.

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**Lemma A4** (Operator–norm contraction from logical error)**.** *Under* **??** *6***??** *9 and* **??***, let* $\Phi_{ec}(p)$ *be the noisy QEC cycle on* $(\mathcal{H}_p, \|\cdot\|_W)$ *and* $\Lambda_{log}(p)$ *its effective logical channel. Suppose there are constants* $A > 0, \alpha > 1, p_* \in (0, \delta_0]$ *such that the logical error bound* (A68) *holds. Then there exist constants* $c_0 > 0$ *and* $p_{**} \in (0, p_*]$ *such that*

$$\|\Phi_{ec}(p)\|_{W\to W} \;\le\; 1 - c_0 p_{max}^{\alpha} \quad \textit{for all } 0 \le p_{max} < p_{**}. \tag{A97}$$

**Proof.** We work on the encoded, traceless subspace of $\mathcal{H}_p$, which is finite dimensional. Let $\mathsf{T}_{enc} \subset \mathcal{H}_p$ denote the subspace of traceless operators supported on the encoded code space, and consider the restriction of $\Phi_{ec}(p)$ to $\mathsf{T}_{enc}$. All norms on $\mathsf{T}_{enc}$ are equivalent, and we use $\|\cdot\|_W$ throughout.

Define, for each $p \ge 0$,

$$f(p) := \|\Phi_{ec}(p)\|_{W\to W, \mathsf{T}_{enc}} := \sup\{\|\Phi_{ec}(p)X\|_W : X \in \mathsf{T}_{enc}, \; \|X\|_W = 1\}. \tag{A98}$$

By definition, $0 \le f(p) \le 1$ and $f(0) = 1$, since $\Phi_{ec}(0) = \Phi_{log}^{phys}$ is an isometry on $\mathsf{T}_{enc}$.

Step 1: pointwise strict contraction for $p > 0$.

By Assumption 10, for every nonzero $X \in \mathsf{T}_{enc}$ and every $p \in (0, \delta_0)$, one has $\|\Phi_{ec}(p)X\|_W < \|X\|_W$. Thus for each fixed $p \in (0, \delta_0)$,

$$f(p) = \sup_{\|X\|_W = 1} \|\Phi_{ec}(p)X\|_W < 1. \tag{A99}$$

Moreover, since $\Phi_{ec}(p)$ depends continuously on $p$ in operator norm (by the local noise model and finite dimensionality), the map $p \mapsto f(p)$ is continuous on $[0, \delta_0]$, with $f(0) = 1$ and $f(p) \in [0, 1)$ for $p \in (0, \delta_0]$.

Step 2: contradiction setup.

Suppose, for contradiction, that the conclusion of the lemma is false. Then for every $n \in \mathbb{N}$ there exists $p^{(n)} \in (0, 1/n) \subset (0, \delta_0)$ such that

$$f(p^{(n)}) > 1 - \frac{1}{n}(p^{(n)})^{\alpha}. \tag{A100}$$

Equivalently,

$$1 - f(p^{(n)}) < \frac{1}{n}(p^{(n)})^{\alpha}, \tag{A101}$$

so the "deficit" $1 - f(p)$ is asymptotically *smaller* than any fixed multiple of $p^{\alpha}$ along the sequence $p^{(n)} \downarrow 0$.

For each $n$ choose $X_n \in \mathsf{T}_{enc}$ with $\|X_n\|_W = 1$ such that

$$\|\Phi_{ec}(p^{(n)})X_n\|_W > f(p^{(n)}) - \frac{1}{n}(p^{(n)})^{\alpha}. \tag{A102}$$

Then, using (A100),

$$\|\Phi_{ec}(p^{(n)})X_n\|_W > 1 - \frac{2}{n}(p^{(n)})^{\alpha}. \tag{A103}$$

Step 3: compactness and a limiting direction.

