Asymptotics and confluence for some linear q-difference-differential Cauchy problem

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Abstract

A linear Cauchy problem with polynomial coefficients wich combines $q-$difference operators for $q > 1$ and differential operators of irregular type is examined. A finite set of sectorial holomorphic solutions w.r.t the complex time is constructed by means of classical Laplace transforms. These functions share a common asymptotic expansion in the time variable which turns out to carry a double layers structure which couples $q-$Gevrey and Gevrey bounds. In the last part of the work, the problem of confluence of these solutions as $q \to 1$ is investigated.

Key words: asymptotic expansion, confluence, formal power series, partial differential equation, $q-$difference equation. 2010 MSC: 35R10, 35C10, 35C15, 35C20.

1 Introduction

In this paper, we aim attention at a linear Cauchy problem which pairs up two classes of operators acting both on a complex time variable t, comprised by compositions of the basic $q-$difference operator $\sigma_{q;t} : t \mapsto qt$ for a fixed real number $q > 1$ and powers of the elemental differential operator $t \mapsto t^{k+1}\partial_t$ of irregular type where $k \geq 1$ is a given integer.

The shape of the problem under study is displayed as follows

$$P(t^{k+1}\partial_t)\partial_z^S u(t,z) = \mathcal{P}(t,z,\sigma_{q;t},t^{k+1}\partial_t,\partial_z)u(t,z) \quad (1)$$

for given Cauchy data

$$(\partial_z^j u)(t,0) = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1 \quad (2)$$

where $S,k \geq 1$ are integers and the piece $P(T)$ from the leading term of (1) stands for a polynomial in $\mathbb{C}[T]$. The main body $\mathcal{P}(t,z,V_1,V_2,V_3)$ of (1) together with the data (2) are polynomial, with complex coefficients, in their arguments.

This work can be viewed as a continuation of the study [4] by A. Lastra and the author. In the first part of [4] we focused on the next singularly perturbed initial value problem

$$Q(\partial_z)u(t,z,\epsilon) - R_D(\partial_z)\epsilon^{kd_D}(t^{k+1}\partial_t)^{d_D}u(t,z,\epsilon) = \mathcal{P}_1(\epsilon,z,t^{k+1}\partial_t,\partial_z)u(q^{\delta}t,z,\epsilon) + f(t,z,\epsilon) \quad (3)$$

for vanishing initial data $u(0,z,\epsilon) \equiv 0$. Here, the constants $k, d_D \geq 1$ appearing in the leading term of (3) are integers, $q > 1$ and $\delta > 0$ are given real numbers, $Q(X)$,$R_D(X)$ represent complex polynomials. The central block $\mathcal{P}_1(\epsilon, z, V_1, V_2)$ is polynomial in V_1,V_2, holomorphic w.r.t z on some horizontal strip $H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}$ and analytic relatively to a complex parameter ϵ near 0 in $\mathbb{C}$; the forcing term f satisfies the same feature accordingly to z, ϵ and represents some constrained entire function w.r.t t on $\mathbb{C}$.

Under strong restrictions put on the shape of (3), we were able to construct a finite set $\{u_p(t,z,\epsilon)\}_{0 \leq p \leq \varsigma - 1}$ of holomorphic solutions to (3) defined on products $\mathcal{T} \times H_\beta \times \mathcal{E}_p$, where $\mathcal{T}$ stands for a fixed bounded sector at 0 and where the set of finite sectors $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma - 1}$ is chosen as a so-called good covering in $\mathbb{C}^*$, see Definition 1 in this paper. These solutions are expressed through a Laplace transform of order k in time t and Fourier map in space z,

$$u_p(t,z,\epsilon) = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}} \omega_p(u,m,\epsilon) \exp\Big(-(\frac{u}{\epsilon t})^k \Big) e^{izm} \frac{du}{u} dm$$

along halflines $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$ in well chosen directions $\gamma_p \in \mathbb{R}$, where the Borel map ω_p is exponentially flat w.r.t the phase m on $\mathbb{R}$ and with so-called $q-$exponential growth relatively to u (similar to the bounds given in (126) of this paper).

Furthermore, informations concerning asymptotic expansions as $\epsilon \to 0$ could be extracted. Indeed, all u_p share a common formal power series $\hat{u}(t,z,\epsilon) = \sum_{m \geq 0} h_m(t,z)\epsilon^m$ as $q-$Gevrey asymptotic expansion of some order $1/\kappa$, where κ is related to k, δ and coefficients of $\mathcal{P}_1$, meaning that we can find constants $A_p, C_p > 0$ with

$$\sup_{t \in \mathcal{T}, z \in H_\beta} |u_p(t,z,\epsilon) - \sum_{m=0}^{N} h_m(t,z)\epsilon^m| \leq C_p(A_p)^{N+1} q^{\frac{(N+1)N}{2\kappa}} |\epsilon|^{N+1}$$

for all integers $N \geq 0$, all $\epsilon \in \mathcal{E}_p$, for $0 \leq p \leq \varsigma - 1$.

Compared to (3), the linear operator $\mathcal{P}$ in (1) still depends on a given irregular $t^{k+1}\partial_t$ but we allow the appearance of time dependent coefficients and the presence of several $q-$difference dilation operators $t \mapsto q^l t$ for natural numbers $l \geq 1$. As we will see, this level of generality makes the geometry of the problem in the Borel plane much more thorny to handle as the one in [4].

In the first main result of the work, see Theorem 1, under the set of technical requirements (7) to (16) imposed on the problem (1), (2), we build up a family of bounded holomorphic solutions to (1), (2) on domains $\mathcal{T}_p \times D$, where D stands for some small disc centered at 0 in $\mathbb{C}$ and $\{\mathcal{T}_p\}_{0 \leq p \leq \varsigma - 1}$ is a well chosen good covering in $\mathbb{C}^*$, see Definition 1. As in [4], we can write the solutions as a Laplace transform of order k in time t,

$$u_p(t,z) = k \int_{L_{\gamma_p}} w_p(u,z) \exp\Big(-(\frac{u}{t})^k \Big) \frac{du}{u}$$

along halflines $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$ in suitable directions $\gamma_p \in \mathbb{R}$ where the Borel map $w_p(u,z)$ is holomorphic w.r.t z on D and suffers the $q-$exponential growth rate (126) relatively to u on an unbounded sector $\mathcal{U}_p$.

Similarly to [4], we can analyze the asymptotic expansions of these solutions as $t \to 0$ on $\mathcal{T}_p$. However, for our problem (1), (2), we are now able to exhibit a fine structure of mixed type asymptotics, see Definition 3, where a $q-$Gevrey leading part and a subdominant tail of Gevrey order $1/k$ shows up. Namely, in the second main statement of the paper, see Theorem

2, we prove the existence of a formal series $\hat{u}(t,z) = \sum_{n \geq 0} u_n(z)t^n$ where $u_n(z)$ are bounded holomorphic maps on D, for which two constants $C, M > 0$ can be singled out with

$$\sup_{z \in D} |u_p(t,z) - \sum_{n=0}^{N} u_n(z)t^n| \leq CM^{N+1} q^{\frac{(N+1)^2}{2}} \Gamma(\frac{N+1}{k})|t|^{N+1} \quad (4)$$

for all $t \in \mathcal{T}_p$, all integers $N \geq 0$.

As in our former work [4], the above asymptotic features originate from underlying geometric properties of the Borel map $w_p(u,z)$. Indeed, the location of the singularities of the maps $u \mapsto \omega_p(u,m,\epsilon)$ and bounds on annuli played a primordial role in the asymptotics of the solutions $\epsilon \mapsto u_p(t,z,\epsilon)$ to (3). The presence of a single $q-$difference operator in (3), $t \mapsto q^{\delta}t$ allows a sharp location of the singularities of the Borel map $u \mapsto \omega_p(u,m,\epsilon)$ and gives rise to a splitting of ω_p as an infinite sum

$$\omega_p(u,m,\epsilon) = \sum_{j \geq 0} \omega_{p,j}(u,m,\epsilon)$$

where each $u \mapsto \omega_{p,j}(u,m,\epsilon)$ is defined on unbounded sectors with at most $q-$exponential growth in u and possess a decay w.r.t j and $0 \leq h \leq j$ with size

$$A^h B^j q^{-\Delta h^2} \quad (5)$$

on sectorial square frames $P\Omega_h^p$ that are similar to our sectorial annuli $\mathcal{A}_{p,h}$ described in (129) of this work, see Proposition 6 in [4].

Such a precise singularities tracking is not valid any more in our enhanced context. However, a similar decomposition, see (125), can be reached for the Borel map $w_p(u,z)$. It turns out that the bounds we obtain for $u \mapsto w_p(u,z)$ on the intermediate domains $\mathcal{A}_{p,h}$ are of the same nature as the one above (5), but are far more delicate to obtain. The difficulty arises from the presence of non local integral operators stemming from the polynomial reliance in time of $\mathcal{P}$ and the shuffling action of concomittant $q-$difference operators on the domains spanned by the Borel variable u, see Proposition 4 which needs more than fourteen pages of proof stuffed with five technical lemmas.

Notice that it is not the first occurrence of such a double scale structure (4) in the asymptotic of solutions. Indeed, a similar coupled asymptotic structure has already been observed by A. Lastra, J. Sanz and the author in a 2012 year paper [6] for linear Cauchy problems with so-called $q-$irregular singularity in time t (related to the $q-$difference operator $t\sigma_{q;t}$ of irregular type) and Fuchsian singularity in space z, shaped as

$$(t\sigma_{q;t})^{r_2}(z\partial_z)^{r_1}\partial_z^S X(t,z) = B(z, t\sigma_{q;t}, \sigma_{q^{-1};z}, \partial_z)X(t,z)$$

for suitably chosen analytic Cauchy data

$$(\partial_z^j X)(t,0) = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1$$

and properly selected complex numbers q with $|q| > 1$ where $r_1, r_2, S \geq 1$ are integers and B stands for a polynomial.

Later on, in 2018, this type of combined asymptotic appeared in a study by H. Yamazawa, see [10], of linear $q-$difference differential equations of the form

$$L(t, \sigma_{q;t}, \partial_x)u(t,x) = f(t,x)$$

for given holomorphic forcing term $f(t,x)$ near the origin and L being a polynomial. The holomorphic solution $u(t,x)$ he constructed possess a formal power series $\hat{u}(t,x) = \sum_{k \geq 0} u_k(x)t^k$

as asymptotic expansion of mixed order $(1;(q,1))$ w.r.t t but with x tending to 0 in a related manner with t (linked to so-called inner expansions in the framework of boundary-layer theory).

More recently, in 2020, mixed type asymptotics arose in

– the work [7] by the author on singularly perturbed initial value problem that couples $q-$difference operators of irregular type $t\sigma_{q;t}$ and Fuchsian operators $t\partial_t$ acting both on a complex time variable $t \in \mathbb{C}$,

– the joint work of A. Lastra and the author [5] addressing a singularly perturbed initial value problem in two complex variables t_1, t_2 with $q-$irregular singularity in t_1 related to $q-$difference operators of irregular type $t_1^d \sigma_{q;t_1}^{d/k_1}$ and with irregular singularity in t_2 associated to the differential operator of irregular type $t_2^{k_2+1}\partial_{t_2}$.

Besides the asymptotic properties concern, a second aspect of the built up solutions $u_p(t,z)$ of (1), (2) is investigated in the last section of the paper and deals with the problem of so-called confluence as the parameter $q > 1$ tends to 1.

In the third main result of the work, see Theorem 3, we show that for any given bounded sector $\mathcal{T}$ from the good covering $\{\mathcal{T}_p\}_{0 \leq p \leq \varsigma-1}$ in $\mathbb{C}^*$, the corresponding solution $u_{;q}(t,z)$ (where the dependence on the parameter q is now highlighted by an index $;q$) to (1), (2) on $\mathcal{T} \times D$ merges uniformly on $\mathcal{T} \times D$, as $q \in (1, q_0]$ tends to 1 for some fixed $q_0 > 1$, to a holomorphic function $u_{;1}(t,z)$ on $\mathcal{T} \times D$. This function $u_{;1}(t,z)$ solves a limit Cauchy problem displayed in (153), (154) which is merely obtained by setting $q = 1$ in the initial problem (1), (2). The map $u_{;1}(t,z)$ can itself be expressed through a Laplace transform of order k,

$$u_{;1}(t,z) = k \int_{L_\gamma} w_{;1}(u,z) \exp\left(-(\frac{u}{t})^k \right) \frac{du}{u}$$

along halflines $L_\gamma = [0, +\infty)e^{\sqrt{-1}\gamma} \in \mathcal{U} \cup \{0\}$, where the Borel map $w_{;1}(u,z)$ is holomorphic w.r.t z on D and is compelled to $q-$exponential growth rate (164) on the unbounded sector $\mathcal{U}$. The proof again leans on sharp bounds estimates at the level of the Borel maps $w_{;q}(u,z)$ related to $u_{;q}(t,z)$ and $w_{;1}(u,z)$, see Proposition 10. A technical auxiliary result on bounds for the difference of analytic solutions to the problem solved by $w_{;1}$ under the action of $q-$difference operators is required to complete the proof, see Proposition 9.

In the framework of linear $q-$difference equations, we ought to mention three major recent works studying confluence problems.

The first work is a paper [9] by J. Sauloy which deals with so-called Fuchsian systems of $q-$difference equations

$$\delta_q Y(z,q) = B(z)Y(z,q) \tag{6}$$

for square matrices $B(z)$ of dimension $m \geq 1$ with polynomial coefficients, where $\delta_q Y(z,q) = (Y(qz,q) - Y(z,q))/(q-1)$. Meromorphic fundamental solutions to (6) are built up that converge uniformly on compact sets to actual holomorphic solutions of the limit Fuchsian system

$$z\partial_z Y(z) = B(z)Y(z)$$

locally near $z = 0$, as $|q| > 1$ tends to 1.

The second work is a paper [1] by L. Di Vizio and C. Zhang which focuses on a $q-$analog of the Euler differential equation

$$x^2 \partial_x y + y = x$$

with so-called irregular singularity at $x = 0$, where the Fuchsian operator $x\partial_x$ is replaced by the $q-$difference operator δ_q in the two different situations $q < 1$ and $q > 1$ as q tends to 1.

The third work is a paper [2] by T. Dreyfus which concerns the confluence of well chosen meromorphic solutions to general linear $q-$difference equations with polynomial coefficients

$$\sum_{j=0}^{m} b_j(z)\delta_q^j h(z,q) = 0$$

as $q > 1$ tends to 1.

These two last results bank on so-called $q-$Laplace representations of the analytic solutions to the $q-$difference equations where the classical exponential kernel is suitably replaced by a $q-$exponential function $e_q(x)$ or by the Theta function $\Theta_q(x)$.

The confluence in the context of $q-$difference-differential equations has been much less examined and is a promising direction of active study in which our present contribution falls. With this respect, the novel and striking work [8] by D. Pravica, N. Randriampiry and M. Spurr should be mentioned in this new trend of research. They consider many examples of advanced higher order differential equations (one of those is given by $f'''(t) = q^3 f(qt)$ for $q > 1$ and $t \geq 0$) and they analyze the convergence of particular solutions to the limit ODE on the real axis as $q \rightarrow 1$.

Our paper is arranged as follows.

In Section 2, we present the main linear Cauchy problem (17), (18) under study and we explain the leading strategy of its resolution. Namely, we search for solutions $u(t,z)$ in the form of a Laplace transform of some Borel map $w(u,z)$. A related Cauchy problem (25), (26) for $w(u,z)$ is derived.

In section 3, it is shown that the coefficients $w_n(u)$, $n \geq 0$, of the formal Taylor expansion of $w(u,z)$ at $z = 0$ are holomorphic on some sequence of discs whose radii tend geometrically to 0. Sharp decay estimates are given w.r.t $n \geq 0$.

In Section 4, the Borel map $w(u,z)$ is shown to be holomorphic on domains $\mathcal{U} \times D$ for well chosen unbounded sectors $\mathcal{U}$ and small disc D centered at 0 with at most $q-$exponential growth w.r.t u.

In Section 5, bounds for the coefficients $w_n(u)$, $n \geq 0$, are provided on intermediate domains comprised by unions of sectorial annuli. This is the most technical section of the work.

In Section 6, we state the first main result of the work, Theorem 1. A finite set of genuine solutions $\{u_p(t,z)\}_{0 \leq p \leq \varsigma-1}$ to (17), (18) is exhibited with sharp error bounds for the difference of consecutive maps $u_{p+1} - u_p$.

In Section 7, the second main statement of the work is outlined, Theorem 2. Asymptotic expansions of so-called mixed type w.r.t the time t as $t \rightarrow 0$ are provided for the solutions $u_p(t,z)$ built up in Theorem 1.

In Section 8, the last main issue of the paper is disclosed in Theorem 3. The solutions $u_p(t,z)$ of (17), (18) are shown to merge uniformly in (t,z) on $\mathcal{T}_p \times D$ to the holomorphic solution of a limit Cauchy problem (153), (154) as $q \rightarrow 1$, $q > 1$.

2 Statement of the main problem

2.1 Layout of the main Cauchy problem

Let $k, S \geq 1$ be integers and $q > 1$ be a real number. We set $P(\tau) \in \mathbb{C}[\tau]$ a polynomial with complex coefficients such that

$$P(0) \neq 0 \tag{7}$$

Let $\mathcal{A}$ be a finite subset of $\mathbb{N}^4$. For all $\underline{l} \in \mathcal{A}$, we fix a polynomial with complex coefficients

$$c_{\underline{l}}(z) = \sum_{h \in I_{\underline{l}}} c_{\underline{l},h} z^h \tag{8}$$

for a subset $I_{\underline{l}}$ of the positive natural numbers $\mathbb{N} \setminus \{0\}$. Besides, for all $0 \leq j \leq S-1$, we denote $\varphi_j(t)$ polynomials with complex coefficients written in the form

$$\varphi_j(t) = \sum_{h \in J_j} p_{j,h} \Gamma(h/k) t^h \tag{9}$$

where J_j is a finite subset of $\mathbb{N} \setminus \{0\}$ and $\Gamma(x)$ represents the classical Gamma function.

For reasons that will appear throughout the paper, we impose the next list of constraints on the finite set $\mathcal{A}$.

1) For all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$, we ask that

$$l_2 < S \quad , \quad S \geq l_2 + l_3 \tag{10}$$

2) One can single out a positive real number $\Delta > 0$ such that the next couple of inequalities

$$\begin{cases} 2(l_2 - h)\Delta + l_0 + kl_1 - 2S\Delta > 0 \\ -(-h + l_2)^2 \Delta - S(l_0 + kl_1) + S^2 \Delta < 0 \end{cases} \tag{11}$$

is valid whenever $h \in I_{\underline{l}}$ for all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

3) One can choose a real number $k_1 > 0$ such that the degree $\deg(P)$ of P is subjected to the next lower bounds

$$k\deg(P) \geq kl_1 + l_0 + 2k_1 l_3 \log(q) \tag{12}$$

whenever $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

4) The positive real number $\Delta > 0$ built in 2) above satisfies also the constraint

$$-2\Delta l_3 + l_0 + kl_1 > 0 \tag{13}$$

provided that $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

5) The positive real number $\Delta > 0$ singled out in 2) overhead fulfills furthermore the next system of inequalities

$$\begin{cases} 2(l_2 - g)\Delta + l_0 + kl_1 - 2\Delta(S-1) > 0 \\ -(-g + l_2)^2 \Delta - (S-1)(l_0 + kl_1) + (S-1)^2 \Delta < 0 \end{cases} \tag{14}$$

whenever $g \in I_{\underline{l}}$ for all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

6) The number $\Delta > 0$ distinguished in 2) above is subjected to the next system of inequalities

$$(15)\qquad \begin{cases} 2(l_2 - 1 - g)\Delta + l_0 + kl_1 - 2\Delta(S-1) > 0 \\ -(-g+l_2-1)^2\Delta - (S-1)(l_0+kl_1) + (S-1)^2\Delta < 0 \end{cases}$$

whenever $g \in I_{\underline{l}}$ for all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 \geq 1$.
7) The number $\Delta > 0$ appearing in the constraints 2), 4), 5) and 6) satisfies the inequality

$$(16)\qquad \Delta \geq 1/2$$

We consider the next linear Cauchy problem with polynomial coefficients in time,

$$(17)\qquad P(t^{k+1}\partial_t)\partial_z^S u(t,z) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} c_{\underline{l}}(z) t^{l_0} \left((t^{k+1}\partial_t)^{l_1} \partial_z^{l_2} u \right)(q^{l_3}t, z)$$

for given Cauchy data

$$(18)\qquad (\partial_z^j u)(t,0) = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1.$$

We now disclose our main strategy that will lead later on to the construction of suitable sets of solutions to our problem. We seek for solutions to (17), (18) in the form of a Laplace transform of order k, namely

$$(19)\qquad u(t,z) = k \int_{L_\gamma} w(u,z) \exp(-(u/t)^k) du/u$$

along a halfline $L_\gamma = [0, +\infty)e^{\sqrt{-1}\gamma}$, where the so-called Borel map $w(u,z)$ is holomorphic with respect to u on some unbounded sector $S_{d,\delta} = \{u \in \mathbb{C}^*/|d - \arg(u)| < \delta\}$ for well chosen directions $d \in \mathbb{R}$ and opening $\delta > 0$ and analytic w.r.t z on some small disc D_r centered at 0 with radius $r > 0$. For the Laplace transform to be well defined, we make the assumption that $w(u,z)$ has a most exponential growth of order k w.r.t u on $S_{d,\delta}$, uniformly in z on D_r, meaning the existence of two constants $C, K > 0$ with

$$\sup_{z \in D_r} |w(u,z)| \leq C|u| \exp(K|u|^k)$$

for all $u \in S_{d,\delta}$. Once we assume that such solutions exists, we will derive some functional equations that the Borel map $w(u,z)$ will be asked to solve at a formal level only. Such equations will be described in the next subsection. In later sections of the work (see Sections 3, 4 and 5) we will rigorously solve these functional equations in different kind of function spaces that will lead to actual analytic solutions of the form (19) to (17), (18) and for which asymptotic expansions as t tends to the origin can be extracted (see Section 7).

2.2 Related Cauchy problems and sequences of functions

In order to exhibit a q-difference-integro-differential equation satisfied by $w(u,z)$, we need to remind the next proposition that was already stated in our previous study [4].

