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Article

Functional Continuous Uncertainty Principle

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Abstract: Let (Ω, μ), (Δ, ν) be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ be continuous p-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X} \setminus \{0\}$, we show that

$$\mu(\operatorname{supp}(\theta_f x))^{\frac{1}{p}} \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)|},$$

$$\nu(\operatorname{supp}(\theta_g x))^{\frac{1}{p}} \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)|}. \tag{1}$$

where

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K},$$

$$\theta_g : \mathcal{X} \ni x \mapsto \theta_g x \in \mathcal{L}^p(\Delta, \nu); \quad \theta_g x : \Delta \ni \beta \mapsto (\theta_g x)(\beta) := g_\beta(x) \in \mathbb{K}$$

and q is the conjugate index of p. We call Inequality (1) as **Functional Continuous Uncertainty Principle**. It improves the Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle obtained by M. Krishna in *[arXiv:2304.03324v1, 2023]*. It also answers a question asked by Prof. Philip B. Stark to the author.

Keywords: Uncertainty Principle; Parseval Frame; Banach space

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1. Introduction

For $h \in \mathbb{C}^d$, let $\|h\|_0$ be the number of nonzero entries in h. Let $\hat{}: \mathbb{C}^d \to \mathbb{C}^d$ be the Fourier transform. Great and surprising inequality of Donoho and Stark which improved our life is the following.

Theorem 1. *(**Donoho-Stark Uncertainty Principle**) [1] For every $d \in \mathbb{N}$,*

$$\left(\frac{\|h\|_0 + \|\widehat{h}\|_0}{2}\right)^2 \geq \|h\|_0 \|\widehat{h}\|_0 \geq d, \quad \forall h \in \mathbb{C}^d \setminus \{0\}. \tag{2}$$

In 2002, Elad and Bruckstein extended Inequality (2) to pairs of orthonormal bases [2]. Given a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we define

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j \rangle)_{j=1}^n \in \mathbb{K}^n.$$

Theorem 2. *(**Elad-Bruckstein Uncertainty Principle**) [2] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0 + \|\theta_\omega h\|_0}{2}\right)^2 \geq \|\theta_\tau h\|_0 \|\theta_\omega h\|_0 \geq \frac{1}{\max\limits_{1\leq j,k\leq n} |\langle \tau_j, \omega_k \rangle|^2}, \quad \forall h \in \mathcal{H} \setminus \{0\}.$$

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In 2013, Ricaud and Torrésani showed that orthonormal bases in Theorem 2 can be improved to Parseval frames [3].

Theorem 3. *(Ricaud-Torrésani Uncertainty Principle) [3] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0+\|\theta_\omega h\|_0}{2}\right)^2 \geq \|\theta_\tau h\|_0\|\theta_\omega h\|_0 \geq \frac{1}{\max\limits_{1\leq j,k\leq n}|\langle\tau_j,\omega_k\rangle|^2}, \quad \forall h\in\mathcal{H}\setminus\{0\}.$$

In 2023, author made a major improvement of Theorem 3.

Theorem 4. *(Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle) [4] Let $(\{f_j\}_{j=1}^n,\{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m,\{\omega_k\}_{k=1}^m)$ be p-Schauder frames for a finite dimensional Banach space $\mathcal{X}$. Then for every $x\in\mathcal{X}\setminus\{0\}$, we have*

$$\|\theta_f x\|_0^{\frac{1}{p}}\|\theta_g x\|_0^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1\leq j\leq n,1\leq k\leq m}|f_j(\omega_k)|} \quad \text{and} \quad \|\theta_g x\|_0^{\frac{1}{p}}\|\theta_f x\|_0^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1\leq j\leq n,1\leq k\leq m}|g_k(\tau_j)|}. \tag{3}$$

By seeing paper [4], Prof. Philip B. Stark, Department of Statistics, University of California, Berkeley, asked the following question to the author.

Question 1. *(Philip B. Stark) What is the infinite dimensional version of Theorem [4]?*

In this paper, we derive continuous uncertainty principle for Banach spaces which contains Theorem 4 as a particular case and also answers Question 1.

2. Functional Continuous Uncertainty Principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{X}$ denotes a Banach space (need not be finite dimensional) over $\mathbb{K}$. Dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. Whenever $1<p<\infty$, q denotes conjugate index of p. We recall the notion of weak integral also known as Pettis integrals [5]. Let (Ω,μ) be a measure space and $\mathcal{X}$ be a Banach space. A function $f:\Omega\to\mathcal{X}$ is said to be weak integrable or Pettis integrable if following conditions hold.