Since $\mathsf{T}_{enc}$ is finite dimensional and $\|X_n\|_W = 1$ for all $n$, there is a subsequence (still denoted $X_n$) and a unit vector $X_\infty \in \mathsf{T}_{enc}$ such that

$$X_n \to X_\infty \quad \text{in the } \|\cdot\|_W \text{ norm as } n \to \infty. \tag{A104}$$

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Moreover $p^{(n)} \to 0$, and $\Phi_{\mathrm{ec}}(p)$ depends continuously on $p$ in the operator norm induced by $\|\cdot\|_W$, so

$$\Phi_{\mathrm{ec}}(p^{(n)})X_n \;\to\; \Phi_{\mathrm{ec}}(0)X_\infty \quad \text{in } \|\cdot\|_W. \tag{A105}$$

Taking norms and using (A103), we obtain

$$\|\Phi_{\mathrm{ec}}(0)X_\infty\|_W = \lim_{n\to\infty} \|\Phi_{\mathrm{ec}}(p^{(n)})X_n\|_W \geq \limsup_{n\to\infty}\Big(1 - \frac{2}{n}(p^{(n)})^\alpha\Big) = 1. \tag{A106}$$

On the other hand, $\Phi_{\mathrm{ec}}(0) = \Phi_{\log}^{\mathrm{phys}}$ is an isometry on $\mathsf{T}_{\mathrm{enc}}$ (ideal, noiseless QEC gadget), so also $\|\Phi_{\mathrm{ec}}(0)X_\infty\|_W \leq \|X_\infty\|_W = 1$. Thus

$$\|\Phi_{\mathrm{ec}}(0)X_\infty\|_W = \|X_\infty\|_W = 1. \tag{A107}$$

In words, $X_\infty$ is a unit vector on which the ideal gadget acts isometrically, as expected.

Step 4: relating logical error to isometry loss.

The QEC logical error bound (A68) implies that the *encoded logical channel* $\Lambda_{\log}(p)$ remains close, in trace norm, to the ideal logical map $\Phi_{\log}$ with an error of order $p_{\max}^\alpha$. By Assumption 9, $\|\cdot\|_W$ and $\|\cdot\|_1$ are equivalent on encoded states: there exist constants $0 < c_1 \leq c_2$ such that

$$c_1\|X\|_1 \leq \|X\|_W \leq c_2\|X\|_1 \tag{A108}$$

for all (differences of) encoded states $X$. Combined with the structure of the encoding map $\mathbb{I}$ and the definition of $\Lambda_{\log}$, this implies the encoded–physical bound (A108), and hence the operator–norm bound on the deviation from the ideal physical logical map $G(p)$ proved in Lemma G.3.0.4:

$$\|G(p)\|_{W\to W,\mathsf{T}_{\mathrm{enc}}} \leq K\, p_{\max}^\alpha \tag{A109}$$

for some constant $K > 0$. Here $G(p)$ is the restriction to $\mathsf{T}_{\mathrm{enc}}$ of

$$G(p) := \Phi_{\mathrm{ec}}(p) - \Phi_{\log}^{\mathrm{phys}}. \tag{A110}$$

On $\mathsf{T}_{\mathrm{enc}}$ we can therefore write

$$\Phi_{\mathrm{ec}}(p) = \Phi_{\log}^{\mathrm{phys}} + G(p), \tag{A111}$$

with $\|G(p)\|_{W\to W} \leq Kp^\alpha$.

Step 5: quantitative lower bound on the deficit.

Fix $X_\infty$ as above with $\|X_\infty\|_W = 1$ and $\|\Phi_{\mathrm{ec}}(0)X_\infty\|_W = 1$. For $p$ small,

$$\begin{aligned}\|\Phi_{\mathrm{ec}}(p)X_\infty\|_W &= \|\Phi_{\log}^{\mathrm{phys}}X_\infty + G(p)X_\infty\|_W \\ &\leq \|\Phi_{\log}^{\mathrm{phys}}X_\infty\|_W + \|G(p)X_\infty\|_W \\ &\leq 1 + Kp^\alpha.\end{aligned}$$

Likewise,

$$\begin{aligned}\|\Phi_{\mathrm{ec}}(p)X_\infty\|_W &\geq \|\Phi_{\log}^{\mathrm{phys}}X_\infty\|_W - \|G(p)X_\infty\|_W \\ &\geq 1 - Kp^\alpha.\end{aligned}$$

Thus, for $p$ small,

$$\big|\,\|\Phi_{\mathrm{ec}}(p)X_\infty\|_W - 1\big| \leq Kp^\alpha. \tag{A112}$$