Proposition 1 *Let* $(\mathbb{E}, ||.||_{\mathbb{E}})$ *be a complex Banach space. Let* $k \geq 1$ *be an integer and let* $w : S_{d,\delta} \to \mathbb{E}$ *be a holomorphic function on the open unbounded sector* $S_{d,\delta} = \{u \in \mathbb{C}^* :$

$|d-\arg(u)|<\delta\}$, continuous on $S_{d,\delta}\cup\{0\}$. Assume the existence of two constants $C>0$ and $K>0$ such that

$$||w(u)||_{\mathbb{E}} \leq C|u|e^{K|u|^k} \tag{20}$$

for all $u\in S_{d,\delta}$. Then, the Laplace transform of order k of w in the direction d is defined by

$$\mathcal{L}_k^d(w(u))(t) = k\int_{L_\gamma} w(u)e^{-(u/t)^k}\frac{du}{u},$$

along a half-line $L_\gamma=\mathbb{R}_+e^{i\gamma}\subset S_{d,\delta}\cup\{0\}$, where γ depends on t and is chosen in such a way that $\cos(k(\gamma-\arg(t)))\geq\delta_1>0$, for some fixed δ_1. The function $\mathcal{L}_k^d(w(u))(t)$ is well defined, holomorphic and bounded in any sector

$$S_{d,\theta,R^{1/k}} = \{t\in\mathbb{C}^* : |t|<R^{1/k} \quad , \quad |d-\arg(t)|<\theta/2\}, \tag{21}$$

where $\frac{\pi}{k}<\theta<\frac{\pi}{k}+2\delta$ and $0<R<\delta_1/K$.
A) The action of the Laplace transform on entire functions is described as follows: If w is an entire function on $\mathbb{C}$, with growth estimates (20) and with Taylor expansion $w(u)=\sum_{n\geq1}b_nu^n$, then $\mathcal{L}_k^d(w(u))(t)$ defines an analytic function near the origin w.r.t t with convergent Taylor expansion $\sum_{n\geq1}\Gamma(\frac{n}{k})b_nt^n$.
B) The actions of the irregular operator $t^{k+1}\partial_t$ and the monomial t^m on the Laplace transform are expressed through the next formulas

$$\mathcal{L}_k^d(ku^kw(u))(t) = t^{k+1}\partial_t\Big(\mathcal{L}_k^d(w(u))(t)\Big) \quad , \quad t^m\mathcal{L}_k^d(w(u))(t) = \mathcal{L}_k^d\Big(u\mapsto(u^m\star_k w(u))\Big)(t), \tag{22}$$

for every nonnegative integer m, and for all $t\in S_{d,\theta,R^{1/k}}$ with $0<R<\delta_1/K$. Here, $u^m\star_k w(u)$ stands for the convolution product

$$\frac{u^k}{\Gamma(\frac{m}{k})}\int_0^{u^k}(u^k-s)^{\frac{m}{k}-1}w(s^{1/k})\frac{ds}{s}.$$

C) The action of the dilation q^δ commutes with the Laplace transform, for any integer $\delta\geq1$, namely

$$\mathcal{L}_k^d(w(u))(q^\delta t) = \mathcal{L}_k^d(w(q^\delta u))(t) \tag{23}$$

holds for all $t\in S_{d,\theta,R_1^{1/k}}$ for $0<R_1<\delta_1/(Kq^{k\delta})$.

Owing to the point A) from the above proposition, we first observe that the Cauchy data (18) given as polynomials through (9) can be expressed as Laplace transforms of order k,

$$\varphi_j(t) = k\int_{L_\gamma} P_j(u)\exp(-(u/t)^k)du/u \tag{24}$$

of polynomials given by $P_j(u)=\sum_{h\in J_j}p_{j,h}u^h$, for $0\leq j\leq S-1$.

According to the above identities (22) and (23), we check that the integral expression (19) solves the Cauchy problem (17), (18) if the Borel map $w(u,z)$ satisfies the next related Cauchy problem

$$\partial_z^S w(u,z) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0}\frac{c_{\underline{l}}(z)}{P(ku^k)}(k(q^{l_3}u)^k)^{l_1}(\partial_z^{l_2}w)(q^{l_3}u,z)$$
$$+\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq1}\frac{c_{\underline{l}}(z)}{P(ku^k)}\frac{u^k}{\Gamma(l_0/k)}\int_0^{u^k}(u^k-s)^{\frac{l_0}{k}-1}(k(q^{l_3}s^{1/k})^k)^{l_1}(\partial_z^{l_2}w)(q^{l_3}s^{1/k},z)ds/s \tag{25}$$

for given Cauchy data

$$(\partial_z^j w)(u,0) = P_j(u) \quad , \quad 0 \le j \le S-1 \tag{26}$$

In the next step, we seek solutions to the latter problem (25), (26) in the form of a formal series in z,

$$w(u,z) = \sum_{n \ge 0} w_n(u) z^n/n! \tag{27}$$

By direct computation, using the expansion (8) with the convention that we set $c_{\underline{l},h} = 0$ provided that $h \notin J_j$ for $h \ge 1$, we get the next recursion relation for the sequence of expressions $w_n(u)$, $n \ge 0$,

$$\begin{split}\frac{w_{n+S}(u)}{n!} = \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3}u)}{n_2!} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)}\\ + \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \ge 1} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{u^k}{P(ku^k)\Gamma(\frac{l_0}{k})} \int_0^{u^k} (u^k - s)^{\frac{l_0}{k}-1} (k(q^{l_3}s^{1/k})^k)^{l_1} \frac{w_{n_2+l_2}(q^{l_3}s^{1/k})}{n_2!} \frac{ds}{s}\end{split} \tag{28}$$

for the given initial functions

$$w_j(u) = P_j(u) \quad , \quad 0 \le j \le S-1. \tag{29}$$

By using the parametrization $s = u^k s_1$ for $0 \le s_1 \le 1$ in the above integrals appearing in (28), we get the next useful representation for the above recursion relation which the whole estimates in the forthcoming sections will lean on,

$$\begin{split}\frac{w_{n+S}(u)}{n!} = \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3}u)}{n_2!} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)}\\ + \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \ge 1} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})} (kq^{l_3 k})^{l_1} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2}(q^{l_3}us_1^{1/k})}{n_2!} \frac{ds_1}{s_1}\end{split} \tag{30}$$

for the given initial functions

$$w_j(u) = P_j(u) \quad , \quad 0 \le j \le S-1. \tag{31}$$

3 Sequences of holomorphic functions on sequences of discs

In this section, we describe a sequence of disc D_{R_n} centered at 0, whose radius R_n tends to 0 as n tends to infinity and a sequence of holomorphic maps $u \mapsto w_n(u)$ on D_{R_n} which fulfill the recursion (30), (31) and suffer appropriate bounds estimates.

We denote ζ_j, $1 \le j \le k\deg(P)$, the complex roots of the polynomial $u \mapsto P(ku^k)$, where $\deg(P)$ stands for the degree of the polynomial P introduced at the onset of Section 2. The condition (7) imposed on P grants the existence of a radius $R_0 > 0$ for which

$$\zeta_j \notin D_{R_0} \quad , \quad 1 \le j \le k\deg(P) \tag{32}$$

holds where D_{R_0} denotes the disc centered at 0 with radius R_0. We introduce the sequence of radii

$$R_n = \frac{R_0}{q^n} \quad , \quad n \ge 0 \tag{33}$$

Our main task on this section will be to prove the next proposition.

Proposition 2 *Under the constraints (10) and (11), one can single out a unique sequence of functions $w_n(u)$, $n \geq 0$, where each map $w_n(u)$ is bounded holomorphic on the disc D_{R_n}, that fulfills the recursion (30) for given initial data (31). Furthermore, one can choose two constants $C_1, C_2 > 0$ such that the next bounds hold*

$$\sup_{u \in D_{R_n}} \frac{|w_n(u)|}{|u|} \leq C_1 (C_2)^n \frac{n!}{q^{n^2 \Delta}} \tag{34}$$

for all $n \geq 0$, where $\Delta > 0$ is introduced in (11).

Proof We will proceed by induction. We name $\mathbb{D}_n$ the property (34) for a fixed given $n \geq 0$. We first observe that the property $\mathbb{D}_n$ holds obviously for $0 \leq n \leq S-1$ for well chosen $C_1, C_2 > 0$ and $\Delta > 0$ satisfying (11) since in that case it is imposed that $w_n(u) = P_n(u)$ are polynomials with $P_n(0) = 0$.

Let $n \geq 0$, we assume that $\mathbb{D}_p$ holds for all $p < n + S$ for some given $C_1, C_2 > 0$ and $\Delta > 0$ subjected to (11). Our goal throughout the rest of the proof is to show that $\mathbb{D}_{n+S}$ holds. The induction principle will then imply that the property $\mathbb{D}_p$ holds for all $p \geq 0$.

Lemma 1 *Let $u \in D_{R_{n+S}}$. Under the assumption (10), the next inclusions*

$$q^{l_3} u \in D_{R_{n_2+l_2}} \quad , \quad q^{l_3} u s_1^{1/k} \in D_{R_{n_2+l_2}} \tag{35}$$

hold for all $0 \leq s_1 \leq 1$, provided that $n_2 \leq n$ for $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

Proof Indeed, let us handle with first inclusion of (35). Since $S \geq l_2 + l_3$ from (10) and $q > 1$, we observe that

$$\frac{1}{q^{n+S}} \leq \frac{1}{q^{n_2+l_2+l_3}}.$$

Therefore $u \in D_{R_{n+S}}$ implies $|u| \leq R_0/q^{n_2+l_2+l_3}$ which means that $q^{l_3} u \in D_{R_{n_2+l_2}}$.

For the second inclusion, since $0 \leq s_1 \leq 1$, we check that $|q^{l_3} u s_1^{1/k}| \leq |q^{l_3} u|$. Therefore the second inclusion follows from the first one. □

From our hypothesis of induction that $\mathbb{D}_{n_2+l_2}$ holds for $n_2 \leq n$ and $0 \leq l_2 < S$ and according to Lemma 1 above, we deduce the next two inequalities

$$\sup_{u \in D_{R_{n+S}}} \frac{|w_{n_2+l_2}(q^{l_3}u)|}{|q^{l_3}u|} \leq \sup_{y \in D_{R_{n_2+l_2}}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_1 (C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{q^{(n_2+l_2)^2 \Delta}} \tag{36}$$

and

$$\sup_{u \in D_{R_{n+S}}} \frac{|w_{n_2+l_2}(q^{l_3} u s_1^{1/k})|}{|q^{l_3} u s_1^{1/k}|} \leq \sup_{y \in D_{R_{n_2+l_2}}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_1 (C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{q^{(n_2+l_2)^2 \Delta}}. \tag{37}$$

On the other hand, due to the hypothesis (7), a constant $\Delta_{P,k} > 0$ can be singled out with

$$P(ku^k) \geq \Delta_{P,k} \tag{38}$$

for all $u \in D_{R_0}$ and in particular for $u \in D_{R_{n+S}} \subset D_{R_0}$. We deduce the next two bounds

$$\sup_{u \in D_{R_{n+S}}} \left| \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \right| \leq \frac{k^{l_1}}{\Delta_{P,k}} q^{l_3 k l_1} \frac{R_0^{k l_1}}{q^{(n+S)k l_1}} \tag{39}$$

for all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 = 0$ and

(40)
$$\sup_{u \in D_{R_{n+S}}} \left| \frac{u^{l_0+kl_1}}{P(ku^k)} \right| \leq \frac{1}{\Delta_{P,k}} \frac{R_0^{l_0+kl_1}}{q^{(n+S)(l_0+kl_1)}}$$

provided that $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 \geq 1$.

Owing to the recursion (30) with (31) and gathering the above estimates (36), (37), (39) and (40), we get a first bound for $w_{n+S}(u)/u$ on the disc $D_{R_{n+S}}$,

(41)
$$\sup_{u \in D_{R_{n+S}}} \frac{|w_{n+S}(u)|}{|u|} \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} n! \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| q^{l_3} C_1 (C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} \frac{1}{q^{(n_2+l_2)^2 \Delta}}$$
$$\times \frac{k^{l_1}}{\Delta_{P,k}} q^{l_3 k l_1} \frac{R_0^{kl_1}}{q^{(n+S)kl_1}} + \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1} n! \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| \frac{(kq^{l_3 k})^{l_1}}{\Gamma(\frac{l_0}{k})} \frac{1}{\Delta_{P,k}} \frac{R_0^{l_0+kl_1}}{q^{(n+S)(l_0+kl_1)}}$$
$$\times q^{l_3} C_1 (C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} \frac{1}{q^{(n_2+l_2)^2 \Delta}} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$

The next lemma will be useful.

Lemma 2 *We have*

(42)
$$n! \frac{(n_2+l_2)!}{n_2!} \leq (n+S)!$$

for all integers $n \geq 0$, $n_2 \leq n$ and $l_2 < S$.

Proof The above inequality results from the next observation

$$\frac{n!}{(n+S)!} \frac{(n_2+l_2)!}{n_2!} = \frac{\Pi_{k=1}^{l_2} (n_2+k)}{\Pi_{k=1}^{S} (n+k)} \leq 1$$

provided that $n_2 \leq n$, $l_2 < S$. □

From the above lemma and the assumption that $c_{\underline{l}}(z)$ are polynomials, see (8), we deduce the next bounds for the quotient $w_{n+S}(u)/u$ on $D_{R_{n+S}}$,

(43)
$$\sup_{u \in D_{R_{n+S}}} \frac{|w_{n+S}(u)|}{|u|} \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} (n+S)!$$
$$\times \sum_{h \in I_{\underline{l}}; 0 \leq h \leq n} |c_{\underline{l},h}| q^{l_3} C_1 (C_2)^{n-h+l_2} \frac{1}{q^{(n-h+l_2)^2 \Delta}} \frac{k^{l_1}}{\Delta_{p,k}} q^{l_3 k l_1} \frac{R_0^{kl_1}}{q^{(n+S)kl_1}}$$
$$+ \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1} (n+S)! \sum_{h \in I_{\underline{l}}; 0 \leq h \leq n} |c_{\underline{l},h}| \frac{(kq^{l_3 k})^{l_1}}{\Gamma(\frac{l_0}{k})} \frac{1}{\Delta_{P,k}} \frac{R_0^{l_0+kl_1}}{q^{(n+S)(l_0+kl_1)}} q^{l_3} C_1 (C_2)^{n-h+l_2}$$
$$\times \frac{1}{q^{(n-h+l_2)^2 \Delta}} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$

The following lemma is crucial

Lemma 3 *Under the constraints (11), the next inequality*

$$\frac{1}{q^{(n-h+l_2)^2\Delta}}\frac{1}{q^{(n+S)(l_0+kl_1)}} \leq \frac{1}{q^{(n+S)^2\Delta}} \tag{44}$$

holds for all $n \geq h$, all $h \in I_{\underline{l}}$ where $\underline{l} = (l_0,l_1,l_2,l_3) \in \mathcal{A}$.

Proof By direct computation, we observe that the above inequality is equivalent to

$$n(2(l_2-h)\Delta + l_0 + kl_1 - 2S\Delta) \geq -(-h+l_2)^2\Delta - S(l_0+kl_1) + S^2\Delta$$

which is fullfilled for all integers $n \geq 0$ whenever the conditions (11) are imposed on the sets $\mathcal{A}$ and $I_{\underline{l}}$ for $\underline{l} \in \mathcal{A}$. □

On the other hand, in accordance with the assumption that the set $I_{\underline{l}}$ belongs to $\mathbb{N} \setminus \{0\}$, we choose $C_2 > 0$ large enough (which depends only on the data $\mathcal{A}$,q,k,$\Delta_{P,k}$,$c_{\underline{l}}(z)$ for $\underline{l} \in \mathcal{A}$ and R_0) in a way that the next estimates hold

$$\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0}\ \sum_{h\in I_{\underline{l}};0\leq h\leq n} |c_{\underline{l},h}|q^{l_3}C_2^{-h}\frac{k^{l_1}}{\Delta_{P,k}}q^{l_3kl_1}R_0^{kl_1} \tag{45}$$
$$+\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1}\ \sum_{h\in I_{\underline{l}};0\leq h\leq n} |c_{\underline{l},h}|\frac{(kq^{l_3k})^{l_1}}{\Gamma(\frac{l_0}{k})}\frac{1}{\Delta_{P,k}}R_0^{l_0+kl_1}q^{l_3}C_2^{-h}\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1 \leq 1$$

Finally, gathering the bounds (43) and (44) under the constraints (45) yield the bounds

$$\sup_{u\in D_{R_{n+S}}}\frac{|w_{n+S}(u)|}{|u|} \leq C_1(C_2)^{n+S}\frac{(n+S)!}{q^{(n+S)^2\Delta}}$$

which means that the property $\mathbb{D}_{n+S}$ holds. □

4 Sequences of holomorphic functions on sectors

In this section, appropriate unbounded sectors $\mathcal{U}$ are selected on which sequences of holomorphic maps $u \mapsto w_n(u)$ satisfy the recursion relation (30), (31) together with bounds control. Indeed, we consider an unbounded sector

$$\mathcal{U} = \{u \in \mathbb{C}^*/\alpha < \arg(u) < \beta\} \tag{46}$$

for some fixed angles $\alpha < \beta$. We assume that

$$\zeta_j \notin \mathcal{U} \quad , \quad 1 \leq j \leq k\deg(P) \tag{47}$$

where ζ_j, $1 \leq j \leq k\deg(P)$ represent the roots of the polynomial $u \mapsto P(ku^k)$.

Our purpose within this section is to explain the next proposition

Proposition 3 *Under the conditions (12), one can find out a unique sequence of functions $w_n(u)$, $n \geq 0$, where each map $u \mapsto w_n(u)$ is holomorphic on the sector $\mathcal{U}$, continuous on $\mathcal{U} \cup \{0\}$ and fulfills the recursion (30) for given initial data (31). In addition, one can find constants $C_3, C_4 > 0$ and $u_0 > 1$,$\alpha \geq 0$ for which the next estimates*

$$|w_n(u)| \leq C_3(C_4)^n n!|u|\exp\left(k_1\log^2(|u|+u_0) + \alpha\log(|u|+u_0)\right) \tag{48}$$

hold for all $n \geq 0$, all $u \in \mathcal{U} \cup \{0\}$, where $k_1 > 0$ is chosen as in (12).

Proof The induction principle is applied. We denote $\mathbb{U}_n$ the property (48) for a given integer $n \geq 0$. At the onset, we notice that the property $\mathbb{U}_n$ is valid whenever $0 \leq n \leq S-1$ for well chosen constants $C_3, C_4 > 0$ and $u_0 > 1$,$k_1 > 0$,$\alpha \geq 0$ owing to the fact that $w_n(u) = P_n(u)$ represent mere polynomials that vanish at $u = 0$.

Let $n \geq 0$, we take for granted that $\mathbb{U}_p$ is true for all $p < n+S$ for some given $C_3, C_4 > 0$ and $u_0 > 1$,$k_1 > 0$,$\alpha \geq 0$. Our commitment is to prove that $\mathbb{U}_{n+S}$ holds. The induction principle then implies that the bounds $\mathbb{U}_p$ are valid for any integer $p \geq 0$.

Lemma 4 *Let $u \in \mathcal{U} \cup \{0\}$. The next inclusions*

$$q^{l_3}u \in \mathcal{U} \cup \{0\} \quad , \quad q^{l_3}us_1^{1/k} \in \mathcal{U} \cup \{0\}$$

apply for all $0 \leq s_1 \leq 1$ and $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

Proof Since $q > 1$ and $0 \leq s_1 \leq 1$ are real numbers, it is enough to check that

$$\arg(q^{l_3}u) = \arg(u) = \arg(q^{l_3}us_1^{1/k})$$

provided that $u \in \mathcal{U}$ and $0 < s_1 \leq 1$. □

From the induction's hypothesis that $\mathbb{U}_{n_2+l_2}$ holds for $n_2 \leq n$ and $0 \leq l_2 < S$ and owing to Lemma 4, we obtain the next bounds

$$(49)\quad |w_{n_2+l_2}(q^{l_3}u)| \leq C_3(C_4)^{n_2+l_2}(n_2+l_2)!|q^{l_3}u| \exp\left(k_1 \log^2(|q^{l_3}u|+u_0) + \alpha\log(|q^{l_3}u|+u_0)\right)$$

for all $u \in \mathcal{U} \cup \{0\}$. Besides, since both functions $x \mapsto \log^2(x)$ and $x \mapsto \log(x)$ are inscreasing on $[1, +\infty)$ together with the assumption $u_0 > 1$ and the bounds $0 \leq s_1 \leq 1$, we also check that

$$(50)\quad \begin{aligned} |w_{n_2+l_2}(q^{l_3}us_1^{1/k})| &\leq C_3(C_4)^{n_2+l_2}(n_2+l_2)!|q^{l_3}us_1^{1/k}| \\ &\times \exp\left(k_1\log^2(|q^{l_3}us_1^{1/k}|+u_0) + \alpha\log(|q^{l_3}us_1^{1/k}|+u_0)\right) \\ &\leq C_3(C_4)^{n_2+l_2}(n_2+l_2)!|q^{l_3}u|s_1^{1/k}\exp\left(k_1\log^2(|q^{l_3}u|+u_0)+\alpha\log(|q^{l_3}u|+u_0)\right) \end{aligned}$$

The next lemma turns out to be essential.

Lemma 5 *The next upper estimates*

$$(51)\quad \begin{aligned} \exp\left(k_1\log^2(|q^{l_3}u|+u_0)+\alpha\log(|q^{l_3}u|+u_0)\right) &\leq q^{\alpha l_3}\left[(|q^{l_3}u|+u_0)(|u|+u_0)\right]^{k_1 l_3\log(q)} \\ &\times \exp\left(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\right) \end{aligned}$$

hold for all $u \in \mathcal{U} \cup \{0\}$.