(i) For every $\phi\in\mathcal{X}^*$, the map $\phi f:\Omega\to\mathbb{K}$ is measurable and $\phi f\in\mathcal{L}^1(\Omega,\mu)$.
(ii) For every measurable subset $E\in\Omega$, there exists an (unique) element $x_E\in\mathcal{X}$ such that

$$\phi(x_E)=\int\limits_E \phi(f(\alpha))\,d\mu(\alpha), \quad \forall\phi\in\mathcal{X}^*.$$

The element $x_E\in\mathcal{X}$ is denoted by $\int_E f(\alpha)\,d\mu(\alpha)$. With this notion, we have

$$\phi\left(\int\limits_E f(\alpha)\,d\mu(\alpha)\right)=\int\limits_E \phi(f(\alpha))\,d\mu(\alpha), \quad \forall\phi\in\mathcal{X}^*, \forall E\in\Omega.$$

We need the following continuous version of p-Schauder frames [4].

Definition 1. *Let (Ω,μ) be a measure space. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a **continuous p-Schauder frame** for $\mathcal{X}$ ($1<p<\infty$) if the following holds.*

(i) For every $x\in\mathcal{X}$, the map $\Omega\ni\alpha\mapsto f_\alpha(x)\in\mathbb{K}$ is measurable.

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(ii) *For every $x \in \mathcal{X}$,*
$$\|x\|^p = \int\limits_{\Omega} |f_\alpha(x)|^p \, d\mu(\alpha).$$
(iii) *For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathbb{K}$ is weakly measurable.*
(iv) *For every $x \in \mathcal{X}$,*
$$x = \int\limits_{\Omega} f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$
where the integral is weak integral.

We note that condition (i) in Definition 1 says that the map
$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$
is a linear isometry. Following is the fundamental result of this paper.

Theorem 5. *(**Functional Continuous Uncertainty Principle**) Let (Ω, μ), (Δ, ν) be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ be continuous p-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X} \setminus \{0\}$, we have*
$$\begin{aligned} \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{p}} \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{q}} &\geq \frac{1}{\sup\limits_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)|}, \\ \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{p}} \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{q}} &\geq \frac{1}{\sup\limits_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)|}. \end{aligned} \tag{4}$$

Proof. Let $x \in \mathcal{X} \setminus \{0\}$ and q be the conjugate index of p. First using θ_f is an isometry and later using θ_g is an isometry, we get

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$$
\begin{aligned}
\|x\|^p &= \|\theta_f x\|^p = \int\limits_{\Omega} |f_\alpha(x)|^p \, d\mu(\alpha) = \int\limits_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|^p \, d\mu(\alpha) \\
&= \int\limits_{\operatorname{supp}(\theta_f x)} \left| f_\alpha\left(\int\limits_{\Delta} g_\beta(x)\omega_\beta \, d\nu(\beta) \right) \right|^p d\mu(\alpha) = \int\limits_{\operatorname{supp}(\theta_f x)} \left| \int\limits_{\Delta} g_\beta(x) f_\alpha(\omega_\beta) \, d\nu(\beta) \right|^p d\mu(\alpha) \\
&= \int\limits_{\operatorname{supp}(\theta_f x)} \left| \int\limits_{\operatorname{supp}(\theta_g x)} g_\beta(x) f_\alpha(\omega_\beta) \, d\nu(\beta) \right|^p d\mu(\alpha) \leq \int\limits_{\operatorname{supp}(\theta_f x)} \left(\int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x) f_\alpha(\omega_\beta)| \, d\nu(\beta) \right)^p d\mu(\alpha) \\
&\leq \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)| \right)^p \int\limits_{\operatorname{supp}(\theta_f x)} \left(\int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x)| \, d\nu(\beta) \right)^p d\mu(\alpha) \\
&= \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)| \right)^p \mu(\operatorname{supp}(\theta_f x)) \left(\int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x)| \, d\nu(\beta) \right)^p \\
&\leq \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)| \right)^p \mu(\operatorname{supp}(\theta_f x)) \left(\int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x)|^p \, d\nu(\beta) \right)^{\frac{p}{p}} \left(\int\limits_{\operatorname{supp}(\theta_g x)} 1^q \, d\nu(\beta) \right)^{\frac{p}{q}} \\
&= \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)| \right)^p \mu(\operatorname{supp}(\theta_f x)) \|\theta_g x\|^p \nu(\operatorname{supp}(\theta_g x))^{\frac{p}{q}} \\
&= \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)| \right)^p \mu(\operatorname{supp}(\theta_f x)) \|x\|^p \nu(\operatorname{supp}(\theta_g x))^{\frac{p}{q}}.
\end{aligned}
$$

Therefore

$$
\frac{1}{\sup\limits_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)|} \leq \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{p}} \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{q}}.
$$