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Now, by continuity of $X \mapsto \|\Phi_{\mathrm{ec}}(p)X\|_W$ on the unit sphere in $\mathsf{T}_{\mathrm{enc}}$, there exists a neighbourhood $\mathcal{U}$ of $X_\infty$ in the unit sphere and a constant $c' > 0$ such that for all $X \in \mathcal{U}$ and all sufficiently small $p$,

$$1 - \|\Phi_{\mathrm{ec}}(p)X\|_W \geq c' p^\alpha. \tag{A113}$$

If this were not the case, we could construct a sequence $X_m \to X_\infty$ and $p_m \downarrow 0$ with $1 - \|\Phi_{\mathrm{ec}}(p_m)X_m\|_W \leq \varepsilon_m p_m^\alpha$ for some $\varepsilon_m \to 0$, which in the limit would contradict either the nontrivial–noise assumption (that every nontrivial encoded direction is strictly contracting for $p > 0$) or the bound (A112) combined with (A108) and the logical error scaling $\sim p^\alpha$.

Step 6: contradiction with the assumed sequence $p^{(n)}$.

Recall that $X_n \to X_\infty$ and $p^{(n)} \to 0$. For $n$ large enough, $X_n \in \mathcal{U}$ and $p^{(n)}$ is small enough that the lower bound just described applies. Hence for all sufficiently large $n$,

$$1 - \|\Phi_{\mathrm{ec}}(p^{(n)})X_n\|_W \geq c'\left(p^{(n)}\right)^\alpha. \tag{A114}$$

In particular,

$$1 - f(p^{(n)}) \geq 1 - \|\Phi_{\mathrm{ec}}(p^{(n)})X_n\|_W \geq c'\left(p^{(n)}\right)^\alpha, \tag{A115}$$

contradicting (A100), which asserts that along the sequence $p^{(n)}$ we have $1 - f(p^{(n)}) < \frac{1}{n}\left(p^{(n)}\right)^\alpha$.

This contradiction shows that there exist $c_0 > 0$ and $p_{**} \in (0, \delta_0]$ such that for all $0 < p_{\max} < p_{**}$,

$$f(p_{\max}) \leq 1 - c_0 p_{\max}^\alpha, \tag{A116}$$

i.e.

$$\|\Phi_{\mathrm{ec}}(p)\|_{W\to W, \mathsf{T}_{\mathrm{enc}}} \leq 1 - c_0 p_{\max}^\alpha. \tag{A117}$$

Since $\mathsf{T}_{\mathrm{enc}}$ contains all nontrivial encoded directions and the identity component can be treated separately, this yields the claimed bound (A97) on the relevant part of $\mathcal{H}_p$, completing the proof. □

**Assumption 10** (Nontrivial noise on encoded directions)**.** *There exists a constant $\delta_0 > 0$ such that for every nonzero encoded, traceless operator $X \in \mathcal{H}_p$, the map $\Phi_{\mathrm{ec}}(p)$ is not an isometry on the ray spanned by $X$ whenever $p_{\max} \in (0, \delta_0)$, i.e.*

$$\|\Phi_{\mathrm{ec}}(p)X\|_W < \|X\|_W \qquad \textit{for all } p_{\max} \in (0, \delta_0). \tag{A118}$$

*Equivalently, once the local noise rates are strictly positive on all physical locations, no nontrivial encoded direction is perfectly preserved by the full noisy QEC gadget.*

**Theorem A8** (Finite–dimensional DSFL envelope and QEC gap)**.** *Under* **??** *6***??** *7***??** *10 and* **??** *there exist constants $c_0 > 0$, $\alpha > 1$ and $p_{**} \in (0, 1)$ such that for all $0 \leq p_{\max} < p_{**}$,*

$$\|\Phi_{\mathrm{ec}}(p)\|_{W\to W}^2 = \lambda_{\max}\big(C_W(\Phi_{\mathrm{ec}}(p))\big) \;\leq\; \left(1 - c_0 p_{\max}^\alpha\right)^2. \tag{A119}$$

*Equivalently, the QEC cycle has a DSFL gap*

$$\gamma_{\mathrm{ec}}(p_{\max}) := 1 - \|\Phi_{\mathrm{ec}}(p)\|_{W\to W} \;\geq\; c_0 p_{\max}^\alpha. \tag{A120}$$

**Proof.** Immediate from Lemma A4 and (A95). □

*Appendix G.4. Logical Lyapunov Inequality*

We can now complete the DSFL Lyapunov picture for the logical residual of sameness.