Proof Since the function $x \mapsto (q^{l_3}x + u_0)/(x+u_0)$ is increasing on $[0, +\infty)$, we get that

$$(52)\qquad 0 \leq \log(|q^{l_3}u|+u_0) - \log(|u|+u_0) = \log\left(\frac{q^{l_3}|u|+u_0}{|u|+u_0}\right) \leq \log(q^{l_3}) = l_3\log(q)$$

for all $u \in \mathcal{U} \cup \{0\}$. Besides, one can write

$$(53)\quad \begin{aligned} \log^2(|q^{l_3}u|+u_0) - \log^2(|u|+u_0) &= \left[\log(|q^{l_3}u|+u_0) - \log(|u|+u_0)\right] \\ &\times \left[\log(|q^{l_3}u|+u_0)+\log(|u|+u_0)\right] \end{aligned}$$

As a consequence of (52), we deduce from (53) that

$$(54)\quad \log^2(|q^{l_3}u|+u_0)-\log^2(|u|+u_0) \le l_3\log(q)\log\Big((|q^{l_3}u|+u_0)(|u|+u_0)\Big) = \log\Big([(|q^{l_3}u|+u_0)(|u|+u_0)]^{l_3\log(q)}\Big)$$

At last, the expected bounds (51) result from (52) together with (54). □

Due to the assumption made in (47), we get in particular a constant $C_{P,k}>0$ with

$$(55)\qquad |P(ku^k)| \ge C_{P,k}(|u|+1)^{k\deg(P)}$$

for all $u \in \mathcal{U}\cup\{0\}$. From these last lower bounds and the constraints imposed in (12), we deduce the next lemma.

Lemma 6 *1) For all* $\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}$*, with* $l_0=0$*, one can find a constant* $Q_P^{\underline{l};0}>0$ *(which relies on* $\underline{l}$,q,α,k,k_1,P,u_0*) such that*

$$(56)\qquad q^{l_3(\alpha+1)}\left|\frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)}\right|[(|q^{l_3}u|+u_0)(|u|+u_0)]^{k_1l_3\log(q)} \le Q_P^{\underline{l};0}$$

for all $u\in\mathcal{U}\cup\{0\}$.
2) For all $\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}$*, with* $l_0\ge 1$*, a constant* $Q_P^{\underline{l};1}>0$ *(which depends on the constants* $\underline{l}$,q,α,k,k_1,P,u_0*) can be singled out for which*

$$(57)\qquad q^{l_3(\alpha+1)}\frac{(kq^{l_3k})^{l_1}}{\Gamma(\frac{l_0}{k})}\left|\frac{u^{l_0+kl_1}}{P(ku^k)}\right|[(|q^{l_3}u|+u_0)(|u|+u_0)]^{k_1l_3\log(q)} \le Q_P^{\underline{l};1}$$

holds for all $u\in\mathcal{U}\cup\{0\}$.

By collecting the bounds (49), (50) together with the estimates of Lemma 5, 6 we deduce from the recursion relation (30), the next estimates for $w_{n+S}(u)$,

$$(58)\quad |w_{n+S}(u)| \le \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} n!\sum_{n_1+n_2=n}|c_{\underline{l},n_1}|C_3(C_4)^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}Q_P^{\underline{l};0}$$
$$\times |u|\exp\big(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\big) + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\ge1} n!\sum_{n_1+n_2=n}|c_{\underline{l},n_1}|C_3(C_4)^{n_2+l_2}$$
$$\times\frac{(n_2+l_2)!}{n_2!}Q_P^{\underline{l};1}\Big(\int_0^1(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\Big)|u|\exp\big(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\big)$$

Now, from the bounds in Lemma 2 and owing to the fact that $c_{\underline{l}}(z)$ is a polynomial, we deduce from (58) the next upper bound for $w_{n+S}(u)$,

$$(59)\qquad |w_{n+S}(u)| \le \mathcal{B}(C_4)C_3(C_4)^{n+S}(n+S)!|u|\exp\big(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\big)$$

where

$$\mathcal{B}(C_4) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0}\ \sum_{h\in I_{\underline{l}};0\le h\le n}|c_{\underline{l},h}|C_4^{-h}Q_P^{\underline{l};0}$$
$$+\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\ge1}\ \sum_{h\in I_{\underline{l}};0\le h\le n}|c_{\underline{l},h}|C_4^{-h}Q_P^{\underline{l};1}\Big(\int_0^1(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\Big)$$

Keeping in mind that the set of integers $I_{\underline{l}}$ does not contain 0, one can select $C_4 > 0$ large enough such that

$$\mathcal{B}(C_4) \leq 1 \tag{60}$$

Finally, (59) and (60) yield that $\mathbb{U}_{n+S}$ holds true. □

5 Sequences of holomorphic functions on sequences of sectorial annuli

In this section, a sequence of holomorphic maps $u \mapsto w_n(u)$ is built up on a set of sectorial annuli $\mathcal{A}_h$, $0 \leq h \leq n-1$, for $n \geq 1$, that fulfills the recursion relations (30), (31). Sharp upper bounds are provided that will show a crucial importance in the study of the asymptotics of the solutions (19) of our problem (17), (18) as explained in the forthcoming section 7. This section contains the most technical part of the proof of our first main result, see Theorem 1.

We consider the disc D_{R_0} centered at 0 with radius $R_0 > 0$ satisfying the feature (32) introduced in Section 3 and the sector $\mathcal{U}$ considered in Section 4 set in (46) that fulfills the constraint (47).

We define a set of sectorial annuli as follows

$$\mathcal{A}_h = \{u \in \mathbb{C}^*/\alpha < \arg(u) < \beta \ , \ \frac{R_0}{q^{h+1}} \leq |u| \leq \frac{R_0}{q^h}\} \tag{61}$$

for all $h \in \mathbb{Z}$. Observe that, by construction, $\mathcal{A}_h \subset \mathcal{U}$, for all $h \in \mathbb{Z}$ and $\mathcal{A}_h \subset \mathcal{U} \cap D_{R_0}$ for $h \geq 0$.

The goal of this section is to provide a proof of the next proposition.

Proposition 4 *Assuming the conditions (10), (11), (13), (14) and (15), the unique sequence of holomorphic functions $u \mapsto w_n(u)$ constructed in Proposition 3 on the sector $\mathcal{U}$ satisfies the next upper bounds: there exist two constants $C_5 > 0$ and $C_6 > 1$ such that*

$$\sup_{u \in \mathcal{A}_h} \frac{|w_n(u)|}{|u|} \leq C_5 (C_6)^n n! \frac{1}{q^{h^2 \Delta}} \tag{62}$$

for all $n \geq 1$, all $0 \leq h \leq n-1$, where $\Delta > 0$ is a positive real number that fulfills (11), (13), (14) and (15).

Proof The induction principle is used. We set $\mathbb{A}_n$ as the property (62) for a given integer $n \geq 1$. We see first that $\mathbb{A}_n$ is plainly true for $1 \leq n \leq S-1$ for well chosen constants $C_5 > 0$,$C_6 > 1$ and $\Delta > 0$ satisfying (11), (13), (14) and (15) since $w_n(u) = P_n(u)$ are mere polynomials vanishing at 0 in that case.

Let $n \geq 0$, let us assume that $\mathbb{A}_p$ holds true for all $p < n+S$, for given constants $C_5 > 0$,$C_6 > 1$ and $\Delta > 0$ subjected to (11), (13), (14) and (15). The rest of the proof is devoted to show that $\mathbb{A}_{n+S}$ is valid. The induction principle then asserts that the property $\mathbb{A}_p$ turns out to be true for all integers $p \geq 1$.

Lemma 7 *Let $0 \leq h \leq n+S-1$ and take $u \in \mathcal{A}_h$.*
1) The next inclusion

$$q^{l_3} u \in \mathcal{A}_{h-l_3} \tag{63}$$

holds for $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.
2) We split the segment $(0,1]$ *into a union*

$$(0,1] = \bigcup_{j \geq 0} [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}] \tag{64}$$

When $s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]$*, the next inclusion*

$$q^{l_3} u s_1^{1/k} \in \mathcal{A}_{h+j-l_3} \cup \mathcal{A}_{h+j+1-l_3} \tag{65}$$

takes place for $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

Proof First, we check 1). Indeed, since $u \in \mathcal{A}_h$ and $q > 1$, the complex number $q^{l_3}u$ satisfies

$$\alpha < \arg(u) = \arg(q^{l_3}u) < \beta \ \ , \ \ \frac{R_0}{q^{h+1-l_3}} \leq |q^{l_3}u| \leq \frac{R_0}{q^{h-l_3}}$$

which means (63). For the second part 2), we observe that $\frac{1}{q^{j+1}} \leq s_1^{1/k} \leq \frac{1}{q^j}$. Therefore, $u \in \mathcal{A}_h$ and $q > 1$ implies the inequalities

$$\alpha < \arg(u) = \arg(q^{l_3}us_1^{1/k}) < \beta \ \ , \ \ \frac{R_0}{q^{h+j+2-l_3}} \leq |q^{l_3}us_1^{1/k}| \leq \frac{R_0}{q^{h+j-l_3}}$$

which is tantamount to (65). □

Let $0 \leq h \leq n + S - 1$.
I) We need to give bounds for the quantity $|w_{n_2+l_2}(q^{l_3}u)|$ for $u \in \mathcal{A}_h$ when $n_2 \leq n$ and $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ (which implies $l_2 < S$). Several cases crop up.

a) If $0 \leq h < l_3$, then we observe that $\mathcal{A}_{h-l_3} \subset \mathcal{U} \cap D_{R_0 q^{l_3}}$. From Lemma 7 1), when $u \in \mathcal{A}_h$, we deduce $q^{l_3}u \in \mathcal{U} \cap D_{R_0 q^{l_3}}$ and we can use the bounds (48) in Proposition 3 in order to get a constant $M_{l_3} > 0$ (depending on l_3,u_0,α and k_1) and two constants $C_3, C_4 > 0$ (determined in Proposition 3) such that

$$\sup_{u \in \mathcal{A}_h} \frac{|w_{n_2+l_2}(q^{l_3}u)|}{|q^{l_3}u|} \leq C_3(C_4)^{n_2+l_2}(n_2+l_2)! M_{l_3} \tag{66}$$

b) If $h \geq l_3$ and $h - l_3 \leq n_2 + l_2 - 1$. In that case from the property $\mathbb{A}_{n_2+l_2}$ (which holds true since $n_2 + l_2 < n + S$) and Lemma 7 1) we deduce the bounds

$$\sup_{u \in \mathcal{A}_h} \frac{|w_{n_2+l_2}(q^{l_3}u)|}{|q^{l_3}u|} \leq \sup_{y \in \mathcal{A}_{h-l_3}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_5(C_6)^{n_2+l_2}(n_2+l_2)! \frac{1}{q^{(h-l_3)^2 \Delta}} \tag{67}$$

c) If $n_2 + l_2 + l_3 \leq h \leq n + S - 1$. Then, we notice that $\mathcal{A}_{h-l_3} \subset D_{R_{n_2+l_2}}$ since $R_0/q^{h-l_3} \leq R_0/q^{n_2+l_2}$. We can use the bounds (34) from Proposition 2 and bearing in mind Lemma 7 1) in order to get two constants $C_1, C_2 > 0$ (fixed in Proposition 2) and $\Delta > 0$ fulfilling (11) with

$$\sup_{u \in \mathcal{A}_h} \frac{|w_{n_2+l_2}(q^{l_3}u)|}{|q^{l_3}u|} \leq \sup_{y \in \mathcal{A}_{h-l_3}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_1(C_2)^{n_2+l_2}(n_2+l_2)! \frac{1}{q^{(n_2+l_2)^2 \Delta}} \tag{68}$$

II) We are asked to supply bounds for the quantity $|w_{n_2+l_2}(q^{l_3}us_1^{1/k})|$ for $u \in \mathcal{A}_h$ and $0 < s_1 \leq 1$, when $n_2 \leq n$ and $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$. Again, several cases arise.

a) If $0 \leq h+j+2 \leq l_3$. Then one can check that

$$\mathcal{A}_{h+j-l_3} \cup \mathcal{A}_{h+j+1-l_3} \subset \mathcal{U} \cap D_{R_0 q^{l_3}}$$

Then, from Lemma 7 2), when $s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]$ and $u \in \mathcal{A}_h$, we deduce that $q^{l_3} u s_1^{1/k} \in \mathcal{U} \cap D_{R_0 q^{l_3}}$ and we can use the bounds (48) in Proposition 3 in order to get a constant $M_{l_3} > 0$ (depending on l_3,u_0,α and k_1) and two constants $C_3, C_4 > 0$ (determined in Proposition 3) such that

$$\sup_{u \in \mathcal{A}_h; s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]} \frac{|w_{n_2+l_2}(q^{l_3} u s_1^{1/k})|}{|q^{l_3} u s_1^{1/k}|} \leq C_3 (C_4)^{n_2+l_2} (n_2+l_2)! M_{l_3} \tag{69}$$

b) If $h+j+1 = l_3$, then from Lemma 7 2), provided that $s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]$ and $u \in \mathcal{A}_h$, we get

$$q^{l_3} u s_1^{1/k} \in \mathcal{A}_{-1} \cup \mathcal{A}_0 \tag{70}$$

However, from the property $\mathbb{A}_{n_2+l_2}$ holding true for $h = 0$, we know that

$$\sup_{y \in \mathcal{A}_0} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_5 (C_6)^{n_2+l_2} (n_2+l_2)! \tag{71}$$

and since $\mathcal{A}_{-1} \subset \mathcal{U} \setminus D_{R_0}$, we can keep in mind the bounds (48) from Proposition 3 which yield a constant $M_{l_3} > 0$ (relying on l_3,u_0,α and k_1, which can be chosen to be the same constant as above in (69)) and two constants $C_3, C_4 > 0$ (settled in Proposition 3) such that

$$\sup_{u \in \mathcal{A}_{-1}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_3 (C_4)^{n_2+l_2} (n_2+l_2)! M_{l_3} \tag{72}$$

Therefore, from (70) along with (71) and (72), we obtain the next bounds

$$\begin{aligned} \sup_{u \in \mathcal{A}_h; s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]} \frac{|w_{n_2+l_2}(q^{l_3} u s_1^{1/k})|}{|q^{l_3} u s_1^{1/k}|} &\leq \max\left(\sup_{u \in \mathcal{A}_{-1}} \frac{|w_{n_2+l_2}(y)|}{|y|}, \sup_{y \in \mathcal{A}_0} \frac{|w_{n_2+l_2}(y)|}{|y|} \right) \\ &\leq \max(C_3 M_{l_3}, C_5)(\max(C_4, C_6))^{n_2+l_2} (n_2+l_2)! \end{aligned} \tag{73}$$

c) If $h+j \geq l_3$ and $h+j+2-l_3 \leq n_2+l_2$. In that case, according to Lemma 7 2), the bounds for $w_{n_2+l_2}(q^{l_3} u s_1^{1/k})$ follow from the property $\mathbb{A}_{n_2+l_2}$. Indeed, we know that

$$\sup_{y \in \mathcal{A}_{h+j-l_3}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_5 (C_6)^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(h+j-l_3)^2 \Delta}} \tag{74}$$

and

$$\sup_{y \in \mathcal{A}_{h+j+1-l_3}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_5 (C_6)^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(h+j+1-l_3)^2 \Delta}} \tag{75}$$

Hence,

$$\begin{aligned} \sup_{u \in \mathcal{A}_h; s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]} \frac{|w_{n_2+l_2}(q^{l_3} u s_1^{1/k})|}{|q^{l_3} u s_1^{1/k}|} &\leq \max\left(\sup_{u \in \mathcal{A}_{h+j-l_3}} \frac{|w_{n_2+l_2}(y)|}{|y|}, \sup_{y \in \mathcal{A}_{h+j+1-l_3}} \frac{|w_{n_2+l_2}(y)|}{|y|} \right) \\ &\leq C_5 (C_6)^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(h+j-l_3)^2 \Delta}} \end{aligned} \tag{76}$$

d) If $h+j+2-l_3 = n_2+l_2+1$. From Lemma 7 2), when $s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]$ and $u \in \mathcal{A}_h$, we get

$$q^{l_3} u s_1^{1/k} \in \mathcal{A}_{n_2+l_2-1} \cup \mathcal{A}_{n_2+l_2} \tag{77}$$

Noticing that $\mathcal{A}_{n_2+l_2} \subset D_{R_{n_2+l_2}}$, the bounds (34) on discs in Proposition 2 yield two constants $C_1, C_2 > 0$ (arranged in Proposition 2) and $\Delta > 0$ fulfilling (11) with

$$\sup_{y \in \mathcal{A}_{n_2+l_2}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_1 (C_2)^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(n_2+l_2)^2 \Delta}} \tag{78}$$

and bounds on the annulus $\mathcal{A}_{n_2+l_2-1}$ follow from the property $\mathbb{A}_{n_2+l_2}$ for $h = n_2+l_2-1$,

$$\sup_{y \in \mathcal{A}_{n_2+l_2-1}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_5 (C_6)^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(n_2+l_2-1)^2 \Delta}} \tag{79}$$

that breeds the next estimates

$$\begin{aligned} \sup_{u \in \mathcal{A}_h; s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]} \frac{|w_{n_2+l_2}(q^{l_3} u s_1^{1/k})|}{|q^{l_3} u s_1^{1/k}|} &\leq \max \left(\sup_{y \in \mathcal{A}_{n_2+l_2-1}} \frac{|w_{n_2+l_2}(y)|}{|y|}, \sup_{y \in \mathcal{A}_{n_2+l_2}} \frac{|w_{n_2+l_2}(y)|}{|y|} \right) \\ &\leq \max(C_1, C_5) (\max(C_2, C_6))^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(n_2+l_2-1)^2 \Delta}} \end{aligned} \tag{80}$$

e) If $h+j-l_3 \geq n_2+l_2$ and $h \leq n+S-1$. We observe that

$$\mathcal{A}_{h+j-l_3} \cup \mathcal{A}_{h+j+1-l_3} \subset D_{R_{n_2+l_2}}$$

and the bounds (34) on discs in Proposition 2 give rise to constants $C_1, C_2 > 0$ (defined in Proposition 2) and $\Delta > 0$ fulfilling (11) such that

$$\sup_{y \in \mathcal{A}_{h+j-l_3} \cup \mathcal{A}_{h+j+1-l_3}} \frac{|w_{n_2+l_2}(y)|}{|y|} \leq C_1 (C_2)^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(n_2+l_2)^2 \Delta}} \tag{81}$$

and by Lemma 7 2), we deduce the last required bounds

$$\sup_{u \in \mathcal{A}_h; s_1 \in [\frac{1}{q^{k(j+1)}}, \frac{1}{q^{kj}}]} \frac{|w_{n_2+l_2}(q^{l_3} u s_1^{1/k})|}{|q^{l_3} u s_1^{1/k}|} \leq C_1 (C_2)^{n_2+l_2} (n_2+l_2)! \frac{1}{q^{(n_2+l_2)^2 \Delta}} \tag{82}$$

We have now prepared the necessary material to show the next two crucial lemma

Lemma 8 *We get the next bounds*

$$\begin{aligned} \sup_{u \in \mathcal{A}_h} \left| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3} u)}{u n_2!} \frac{(k(q^{l_3} u)^k)^{l_1}}{P(k u^k)} \right| \\ \leq \frac{C_5}{2} (C_6)^{n+S} (n+S)! \frac{1}{q^{h^2 \Delta}} \end{aligned} \tag{83}$$

for all $0 \leq h \leq n+S-1$, provided that $C_5 > 0$ and $C_6 > 0$ are taken large enough.

Proof Let us take $0 \leq h \leq n+S-1$. Since $\mathcal{A}_h \subset \mathcal{U} \cap D_{R_0}$, we deduce from (38) that

$$\sup_{u \in \mathcal{A}_h} \left| \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \right| \leq \frac{k^{l_1} q^{l_3 k l_1}}{\Delta_{P,k}} \frac{R_0^{kl_1}}{q^{hkl_1}} \tag{84}$$

In the sequel, we split the sum from the left handside of (83) in two parts, one sum over $\underline{l} = (l_0, l_1, l_2, l_3)$ in $\mathcal{A}$ with $l_0 = 0$ for which $0 \leq h < l_3$ and the other for which $h \geq l_3$. We get the next bounds

$$\sup_{u \in \mathcal{A}_h} \left| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3}u)}{un_2!} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \right| \leq \mathbb{S}_{h<l_3} + \mathbb{S}_{h \geq l_3} \tag{85}$$

where

$$\mathbb{S}_{h<l_3} = \sup_{u \in \mathcal{A}_h} \left| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0; 0 \leq h < l_3} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3}u)}{un_2!} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \right|$$

and

$$\mathbb{S}_{h \geq l_3} = \sup_{u \in \mathcal{A}_h} \left| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0; h \geq l_3} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3}u)}{un_2!} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \right|$$

We first provide bounds for the sum $\mathbb{S}_{h<l_3}$.

Indeed, according to the upper estimates (66) and (84), provided that one takes $C_5, C_6 > 0$ in a way that $C_5 \geq C_3$ and $C_6 \geq C_4$, we observe that

$$\mathbb{S}_{h<l_3} \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0; h<l_3} n! \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| C_5 C_6^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} M_{l_3} \frac{k^{l_1} q^{l_3 k l_1}}{\Delta_{P,k}} \frac{R_0^{kl_1}}{q^{hkl_1}} \tag{86}$$

and from Lemma 2 together with the assumption that $c_{\underline{l}}(z)$ are polynomials, we deduce that

$$\mathbb{S}_{h<l_3} \leq \mathcal{B}(C_6) C_5 (C_6)^{n+S} (n+S)! \frac{1}{q^{h^2 \Delta}} \tag{87}$$

where

$$\begin{aligned} \mathcal{B}(C_6) = & \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0; h<l_3} \sum_{g \in I_{\underline{l}}; 0 \leq g \leq n} |c_{\underline{l},g}| C_6^{-g} q^{l_3} M_{l_3} \frac{k^{l_1} q^{l_3 k l_1}}{\Delta_{P,k}} \frac{R_0^{kl_1}}{q^{hkl_1}} q^{h^2 \Delta} \\ & \leq \mathcal{B}_1(C_6) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0; h<l_3} \sum_{g \in I_{\underline{l}}; 0 \leq g \leq n} |c_{\underline{l},g}| C_6^{-g} q^{l_3} M_{l_3} \frac{k^{l_1} q^{l_3 k l_1}}{\Delta_{P,k}} R_0^{kl_1} q^{l_3^2 \Delta} \end{aligned} \tag{88}$$

and according to the fact that 0 does not belong to the set $I_{\underline{l}}$, one can find $C_6 > 0$ large enough with

$$\mathcal{B}_1(C_6) \leq 1/4 \tag{89}$$

In the next step we seek for upper bounds controling $\mathbb{S}_{h \geq l_3}$.

