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[Frank Vega](#) *

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Article

A Note on Oppermann's Conjecture

Frank Vega

Information Physics Institute, Miami, Florida, United States; vega.frank@gmail.com

Abstract: A prime gap is the difference between consecutive prime numbers. The n^{th} prime gap, denoted g_n , is calculated by subtracting the n^{th} prime from the $(n + 1)^{\text{th}}$ prime: $g_n = p_{n+1} - p_n$. Oppermann's conjecture is a prominent unsolved problem in pure mathematics concerning prime gaps. Despite verification for numerous primes, a general proof remains elusive. If true, the conjecture implies that prime gaps grow at a rate bounded by $g_n < \sqrt{p_n}$. This note presents a proof of Oppermann's conjecture using the square root function on inequalities involving prime numbers. This proof simultaneously establishes Andrica's, Legendre's, and Brocard's conjectures.

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MSC: 11A41; 11A25

1. Introduction

Prime numbers, the fundamental building blocks of integers, have captivated mathematicians for centuries. Their erratic distribution, punctuated by seemingly random gaps, remains a captivating enigma. Several conjectures, including those related to large prime gaps, attempt to elucidate patterns within this irregularity by correlating prime gap sizes with the primes themselves.

Andrica's conjecture, attributed to Dorin Andrica, posits a specific relationship between consecutive primes [1]. It asserts that the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds true for all positive integers n , where p_n represents the n^{th} prime number. Equivalently, if g_n denotes the n^{th} prime gap (the difference between p_{n+1} and p_n), Andrica's conjecture can be expressed as

$$g_n < 2 \cdot \sqrt{p_n} + 1.$$

Legendre's conjecture, attributed to Adrien-Marie Legendre, posits the existence of at least one prime number between the squares of any consecutive positive integers [2]. This unsolved problem is classified as one of Landau's problems and implies that the gap between a prime and its successor is on the order of the square root of the prime (expressed as $O(\sqrt{p})$).

Oppermann's conjecture, another open question related to prime distribution, is a stronger assertion than both Legendre's and Andrica's conjectures. Proposed by Danish mathematician Ludvig Oppermann in 1877, it suggests an upper bound for prime gaps of $g_n < \sqrt{p_n}$ [3]. The conjecture states that, for every integer $x > 1$, there is at least one prime number between

$$x \cdot (x - 1) \text{ and } x^2,$$

and at least another prime between

$$x^2 \text{ and } x \cdot (x + 1).$$

If true, this would also entail Brocard's conjecture, which states that there are at least four primes between the squares of consecutive odd primes [2].

Despite its seemingly straightforward formulation, Oppermann's conjecture has far-reaching implications for our comprehension of prime number distribution. Although extensively verified for countless primes, a general proof remains elusive. This unproven conjecture nonetheless serves as a

compelling focal point, driving research to uncover deeper patterns in the prime number sequence. By resolving Oppermann's conjecture, this work aims to significantly advance our understanding of this fundamental mathematical enigma.

2. Background and Ancillary Results

We based our proof on the following Propositions:

Proposition 1. For $x \geq 1$ [4, pp. 1]:

$$2 \cdot \sqrt{x+1} - 2 \cdot \sqrt{x} < \frac{1}{\sqrt{x}}.$$

An alternate form is

$$1 < \frac{1}{2 \cdot (\sqrt{x^2+x} - x)}.$$

Proposition 2. We know that if $x \geq 89693$, there is at least one prime p in the interval $x < p \leq \left(1 + \frac{1}{\log^3(x)}\right) \cdot x$ [5, Proposition 5.4 pp. 242].

This is a key finding.

Lemma 1. For every two consecutive primes p_n and p_{n+1} , if the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

holds then $g_n < \sqrt{p_n}$.

Proof. The inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

is the same as

$$\sqrt{p_{n+1}} < \left(\sqrt{p_n} + \frac{1}{3}\right)$$

and

$$p_{n+1} < \left(\sqrt{p_n} + \frac{1}{3}\right)^2$$

after raising both sides to the square and distributing the terms. We know that

$$\left(\sqrt{p_n} + \frac{1}{3}\right)^2 = p_n + \frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$$

which is

$$g_n = p_{n+1} - p_n < \frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$$

and so,

$$\frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9} < \sqrt{p_n}$$

for all $p_n \geq 2$. $\square$

This is a main insight.

Lemma 2. The real function $t \mapsto \frac{\sqrt{1 + \frac{1}{\log^3(t)}} - 1}{\sqrt{t+1} - \sqrt{t}} - \frac{2}{3}$ is decreasing and lesser than zero in the range $t \geq 5$.

Proof. The greatest root value of

$$f(x) = \frac{\sqrt{1 + \frac{1}{\log^3(x)}} - 1}{\sqrt{x+1} - \sqrt{x}} - \frac{2}{3}$$

is located at

$$x \approx 4.25280413354306 \dots$$

For all $x \geq 5$, the function $f(x)$ is strictly decreasing. Certainly, the derivative of the function

$$f'(x) = \frac{-3 \cdot \sqrt{x+1} - \sqrt{x} \cdot \left(\sqrt{1 + \frac{1}{\log^3(x)}} - 1 \right) \cdot \log^4(x) + \sqrt{x} \cdot \log(x)}{2 \cdot x \cdot \sqrt{x+1} \cdot (\sqrt{x+1} - \sqrt{x}) \cdot \sqrt{1 + \frac{1}{\log^3(x)}} \cdot \log^4(x)}$$

is less than zero for all $x \geq 5$ (Derivatives can be used to determine whether a function is decreasing on an interval: $f(x)$ is decreasing if derivative $f'(x) < 0$). Moreover, $f(x)$ is lesser than zero in the range $x \geq 5$ because of $f(5) \approx -0.1348800327 \dots < 0$. $\square$

By combining these results, we present a proof of Oppermann's conjecture.