Let $(s_k, p_k)$ denote the blueprint and physical states after $k$ QEC cycles, with $e_k = p_k - \mathbb{I}s_k$ and $R_k = R(s_k, p_k) = \|e_k\|_W^2$. Let $\widetilde{\Phi}_{\mathrm{ec}} = \Phi_{\log}$ be the ideal logical map and $\Phi_{\mathrm{ec}}(p)$ the noisy QEC map.

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**Proposition A6** (One–step logical Lyapunov inequality)**.** *Under the assumptions of Theorem A8 and for* $0 \le p_{\max} < p_{**}$*,*

$$R_{k+1} = R\big(\widetilde{\Phi}_{\rm ec} s_k, \Phi_{\rm ec}(p) p_k\big) \;\le\; \|\Phi_{\rm ec}(p)\|^2_{W\to W}\, R_k \;\le\; e^{-2\lambda_{\rm ec}(p_{\max})}\, R_k, \tag{A121}$$

*where*

$$\lambda_{\rm ec}(p_{\max}) := -\tfrac{1}{2}\log\big(1-\gamma_{\rm ec}(p_{\max})\big) \;\ge\; c_1 p_{\max}^{\alpha} \tag{A122}$$

*for some constant* $c_1 > 0$ *and* $\alpha > 1$*.*

**Proof.** By the DSFL DPI (Theorem 1) and the definition of the operator norm,

$$R_{k+1} = R(\widetilde{\Phi}_{\rm ec} s_k, \Phi_{\rm ec}(p) p_k) \le \|\Phi_{\rm ec}(p)\|^2_{W\to W}\, R_k. \tag{A123}$$

Using $\|\Phi_{\rm ec}(p)\|_{W\to W} \le 1-\gamma_{\rm ec}$ and defining $\lambda_{\rm ec}$ from $\gamma_{\rm ec}$ as in the statement yields

$$\|\Phi_{\rm ec}(p)\|^2_{W\to W} \le \big(1-\gamma_{\rm ec}(p_{\max})\big)^2 = e^{-2\lambda_{\rm ec}(p_{\max})}, \tag{A124}$$

and hence the desired inequality. The bound $\lambda_{\rm ec}(p_{\max}) \ge c_1 p_{\max}^{\alpha}$ follows from $\gamma_{\rm ec}(p_{\max}) \ge c_0 p_{\max}^{\alpha}$ and the Taylor expansion of the logarithm near 1. □

Iterating Proposition A6 immediately yields:

**Corollary A2** (Exponential decay of the logical residual)**.** *Under the assumptions above, the logical residual of sameness obeys*

$$R_k \;\le\; \exp\big(-2k\,\lambda_{\rm ec}(p_{\max})\big)\, R_0, \tag{A125}$$

*for all integers* $k \ge 0$ *and all* $0 \le p_{\max} < p_{**}$*. In particular, a strictly positive QEC DSFL gap* $\gamma_{\rm ec}(p_{\max}) > 0$ *is equivalent to exponential decay of the logical residual and hence to logical stability under arbitrarily long fault–tolerant computation.*

This completes the DSFL reformulation of the QEC accuracy threshold: *below threshold, the QEC cycle is a DSFL–admissible channel with a strictly positive Lyapunov rate for the residual of sameness*.

## Appendix H. DSFL Residuals in Canonical Quantum Channels

This appendix gives explicit finite–dimensional demonstrations of the DSFL spectral envelope in one–qubit CPTP maps. Although not part of our gravitational results, these examples show that DSFL residuals, rates, and Lyapunov structure already appear in textbook quantum channels.

We consider three channels:

- the **amplitude damping** channel $\mathcal{E}_\gamma$,
- the **depolarising** channel $\mathcal{D}_p$,
- the **pure dephasing** channel $\mathcal{F}_p$.

Each acts on $\mathcal{B}(\mathbb{C}^2)$, and is analysed in the Hilbert–Schmidt norm $\|\cdot\|_2$, corresponding to a constant weight $W = \mathbb{I}$.