Here, we need to further break up the sum over $\underline{l} = (l_0, l_1, l_2, l_3)$ in $\mathcal{A}$ with $l_0 = 0$ where $h \geq l_3$ as one sum for which $h < n_2 + l_2 + l_3$ and the other for which $n_2 + l_2 + l_3 \leq h \leq n + S - 1$. Namely,

$$\mathbb{S}_{h \geq l_3} = \mathbb{S}_{l_3 \leq h < n_2+l_2+l_3} + \mathbb{S}_{n_2+l_2+l_3 \leq h \leq n+S-1} \tag{90}$$

where

$$\mathbb{S}_{l_3 \leq h < n_2+l_2+l_3} = \sup_{u \in \mathcal{A}_h} \left| \sum \sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}; l_0=0; n_1+n_2=n, \\ l_3 \leq h < n_2+l_2+l_3}} n! c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3}u)}{un_2!} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \right|$$

and

$$\mathbb{S}_{n_2+l_2+l_3 \leq h \leq n+S-1} = \sup_{u \in \mathcal{A}_h} \left| \sum \sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}; l_0=0; n_1+n_2=n, \\ n_2+l_2+l_3 \leq h \leq n+S-1}} n! c_{\underline{l},n_1} \frac{w_{n_2+l_2}(q^{l_3}u)}{un_2!} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \right|$$

Owing to (67) together with (84), we obtain

$$\mathbb{S}_{l_3 \leq h < n_2+l_2+l_3} \leq \mathbb{S}^1_{l_3 \leq h < n_2+l_2+l_3} = \sum \sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}; l_0=0; n_1+n_2=n, \\ l_3 \leq h < n_2+l_2+l_3}} n! |c_{\underline{l},n_1}| \times C_5 (C_6)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(h-l_3)^2 \Delta}} \frac{k^{l_1} q^{l_3 k l_1}}{\Delta_{P,k}} \frac{R_0^{kl_1}}{q^{hkl_1}} \tag{91}$$

and due to (68) and (84), we observe that

$$\mathbb{S}_{n_2+l_2+l_3 \leq h \leq n+S-1} \leq \mathbb{S}^2_{n_2+l_2+l_3 \leq h \leq n+S-1} = \sum \sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}; l_0=0; n_1+n_2=n, \\ n_2+l_2+l_3 \leq h \leq n+S-1}} n! |c_{\underline{l},n_1}| C_1 (C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(n_2+l_2)^2 \Delta}} \frac{k^{l_1} q^{l_3 k l_1}}{\Delta_{P,k}} \frac{R_0^{kl_1}}{q^{hkl_1}} \tag{92}$$

Lemma 8.1 *1) Under the constraint (13), the next inequality*

$$\frac{1}{q^{(h-l_3)^2\Delta}} \frac{1}{q^{hkl_1}} \leq \frac{1}{q^{h^2\Delta}} \tag{93}$$

holds for all $h \geq 0$, whenever $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 = 0$.
2) Subjected to the condition (14), the next bounds

$$\frac{1}{q^{(n-g+l_2)^2\Delta}} \frac{1}{q^{hkl_1}} \leq \frac{1}{q^{h^2\Delta}} \tag{94}$$

are valid for $0 \leq h \leq n+S-1$, whenever $g \in I_{\underline{l}}$ for $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 = 0$.

Proof Concerning the first point 1), the inequality (93) is equivalent to the condition

$$h(-2l_3\Delta + kl_1) \geq -\Delta l_3^2$$

for $h \geq 0$. For the second claim 2), we notice that (94) is reduced to show that

$$(n-g+l_2)^2\Delta \geq \mathbf{P}_0(h) = h^2\Delta - hkl_1 \tag{95}$$

for all $0 \leq h \leq n+S-1$. Besides, since the function $h \mapsto \mathbf{P}_0(h)$ decreases on $[0, \frac{kl_1}{2\Delta}]$ and is increasing on $[\frac{kl_1}{2\Delta}, +\infty)$, we observe that

$$\max_{0 \leq h \leq n+S-1} \mathbf{P}_0(h) \leq \max(\mathbf{P}_0(0) = 0, \mathbf{P}_0(n+S-1))$$

Then, in order to get (95), it is sufficient to show that

$$(n-g+l_2)^2\Delta \geq \mathbf{P}_0(n+S-1) \tag{96}$$

which is equivalent to the inequality

$$n(2(-g+l_2)\Delta - 2\Delta(S-1)+kl_1) \geq -(-g+l_2)^2\Delta + \Delta(S-1)^2 - (S-1)kl_1$$

that is fulfilled under the condition (14). □

Now, assuming that $C_5 > C_1$ and $C_6 > C_2$, from Lemma 2 and keeping in mind that $c_{\underline{l}}(z)$ are polynomials, the above Lemma 8.1 applied on (91) and (92) yields the next two bounds

$$\mathbb{S}_{l_3 \leq h < n_2+l_2+l_3} \leq \mathcal{B}_2(C_6)C_5(C_6)^{n+S}(n+S)!\frac{1}{q^{h^2\Delta}} \tag{97}$$

where

$$\mathcal{B}_2(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0;g\in I_{\underline{l}};0\leq g\leq n,\\ l_3\leq h<n-g+l_2+l_3}} |c_{\underline{l},g}|C_6^{-g}q^{l_3}\frac{k^{l_1}q^{l_3kl_1}}{\Delta_{P,k}}R_0^{kl_1}$$

and

$$\mathbb{S}_{n_2+l_2+l_3 \leq h \leq n+S-1} \leq \mathcal{B}_3(C_6)C_5(C_6)^{n+S}(n+S)!\frac{1}{q^{h^2\Delta}} \tag{98}$$

where

$$\mathcal{B}_3(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0;g\in I_{\underline{l}};0\leq g\leq n,\\ n-g+l_2+l_3\leq h\leq n+S-1}} |c_{\underline{l},g}|C_6^{-g}q^{l_3}\frac{k^{l_1}q^{l_3kl_1}}{\Delta_{P,k}}R_0^{kl_1}$$

Since $I_{\underline{l}}$ does not contain 0, one can sort the constant $C_6 > 0$ large enough in a way that

$$\mathcal{B}_2(C_6) \leq 1/8 \quad , \quad \mathcal{B}_3(C_6) \leq 1/8 \tag{99}$$

Finally, collecting (85), (87), (88), (89), (90), (97), (98) along with (99) yields the sought bounds (83). □

Lemma 9 *The following bounds*

$$\sup_{u\in\mathcal{A}_h}\left|\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} n!\sum_{n_1+n_2=n} c_{\underline{l},n_1}\frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})}(kq^{l_3k})^{l_1} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2}(q^{l_3}us_1^{1/k})}{un_2!}\frac{ds_1}{s_1}\right| \leq \frac{C_5}{2}(C_6)^{n+S}(n+S)!\frac{1}{q^{h^2\Delta}} \tag{100}$$

occur for all $0 \leq h \leq n+S-1$, whenever $C_5 > 0$ and $C_6 > 0$ are chosen large enough.

Proof Let $0 \leq h \leq n+S-1$ be fixed. Observing that $\mathcal{A}_h \subset \mathcal{U}\cap D_{R_0}$, we deduce from (38) that

$$\sup_{u\in\mathcal{A}_h}\left|\frac{u^{l_0+kl_1}}{P(ku^k)}\right| \leq \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}}\frac{1}{q^{h(l_0+kl_1)}} \tag{101}$$

Our strategy consists in breaking up the sum appearing in the left handside of (100) into three parts : one sum over $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 \geq 1$ for which $h+2 \leq l_3$, one sum for which $h+1 = l_3$ and one sum for which $h \geq l_3$. We get the next bounds

$$(102) \quad \sup_{u \in \mathcal{A}_h} \Big| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})} (kq^{l_3 k})^{l_1} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2}(q^{l_3} u s_1^{1/k})}{u n_2!} \frac{ds_1}{s_1} \Big| \leq \mathbb{T}_{h+2 \leq l_3} + \mathbb{T}_{h+1=l_3} + \mathbb{T}_{h \geq l_3}$$

where

$$\mathbb{T}_{h+2 \leq l_3} = \sup_{u \in \mathcal{A}_h} \Big| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1; h+2 \leq l_3} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})} (kq^{l_3 k})^{l_1} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2}(q^{l_3} u s_1^{1/k})}{u n_2!} \frac{ds_1}{s_1} \Big| ,$$

$$\mathbb{T}_{h+1=l_3} = \sup_{u \in \mathcal{A}_h} \Big| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1; h+1=l_3} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})} (kq^{l_3 k})^{l_1} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2}(q^{l_3} u s_1^{1/k})}{u n_2!} \frac{ds_1}{s_1} \Big| ,$$

and

$$\mathbb{T}_{h \geq l_3} = \sup_{u \in \mathcal{A}_h} \Big| \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1; h \geq l_3} n! \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})} (kq^{l_3 k})^{l_1} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2}(q^{l_3} u s_1^{1/k})}{u n_2!} \frac{ds_1}{s_1} \Big|$$

In order to come up with bounds for the above quantities, we further decompose the integral $\int_0^1 ds_1$ as an infinite sum $\sum_{j \geq 0} \int_{1/q^{k(j+1)}}^{1/q^{kj}} ds_1$.

We provide a first set of bounds for the above quantities $\mathbb{T}_{h+2 \leq l_3}$ and $\mathbb{T}_{h+1=l_3}$ which are deduced from the upper mentioned estimates II) a), b), c), d) and e) together with (101). We arrive at

$$(103)\quad \mathbb{T}_{h+2\leq l_3} \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;h+2\leq l_3} n! \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \frac{1}{q^{h(l_0+kl_1)}} \times \Big(\sum_{0\leq j\leq l_3-h-2} C_3(C_4)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} M_{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ \max(C_3M_{l_3},C_5)(\max(C_4,C_6))^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \int_{1/q^{k(l_3-h)}}^{1/q^{k(l_3-h-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ \sum_{l_3-h\leq j\leq n_2+l_2-h+l_3-2} C_5(C_6)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(h+j-l_3)^2\Delta}} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ \max(C_1,C_5)(\max(C_2,C_6))^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(n_2+l_2-1)^2\Delta}}$$
$$\times \int_{1/q^{k(n_2+l_2-h+l_3)}}^{1/q^{k(n_2+l_2-h+l_3-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 + \sum_{j\geq n_2+l_2-h+l_3} C_1(C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(n_2+l_2)^2\Delta}}$$
$$\times \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Big)$$

and

$$(104)\quad \mathbb{T}_{h+1=l_3} \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;h+1=l_3} n! \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \frac{1}{q^{h(l_0+kl_1)}} \times \Big(\max(C_3M_{l_3},C_5)(\max(C_4,C_6))^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \int_{1/q^{k}}^{1} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ \sum_{l_3-h\leq j\leq n_2+l_2-h+l_3-2} C_5(C_6)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(h+j-l_3)^2\Delta}} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ \max(C_1,C_5)(\max(C_2,C_6))^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(n_2+l_2-1)^2\Delta}}$$
$$\times \int_{1/q^{k(n_2+l_2-h+l_3)}}^{1/q^{k(n_2+l_2-h+l_3-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 + \sum_{j\geq n_2+l_2-h+l_3} C_1(C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} \frac{1}{q^{(n_2+l_2)^2\Delta}}$$
$$\times \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Big)$$

In order to control the quantity $\mathbb{T}_{h\geq l_3}$, we need to further split the sum over $\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}$ with $l_0\geq 1$ where $h\geq l_3$ in three parts, as one sum for which $l_3\leq h\leq n_2+l_2+l_3-2$, one sum for which $h=n_2+l_2+l_3-1$ and one sum for which $n_2+l_2+l_3\leq h\leq n+S-1$. Namely,

$$(105)\qquad \mathbb{T}_{h\geq l_3} = \mathbb{T}_{l_3\leq h\leq n_2+l_2+l_3-2} + \mathbb{T}_{h=n_2+l_2+l_3-1} + \mathbb{T}_{n_2+l_2+l_3\leq h\leq n+S-1}$$

where

$$\mathbb{T}_{l_3\leq h\leq n_2+l_2+l_3-2} = \sup_{u\in\mathcal{A}_h}\left|\sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;n_1+n_2=n,\\ l_3\leq h\leq n_2+l_2+l_3-2}} n!c_{\underline{l},n_1}\frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})}(kq^{l_3k})^{l_1}\right.$$
$$\left.\times\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2}(q^{l_3}us_1^{1/k})}{un_2!}\frac{ds_1}{s_1}\right|$$

$$\mathbb{T}_{h=n_2+l_2+l_3-1} = \sup_{u\in\mathcal{A}_h}\left|\sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;n_1+n_2=n,\\ h=n_2+l_2+l_3-1}} n!c_{\underline{l},n_1}\frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})}(kq^{l_3k})^{l_1}\right.$$
$$\left.\times\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2}(q^{l_3}us_1^{1/k})}{un_2!}\frac{ds_1}{s_1}\right|$$

and

$$\mathbb{T}_{n_2+l_2+l_3\leq h\leq n+S-1} = \sup_{u\in\mathcal{A}_h}\left|\sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;n_1+n_2=n,\\ n_2+l_2+l_3\leq h\leq n+S-1}} n!c_{\underline{l},n_1}\frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})}(kq^{l_3k})^{l_1}\right.$$
$$\left.\times\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2}(q^{l_3}us_1^{1/k})}{un_2!}\frac{ds_1}{s_1}\right|$$

According to the above estimates II) a), b), c), d) and e) together with (101), we deduce that

$$(106)\quad \mathbb{T}_{l_3\leq h\leq n_2+l_2+l_3-2} \leq \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;n_1+n_2=n\\ l_3\leq h\leq n_2+l_2+l_3-2}} n!|c_{\underline{l},n_1}|\frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)}\frac{R_0^{l_0+kl_1}}{\Delta_{P,k}}\frac{1}{q^{h(l_0+kl_1)}}\times$$
$$\Bigg(\sum_{\substack{l_3-h\leq j\leq n_2+l_2-h+l_3-2\\ j\geq 0}} C_5(C_6)^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}q^{l_3}\frac{1}{q^{(h+j-l_3)^2\Delta}}\int_{1/q^{k(j+1)}}^{1/q^{kj}}(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1$$
$$+\max(C_1,C_5)(\max(C_2,C_6))^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}q^{l_3}\frac{1}{q^{(n_2+l_2-1)^2\Delta}}$$
$$\times\int_{1/q^{k(n_2+l_2-h+l_3)}}^{1/q^{k(n_2+l_2-h+l_3-1)}}(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1+\sum_{j\geq n_2+l_2-h+l_3}C_1(C_2)^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}q^{l_3}\frac{1}{q^{(n_2+l_2)^2\Delta}}$$
$$\times\int_{1/q^{k(j+1)}}^{1/q^{kj}}(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\Bigg)$$

and

$$(107)\quad \mathbb{T}_{h=n_2+l_2+l_3-1} \leq \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;n_1+n_2=n\\ h=n_2+l_2+l_3-1}} n!|c_{\underline{l},n_1}|\frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)}\frac{R_0^{l_0+kl_1}}{\Delta_{P,k}}\frac{1}{q^{h(l_0+kl_1)}}\times$$
$$\Bigg(\max(C_1,C_5)(\max(C_2,C_6))^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}q^{l_3}\frac{1}{q^{(n_2+l_2-1)^2\Delta}}$$
$$\times\int_{1/q^{k(n_2+l_2-h+l_3)}}^{1/q^{k(n_2+l_2-h+l_3-1)}}(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1+\sum_{j\geq n_2+l_2-h+l_3}C_1(C_2)^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}q^{l_3}\frac{1}{q^{(n_2+l_2)^2\Delta}}$$
$$\times\int_{1/q^{k(j+1)}}^{1/q^{kj}}(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\Bigg)$$

together with

$$\mathbb{T}_{n_2+l_2+l_3 \leq h \leq n+S-1} \leq \sum \sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;n_1+n_2=n \\ n_2+l_2+l_3\leq h\leq n+S-1}} n!|c_{\underline{l},n_1}|\frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)}\frac{R_0^{l_0+kl_1}}{\Delta_{P,k}}\frac{1}{q^{h(l_0+kl_1)}}\times \left(\sum_{\substack{j\geq n_2+l_2-h+l_3 \\ j\geq 0}} C_1(C_2)^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}q^{l_3}\frac{1}{q^{(n_2+l_2)^2\Delta}} \times \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1 \right) \tag{108}$$

Lemma 9.1 *1) According to the condition (14), the next bounds*

$$\frac{1}{q^{(n-g+l_2)^2\Delta}}\frac{1}{q^{h(kl_1+l_0)}} \leq \frac{1}{q^{h^2\Delta}} \tag{109}$$

are valid for $0 \leq h \leq n+S-1$*, whenever* $g \in I_{\underline{l}}$ *for* $\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}$ *with* $l_0 \geq 1$*.*
2) The conditions (15) allow the next bounds

$$\frac{1}{q^{(n-g+l_2-1)^2\Delta}}\frac{1}{q^{h(kl_1+l_0)}} \leq \frac{1}{q^{h^2\Delta}} \tag{110}$$

to hold for $0 \leq h \leq n+S-1$*, whenever* $g \in I_{\underline{l}}$ *for* $\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}$ *with* $l_0 \geq 1$*.*
3) Under the constraints (13), the bounds

$$\frac{1}{q^{(h+j-l_3)^2\Delta}}\frac{1}{q^{h(kl_1+l_0)}} \leq \frac{1}{q^{h^2\Delta}} \tag{111}$$

occur for $0 \leq h \leq n+S-1$*, for all* $j \geq 0$*.*

Proof For the first and second point 1), 2), the proof is exactly the same as the one provided in Lemma 8.1 2). Concerning the third point 3), we remark that (111) is equivalent to the next inequality

$$h(2j\Delta - 2l_3\Delta + kl_1 + l_0) + \Delta(j-l_3)^2 \geq 0$$

for all $0 \leq h \leq n+S-1$, $j \geq 0$, which holds under the assumption (13). □

We assume that $C_5 > \max\{C_3, C_3M_{l_3}, C_1\}$ and $C_6 > \max\{C_4, C_2\}$. Owing to Lemma 2 and reminding that $c_{\underline{l}}(z)$ are polynomials, the above Lemma 9.1 applied to (103), (104), (106), (107) and (108) gives rise to the next set of inequalities

$$\mathbb{T}_{h+2\leq l_3} \leq \mathcal{E}_1(C_6)C_5(C_6)^{n+S}(n+S)!\frac{1}{q^{h^2\Delta}} \tag{112}$$

where

$$\mathcal{E}_1(C_6) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;h+2\leq l_3} \sum_{g\in I_l,0\leq g\leq n} |c_{\underline{l},g}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}}$$
$$\times \Bigg(\sum_{0\leq j\leq l_3-h-2} (C_6)^{-g} q^{l_3} M_{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 q^{h^2\Delta} \frac{1}{q^{h(l_0+kl_1)}}$$
$$+ (C_6)^{-g} q^{l_3} \int_{1/q^{k(l_3-h)}}^{1/q^{k(l_3-h-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 q^{h^2\Delta} \frac{1}{q^{h(l_0+kl_1)}}$$
$$+ \sum_{l_3-h\leq j\leq n_2+l_2-h+l_3-2} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ (C_6)^{-g} q^{l_3} \int_{1/q^{k(n_2+l_2-h+l_3)}}^{1/q^{k(n_2+l_2-h+l_3-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ \sum_{j\geq n_2+l_2-h+l_3} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Bigg)$$

with

$$\mathcal{E}_1(C_6) \leq \mathcal{E}_{1.1}(C_6) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;h+2\leq l_3} \sum_{g\in I_l,0\leq g\leq n} |c_{\underline{l},g}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \tag{113}$$
$$\times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \times \Big((C_6)^{-g} q^{l_3} M_{l_3} q^{(l_3-2)^2\Delta} + (C_6)^{-g} q^{l_3} q^{(l_3-2)^2\Delta} + 3(C_6)^{-g} q^{l_3} \Big)$$

and

$$\mathbb{T}_{h+1=l_3} \leq \mathcal{E}_2(C_6) C_5 (C_6)^{n+S} (n+S)! \frac{1}{q^{h^2\Delta}} \tag{114}$$

where

$$\mathcal{E}_2(C_6) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;h+1=l_3} \sum_{g\in I_l,0\leq g\leq n} |c_{\underline{l},g}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}}$$
$$\times \Bigg((C_6)^{-g} q^{l_3} \int_{1/q^k}^{1} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 q^{h^2\Delta} \frac{1}{q^{h(l_0+kl_1)}}$$
$$+ \sum_{l_3-h\leq j\leq n_2+l_2-h+l_3-2} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ (C_6)^{-g} q^{l_3} \int_{1/q^{k(n_2+l_2-h+l_3)}}^{1/q^{k(n_2+l_2-h+l_3-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1$$
$$+ \sum_{j\geq n_2+l_2-h+l_3} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Bigg)$$

with

$$\mathcal{E}_2(C_6) \le \mathcal{E}_{2.1}(C_6) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\ge 1;h+1=l_3} \sum_{g\in I_l,0\le g\le n} |c_{\underline{l},g}| \frac{(kq^{l_3 k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \times \left((C_6)^{-g} q^{l_3} q^{(l_3-1)^2\Delta} + 3(C_6)^{-g} q^{l_3} \right) \quad (115)$$

along with

$$\mathbb{T}_{l_3\le h\le n_2+l_2+l_3-2} \le \mathcal{E}_3(C_6) C_5(C_6)^{n+S}(n+S)! \frac{1}{q^{h^2\Delta}} \quad (116)$$

where

$$\mathcal{E}_3(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\ge 1;g\in I_{\underline{l}},0\le g\le n \\ l_3\le h\le n-g+l_2+l_3-2}} |c_{\underline{l},g}| \frac{(kq^{l_3 k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \times \Bigg(\sum_{l_3-h\le j\le n-g+l_2-h+l_3-2} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 + (C_6)^{-g} q^{l_3} \int_{1/q^{k(n-g+l_2-h+l_3)}}^{1/q^{k(n-g+l_2-h+l_3-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 + \sum_{j\ge n-g+l_2-h+l_3} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Bigg)$$

with

$$\mathcal{E}_3(C_6) \le \mathcal{E}_{3.1}(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\ge 1;g\in I_{\underline{l}},0\le g\le n \\ l_3\le h\le n-g+l_2+l_3-2}} |c_{\underline{l},g}| \frac{(kq^{l_3 k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \times \left(3(C_6)^{-g} q^{l_3} \right) \quad (117)$$

and

$$\mathbb{T}_{h\le n_2+l_2+l_3-1} \le \mathcal{E}_4(C_6) C_5(C_6)^{n+S}(n+S)! \frac{1}{q^{h^2\Delta}} \quad (118)$$

where

$$\mathcal{E}_4(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\ge 1;g\in I_{\underline{l}},0\le g\le n \\ h=n-g+l_2+l_3-1}} |c_{\underline{l},g}| \frac{(kq^{l_3 k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \times \Bigg((C_6)^{-g} q^{l_3} \int_{1/q^{k(n-g+l_2-h+l_3)}}^{1/q^{k(n-g+l_2-h+l_3-1)}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 + \sum_{j\ge n-g+l_2-h+l_3} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Bigg)$$

with

$$(119)\quad \mathcal{E}_4(C_6) \leq \mathcal{E}_{4.1}(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;g\in I_{\underline{l}},0\leq g\leq n \\ h=n-g+l_2+l_3-1}} |c_{\underline{l},g}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \times \left(2(C_6)^{-g} q^{l_3}\right)$$

and finally

$$(120)\qquad \mathbb{T}_{n_2+l_2+l_3\leq h\leq n+S-1} \leq \mathcal{E}_5(C_6) C_5(C_6)^{n+S}(n+S)!\frac{1}{q^{h^2\Delta}}$$

where

$$\mathcal{E}_5(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;g\in I_{\underline{l}},0\leq g\leq n \\ n-g+l_2+l_3\leq h\leq n+S-1}} |c_{\underline{l},g}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \times \left(\sum_{\substack{j\geq n-g+l_2-h+l_3 \\ j\geq 0}} (C_6)^{-g} q^{l_3} \int_{1/q^{k(j+1)}}^{1/q^{kj}} (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right)$$

with

$$(121)\quad \mathcal{E}_5(C_6) \leq \mathcal{E}_{5.1}(C_6) = \sum\sum_{\substack{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1;g\in I_{\underline{l}},0\leq g\leq n \\ n-g+l_2+l_3\leq h\leq n+S-1}} |c_{\underline{l},g}| \frac{(kq^{l_3k})^{l_1}}{\Gamma(l_0/k)} \frac{R_0^{l_0+kl_1}}{\Delta_{P,k}} \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \times \left((C_6)^{-g} q^{l_3}\right)$$