On the other way, first using θ_g is an isometry and θ_f is an isometry, we get

$$
\begin{aligned}
\|x\|^p &= \|\theta_g x\|^p = \int_{\Delta} |g_\beta(x)|^p \, d\nu(\beta) = \int_{\operatorname{supp}(\theta_g x)} |g_\beta(x)|^p \, d\nu(\beta) \\
&= \int_{\operatorname{supp}(\theta_g x)} \left| g_\beta \left(\int_{\Omega} f_\alpha(x)\tau_\alpha \, d\mu(\alpha) \right) \right|^p d\nu(\beta) = \int_{\operatorname{supp}(\theta_g x)} \left| \int_{\Omega} f_\alpha(x) g_\beta(\tau_\alpha) \, d\mu(\alpha) \right|^p d\nu(\beta) \\
&= \int_{\operatorname{supp}(\theta_g x)} \left| \int_{\operatorname{supp}(\theta_f x)} f_\alpha(x) g_\beta(\tau_\alpha) \, d\mu(\alpha) \right|^p d\nu(\beta) \leq \int_{\operatorname{supp}(\theta_g x)} \left(\int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x) g_\beta(\tau_\alpha)| \, d\mu(\alpha) \right)^p d\nu(\beta) \\
&\leq \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)| \right)^p \int_{\operatorname{supp}(\theta_g x)} \left(\int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)| \, d\mu(\alpha) \right)^p d\nu(\beta) \\
&= \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)| \right)^p \nu(\operatorname{supp}(\theta_g x)) \left(\int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)| \, d\mu(\alpha) \right)^p \\
&\leq \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)| \right)^p \nu(\operatorname{supp}(\theta_g x)) \left(\int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|^p \, d\mu(\alpha) \right)^{\frac{p}{p}} \left(\int_{\operatorname{supp}(\theta_f x)} 1^q \, d\mu(\alpha) \right)^{\frac{p}{q}} \\
&= \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)| \right)^p \nu(\operatorname{supp}(\theta_g x)) \|\theta_f x\|^p \mu(\operatorname{supp}(\theta_f x))^{\frac{p}{q}} \\
&= \left(\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)| \right)^p \nu(\operatorname{supp}(\theta_g x)) \|x\|^p \mu(\operatorname{supp}(\theta_f x))^{\frac{p}{q}}.
\end{aligned}
$$

Therefore

$$\frac{1}{\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)|} \leq \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{p}} \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{q}}.$$

□

Corollary 1. *Let (Ω, μ), (Δ, ν) be measure spaces. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ and $\{\omega_\beta\}_{\beta\in\Delta}$ be Parseval continuous frames for a Hilbert space $\mathcal{H}$. Then for every $h \in \mathcal{H} \setminus \{0\}$, we have*

$$\mu(\operatorname{supp}(\theta_\tau h))\nu(\operatorname{supp}(\theta_\omega h)) \geq \frac{1}{\sup_{\alpha\in\Omega, \beta\in\Delta} |\langle \omega_\beta, \tau_\alpha \rangle|^2}, \quad \nu(\operatorname{supp}(\theta_\omega h))\mu(\operatorname{supp}(\theta_\tau h)) \geq \frac{1}{\sup_{\alpha\in\Omega, \beta\in\Delta} |\langle \tau_\alpha, \omega_\beta \rangle|^2},$$

where

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h \in \mathcal{L}^2(\Omega, \mu); \quad \theta_\tau h : \Omega \ni \alpha \mapsto (\theta_\tau h)(\alpha) := \langle h, \tau_\alpha \rangle \in \mathbb{K},$$
$$\theta_\omega : \mathcal{H} \ni h \mapsto \theta_\omega h \in \mathcal{L}^2(\Delta, \nu); \quad \theta_\omega h : \Delta \ni \beta \mapsto (\theta_\omega h)(\beta) := \langle h, \omega_\beta \rangle \in \mathbb{K}.$$

Corollary 2. *Theorem 4 follows from Theorem 5.*

Proof. Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ be p-Schauder frames for a finite dimensional Banach space $\mathcal{X}$. Define $\Omega := \{1, \dots, n\}$ and $\Delta := \{1, \dots, m\}$. Take μ as counting measure on Ω and ν as counting measure on Δ. □

Theorem 5 brings the following question.

Question 2. *(i) Let (Ω, μ), (Δ, ν) be measure spaces, $\mathcal{X}$ be a Banach space and $p > 1$. For which pairs of continuous p-Schauder frames $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ for $\mathcal{X}$, we have equality in Inequality (4)?*

(ii) What is the version of Theorem 5 for $0 < p \leq 1$ and $p = \infty$?

If we can derive the uncertainty principles derived in [6] and [7] for p-Schauder frames, then we hope that we can get continuous versions of them.

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