### *Appendix H.1. Spectral Envelopes in Finite–Dimensional Channels*

Before turning to the black–hole room, it is instructive to note that the DSFL residual law already governs dynamics in simple quantum systems. In particular, the DSFL envelope

$$R_k := \|\Phi^k(\rho_0) - \rho_*\|^2_W \tag{A126}$$

decays at a rate dictated by the spectrum of the channel $\Phi$ in standard one–qubit CPTP maps.

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Example (Amplitude damping).

For the amplitude damping channel $\mathcal{E}_\gamma$, the Liouville spectrum is

$$\sigma(S_\gamma) = \{1, \sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma\}, \tag{A127}$$

with fixed point $\rho_* = |0\rangle\langle 0|$. The DSFL residual decays as

$$R_k \sim C\,|\lambda|^{2k}, \tag{A128}$$

with $|\lambda| = \sqrt{1-\gamma}$ generically, but for certain states (e.g. $\rho_0 = |1\rangle\langle 1|$), one observes a faster decay $(1-\gamma)^{2k}$, saturating the envelope of the fastest mode.

Example (Depolarising channel).

For $\mathcal{D}_p(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{2}$, all eigenvalues except $\lambda = 1$ equal $1-p$, and

$$R_k = (1-p)^{2k} R_0 \tag{A129}$$

for all initial states. The DSFL envelope is always saturated: this is a maximally uniform scrambler.

Example (Pure dephasing).

For the dephasing channel $\mathcal{F}_p(\rho) = (1-p)\rho + pZ\rho Z$, one has

$$\|\mathcal{F}_p\|_{2\to 2} = 1, \tag{A130}$$

and the DSFL residual fails to contract in general. Only off-diagonal terms decay. This is a gapless DSFL example: the envelope flattens, and no global contraction law holds in the $\|\cdot\|_2$ norm.

These finite–dimensional examples confirm that:

- DSFL envelopes are governed by spectral properties of the update map $\Phi$;
- Residuals can saturate or undersaturate the envelope depending on state alignment;
- The Lyapunov rate $\alpha = -\log|\lambda_2|$ corresponds to known mixing rates;
- The same residual structure appears in black hole scrambling and GR sectors, but now driven by near–horizon quasinormal decay and out–of–equilibrium quantum dynamics.

In this sense, the single–residual DSFL law is already realised in textbook quantum channels. A full treatment of these qubit models is given in Appendix B, including proofs and decay mode analysis.

*Appendix H.2. Amplitude Damping*

Let $\mathcal{E}_\gamma$ have Kraus operators

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \tag{A131}$$

The unique fixed point is $\rho_* = |0\rangle\langle 0|$. The residual

$$R_k := \|\mathcal{E}_\gamma^k(\rho_0) - \rho_*\|_2^2 \tag{A132}$$

has state-dependent decay:

- For generic $\rho_0$, $R_k \sim C(1-\gamma)^k$,
- For $\rho_0 = |1\rangle\langle 1|$, $R_k = C(1-\gamma)^{2k}$.

This reflects the Liouville spectrum

$$\sigma(S_\gamma) = \{1,\ \sqrt{1-\gamma},\ \sqrt{1-\gamma},\ 1-\gamma\}, \tag{A133}$$

with coherence modes decaying more slowly than population.

*Appendix H.3. Depolarising Channel*

For $\mathcal{D}_p(\rho) = (1-p)\rho + p\,\frac{\mathbb{I}}{2}$, all states decay uniformly:

$$R_k = (1-p)^{2k} R_0, \quad \text{with rate} \quad \alpha = -\log(1-p). \tag{A134}$$

This is a uniformly contracting DSFL map: the spectral envelope is saturated by all states.

*Appendix H.4. Dephasing Channel*

For $\mathcal{F}_p(\rho) = (1-p)\rho + pZ\rho Z$, the residual decay depends entirely on coherences:

$$R_k(\rho_0) = R_0(\rho_0) \quad \text{if } \rho_0 \text{ is diagonal}, \quad R_k \sim (1-2p)^{2k} \text{ for off-diagonal entries.} \tag{A135}$$

The Liouville spectrum includes 1 with multiplicity, so $\|\mathcal{F}_p\|_{2\to 2} = 1$: there is no global DSFL gap. This is a gapless DSFL example.

Conclusion.

These examples confirm the DSFL envelope in concrete channels, validate its dependence on spectral alignment, and provide an intuitive benchmark before turning to gravitational settings.

## END APPENDIX

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