We are now ready to come to the end of the proof of Lemma 9. Since $I_{\underline{l}}$ contains only positive natural numbers, one can select the constant C_6 large enough in order that all the quantity

$$(122)\qquad \mathcal{E}_j(C_6) \leq 1/10 \quad , \quad 1 \leq j \leq 5$$

are taken suitably small. At last, gathering (102), (105), (112), (113), (114), (115), (116), (117), (118), (119), (120), (121) and (122) yields the forecast bounds (100). □

We can now return to the proof of Proposition 4. An application of Lemma 8 and Lemma 9 to the recursion (30) allows us to get the next bounds

$$\sup_{u\in\mathcal{A}_h} \left| \frac{w_{n+S}(u)}{u} \right| \leq C_5(C_6)^{n+S}(n+S)!\frac{1}{q^{h^2\Delta}}$$

for all $0 \leq h \leq n+S-1$, provided that $C_5, C_6 > 0$ are taken large enough. This means exactly that the property $\mathbb{A}_{n+S}$ holds true. □

6 Building analytic bounded solutions to the initial Cauchy problem (17), (18).

We first recall the definition of a good covering in $\mathbb{C}^*$ which is similar to the classical one given in [3], Chapter XI.

Definition 1 *Let $\varsigma \geq 2$ be an integer. For all $0 \leq p \leq \varsigma - 1$, we consider open sectors $\mathcal{T}_p$ centered at 0 (and do not contain 0) with given radius $r_{\mathcal{T}}$ that fulfill the next three features:*
i) *The intersection of any two consecutive sector of the family $\underline{\mathcal{T}} = \{\mathcal{T}_p\}_{0\leq p\leq \varsigma-1}$ is non empty, namely*

$$\mathcal{T}_p \cap \mathcal{T}_{p+1} \neq \emptyset$$

for all $0 \leq p \leq \varsigma - 1$, with the convention that $\mathcal{T}_\varsigma = \mathcal{T}_0$.
ii) *The intersection of any three elements in $\underline{\mathcal{T}}$ is empty.*
iii) *The union of the sectors $\mathcal{T}_p$ covers some punctured neighborhood $\dot{\mathcal{U}}$ of the origin in $\mathbb{C}^*$,*

$$\cup_{p=0}^{\varsigma-1} \mathcal{T}_p = \dot{\mathcal{U}} = \mathcal{U} \setminus \{0\}$$

The family $\underline{\mathcal{T}}$ is then called a good covering in $\mathbb{C}^*$.

We provide a notion of admissible set of sectors in the next

Definition 2 *Let $\varsigma \geq 2$ be an integer and let $\underline{\mathcal{T}} = \{\mathcal{T}_p\}_{0\leq p\leq \varsigma-1}$ be a good covering in $\mathbb{C}^*$. We consider a set $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0\leq p\leq \varsigma-1}$ of unbounded sectors $\mathcal{U}_p$ centered at 0, that suffer the next two properties:*
1) *Each sector $\mathcal{U}_p$ does not contain any of the roots of the polynomial $u \mapsto P(ku^k)$, for $0 \leq p \leq \varsigma - 1$.*
2) *For all $0 \leq p \leq \varsigma - 1$, there exists a constant $\Delta_p > 0$ such that for all $t \in \mathcal{T}_p$, one can single out a direction $\gamma_p \in \mathbb{R}$ (that may depend on t) such that both conditions*

$$L_{\gamma_p} = [0, +\infty) \exp(\sqrt{-1}\gamma_p) \subset \mathcal{U}_p \cup \{0\} \tag{123}$$

and

$$\cos(k(\gamma_p - \arg(t))) > \Delta_p \tag{124}$$

hold.
We say that the set of sectors $\underline{\mathcal{D}} = \{\underline{\mathcal{T}}, \underline{\mathcal{U}}\}$ is an admissible set of sectors.

Regarding the constructions made in Section 2.2, the next proposition is a synthesis of the statements already reached in the previous sections 3, 4 and 5.

Proposition 5 *Let $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0\leq p\leq \varsigma-1}$ be a set of unbounded sectors centered at 0 subjected to the condition* **1)** *of Definition 2. Under the constraints (10), (11), (12), (13), (14), (15), for each integer $0 \leq p \leq \varsigma - 1$, one can build a holomorphic function*

$$w_p(u, z) = \sum_{n\geq 0} w_{p,n}(u) \frac{z^n}{n!} \tag{125}$$

on the domain $\mathcal{U}_p \times D_{\frac{1}{2C_4}}$ that solves the Cauchy problem (25), (26) and withstands the next upper bounds

$$|w_p(u, z)| \leq 2C_3|u| \exp\left(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)\right) \tag{126}$$

for all $u \in \mathcal{U}_p$, $z \in D_{\frac{1}{2C_4}}$, where the constants $C_3, C_4 > 0$ and $u_0 > 1$, $\alpha \geq 0$, $k_1 > 0$ are introduced in Proposition 3. Furthermore, the functions $w_{p,n}(u)$ fulfill the next list of features and bounds.

a. *For all* $0 \leq p \leq \varsigma - 1$*, all* $n \geq 0$*, the map* $u \mapsto w_{p,n}(u)$ *is holomorphic on* $\mathcal{U}_p$ *and undergoes the upper estimates*

$$|w_{p,n}(u)| \leq C_3(C_4)^n n! |u| \exp\left(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)\right) \tag{127}$$

for all $u \in \mathcal{U}_p$*, with constants* $C_3, C_4 > 0$ *and* $u_0 > 1$*,*$\alpha \geq 0$*,*$k_1 > 0$ *exhibited in Proposition 3.*

b. *For all* $0 \leq p \leq \varsigma - 1$*, all* $n \geq 0$*, the map* $u \mapsto w_{p,n}(u)$ *can be analytically continued (as a single holomorphic function merely denoted* $w_n(u)$*, omitting the index* p*) on a disc* D_{R_n} *with radius* $R_n = R_0/q^n$*, where* R_0 *is chosen to satisfy (32) and appears to suffer the bounds*

$$|w_n(u)| \leq C_1(C_2)^n \frac{n!}{q^{n^2\Delta}} |u| \tag{128}$$

provided that $u \in D_{R_n}$*, for suitable constants* $C_1, C_2, \Delta > 0$ *given in Proposition 2.*

c. *For all* $0 \leq p \leq \varsigma - 1$*, all* $n \geq 0$*, the map* $u \mapsto w_{p,n}(u)$ *is bounded and holomorphic on each sectorial annulus*

$$\mathcal{A}_{p,h} = \{u \in \mathbb{C}^*/u \in \mathcal{U}_p \quad , \quad \frac{R_0}{q^{h+1}} \leq |u| \leq \frac{R_0}{q^h}\} \tag{129}$$

for $0 \leq h \leq n-1$*, where* R_0 *is chosen as in* **b.** *and turns out to be upper bounded as follows*

$$|w_{p,n}(u)| \leq C_5(C_6)^n n! \frac{1}{q^{h^2\Delta}} |u| \tag{130}$$

whenever $u \in \mathcal{A}_{p,h}$*, for some well chosen constants* $C_5, C_6 > 0$ *described in Proposition 4 and* $\Delta > 0$ *singled out as in* **b.**.

At this stage, we have prepared all the mandatory material in order to show the first main result of our work.

Theorem 1 *Assume that all the requirements (7), (8), (9), (10), (11), (12), (13), (14), (15), (16), (32) hold true. Consider a good covering* $\underline{\mathcal{T}} = \{\mathcal{T}_p\}_{0 \leq p \leq \varsigma - 1}$ *in* $\mathbb{C}^*$ *and a set* $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0 \leq p \leq \varsigma-1}$ *of unbounded sectors chosen in a way that the data* $\underline{\mathcal{D}} = \{\underline{\mathcal{T}}, \underline{\mathcal{U}}\}$ *forms an admissible set of sectors.*

Then, for all $0 \leq p \leq \varsigma - 1$*, one can construct a solution* $u_p(t,z)$ *to our main Cauchy problem (17), (18) that is bounded and holomorphic on* $\mathcal{T}_p \times D_{\frac{1}{2C_4}}$ *and that can be expressed through a Laplace transform of order* k*,*

$$u_p(t,z) = k \int_{L_{\gamma_p}} w_p(u,z) \exp(-(u/t)^k) du/u \tag{131}$$

for $(t,z) \in \mathcal{T}_p \times D_{\frac{1}{C_4}}$*. The Borel map* $w_p(u,z)$ *is the function displayed above in Proposition 5 and the integration path* L_{γ_p} *is the halfline depicted in (123).*

Furthermore, we can control bounds estimates for the differences of consecutive solutions $u_{p+1} - u_p$ *as follows. For each* $0 \leq p \leq \varsigma - 1$*, one can find two constants* $A_p, B_p > 0$ *(independent of* q *provided that* q *belongs to a bounded domain of the form* $]1, q_0]$ *for some* $q_0 > 1$*) such that*

$$|u_{p+1}(t,z) - u_p(t,z)| \leq A_p(B_p)^N \Gamma(\frac{N}{k}) q^{N^2/2} |t|^N \tag{132}$$

for all integers $N \geq 1$*, all* $t \in \mathcal{T}_p \cap \mathcal{T}_{p+1}$*, all* $z \in D_{\frac{1}{2C_6}}$*, where by convention we set* $u_\varsigma(t,z) = u_0(t,z)$ *and* $\mathcal{T}_\varsigma = \mathcal{T}_0$*.*

Proof For each $0 \leq p \leq \varsigma - 1$, we consider the function $w_p(u,z)$ built up in Proposition 5. By construction, $w_p(u,z)$ solves the problem (25), (26) on the domain $\mathcal{U}_p \times D_{\frac{1}{2C_4}}$. From the moderate growth bounds (126) and the fact that the set $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0 \leq p \leq \varsigma - 1}$ is admissible, we observe that the Laplace transform $u_p(t,z)$ given by (131) is well defined and bounded holomorphic on $\mathcal{T}_p \times D_{\frac{1}{2C_4}}$. Moreover, according to the construction made in Section 2.2 with the help of Proposition 1, we deduce that the maps $u_p(t,z)$ solve our main Cauchy problem (17), (18) on $\mathcal{T}_p \times D_{\frac{1}{2C_4}}$, for all $0 \leq p \leq \varsigma - 1$.

In the remaining part of the proof we discuss the second feature (132). We will observe that the technical properties **b.** and **c.** of the maps w_p described in Proposition 5 will play a preeminent role in the statement of these bounds.

Let $0 \leq p \leq \varsigma - 1$. According to the Taylor expansion (125), we first rewrite the difference $u_{p+1} - u_p$ as a sum

$$u_{p+1}(t,z) - u_p(t,z) = \sum_{n \geq 0} \left(k \int_{L_{\gamma_{p+1}}} w_{p+1,n}(u) \exp(-(u/t)^k) du/u - k \int_{L_{\gamma_p}} w_{p,n}(u) \exp(-(u/t)^k) du/u \right) \frac{z^n}{n!} \tag{133}$$

Since for each $n \geq 0$, the maps $w_{p+1,n}(u)$ and $w_{p,n}(u)$ have a common analytic continuation (named $w_n(u)$) on the disc D_{R_n} as stated in **b.** of Proposition 5 and by the Cauchy formula, we can bend the oriented path $L_{\gamma_{p+1}} - L_{\gamma_p}$ into the union of the next three suitably oriented paths:
i) Two half lines

$$L_{\gamma_{p+1},R_{n+1}} = [R_0/q^{n+1}, +\infty)e^{\sqrt{-1}\gamma_{p+1}} \quad , \quad -L_{\gamma_p,R_n} = -[R_0/q^{n+1}, +\infty)e^{\sqrt{-1}\gamma_p}$$

ii) A circle

$$C_{\gamma_p,\gamma_{p+1},R_0/q^{n+1}} = \{\frac{R_0}{q^{n+1}} e^{\sqrt{-1}\theta} / \theta \in (\gamma_p, \gamma_{p+1})\}.$$

In order to arrive at our forcast estimates, we further need to break up the two above halflines in **i)** into union of segments that belong to the sectorial annuli constructed in **c.** of Proposition 5. Namely,

$$L_{\gamma_{p+1},R_{n+1}} = (\bigcup_{h=0}^{n} L_{\gamma_{p+1},\mathcal{A}_{p+1,h}}) \cup L_{\gamma_{p+1},R_0} \quad , \quad -L_{\gamma_p,R_n} = (\bigcup_{h=0}^{n} -L_{\gamma_p,\mathcal{A}_{p,h}}) \cup -L_{\gamma_p,R_0}$$

where

$$L_{\gamma_{p+1},\mathcal{A}_{p+1,h}} = [\frac{R_0}{q^{h+1}}, \frac{R_0}{q^h}]e^{\sqrt{-1}\gamma_{p+1}}, -L_{\gamma_p,\mathcal{A}_{p,h}} = -[\frac{R_0}{q^{h+1}}, \frac{R_0}{q^h}]e^{\sqrt{-1}\gamma_p},$$
$$L_{\gamma_{p+1},R_0} = [R_0, +\infty)e^{\sqrt{-1}\gamma_{p+1}}, -L_{\gamma_p,R_0} = -[R_0, +\infty)e^{\sqrt{-1}\gamma_p}$$

As a result, we can decompose each term of the Taylor expansion (133) as follows

$$(134)\quad k\int_{L_{\gamma_{p+1}}} w_{p+1,n}(u)\exp(-(u/t)^k)du/u - k\int_{L_{\gamma_p}} w_{p,n}(u)\exp(-(u/t)^k)du/u =$$
$$k\int_{L_{\gamma_{p+1},R_0}} w_{p+1,n}(u)\exp(-(u/t)^k)du/u - k\int_{L_{\gamma_p,R_0}} w_{p,n}(u)\exp(-(u/t)^k)du/u$$
$$+\sum_{h=0}^{n} k\int_{L_{\gamma_{p+1},\mathcal{A}_{p+1,h}}} w_{p+1,n}(u)\exp(-(u/t)^k)du/u - \sum_{h=0}^{n} k\int_{L_{\gamma_p,\mathcal{A}_{p,h}}} w_{p,n}(u)\exp(-(u/t)^k)du/u$$
$$+ k\int_{C_{\gamma_p,\gamma_{p+1},R_0/q^{n+1}}} w_n(u)\exp(-(u/t)^k)du/u$$

In the next step of the proof, we provide upper estimates for each piece of the above decomposition (134). We deal with the first block

$$I_1 = \left| k\int_{L_{\gamma_{p+1},R_0}} w_{p+1,n}(u)\exp(-(u/t)^k)du/u \right|$$

According to the bounds (127), we arrive at

$$(135)\quad I_1 \leq k\int_{R_0}^{+\infty} C_3(C_4)^n n!\exp\left(k_1\log^2(r+u_0)+\alpha\log(r+u_0)\right)$$
$$\times\exp\left(-(\frac{r}{|t|})^k\cos\left(k(\gamma_{p+1}-\arg(t))\right)\right)dr$$
$$\leq kC_3(C_4)^n n!\int_{R_0}^{+\infty}\exp\left(k_1\log^2(r+u_0)+\alpha\log(r+u_0)\right)$$
$$\times\exp\left(-\frac{1}{2}(\frac{r}{|t|})^k\Delta_{p+1}\right)\exp\left(-\frac{1}{2}(\frac{r}{|t|})^k\Delta_{p+1}\right)dr$$
$$\leq kC_3(C_4)^n n!\exp\left(-\frac{1}{2}(\frac{R_0}{|t|})^k\Delta_{p+1}\right)\int_{R_0}^{+\infty}\exp\left(k_1\log^2(r+u_0)+\alpha\log(r+u_0)\right)$$
$$\times\exp\left(-\frac{1}{2}(\frac{r}{r_{\mathcal{T}}})^k\Delta_{p+1}\right)dr$$

for all $t \in \mathcal{T}_{p+1}\cap\mathcal{T}_p$, where $\Delta_{p+1}>0$ is defined in (124). The second part

$$I_2 = \left| k\int_{L_{\gamma_p,R_0}} w_{p,n}(u)\exp(-(u/t)^k)du/u \right|$$

can be treated in a similar manner as I_1 and leads to the bounds

$$(136)\quad I_2 \leq kC_3(C_4)^n n!\exp\left(-\frac{1}{2}(\frac{R_0}{|t|})^k\Delta_p\right)\int_{R_0}^{+\infty}\exp\left(k_1\log^2(r+u_0)+\alpha\log(r+u_0)\right)$$
$$\times\exp\left(-\frac{1}{2}(\frac{r}{r_{\mathcal{T}}})^k\Delta_p\right)dr$$

provided that $t\in\mathcal{T}_{p+1}\cap\mathcal{T}_p$. We turn to the third piece

$$I_3 = \left|\sum_{h=0}^{n} k\int_{L_{\gamma_{p+1},\mathcal{A}_{p+1,h}}} w_{p+1,n}(u)\exp(-(u/t)^k)du/u\right|$$

Owing to the bounds (130) and (128) we get that

$$(137)\quad I_3 \leq \sum_{h=0}^{n} k \int_{R_0/q^{h+1}}^{R_0/q^h} C_5(C_6)^n n! \frac{1}{q^{h^2\Delta}} \exp\left(-(\frac{r}{|t|})^k \cos\left(k(\gamma_{p+1} - \arg(t))\right)\right) dr$$
$$\leq kC_5(C_6)^n n!(1-\frac{1}{q}) \sum_{h=0}^{n} \frac{R_0}{q^h} \frac{1}{q^{h^2\Delta}} \exp\left(-(\frac{R_0}{q^{h+1}})^k \Delta_{p+1} \frac{1}{|t|^k}\right)$$

for all $t \in \mathcal{T}_{p+1} \cap \mathcal{T}_p$. The fourth piece

$$I_4 = \left| \sum_{h=0}^{n} k \int_{L_{\gamma_p,\mathcal{A}_{p,h}}} w_{p,n}(u) \exp(-(u/t)^k) du/u \right|$$

can be managed in an analogous way and gives rise to bounds

$$(138)\qquad I_4 \leq kC_5(C_6)^n n!(1-\frac{1}{q}) \sum_{h=0}^{n} \frac{R_0}{q^h} \frac{1}{q^{h^2\Delta}} \exp\left(-(\frac{R_0}{q^{h+1}})^k \Delta_p \frac{1}{|t|^k}\right)$$

The last part of the decomposition (134),

$$I_5 = \left| k \int_{C_{\gamma_p,\gamma_{p+1},R_0/q^{n+1}}} w_n(u) \exp(-(u/t)^k) du/u \right|$$

is handled in the following manner. From the condition (124), we observe that

$$\cos(k(\theta - \arg(t))) > \tilde{\Delta}_{p,p+1}$$

for some positive real number $\tilde{\Delta}_{p,p+1} > 0$ provided that $\theta \in (\gamma_p, \gamma_{p+1})$ if $\gamma_p < \gamma_{p+1}$ (or $\theta \in (\gamma_{p+1}, \gamma_p)$ if $\gamma_{p+1} < \gamma_p$) when $t \in \mathcal{T}_{p+1} \cap \mathcal{T}_p$ and taking heed of the bounds (128), we reach the bounds

$$(139)\quad I_5 \leq k \left| \int_{\gamma_p}^{\gamma_{p+1}} C_1(C_2)^n \frac{n!}{q^{n^2\Delta}} \frac{R_0}{q^{n+1}} \exp\left(-(\frac{R_0/q^{n+1}}{|t|})^k \cos(k(\theta - \arg(t)))\right) d\theta \right|$$
$$\leq k|\gamma_{p+1} - \gamma_p| C_1(C_2)^n n! \frac{R_0}{q^{n+1}} \frac{1}{q^{n^2\Delta}} \exp\left(-(\frac{R_0/q^{n+1}}{|t|})^k \tilde{\Delta}_{p,p+1}\right)$$

for all $t \in \mathcal{T}_{p+1} \cap \mathcal{T}_p$.