3. Main Result

This is the main theorem.

Theorem 1. *The Oppermann's conjecture is true.*

Proof. There is not any natural number n' such that

$$\sqrt{p_{n'+1}} - \sqrt{p_{n'}} = \frac{1}{3}$$

since this implies that $g_{n'} = \frac{2}{3} \cdot \sqrt{p_{n'}} + \frac{1}{9}$. For every n , g_n is a natural number and $\frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$ is always irrational. In fact, all square roots of natural numbers, other than of perfect squares, are irrational [6]. We have confirmed the conjecture for p_n up to 10^8 by a numerical computation. Suppose that there exists a prime number $p_{n_0} > 10^8$ such that

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > \frac{1}{3}.$$

That is equivalent to

$$\sqrt{p_{n_0+1} \cdot p_{n_0}} - p_{n_0} > \frac{2}{3} \cdot \sqrt{p_{n_0}} \cdot \frac{1}{2}.$$

after multiplying both sides by $\sqrt{p_{n_0}}$ where $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$. That is the same as

$$\frac{\sqrt{p_{n_0+1} \cdot p_{n_0}} - p_{n_0}}{\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0}} > \frac{2}{3} \cdot \sqrt{p_{n_0}} \cdot \frac{1}{2 \cdot (\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0})}$$

after dividing both sides by $\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0}$ where

$$\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0} > 0.$$

By Proposition 1, we have

$$\frac{2}{3} \cdot \sqrt{p_{n_0}} \cdot \frac{1}{2 \cdot (\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0})} > \frac{2}{3} \cdot \sqrt{p_{n_0}}.$$

Besides, we deduce that

$$\begin{aligned} \frac{\sqrt{p_{n_0+1} \cdot p_{n_0}} - p_{n_0}}{\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0}} &= \frac{\sqrt{(p_{n_0} + g_{n_0}) \cdot p_{n_0}} - p_{n_0}}{\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0}} \\ &= \frac{\sqrt{p_{n_0}^2 + g_{n_0} \cdot p_{n_0}} - p_{n_0}}{\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0}} \\ &\leq \frac{\sqrt{p_{n_0}^2 + \frac{p_{n_0}^2}{\log^3(p_{n_0})}} - p_{n_0}}{\sqrt{p_{n_0}^2 + p_{n_0}} - p_{n_0}} \\ &= \sqrt{p_{n_0}} \cdot \frac{\sqrt{1 + \frac{1}{\log^3(p_{n_0})}} - 1}{\sqrt{p_{n_0} + 1} - \sqrt{p_{n_0}}} \end{aligned}$$

since $g_{n_0} \leq \frac{p_{n_0}}{\log^3(p_{n_0})}$ for all $p_{n_0} > 10^8$ by Proposition 2. So, we would have

$$\sqrt{p_{n_0}} \cdot \frac{\sqrt{1 + \frac{1}{\log^3(p_{n_0})}} - 1}{\sqrt{p_{n_0} + 1} - \sqrt{p_{n_0}}} - \frac{2}{3} \cdot \sqrt{p_{n_0}} > 0$$

which is

$$\frac{\sqrt{1 + \frac{1}{\log^3(p_{n_0})}} - 1}{\sqrt{p_{n_0} + 1} - \sqrt{p_{n_0}}} - \frac{2}{3} > 0.$$

Hence, it is enough to show that

$$\frac{\sqrt{1 + \frac{1}{\log^3(p_{n_0})}} - 1}{\sqrt{p_{n_0} + 1} - \sqrt{p_{n_0}}} - \frac{2}{3} > 0$$

does not hold for all $p_{n_0} > 10^8$ by Lemma 2. Then, the proof is done by Lemma 1. $\square$

4. Conclusion

This paper presents a novel approach to the longstanding Oppermann conjecture, leveraging the properties of prime numbers based on some inequalities using the square root function. By establishing a rigorous framework and employing careful analysis, we have demonstrated that the conjecture holds true for all prime numbers. This result not only resolves a fundamental open problem in number theory but also provides new insights into the distribution of primes. The implications of this work extend beyond prime number theory, potentially impacting areas such as cryptography, computational number theory, and related fields.

References

1. Andrica, D. Note on a conjecture in prime number theory. *Studia Univ. Babes-Bolyai Math* **1986**, 31, 44–48.
2. Wells, D. *Prime Numbers: The Most Mysterious Figures in Math*; Turner Publishing Company, 2011.
3. Oppermann, L. Om vor Kundskab om Primtallenes maengde mellem givne Graendser. *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger og dets Medlemmers Arbejder* **1882**, pp. 169–179.
4. Kozma, L. Useful Inequalities. Kozma's Homepage, Useful inequalities cheat sheet. http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf, 2023. Accessed September 6, 2024.

5. Dusart, P. Explicit estimates of some functions over primes. *The Ramanujan Journal* **2018**, *45*, 227–251. doi:10.1007/s11139-016-9839-4.
6. Jackson, T. 95.42 Irrational square roots of natural numbers - a geometrical approach. *The Mathematical Gazette* **2011**, *95*, 327–330. doi:10.1017/S0025557200003193.

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