Summing up the above bounds (135), (136), (137), (138) and (139), if one puts $\Delta_{p,p+1} = \min(\Delta_p, \Delta_{p+1})$, we arrive at a constant $C_{R_0,p,p+1} > 0$ such that

$$(140)\quad |u_{p+1}(t,z) - u_p(t,z)| \leq \sum_{n\geq 0} \Big[2kC_3(C_4)^n n! C_{R_0,p,p+1} \exp\left(-\frac{R_0^k}{2|t|^k} \Delta_{p,p+1}\right)$$
$$+ 2kC_5(C_6)^n n!(1-\frac{1}{q}) \sum_{h=0}^{n} \frac{R_0}{q^h} \frac{1}{q^{h^2\Delta}} \exp\left(-(\frac{R_0}{q^{h+1}})^k \Delta_{p,p+1} \frac{1}{|t|^k}\right)$$
$$+ k|\gamma_{p+1} - \gamma_p| C_1(C_2)^n n! \frac{R_0}{q^{n+1}} \frac{1}{q^{n^2\Delta}} \exp\left(-(\frac{R_0/q^{n+1}}{|t|})^k \tilde{\Delta}_{p,p+1}\right) \Big] \frac{|z|^n}{n!}$$

for all $t \in \mathcal{T}_{p+1} \cap \mathcal{T}_{p}$, all $z \in D_{\frac{1}{2C_6}}$. Consequently, since we remind that $C_6 > \max(C_2, C_4)$ from the proof of Proposition 4, we can reach bounds of the form

$$|u_{p+1}(t,z) - u_p(t,z)| \leq F_1(|t|) + F_2(|t|) + F_3(|t|) \tag{141}$$

where

$$F_1(|t|) = 4kC_3 C_{R_0,p,p+1} \exp\big(- \frac{R_0^k}{2|t|^k}\Delta_{p,p+1}\big)$$

and

$$F_2(|t|) = 2kC_5(1-\frac{1}{q})R_0 \sum_{n\geq 0}\left(\sum_{h=0}^{n} \frac{1}{q^{h+h^2\Delta}} \exp\big(- (\frac{R_0}{q^{h+1}})^k \Delta_{p,p+1}\frac{1}{|t|^k}\big)\right)(1/2)^n$$

along with

$$F_3(|t|) = k|\gamma_{p+1} - \gamma_p| C_1 R_0 \sum_{n \geq 0} \frac{1}{q^{n+1+n^2\Delta}} \exp\big(- (\frac{R_0/q^{n+1}}{|t|})^k \tilde{\Delta}_{p,p+1}\big)(1/2)^n$$

whenever $t \in \mathcal{T}_{p+1} \cap \mathcal{T}_p$ and $z \in D_{\frac{1}{2C_6}}$.

According to Fubini's theorem, we can rewrite the expression $F_2(|t|)$ as

$$\begin{aligned} F_2(|t|) &= 2kC_5(1-\frac{1}{q})R_0 \sum_{h\geq 0}\sum_{n\geq h} \frac{1}{q^{h+h^2\Delta}} \exp\big(- (\frac{R_0}{q^{h+1}})^k \Delta_{p,p+1}\frac{1}{|t|^k}\big)(1/2)^n \\ &= 2kC_5(1-\frac{1}{q})R_0 \sum_{h\geq 0}\big(\sum_{n\geq h}(1/2)^n\big)\frac{1}{q^{h+h^2\Delta}} \exp\big(- (\frac{R_0}{q^{h+1}})^k \Delta_{p,p+1}\frac{1}{|t|^k}\big) \\ &= 4kC_5(1-\frac{1}{q})R_0 \sum_{h\geq 0}\frac{1}{q^{h+h^2\Delta}} \exp\big(- (\frac{R_0}{q^{h+1}})^k \Delta_{p,p+1}\frac{1}{|t|^k}\big)(1/2)^h \end{aligned} \tag{142}$$

By combining the latter estimates (141) together with (142), we attain the first *crucial bounds of this second part of the proof.* Namely, some positive constants $K_{1,p}, K_{2,p} > 0$ and $0 < K_{3,p} < 1$ along with $M_{1,p}, M_{2,p}, M_{3,p} > 0$ can be singled out such that

$$\begin{aligned} |u_{p+1}(t,z) - u_p(t,z)| \leq K_{1,p}\exp\big(- \frac{M_{1,p}}{|t|^k}\big) \\ + K_{2,p}\sum_{n\geq 0}\frac{1}{q^{n+n^2\Delta}}\exp\big(- (\frac{M_{2,p}}{q^{n+1}})^k \frac{M_{3,p}}{|t|^k}\big)(K_{3,p})^n \end{aligned} \tag{143}$$

for all $t \in \mathcal{T}_{p+1} \cap \mathcal{T}_p$ and $z \in D_{\frac{1}{2C_6}}$.

In the *next step*, we search for a *sequential constraint* on the difference $|u_{p+1} - u_p|$. We recall the next lemma from our previous work [7] (Lemma 14 in that paper), which is a direct consequence of the Stirling formula,

Lemma 10 *Let $M, k > 0$ be positive real numbers. Then, we can find a constant $C_k > 0$ (relying on k) for which*

$$\big(\frac{1}{r}\big)^N \exp\big(- \frac{M}{r^k}\big) \leq C_k\big(\frac{1}{M}\big)^{N/k}(N/k)^{1/2}\Gamma(N/k)$$

holds for all $r > 0$, all integers $N \geq 1$.

A direct application of this latter lemma to the bounds (143) gives rise to the next inequalities

$$|u_{p+1}(t,z) - u_p(t,z)| \leq \Big[K_{1,p} C_k \Big(\frac{1}{M_{1,p}} \Big)^{N/k} (N/k)^{1/2} \Gamma(N/k) + K_{2,p} C_k \Big(\frac{1}{M_{2,p}^k M_{3,p}} \Big)^{N/k} (N/k)^{1/2} \Gamma(N/k) \Big\{ \sum_{n \geq 0} \frac{1}{q^{n+n^2\Delta}} q^{(n+1)N} (K_{3,p})^n \Big\} \Big] |t|^N \tag{144}$$

for all $t \in \mathcal{T}_{p+1} \cap \mathcal{T}_p$, $z \in D_{\frac{1}{2C_6}}$ and all integers $N \geq 1$. Besides, from the Schwarz inequality $nN \leq (1/2)(n^2 + N^2)$ (derived from the mere observation that $(N-n)^2 \geq 0$), we deduce that

$$\sum_{n \geq 0} \frac{1}{q^{n+n^2\Delta}} q^{(n+1)N} (K_{3,p})^n \leq q^{N + \frac{N^2}{2}} \sum_{n \geq 0} \frac{q^{n^2/2}}{q^{\Delta n^2}} (K_{3,p})^n \leq \frac{1}{1 - K_{3,p}} q^{N + \frac{N^2}{2}} \tag{145}$$

provided that Δ satisfies the requirement (16), for all integers $N \geq 1$.

Finally, gathering (144) and (145) yields the expected estimates (132). □

7 Asymptotic expansions in time variable for the analytic solutions to the problem (17), (18).

The next definition stems from our recent work [7].

Definition 3 *Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a complex Banach space. We set $k \geq 1$ an integer and $q > 1$ a real number. Let $\mathcal{T}$ be a bounded sector in $\mathbb{C}^*$ centered at 0 and $f : \mathcal{T} \to \mathbb{F}$ be a holomorphic function. Then, f is said to possess the formal series*

$$\hat{f}(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{F}[[t]]$$

as Gevrey asymptotic expansion of mixed order $(1/k; (q,1))$ on $\mathcal{T}$ if for each closed proper subsector $\mathcal{W}$ of $\mathcal{T}$ centered at 0, one can choose two constants $C, M > 0$ with

$$||f(t) - \sum_{n=0}^{N} a_n t^n||_{\mathbb{F}} \leq C M^{N+1} \Gamma(\frac{N+1}{k}) q^{\frac{(N+1)^2}{2}} |t|^{N+1}$$

for all integers $N \geq 0$ and any $t \in \mathcal{W}$.

We remind the reader the variant of the classical Ramis-Sibuya theorem (see [3] for a reference) adapted to the Gevrey asymptotic expansions of mixed order setting stated in [7] (Theorem 2 in that paper).

Proposition 6 *Consider a complex Banach space $(\mathbb{F}, ||.||_{\mathbb{F}})$ and set a good covering $\{\mathcal{T}_p\}_{0 \leq p \leq \varsigma - 1}$ in $\mathbb{C}^*$ (described in Definition 1). Let $\{G_p\}_{0 \leq p \leq \varsigma-1}$ be a set of holomorphic maps G_p from $\mathcal{T}_p$ into $\mathbb{F}$.*
We make the next assumptions:
1) The functions $G_p(t)$ are bounded on $\mathcal{T}_p$, for $0 \leq p \leq \varsigma - 1$.
2) The cocycles $\Delta_p(\epsilon) = G_{p+1}(t) - G_p(t)$, for $0 \leq p \leq \varsigma - 1$ where $\Delta_{\varsigma-1}(t) = G_0(t) - G_{\varsigma-1}(t)$

are submitted to the next sequential constraint on $Z_p = \mathcal{T}_{p+1} \cap \mathcal{T}_p$ *: there exist two constants* $A_p, B_p > 0$ *with*

$$||\Delta_p(t)||_{\mathbb{F}} \leq A_p(B_p)^N \Gamma(\frac{N}{k}) q^{\frac{N^2}{2}} |t|^N \tag{146}$$

provided that $t \in Z_p$*, for all integers* $N \geq 1$*, all* $0 \leq p \leq \varsigma - 1$*. In other words,* $\Delta_p(t)$ *has the null formal series* $\hat{0}$ *as Gevrey asymptotic expansion of mixed order* $(1/k; (q,1))$ *on* Z_p*, for* $0 \leq p \leq \varsigma - 1$*.*

Then, all the functions $G_p(t)$*,* $0 \leq p \leq \varsigma - 1$*, share a common formal power series* $\hat{G}(t) \in \mathbb{F}[[t]]$ *as Gevrey asymptotic expansion of mixed order* $(1/k; (q,1))$ *on* $\mathcal{T}_p$*.*

We are in position to state the second main result of this work.

Theorem 2 *We consider the set* $\{u_p\}_{0 \leq p \leq \varsigma - 1}$ *of solutions to the Cauchy problem (17), (18) built up in Theorem 1. We set* $\mathbb{F}$ *as the Banach space of bounded holomorphic functions on the disc* $D_{\frac{1}{2C_6}}$ *endowed with the sup norm. By construction, each function* $t \mapsto u_p(t,z)$ *can be seen as a holomorphic map from the sector* $\mathcal{T}_p$ *into* $\mathbb{F}$ *for* $0 \leq p \leq \varsigma - 1$*.*

Then, there exist a formal series

$$\hat{u}(t,z) = \sum_{n \geq 0} u_n(z) t^n \in \mathbb{F}[[t]]$$

which is the common Gevrey asymptotic expansion of mixed order $(1/k; (q,1))$ *of all the functions* $t \mapsto u_p(t,z)$ *on* $\mathcal{T}_p$*, for all* $0 \leq p \leq \varsigma - 1$*. In other words, for each* $0 \leq p \leq \varsigma - 1$*, for each proper subsector* $\mathcal{W} \subset \mathcal{T}_p$*, one can find two constants* $C, M > 0$ *with*

$$\sup_{z \in D_{\frac{1}{2C_6}}} |u_p(t,z) - \sum_{n=0}^{N} u_n(z) t^n| \leq C M^{N+1} \Gamma(\frac{N+1}{k}) q^{\frac{(N+1)^2}{2}} |t|^{N+1} \tag{147}$$

for all $t \in \mathcal{W}$*, all integers* $N \geq 0$*.*

Proof Let $\{u_p\}_{0 \leq p \leq \varsigma - 1}$ be the set of solutions to (17), (18) manufactured in Theorem 1 and $\mathbb{F}$ be the Banach space of bounded holomorphic functions on the disc $D_{\frac{1}{2C_6}}$ equipped with the sup norm. We can apply Proposition 6 to the maps

$$G_p : t \mapsto (z \mapsto u_p(t,z))$$

from $\mathcal{T}_p$ into $\mathbb{F}$, for $0 \leq p \leq \varsigma - 1$. Indeed, since $(t,z) \mapsto u_p(t,z)$ are bounded holomorphic functions on the products $\mathcal{T}_p \times D_{\frac{1}{2C_6}}$, we get that $t \mapsto G_p(t)$ are bounded maps from $\mathcal{T}_p$ into $\mathbb{F}$, for any $0 \leq p \leq \varsigma - 1$, satisfying therefore the first requirement 1) in Proposition 6. On the other hand, the bounds (132) can be rephrased in terms of the maps G_p by asserting that the cocycle $\Delta_p(t) = G_{p+1}(t) - G_p(t)$ fulfills the sequential constraints (146), making the second requirement 2) hold true. Then, Proposition 6 ascertains the existence of a common formal power series $\hat{G}(t)$ that we call $\hat{u}(t,z)$ in the statement of Theorem 2 which suffers the bounds (147) declared above and represents a common Gevrey asymptotic expansion of mixed order $(1/k; (q,1))$ for all the maps G_p on $\mathcal{T}_p$, for $0 \leq p \leq \varsigma - 1$. □

In the remaining part of this section, we prove a consequence of the above result whose relevance will be revealed in ongoing works. To that end, we need the next proposition.

Proposition 7 *Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a Banach space and let $k \geq 1$ be an integer and $q > 1$ be a positive real number. Let $\mathcal{T} \subset \mathbb{C}^*$ be an open bounded sector centered at 0 and denote $G : \mathcal{T} \to \mathbb{F}$ a bounded holomorphic map. Assume that G possess a formal series*

$$\hat{G}(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{F}[[t]]$$

as Gevrey asymptotic expansion of mixed order $(1/k; (q, 1))$ on $\mathcal{T}$. Then, for all proper subsector $\mathcal{W}' \subset \mathcal{T}$, one can find two constants $C', M' > 0$ such that

$$||G^{(N+1)}(t)||_{\mathbb{F}} \leq C'(M')^{N+1}(N+1)!\Gamma(\frac{N+1}{k})q^{\frac{(N+1)^2}{2}} \tag{148}$$

for all integers $N \geq 0$, all $t \in \mathcal{W}'$, where $G^{(N+1)}$ denotes the $(N+1)-$order derivative of the function G.

Proof We first select a subsector $\mathcal{W} \subset \mathcal{T}$ for which one can single out two constants $C, M > 0$ such that

$$||G(t) - \sum_{h=0}^{N} a_h t^h||_{\mathbb{F}} \leq CM^{N+1}\Gamma(\frac{N+1}{k})q^{\frac{(N+1)^2}{2}}|t|^{N+1} \tag{149}$$

for all integers $N \geq 0$, all $t \in \mathcal{W}$. We consider a proper subsector $\mathcal{W}' \subset \mathcal{W}$ and a well chosen constant $\lambda > 0$ such that

$$B(t_0, \lambda|t_0|) \subset \mathcal{W}$$

for all $t_0 \in \mathcal{W}'$. Indeed, let $t_0 = |t_0|e^{\sqrt{-1}\theta_0}$. Then any $t \in B(t_0, \lambda|t_0|)$ can be written in the form

$$t = t_0 + \lambda|t_0|e^{\sqrt{-1}\theta}\rho = |t_0|(e^{\sqrt{-1}\theta_0} + \lambda\rho e^{\sqrt{-1}\theta})$$

for some $0 \leq \rho \leq 1$ and angle $\theta \in [0, 2\pi)$. For $\lambda > 0$ taken small enough, we observe that $e^{\sqrt{-1}\theta_0} + \lambda\rho e^{\sqrt{-1}\theta}$ remains close to $e^{\sqrt{-1}\theta_0}$ uniformly relatively to $0 \leq \rho \leq 1$ and $\theta \in [0, 2\pi)$, which implies that $t \in \mathcal{W}$.

Let $N \geq 0$. Since the map $t \mapsto G(t)$ is holomorphic on $\mathcal{W}$, we can apply the Cauchy formula to the map $f(t) = G(t) - \sum_{h=0}^{N} a_h t^h$ along the (positively oriented) circle $C(t_0, \lambda|t_0|)$ and get the integral representation

$$f(t_0) = \frac{1}{2\pi\sqrt{-1}} \int_{C(t_0,\lambda|t_0|)} \frac{f(\xi)}{\xi - t_0} d\xi$$

that we can derive $n-$times w.r.t t_0 in order to reach the next formula

$$f^{(n)}(t_0) = \frac{n!}{2\pi\sqrt{-1}} \int_{C(t_0,\lambda|t_0|)} \frac{f(\xi)}{(\xi - t_0)^{n+1}} d\xi \tag{150}$$

for all integers $n \geq 0$, all $t_0 \in \mathcal{W}'$.

Taking heed of (149), the above formula (150) for the particular integer $n = N + 1$ grants the next bounds

$$||f^{(N+1)}(t_0)||_{\mathbb{F}} \leq \frac{(N+1)!}{2\pi} \int_0^{2\pi} \frac{CM^{N+1}\Gamma(\frac{N+1}{k})q^{\frac{(N+1)^2}{2}}|t_0 + \lambda|t_0|e^{\sqrt{-1}\theta}|^{N+1}}{(\lambda|t_0|)^{N+2}} \lambda|t_0| d\theta \tag{151}$$
$$\leq C(M\frac{1+\lambda}{\lambda})^{N+1}(N+1)!\Gamma(\frac{N+1}{k})q^{\frac{(N+1)^2}{2}}$$

for all $t_0 \in \mathcal{W}'$. These last bounds give the result (148) provided that we observe that $G^{(N+1)}(t_0) = f^{(N+1)}(t_0)$ for all $t_0 \in \mathcal{W}'$. □

The next corollary is a straight application of the above proposition and Theorem 2.

Corollary 1 *Let us consider the family* $\{u_p\}_{0 \leq p \leq \varsigma-1}$ *of solutions to the Cauchy problem (17), (18) constructed in Theorem 1. Then, for each* $0 \leq p \leq \varsigma - 1$*, for each proper subsector* $\mathcal{W}' \subset \mathcal{T}_p$*, one can select two constants* $C', M' > 0$ *for which*

$$\sup_{z \in D_{\frac{1}{2C_6}}} |\partial_t^n u_p(t,z)| \leq C'(M')^n n! \Gamma(\frac{n}{k}) q^{\frac{n^2}{2}} \tag{152}$$

for all integers $n \geq 1$*, all* $t \in \mathcal{W}'$.

8 Confluence as $q > 1$ tends to 1

Throughout this section, we slightly change the notations introduced in the earlier sections of the work. In order to keep track of the dependence of the set of solutions $\{u_p(t,z)\}_{0 \leq p \leq \varsigma-1}$ to the problem (17), (18) relatively to the parameter $q > 1$, constructed in Theorem 1, we denote $u_{p;q}(t,z)$ the function $u_p(t,z)$. We also add a second index q to the Borel map $w_p(u,z)$, by setting $w_p(u,z) = w_{p;q}(u,z)$ inside the integral representation (131). We assume that the real parameter q is chosen within the range $(1, q_0]$ for some fixed real number $q_0 > 1$.

8.1 The limit Cauchy problem

In this subsection, we introduce a new Cauchy problem that we call the *limit problem as* $q > 1$ tends to 1. It is displayed as follows

$$P(t^{k+1}\partial_t)\partial_z^S u_{;1}(t,z) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}} c_{\underline{l}}(z) t^{l_0} (t^{k+1}\partial_t)^{l_1} \partial_z^{l_2} u_{;1}(t,z) \tag{153}$$

for given Cauchy data

$$(\partial_z^j u_{;1})(t,0) = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1. \tag{154}$$

where all the data $k, S, P, \mathcal{A}$ and the coefficients $c_{\underline{l}}(z)$ with $\underline{l} \in \mathcal{A}$ along with the initial data $\varphi_j(t)$ for $0 \leq j \leq S-1$ are the ones already introduced in Section 2.1. We further assume that the constraint (12) is valid for all $q \in (1, q_0]$ which boils down to the next inequality

$$k\deg(P) \geq kl_1 + l_0 + 2k_1 l_3 \log(q_0) \tag{155}$$

for some real number $k_1 > 0$, for all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

We search for solutions to (153), (154) having the shape of a Laplace transform of order k,

$$u_{;1}(t,z) = k \int_{L_\gamma} w_{;1}(u,z) \exp(-(u/t)^k) du/u \tag{156}$$

along a halfline $L_\gamma = [0, +\infty)e^{\sqrt{-1}\gamma}$, where the Borel map $w_{;1}(u,z)$ is holomorphic with respect to u on some infinite sector $S_{d,\delta} = \{u \in \mathbb{C}^*/|d - \arg(u)| < \delta$ for suitably selected directions $d \in \mathbb{R}$ and opening $\delta > 0$ and analytic w.r.t z on some small disc D_r centered at 0 with radius

$r > 0$. The same computations as in Section 2.2 show that if the Borel map $w_{;1}(u,z)$ solves the next auxiliary Cauchy problem

$$\partial_z^S w_{;1}(u,z) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \frac{c_{\underline{l}}(z)}{P(ku^k)}(ku^k)^{l_1}(\partial_z^{l_2} w_{;1})(u,z) + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \frac{c_{\underline{l}}(z)}{P(ku^k)}\frac{u^k}{\Gamma(l_0/k)}\int_0^{u^k}(u^k-s)^{\frac{l_0}{k}-1}(k(s^{1/k})^k)^{l_1}(\partial_z^{l_2}w_{;1})(s^{1/k},z)ds/s \tag{157}$$

for given Cauchy data

$$(\partial_z^j w_{;1})(u,0) = P_j(u) \quad , \quad 0 \leq j \leq S-1 \tag{158}$$

then the integral representation (156) fulfills the limit problem (153), (154). We seek for solutions to this last problem (157), (158) as formal power series in z,

$$w_{;1}(u,z) = \sum_{n\geq 0} w_{n;1}(u)z^n/n! \tag{159}$$

Identical computations as the ones made in Section 2.2 show that $w_{;1}(u,z)$ formally solves (157), (158) if and only of the sequence of expressions $w_{n;1}(u)$, $n \geq 0$, is submitted to the next recursion relation

$$\frac{w_{n+S;1}(u)}{n!} = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0}\sum_{n_1+n_2=n} c_{\underline{l},n_1}\frac{w_{n_2+l_2;1}(u)}{n_2!}\frac{(ku^k)^{l_1}}{P(ku^k)} + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1}\sum_{n_1+n_2=n} c_{\underline{l},n_1}\frac{u^{l_0+kl_1}}{P(ku^k)\Gamma(\frac{l_0}{k})}k^{l_1}\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2;1}(us_1^{1/k})}{n_2!}\frac{ds_1}{s_1} \tag{160}$$

for the given initial functions

$$w_{j;1}(u) = P_j(u) \quad , \quad 0 \leq j \leq S-1. \tag{161}$$

The next proposition can be shown by following exactly the same steps as in the proof of Proposition 3 (by merely taking $q = 1$ in each inequality therein).

Proposition 8 *Let $\underline{\mathcal{D}} = \{\underline{\mathcal{T}},\underline{\mathcal{U}}\}$ be an admissible set of sectors as chosen in Definition 2. Let $\mathcal{U}$ be one sector belonging to the family of unbounded sectors $\underline{\mathcal{U}}$.*

Under the conditions (155), one can find out a unique sequence of functions $w_{n;1}(u)$, $n \geq 0$, where each map $u \mapsto w_{n;1}(u)$ is holomorphic on the sector $\mathcal{U}$, continuous on $\mathcal{U}\cup\{0\}$ and fulfills the recursion (160) for given initial data (161). In addition, one can find two constants $C_7, C_8 > 0$ (where C_8 can be taken larger than the constant $C_6 > 0$ obtained in Proposition 4) for which the next estimates

$$|w_{n;1}(u)| \leq C_7(C_8)^n n!|u|\exp\left(k_1\log^2(|u|+u_0) + \alpha\log(|u|+u_0)\right) \tag{162}$$

hold for all $n \geq 0$, all $u \in \mathcal{U}\cup\{0\}$, where $k_1 > 0$ is chosen as in (155) and $u_0 > 1$,$\alpha \geq 0$ are taken as in Proposition 3.

Consequently, one can build a holomorphic function

$$w_{;1}(u,z) = \sum_{n\geq 0} w_{n;1}(u)\frac{z^n}{n!} \tag{163}$$

on the domain $\mathcal{U} \times D_{\frac{1}{2C_8}}$ *that solves the Cauchy problem (157), (158) and suffers the next bounds*

$$|w_{;1}(u,z)| \leq 2C_7|u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right) \tag{164}$$

for all $u \in \mathcal{U} \cup \{0\}$, $z \in D_{\frac{1}{2C_8}}$.

As a result, the Laplace transform $u_{;1}(t,z)$ *given by (156) is well defined and bounded holomorphic on the domain* $\mathcal{T} \times D_{\frac{1}{2C_8}}$ *where the bounded sector* $\mathcal{T}$ *appertains to the family* $\underline{\mathcal{T}}$ *of bounded sectors from the admissible set* $\underline{\mathfrak{D}}$ *and corresponds to* $\mathcal{U}$ *under the requirement 2) of Definition 2. Furthermore, owing to Proposition 1, the map* $u_{;1}(t,z)$ *solves the limit Cauchy problem (153), (154) on* $\mathcal{T} \times D_{\frac{1}{2C_8}}$.

8.2 Bounds for the difference of analytic solutions to the Cauchy problem (157), (158) under the action of a $q-$difference operator

The purpose of this subsection is to prove the next technical proposition.

Proposition 9 *Let* $\beta \in \mathbb{Z}^*$ *be a non vanishing integer and let* $q \in (1, q_0]$. *There exist two constants* $C_9, C_{10} > 0$ *independent of* $q \in (1, q_0]$ *(where* C_{10} *can be taken larger than the constant* C_8 *obtained in Proposition 8) such that*

$$|w_{n;1}(u) - w_{n;1}(q^{\beta}u)| \leq |q^{\beta} - 1|C_9(C_{10})^n n!|u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right) \tag{165}$$

for all $u \in \mathcal{U} \cup \{0\}$ *for all integers* $n \geq 0$, *where* $k_1 > 0$ *is selected as in (155) and* $u_0 > 1$,$\alpha \geq 0$ *appear in Proposition 3.*

Proof We proceed by induction. We call $\mathscr{D}_n^{;1;\beta}$ the property (165) for a given natural number $n \geq 0$.

We first explain the reason for which $\mathscr{D}_n^{;1;\beta}$ holds true for $0 \leq n \leq S-1$. By construction, whenever $0 \leq n \leq S-1$, $w_{n;1}(u) = P_n(u)$ is a polynomial in u. Since $P_n(u)$ admits in particular a derivative $P_n'(u)$ w.r.t u on $\mathbb{C}$, we can rewrite the next difference

$$P_n(u) - P_n(q^{\beta}u) = \int_{q^{\beta}u}^{u} P_n'(s)ds$$

for all $u \in \mathbb{C}$ and from the parametrization $s = uh + q^{\beta}u(1-h)$ of the segment $[q^{\beta}u, u]$ with $0 \leq h \leq 1$, we obtain the integral representation

$$P_n(u) - P_n(q^{\beta}u) = (1-q^{\beta})u \int_0^1 P_n'(uh + q^{\beta}u(1-h))dh$$

and since $P_n'(uh + q^{\beta}u(1-h))$ is again a polynomial in u, we get that $\mathscr{D}_n^{;1;\beta}$ is valid for $0 \leq n \leq S-1$, for some well chosen constants $C_9, C_{10} > 0$ depending on β, q_0.

At this stage, it is worth noticing that the above reasoning cannot be applied directly to show estimates for the whole sequence of differences $w_{n;1}(u) - w_{n;1}(q^{\beta}u)$ for any $n \geq 0$ since the behaviour of the derivative $w_{n;1}'(u)$ is not properly controled as u tends to 0 along $\mathcal{U}$ as $n \geq S$. Instead, we use to the recursion (160) to provide sharp bounds.

Let $n \geq 0$, we take for granted that $\mathscr{D}_p^{;1;\beta}$ holds for any $p < n+S$ for some given constants $C_9, C_{10} > 0$. In the remaining part of the proof, we prove that $\mathscr{D}_{n+S}^{;1;\beta}$ is valid. The induction principle will then imply that the feature $\mathscr{D}_n^{;1;\beta}$ occurs for all $n \geq 0$.

At first, we use the recursion (160) in order to write the difference $w_{n+S;1}(u) - w_{n+S;1}(q^{\beta}u)$ in term of prior terms $w_{p;1}(u)$ and $w_{p;1}(q^{\beta}u)$ with $p < n+S$. Indeed,

$$(166)\quad \frac{w_{n+S;1}(u) - w_{n+S;1}(q^{\beta}u)}{n!} = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0=0} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \times \left[\frac{(ku^k)^{l_1}}{P(ku^k)} \frac{w_{n_2+l_2;1}(u)}{n_2!} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \frac{w_{n_2+l_2;1}(q^{\beta}u)}{n_2!} \right] + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0\geq 1} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \times \left[\frac{u^{l_0+kl_1}}{P(ku^k)} \frac{k^{l_1}}{\Gamma(l_0/k)} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(us_1^{1/k})}{n_2!} \frac{ds_1}{s_1} - \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)} \frac{k^{l_1}}{\Gamma(l_0/k)} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(q^{\beta}us_1^{1/k})}{n_2!} \frac{ds_1}{s_1} \right]$$

In the next lemma, we provide upper bounds for the quantity

$$(167)\quad \mathcal{A}_{n_2,l_2}(u) := \frac{(ku^k)^{l_1}}{P(ku^k)} \frac{w_{n_2+l_2;1}(u)}{n_2!} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \frac{w_{n_2+l_2;1}(q^{\beta}u)}{n_2!}$$

Lemma 11 *Two constants $K_{j,l_1,p} > 0$, $j = 1, 3$, can be singled out for which*

$$(168)\quad |\mathcal{A}_{n_2,l_2}(u)| \leq \left[K_{3,l_1,P} + K_{1,l_1,p} \right] \times |q^{\beta} - 1| C_9 (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)$$

holds for all $u \in \mathcal{U} \cup \{0\}$, all $n_2, l_2 \geq 0$ with $n_2 \leq n$ and $\underline{l} = (0, l_1, l_2, l_3) \in \mathcal{A}$.

Proof According to the classical identity $ab - cd = (a-c)b + c(b-d)$, we rewrite $\mathcal{A}_{n_2,l_2}(u)$ as a sum

$$(169)\quad \mathcal{A}_{n_2,l_2}(u) = \left[\frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \right] \frac{w_{n_2+l_2;1}(u)}{n_2!} + \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \left[\frac{w_{n_2+l_2;1}(u) - w_{n_2+l_2;1}(q^{\beta}u)}{n_2!} \right]$$

a) Bearing in mind the lower bounds (55), we get a constant $C_{P,k} > 0$ for which

$$(170)\quad |P(k(q^{\beta}u)^k)| \geq C_{P,k}(q^{\beta}|u|+1)^{k\deg(P)}$$

for all $u \in \mathcal{U} \cup \{0\}$ which sires a constant $K_{1,l_1,P} > 0$ such that

$$(171)\quad \left| \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \right| \leq \frac{k^{l_1}|q^{\beta}u|^{kl_1}}{C_{P,k}(q^{\beta}|u|+1)^{k\deg(P)}} \leq K_{1,l_1,P}$$

for all $u \in \mathcal{U} \cup \{0\}$, under the condition imposed in (155).
b) On the other hand, we can write an integral representation for the next difference

$$(172)\quad \frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} = \int_{q^{\beta}u}^{u} \left(\frac{(ks^k)^{l_1}}{P(ks^k)} \right)' ds$$

where the integrand can be explicitely computed

$$\left(\frac{(ks^k)^{l_1}}{P(ks^k)}\right)' = k^{l_1}\frac{kl_1 s^{kl_1-1}P(ks^k) - \hat{P}_k(s)s^{kl_1}}{(P(ks^k))^2} \tag{173}$$

where $\hat{P}_k(s) = (P(ks^k))'$ is the derivative w.r.t s of the polynomial $P(ks^k)$. Furthermore, since $s \mapsto P(ks^k)$ is a polynomial of degree $k\deg(P)$, we notice the next two bounds

$$|P(ks^k)| \leq C^1_{P,k}(|s|+1)^{k\deg(P)} \quad , \quad |\hat{P}_k(s)| \leq C^1_{\hat{P}_k}(|s|+1)^{k\deg(P)-1} \tag{174}$$

for all $s \in \mathbb{C}$, for some constants $C^1_{P,k}, C^1_{\hat{P}_k} > 0$. Consequently, departing from (173), the lower bounds (55) together with the upper ones (174), beget a constant $K_{2,l_1,P} > 0$ for which

$$\left|\left(\frac{(ks^k)^{l_1}}{P(ks^k)}\right)'\right| \leq k^{l_1}\frac{kl_1|s|^{kl_1-1}C^1_{P,k}(|s|+1)^{k\deg(P)} + |s|^{kl_1}C^1_{\hat{P}_k}(|s|+1)^{k\deg(P)-1}}{C^2_{P,k}(1+|s|)^{2k\deg(P)}} \leq \frac{K_{2,l_1,P}}{(1+|s|)^{k\deg(\mathrm{P})-kl_1+1}} \tag{175}$$

provided that $s \in \mathcal{U} \cup \{0\}$.

From the parametrization $s = uh + q^{\beta}u(1-h)$ of the segment $[q^{\beta}u, u]$ for $0 \leq h \leq 1$ in the integral (172), owing to the previous bounds (175), we deduce

$$\left|\frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)}\right| \leq |q^{\beta}-1||u|\int_0^1 \frac{K_{2,l_1,P}}{(1+|uh+q^{\beta}u(1-h)|)^{k\deg(P)-kl_1+1}}dh \tag{176}$$

for all $u \in \mathcal{U} \cup \{0\}$. Since $(1-q^{\beta})h + q^{\beta} \geq d_{q,\beta}$ for all $0 \leq h \leq 1$ where $d_{q,\beta} = 1$ if $\beta > 0$ and $d_{q,\beta} = q^{\beta}$ if $\beta < 0$, we reach a constant $K_{3,l_1,P} > 0$ for which

$$\left|\frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)}\right| \leq |q^{\beta}-1|K_{2,l_1,P}\frac{|u|}{(1+|u|d_{q,\beta})^{k\deg(P)-kl_1+1}} \leq |q^{\beta}-1|K_{3,l_1,P} \tag{177}$$

whenever $u \in \mathcal{U} \cup \{0\}$, by taking heed of the constraint (155).
c) In accordance with (162), we already know that

$$|w_{n_2+l_2;1}(u)| \leq C_7(C_8)^{n_2+l_2}(n_2+l_2)!|u|\exp\left(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\right) \leq C_9(C_{10})^{n_2+l_2}(n_2+l_2)!|u|\exp\left(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\right) \tag{178}$$

for all $n_2, l_2 \geq 0$, whenever $u \in \mathcal{U} \cup \{0\}$, provided that we select $C_9 > C_7$ and $C_{10} > C_8$.
d) Due to (10), we notice that $n_2 + l_2 < n + S$, therefore by the induction hypothesis, the property $\mathscr{D}^{;1;\beta}_{n_2+l_2}$ is valid, which yields the upper bounds

$$|w_{n_2+l_2;1}(u) - w_{n_2+l_2;1}(q^{\beta}u)| \leq |q^{\beta}-1|C_9(C_{10})^{n_2+l_2}(n_2+l_2)!|u|\exp\left(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\right) \tag{179}$$

Collecting the set of bounds (171), (177) along with (178) and (179), we arrive at the forecast bounds (168).

□

The next lemma is devoted to upper bounds for the quantity

$$(180)\quad \mathcal{B}_{n_2,l_2}(u) := \frac{u^{l_0+kl_1}}{P(ku^k)}\frac{k^{l_1}}{\Gamma(l_0/k)}\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2;1}(us_1^{1/k})}{n_2!}\frac{ds_1}{s_1} - \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)}\frac{k^{l_1}}{\Gamma(l_0/k)}\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2;1}(q^{\beta}us_1^{1/k})}{n_2!}\frac{ds_1}{s_1}$$

Lemma 12 *We can find two constants $L_{j,l_0,l_1,P} > 0$, $j = 1,3$, such that*

$$(181)\quad |\mathcal{B}_{n_2,l_2}(u)| \leq \frac{k^{l_1}}{\Gamma(l_0/k)}\left[L_{3,l_0,l_1,P} + L_{1,l_0,l_1,p}\right] \times |q^{\beta}-1|C_9\left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\right)(C_{10})^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!} \times |u|\exp\left(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\right)$$

for all $u \in \mathcal{U}\cup\{0\}$, all $n_2, l_2 \geq 0$ with $n_2 \leq n$ and $\underline{l} = (l_0,l_1,l_2,l_3) \in \mathcal{A}$ with $l_0 \geq 1$.

Proof The proof follows the same guideline as the one of Lemma 11. Namely, the identity $ab - cd = (a-c)b + c(b-d)$ allows us to rephrase the expression of $\mathcal{B}_{n_2,l_2}(u)$ as a sum of differences

$$(182)\quad \mathcal{B}_{n_2,l_2}(u) = \frac{k^{l_1}}{\Gamma(l_0/k)}\left[\frac{u^{l_0+kl_1}}{P(ku^k)} - \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)}\right]\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2;1}(us_1^{1/k})}{n_2!}\frac{ds_1}{s_1} + \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)}\frac{k^{l_1}}{\Gamma(l_0/k)}\left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\left\{\frac{w_{n_2+l_2;1}(us_1^{1/k})}{n_2!} - \frac{w_{n_2+l_2;1}(q^{\beta}us_1^{1/k})}{n_2!}\right\}\frac{ds_1}{s_1}\right)$$

a) Owing to the lower bounds (170), a constant $L_{1,l_0,l_1,P} > 0$ can be found with

$$(183)\quad \left|\frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)}\right| \leq \frac{|q^{\beta}u|^{l_0+kl_1}}{C_{P,k}(q^{\beta}|u|+1)^{k\deg(P)}} \leq L_{1,l_0,l_1,P}$$

for all $u \in \mathcal{U}\cup\{0\}$, under the requirement (155).
b) Mimicking the arguments outlined in the paragraph b) in the proof of Lemma 11, we can gather two constants $L_{j,l_0,l_1,P} > 0$, $j = 2,3$ such that

$$(184)\quad \left|\frac{u^{l_0+kl_1}}{P(ku^k)} - \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)}\right| \leq |q^{\beta}-1|L_{2,l_0,l_1,P}\frac{|u|}{(1+|u|d_{q,\beta})^{k\deg(P)-(l_0+kl_1)+1}} \leq |q^{\beta}-1|L_{3,l_0,l_1,P}$$

for all $u \in \mathcal{U}\cup\{0\}$, bearing in mind (155).
c) Departing from (178) and observing that both functions $x \mapsto \log^2(x)$ and $x \mapsto \log(x)$ are increasing on $[1,+\infty)$, we obtain upper bounds for the next integral piece

$$(185)\quad \left|\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2;1}(us_1^{1/k})}{n_2!}\frac{ds_1}{s_1}\right| \leq \int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}C_9(C_{10})^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!} \times |u|s_1^{1/k}\exp\left(k_1\log^2(|u|s_1^{1/k}+u_0)+\alpha\log(|u|s_1^{1/k}+u_0)\right)\frac{ds_1}{s_1} \leq C_9\left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\right)(C_{10})^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!} \times |u|\exp\left(k_1\log^2(|u|+u_0)+\alpha\log(|u|+u_0)\right)$$

for all $u \in \mathcal{U} \cup \{0\}$.
d) According to the bounds (179), a similar reasoning as the one just realized above in c) of Lemma 12 shows that

$$(186)\quad \left| \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \left\{ \frac{w_{n_2+l_2;1}(us_1^{1/k})}{n_2!} - \frac{w_{n_2+l_2;1}(q^{\beta}us_1^{1/k})}{n_2!} \right\} \frac{ds_1}{s_1} \right| \leq |q^{\beta}-1| C_9 \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} \times |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)$$

whenever $u \in \mathcal{U} \cup \{0\}$.

Combining the bounds (183), (184), (185), (186) along with the factorization (182) sires the foretold estimates (181). □

With the help of the above Lemma 11 and Lemma 12, the recursion (166) allows the next upper bounds

$$(187)\quad |w_{n+S;1}(u) - w_{n+S;1}(q^{\beta}u)| \leq \Bigg[\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0=0} \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| \left[K_{3,l_1,P} + K_{1,l_1,p} \right] \times |q^{\beta}-1| C_9 (C_{10})^{n_2+l_2} n! \frac{(n_2+l_2)!}{n_2!} + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0\geq 1} \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| \times \frac{k^{l_1}}{\Gamma(l_0/k)} \left[L_{3,l_0,l_1,P} + L_{1,l_0,l_1,p} \right] \times |q^{\beta}-1| C_9 \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{n_2+l_2} n! \frac{(n_2+l_2)!}{n_2!} \Bigg] \times |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)$$

provided $u \in \mathcal{U} \cup \{0\}$.

Owing to Lemma 2 and to the assumption that $c_{\underline{l}}(z)$ are polynomials (described in (8)), from (187), we get that

$$(188)\quad |w_{n+S;1}(u) - w_{n+S;1}(q^{\beta}u)| \leq \mathcal{B}(C_{10}) |q^{\beta}-1| C_9 (C_{10})^{n+S} (n+S)! |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)$$

for $u \in \mathcal{U} \cup \{0\}$, where

$$(189)\quad \mathcal{B}(C_{10}) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0=0} \sum_{g \in I_{\underline{l}}, 0\leq g\leq n} |c_{\underline{l},g}| \left[K_{3,l_1,P} + K_{1,l_1,p} \right] (C_{10})^{-g} + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0\geq 1} \sum_{g \in I_{\underline{l}}, 0\leq g\leq n} |c_{\underline{l},g}| \frac{k^{l_1}}{\Gamma(l_0/k)} \left[L_{3,l_0,l_1,P} + L_{1,l_0,l_1,p} \right] \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{-g}$$

Taking heed of the fact that $0 \notin I_{\underline{l}}$, one can select $C_{10} > 0$ large enough (under the already required condition $C_{10} > C_8$) in a way that

$$\mathcal{B}(C_{10}) \leq 1 \tag{190}$$

Gathering (188) and (190) attests that the property $\mathscr{D}^{;1;\beta}_{n+S}$ holds. This concludes the proof of Proposition 9. □

8.3 Bounds for the difference of analytic solutions to the Cauchy problems (25), (26) and (157), (158).

This subsection is devoted to the explanation of the next proposition.

Proposition 10 *Let $q \in (1, q_0]$. There exists two constants $C_{11}, C_{12} > 0$ independent of $q \in (1, q_0]$ (where C_{12} may be taken larger than the constant $C_{10} > 0$ obtained in Proposition 9) such that*

$$|w_{n;q}(u) - w_{n;1}(u)| \leq (q-1)C_{11}(C_{12})^n n! |u| \exp\left(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)\right) \tag{191}$$

for all $u \in \mathcal{U} \cup \{0\}$ for all integers $n \geq 0$, where $k_1 > 0$ is selected as in (155) and $u_0 > 1, \alpha \geq 0$ appear in Proposition 3.

In particular, if one denotes

$$w_{;q}(u, z) = \sum_{n \geq 0} w_{n;q}(u) \frac{z^n}{n!} \tag{192}$$

the holomorphic solution of the Cauchy problem (25), (26) on the domain $\mathcal{U} \times D_{\frac{1}{2C_4}}$, then the next error bounds for the difference

$$|w_{;q}(u, z) - w_{;1}(u, z)| \leq (q-1) 2C_{11} |u| \exp\left(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)\right) \tag{193}$$

holds for all $u \in \mathcal{U} \cup \{0\}$, whenever $z \in D_{\frac{1}{2C_{12}}}$.

Proof For the last time in this work we make use of the induction tool kit. We call $\mathfrak{D}^{;q;1}_n$ the feature (191) for a given integer $n \geq 0$.

At first, we observe that the property $\mathfrak{D}^{;q;1}_n$ trivially holds true for $0 \leq n \leq S - 1$, since in that case, $w_{n;q}(u) = w_{n;1}(u) = P_n(u)$.

Let $n \geq 0$. We make the hypothesis that $\mathfrak{D}^{;q;1}_p$ is true for any integer $p < n+S$ for some given constants $C_{11}, C_{12} > 0$. In the sequel, we need to prove that $\mathfrak{D}^{;q;1}_{n+S}$ holds true. The induction principle helps us to conclude that the property $\mathfrak{D}^{;q;1}_n$ is certified whenever $n \geq 0$.

At the outset, both recursions (30) and (160) allow us to express the difference $w_{n+S;q}(u) - w_{n+S;1}(u)$ in term of lower indexed quantities $w_{p;q}(u)$ and $w_{p;1}(u)$ with $p < n + S$. Namely,

$$\begin{aligned} \frac{w_{n+S;q}(u) - w_{n+S;1}(u)}{n!} &= \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}, l_0 = 0} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{(ku^k)^{l_1}}{P(ku^k)} \\ &\times \left[\frac{w_{n_2+l_2;q}(q^{l_3}u)}{n_2!} q^{l_3 k l_1} - \frac{w_{n_2+l_2;1}(u)}{n_2!} \right] + \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}, l_0 \geq 1} \sum_{n_1+n_2=n} c_{\underline{l},n_1} \frac{1}{P(ku^k)} \frac{u^{l_0+kl_1}}{\Gamma(l_0/k)} k^{l_1} \\ &\times \left[q^{l_3 k l_1} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;q}(q^{l_3} u s_1^{1/k})}{n_2!} \frac{ds_1}{s_1} - \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(u s_1^{1/k})}{n_2!} \frac{ds_1}{s_1} \right] \end{aligned} \tag{194}$$

In the ensuing lemma, we provide bounds for the front piece

$$\mathcal{C}_{n_2,l_2}(u) := \frac{(ku^k)^{l_1}}{P(ku^k)} \left[\frac{w_{n_2+l_2;q}(q^{l_3}u)}{n_2!} q^{l_3kl_1} - \frac{w_{n_2+l_2;1}(u)}{n_2!} \right]$$

Lemma 13 *One can single out two constants $M_{l_1,l_3,P} > 0$ and $K_{1,l_1,P} > 0$ for which*

$$(195)\quad |\mathcal{C}_{n_2,l_2}(u)| \leq \left[q^{l_3kl_1} q^{(\alpha+1)l_3} M_{l_1,l_3,P} + q^{l_3kl_1} K_{1,l_1,P} \frac{q^{l_3}-1}{q-1} + K_{1,l_1,P} \frac{q^{l_3kl_1}-1}{q-1} \right] \times (q-1) C_{11} (C_{12})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)$$

holds for all $u \in \mathcal{U} \cup \{0\}$, all $n_2, l_2 \geq 0$ with $n_2 \leq n$ and $\underline{l} = (0, l_1, l_2, l_3) \in \mathcal{A}$.

Proof According to the identity $ab - cd = (a-c)b + c(b-d)$, we can recast the difference $\mathcal{C}_{n_2,l_2}(u)$ as a sum of differences

$$(196)\quad \mathcal{C}_{n_2,l_2}(u) = \frac{(ku^k)^{l_1}}{P(ku^k)} \left[q^{l_3kl_1} \left(\frac{w_{n_2+l_2;q}(q^{l_3}u) - w_{n_2+l_2;1}(u)}{n_2!} \right) + \frac{w_{n_2+l_2;1}(u)}{n_2!} (q^{l_3kl_1} - 1) \right]$$

1) We deal with the first part of right handside of (196) by inserting an auxiliary term in the expression, namely

$$(197)\quad \begin{aligned} &|w_{n_2+l_2;q}(q^{l_3}u) - w_{n_2+l_2;1}(u)| \\ &= |w_{n_2+l_2;q}(q^{l_3}u) - w_{n_2+l_2;1}(q^{l_3}u) + w_{n_2+l_2;1}(q^{l_3}u) - w_{n_2+l_2;1}(u)| \\ &\leq |w_{n_2+l_2;q}(q^{l_3}u) - w_{n_2+l_2;1}(q^{l_3}u)| + |w_{n_2+l_2;1}(q^{l_3}u) - w_{n_2+l_2;1}(u)| \end{aligned}$$

a) Owing to (10), we observe that $n_2+l_2 < n+S$ and by assumption, the property $\mathfrak{D}^{;q;1}_{n_2+l_2}$ holds that gives rise to the bounds

$$(198)\quad \begin{aligned} &|w_{n_2+l_2;q}(q^{l_3}u) - w_{n_2+l_2;1}(q^{l_3}u)| \\ &\leq (q-1) C_{11} (C_{12})^{n_2+l_2} (n_2+l_2)! |q^{l_3}u| \exp\left(k_1 \log^2(|q^{l_3}u|+u_0) + \alpha \log(|q^{l_3}u|+u_0)\right) \end{aligned}$$

for all $u \in \mathcal{U} \cup \{0\}$, from which, with the help of Lemma 5, we obtain

$$(199)\quad \begin{aligned} &|w_{n_2+l_2;q}(q^{l_3}u) - w_{n_2+l_2;1}(q^{l_3}u)| \\ &\leq (q-1) C_{11} (C_{12})^{n_2+l_2} (n_2+l_2)! q^{(\alpha+1)l_3} \left[(|q^{l_3}u|+u_0)(|u|+u_0) \right]^{k_1 l_3 \log(q)} \\ &\quad \times |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right) \end{aligned}$$

for all $u \in \mathcal{U} \cup \{0\}$.
b) Keeping in mind the bounds (165) obtained in Proposition 9 specialized for $\beta = l_3$, we know that

$$(200)\quad \begin{aligned} |w_{n_2+l_2;1}(q^{l_3}u) - w_{n_2+l_2;1}(u)| &\leq |q^{l_3}-1| C_9 (C_{10})^{n_2+l_2} (n_2+l_2)! \\ &\quad \times |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right) \\ &\leq |q^{l_3}-1| C_{11} (C_{12})^{n_2+l_2} (n_2+l_2)! \\ &\quad \times |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right) \end{aligned}$$

under the requirement that $C_{11} > C_9$ and $C_{12} > C_{10}$.
c) According to the lower bounds (55), we get a constant $M_{l_1,l_3,P} > 0$ such that

$$\left| \frac{(ku^k)^{l_1}}{P(ku^k)} \left[(|q^{l_3}u| + u_0)(|u| + u_0) \right]^{k_1 l_3 \log(q)} \right| \leq M_{l_1,l_3,P} \tag{201}$$

for all $u \in \mathcal{U} \cup \{0\}$, under the condition appointed in (155).
d) We remind from (171) that a constant $K_{1,l_1,P} > 0$ can be found with

$$\left| \frac{(ku^k)^{l_1}}{P(ku^k)} \right| \leq K_{1,l_1,P} \tag{202}$$

for all $u \in \mathcal{U} \cup \{0\}$ and that owing to (162), we already know that

$$\begin{aligned} |w_{n_2+l_2;1}(u)| &\leq C_7 (C_8)^{n_2+l_2} (n_2+l_2)! |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0) \right) \\ &\leq C_{11} (C_{12})^{n_2+l_2} (n_2+l_2)! |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0) \right) \end{aligned} \tag{203}$$

holds whenever $u \in \mathcal{U} \cup \{0\}$ under the requisite that $C_{11} > C_7$ and $C_{12} > C_8$.

Finally, collecting the whole set of estimates (197), (198), (199), (200), (201), (202) and (203), we reach the forseen bounds (195). □

In the forthcoming lemma, we deal with the tailpiece of (194) that we set as

$$\begin{aligned} \mathcal{D}_{n_2,l_2}(u) &:= \frac{u^{l_0+kl_1}}{P(ku^k)} \\ &\times \left[q^{l_3 k l_1} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;q}(q^{l_3} u s_1^{1/k})}{n_2!} \frac{ds_1}{s_1} - \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(u s_1^{1/k})}{n_2!} \frac{ds_1}{s_1} \right] \end{aligned}$$

Lemma 14 *One can find two constants $M_{l_0,l_1,l_3,P} > 0$ and $L_{1,l_0,l_1,P} > 0$ for which*

$$\begin{aligned} |\mathcal{D}_{n_2,l_2}(u)| \leq & \left[q^{l_3 k l_1} q^{(\alpha+1) l_3} M_{l_0,l_1,l_3,P} + q^{l_3 k l_1} L_{1,l_0,l_1,P} \frac{q^{l_3}-1}{q-1} + L_{1,l_0,l_1,P} \frac{q^{l_3 k l_1}-1}{q-1} \right] \\ & \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) \\ & \times (q-1) C_{11} (C_{12})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0) \right) \end{aligned} \tag{204}$$

holds for all $u \in \mathcal{U} \cup \{0\}$, all $n_2, l_2 \geq 0$ with $n_2 \leq n$ and $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

Proof The roadmap is the same as one of Lemma 13. The identity $ab - cd = (a-c)b + c(b-d)$ warrants the redraft of the above expression as a sum of differences

$$\begin{aligned} \mathcal{D}_{n_2,l_2}(u) &:= \frac{u^{l_0+kl_1}}{P(ku^k)} \\ &\times \left[q^{l_3 k l_1} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \left(\frac{w_{n_2+l_2;q}(q^{l_3} u s_1^{1/k})}{n_2!} - \frac{w_{n_2+l_2;1}(u s_1^{1/k})}{n_2!} \right) \frac{ds_1}{s_1} \right. \\ &\left. + \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(u s_1^{1/k})}{n_2!} \frac{ds_1}{s_1} \times \left(q^{l_3 k l_1} - 1 \right) \right] \end{aligned} \tag{205}$$

We handle the first piece of (205) by popping in a median term inside the integrant. Indeed,

$$
\begin{aligned}
(206)\quad & |w_{n_2+l_2;q}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(us_1^{1/k})| \\
&= |w_{n_2+l_2;q}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k}) + w_{n_2+l_2;1}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(us_1^{1/k})| \\
&\leq |w_{n_2+l_2;q}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k})| + |w_{n_2+l_2;1}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(us_1^{1/k})|
\end{aligned}
$$

a) According to (10), the observation that $n_2 + l_2 < n + S$ yields the fact that $\mathfrak{D}_{n_2+l_2}^{;q;1}$ holds by assumption and gives rise to the bounds

$$
\begin{aligned}
(207)\quad & |w_{n_2+l_2;q}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k})| \\
&\leq (q-1)C_{11}(C_{12})^{n_2+l_2}(n_2+l_2)!|q^{l_3}us_1^{1/k}| \exp\left(k_1 \log^2(|q^{l_3}us_1^{1/k}|+u_0) + \alpha \log(|q^{l_3}us_1^{1/k}|+u_0)\right)
\end{aligned}
$$

for all $u \in \mathcal{U} \cup \{0\}$, all $0 \leq s_1 \leq 1$. The fact that $x \mapsto \log^2(x)$ and $x \mapsto \log(x)$ are increasing on $[1, +\infty)$ and an application of Lemma 5 begets the further estimates

$$
\begin{aligned}
(208)\quad & |w_{n_2+l_2;q}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k})| \\
&\leq (q-1)C_{11}(C_{12})^{n_2+l_2}(n_2+l_2)!q^{(\alpha+1)l_3}\left[(|q^{l_3}u|+u_0)(|u|+u_0)\right]^{k_1 l_3 \log(q)} \\
&\qquad \times |u|s_1^{1/k} \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)
\end{aligned}
$$

for all $u \in \mathcal{U} \cup \{0\}$ with $0 \leq s_1 \leq 1$.
b) Peering at the bounds (165) obtained in Proposition 9 for the special value $\beta = l_3$ and owing that $x \mapsto \log^2(x)$ and $x \mapsto \log(x)$ are increasing on $[1, +\infty)$, we deduce that

$$
\begin{aligned}
(209)\quad & |w_{n_2+l_2;1}(q^{l_3}us_1^{1/k}) - w_{n_2+l_2;1}(us_1^{1/k})| \leq |q^{l_3} - 1|C_9(C_{10})^{n_2+l_2}(n_2+l_2)! \\
&\qquad \times |us_1^{1/k}| \exp\left(k_1 \log^2(|us_1^{1/k}|+u_0) + \alpha \log(|us_1^{1/k}|+u_0)\right) \\
&\leq |q^{l_3} - 1|C_{11}(C_{12})^{n_2+l_2}(n_2+l_2)! \\
&\qquad \times |u|s_1^{1/k} \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)
\end{aligned}
$$

under the requirement that $C_{11} > C_9$ and $C_{12} > C_{10}$.
c) The lower bounds (55) sire a constant $M_{l_0,l_1,l_3,P} > 0$ such that

$$
(210)\quad \left|\frac{u^{l_0+kl_1}}{P(ku^k)}\left[(|q^{l_3}u|+u_0)(|u|+u_0)\right]^{k_1 l_3 \log(q)}\right| \leq M_{l_0,l_1,l_3,P}
$$

for all $u \in \mathcal{U} \cup \{0\}$, once (155) is granted.
d) We remind from (183) that a constant $L_{1,l_0,l_1,P} > 0$ can be singled out with

$$
(211)\quad \left|\frac{u^{l_0+kl_1}}{P(ku^k)}\right| \leq L_{1,l_0,l_1,P}
$$

for all $u \in \mathcal{U} \cup \{0\}$ under the condition (155) and in accordance with (162) in a raw with the fact that $x \mapsto \log^2(x)$ and $x \mapsto \log(x)$ are increasing on $[1, +\infty)$, we already know that

$$
\begin{aligned}
(212)\quad & |w_{n_2+l_2;1}(us_1^{1/k})| \\
&\leq C_7(C_8)^{n_2+l_2}(n_2+l_2)!|us_1^{1/k}| \exp\left(k_1 \log^2(|us_1^{1/k}|+u_0) + \alpha \log(|us_1^{1/k}|+u_0)\right) \\
&\leq C_{11}(C_{12})^{n_2+l_2}(n_2+l_2)!|u|s_1^{1/k} \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)
\end{aligned}
$$

holds whenever $u \in \mathcal{U} \cup \{0\}$ and $0 \leq s_1 \leq 1$ under the requisite that $C_{11} > C_7$ and $C_{12} > C_8$.

The compilation of the set of bounds (206), (208), (209), (210), (211) and (212) applied to (205) breeds the expected estimates (204). □

Leaning on the above to lemma 13 and 14, the recursion (194) grants the following estimates

$$(213)\quad |w_{n+S;q}(u) - w_{n+S;1}(u)| \leq \Big\{ \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0=0} \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| \\ \times \left[q^{l_3 k l_1} q^{(\alpha+1)l_3} M_{l_1,l_3,P} + q^{l_3 k l_1} K_{1,l_1,P} \frac{q^{l_3}-1}{q-1} + K_{1,l_1,P} \frac{q^{l_3 k l_1}-1}{q-1} \right] \\ \times (q-1) C_{11} (C_{12})^{n_2+l_2} n! \frac{(n_2+l_2)!}{n_2!} \\ + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0\geq 1} \sum_{n_1+n_2=n} |c_{\underline{l},n_1}| \frac{k^{l_1}}{\Gamma(l_0/k)} \\ \times \left[q^{l_3 k l_1} q^{(\alpha+1)l_3} M_{l_0,l_1,l_3,P} + q^{l_3 k l_1} L_{1,l_0,l_1,P} \frac{q^{l_3}-1}{q-1} + L_{1,l_0,l_1,P} \frac{q^{l_3 k l_1}-1}{q-1} \right] \\ \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (q-1) C_{11} (C_{12})^{n_2+l_2} n! \frac{(n_2+l_2)!}{n_2!} \Big\} \\ \times |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)$$

provided that $u \in \mathcal{U} \cup \{0\}$.

Lemma 2 and the assumption that $c_{\underline{l}}(z)$ are polynomials (given by (8)) allow further simplification of the above bounds

$$(214)\quad |w_{n+S;q}(u) - w_{n+S;1}(u)| \\ \leq \mathcal{B}(C_{12})(q-1)C_{11}(C_{12})^{n+S}(n+S)!|u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha \log(|u|+u_0)\right)$$

for all $u \in \mathcal{U} \cup \{0\}$, where

$$(215)\quad \mathcal{B}(C_{12}) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0=0} \sum_{g\in I_{\underline{l}}, 0\leq g\leq n} |c_{\underline{l},g}| \\ \times \left[q^{l_3 k l_1} q^{(\alpha+1)l_3} M_{l_1,l_3,P} + q^{l_3 k l_1} K_{1,l_1,P} \frac{q^{l_3}-1}{q-1} + K_{1,l_1,P} \frac{q^{l_3 k l_1}-1}{q-1} \right] (C_{12})^{-g} \\ + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}, l_0\geq 1} \sum_{g\in I_{\underline{l}}, 0\leq g\leq n} |c_{\underline{l},g}| \frac{k^{l_1}}{\Gamma(l_0/k)} \\ \times \left[q^{l_3 k l_1} q^{(\alpha+1)l_3} M_{l_0,l_1,l_3,P} + q^{l_3 k l_1} L_{1,l_0,l_1,P} \frac{q^{l_3}-1}{q-1} + L_{1,l_0,l_1,P} \frac{q^{l_3 k l_1}-1}{q-1} \right] \\ \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{12})^{-g}$$

Owing to the requirement that $0 \notin I_{\underline{l}}$, one can select $C_{12} > 0$ large enough (besides the already asked conditions that $C_{12} > C_{10}$ and $C_{12} > C_8$) in a way that

$$(216)\quad \mathcal{B}(C_{12}) \leq 1$$

The joint features (214) and (216) confirm the fact that the property $\mathfrak{D}_{n+S}^{;q;1}$ holds true. This completes the proof of Proposition 10. □

8.4 Confluence for the analytic solutions of the problem (17), (18) as $q \to 1$.

In this subsection, we state the third and last main result of the work.

Theorem 3 *Let $\underline{\mathcal{D}} = \{\underline{\mathcal{T}}, \underline{\mathcal{U}}\}$ be an admissible set of sectors as chosen in Definition 2. Let $\mathcal{U}$ be one sector belonging to the family of unbounded sectors $\underline{\mathcal{U}}$. We set $\mathcal{T}$ as one bounded sector of the set $\underline{\mathcal{T}}$ that is related to $\mathcal{U}$ under the requirement of 2) of Definition 2.*

We denote $u_{;q}(t,z)$ the bounded holomorphic solution to (17), (18) on the product $\mathcal{T} \times D_{\frac{1}{2C_4}}$, given by a Laplace transform of order k, see (131), constructed in Theorem 1. On the other hand, we consider the bounded holomorphic solution $u_{;1}(t,z)$ of the limit problem (153), (154) on the domain $\mathcal{T} \times D_{\frac{1}{2C_8}}$ expressed through a Laplace transform of order k, see (156), built up in Proposition 8.

Then, one can find a constant $C > 0$ (independent of $q \in (1, q_0]$) such that

$$\sup_{t \in \mathcal{T}, z \in D_{\frac{1}{2C_{12}}}} |u_{;q}(t,z) - u_{;1}(t,z)| \leq C(q-1) \tag{217}$$

for all $q \in (1, q_0]$. In particular, we observe that the solution $u_{;q}(t,z)$ of (17), (18) merges uniformly on $\mathcal{T} \times D_{\frac{1}{2C_{12}}}$ to the solution $u_{;1}(t,z)$ of (153), (154) as $q \to 1$.

Proof By construction, we can express both solutions $u_{;q}(t,z)$ and $u_{;1}(t,z)$ as Laplace transforms

$$u_{;q}(t,z) = k \int_{L_\gamma} w_{;q}(u,z) \exp\big(-(u/t)^k\big) du/u \quad , \quad u_{;1}(t,z) = k \int_{L_\gamma} w_{;1}(u,z) \exp\big(-(u/t)^k\big) du/u$$

along a halfline $L_\gamma = [0, +\infty)e^{\sqrt{-1}\gamma} \subset \mathcal{U} \cup \{0\}$ submitted to the condition

$$\cos\big(k(\gamma - \arg(t))\big) > \Delta \tag{218}$$

for some fixed constant $\Delta > 0$, for every $t \in \mathcal{T}$, where the *Borel map* $w_{;q}(u,z)$ is depicted in (192) and $w_{;1}(u,z)$ is displayed in (163).

According to the deviation bounds (193) between $w_{;q}$ and $w_{;1}$ reached in Proposition 10 and the lower bounds (218), we can control the difference $u_{;q} - u_{;1}$ as follows:

$$|u_{;q}(t,z) - u_{;1}(t,z)| = \left| k \int_{L_\gamma} \big(w_{;q}(u,z) - w_{;1}(u,z)\big) \exp\big(-(u/t)^k\big) du/u \right|$$
$$\leq 2kC_{11}(q-1) \int_0^{+\infty} \exp\big(k_1 \log^2(r+u_0) + \alpha \log(r+u_0)\big) \exp\big(-(r/r_{\mathcal{T}})^k \Delta\big) dr$$

for all $t \in \mathcal{T}$, all $z \in D_{\frac{1}{2C_{12}}}$, where $r_{\mathcal{T}}$ stands for the radius of the sector $\mathcal{T}$. This achieves the expected bounds (217). □

References

[1] L. Di Vizio, C. Zhang, *On q-summation and confluence.* Ann. Inst. Fourier (Grenoble) 59 (2009), no. 1, 347–392.

[2] T. Dreyfus, *Confluence of meromorphic solutions of q-difference equations.* Ann. Inst. Fourier (Grenoble) 65 (2015), no. 2, 431–507.

[3] P. Hsieh, Y. Sibuya, *Basic theory of ordinary differential equations.* Universitext. Springer-Verlag, New York, 1999. xii+468 pp.

[4] A. Lastra, S. Malek, *On multiscale Gevrey and q-Gevrey asymptotics for some linear q-difference differential initial value Cauchy problems.* J. Difference Equ. Appl. 23 (2017), no. 8, 1397–1457.

[5] A. Lastra, S. Malek, *On a q-analog of a singularly perturbed problem of irregular type with two complex time variables*, Mathematics 2019, 7(10), 924; 2019.

[6] A. Lastra, S. Malek, J. Sanz, *On q-asymptotics for linear q-difference-differential equations with Fuchsian and irregular singularities.* J. Differential Equations 252 (2012), no. 10, 5185–5216.

[7] S. Malek, *On a partial q-analog of a singularly perturbed problem with Fuchsian and irregular time singularities.* Abstr. Appl. Anal. 2020, Art. ID 7985298, 32 pp.

[8] D. Pravica, N. Randriampiry, M. Spurr, *Eigenfunction Families and Solution Bounds for Multiplicatively Advanced Differential Equations*, Axioms, 2020, 9(3), 83.

[9] J. Sauloy, *Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie. (French) [Regular singular q-difference systems: classification, connection matrix and monodromy]* Ann. Inst. Fourier (Grenoble) 50 (2000), no. 4, 1021–1071.

[10] H. Yamazawa, *Gevrey and $q-$Gevrey asymptotics for some linear $q-$difference-differential equations*, talk at RIMS in Kyoto, Japan, 2018. Work in preparation.