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Article

Separation Axioms in Fuzzy Closure Spaces

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Abstract: Fuzzy closure spaces are an extension of classical closure spaces in topology, where the concept of closure is defined in terms of fuzzy sets. This article introduces interior operators and neighborhood systems in fuzzy closure spaces. Using that, we have redefined ČF-continuity. Separation axioms such as ČFT₀, ČFT₁, and ČFT₂, ČF-Urysohn, ČF-regular, and ČF-normal in fuzzy closure spaces are introduced using these neighborhood systems. Additive, productive, hereditary, and other properties of these axioms have been observed. Relationships between these axioms are also investigated.

Keywords: fuzzy closure spaces; separation axioms; neighborhood systems; interior operator; ČF-continuity

MSC: 54A40; 54A05; 54D10; 54D15; 03E72

1. Introduction

The concept of a topological space is generally introduced in terms of the axioms of the open sets. However, there are different methods to describe a topology on the set X are often used in terms of neighborhood systems, the family of closed sets, the closure operator, the interior operator, etc. To each topological space (X, τ) we can associate a closure operator $c : P(X) \rightarrow P(X)$, where $c(A)$ is the smallest closed set which contains A for each $A \subseteq X$. This closure operator have the following properties, (i) $c(\phi) = \phi$ (ii) $A \subseteq c(A)$ (iii) c commutes with finite unions (iv) c is idempotent. Such a closure operator is known as a topological closure operator or Kuratowski's closure operator. In 1966, Edward Čech [1] introduced Čech closure operators by weakening the idempotent law of topological closure operators. A set X together with a Čech closure operator is called a Čech closure space. Thus Čech closure space can be considered as a generalization of topological spaces. Subsequently, Birkoff [2] defined closure operator in another way by weakening the finite additivity property of the topological closure operator.

By extending the idea of crisp sets, Zadeh [3] established the concept of fuzzy sets in 1965. Chang [4] later gave fuzzy sets a topological structure. Later, the concept of fuzzy closure operators on a set X was introduced separately by Mashhour, Ghanim [5] and Srivastava et al. [6]. The definition of Mashhour and Ghanim is an analogue of Čech closure operators. Srivastava et al. [6] have introduced fuzzy closure operators as an analogue of Birkoff closure operator [2]. Srivastava et al. [7] introduced T_0 and T_1 separation properties in their fuzzy closure spaces. In their definition of fuzzy closure spaces, most of the properties will coincide with the fuzzy topology associated with it. Even the continuous maps of fuzzy closure spaces will be the same as that of it's associated fuzzy topological spaces. So we are working on fuzzy closure spaces introduced by Mashhour and Ghanim [5].

In 1985, Mashhour and Ghanim [5] have introduced the fuzzy closure spaces and extended classical concepts such as subspaces, sums, and products. They have introduced ČF-continuity, compactness, and regularity and studied its properties. Also, they have investigated the interaction between fuzzy proximity spaces and fuzzy closure spaces. Johnson [8] studied completely homogenous fuzzy closure

spaces and lattices of fuzzy closure operators. In 2002, Sunil[9] studied generalised-closed sets, and well closed fuzzy points in fuzzy closure spaces. He paid some attention to separation axioms such as FT_1 and quasi separation. But according to his definition, a fuzzy closure space is FT_1 (respectively, quasi separated) if its associated fuzzy topology is FT_1 (respectively, quasi separated). Further, he defined static fuzzy closure spaces, D-fuzzy closure spaces, and adjacent fuzzy closure operators. In 2007, Bloomy Joseph[10] studied denseness in fuzzy closure spaces and fuzzy closure fuzzy convexity spaces. In 2014, Tapi and Deole ([11][12]) introduced various forms of connectedness in fuzzy closure spaces and fuzzy biclosure spaces. Rachna Navalakhe [13] introduced separation axioms such as Hausdorff, regular and normal in fuzzy closure spaces. They have used fuzzy open sets for the separation. So according to their definition, a fuzzy closure space satisfies a certain separation property if its associated fuzzy topology also posses the same property. Their definitions are not an extension of separation axioms in Čech closure spaces. Also, she has studied generalised-closed sets, generalised forms of separation axioms, δ -fuzzy closed sets, separation axioms using δ -fuzzy closed sets, and studied analogous concepts in fuzzy biclosure spaces.

In this paper, we have introduced the interior operator and hence the neighborhood systems in fuzzy closure spaces. We have defined the notion of ČF-continuity at a fuzzy point. While defining separation axioms on fuzzy closure spaces, we want to make it an extension of that of Čech closure spaces and fuzzy topological spaces. Many authors have introduced separation axioms in fuzzy topological spaces differently. In L-topology [14], it was introduced by using Q-neighborhoods. If we introduce separation axioms on fuzzy closure spaces using Q-neighborhoods, the closure operator will have no role. So we introduce it as an extension of separation properties of fuzzy topological spaces as introduced by Mashhur et al. [15]. Using the neighborhood systems, we have introduced separation properties such as ČFT₀, ČFT₁, ČFT₂, ČF-Urysohn, ČF-regular, and ČF-normal. Additive and productive properties of these separation axioms have been studied. All these separation properties are fuzzy closure properties. The relationships between these separation axioms are also discussed.

2. Preliminaries

This section contains some basic concepts and theorems about fuzzy topological spaces, Čech closure spaces, and fuzzy closure spaces which will be needed in the sequel. Throughout this paper X, Y, Z , etc. denotes non-empty ordinary sets, f, g, h , etc. denotes fuzzy sets defined on an ordinary set, and x, y, x^1, x^2, x^3 , etc. denotes elements of ordinary sets.

Definition 1. [1] A Čech closure operator (also called a Č-closure operator, or simply a closure operator) on a set X is a function $c : \{0, 1\}^X \rightarrow \{0, 1\}^X$ satisfying the following axioms:

- (i) $c(\emptyset) = \emptyset$,
- (ii) $A \subset c(A)$, for all $A \subseteq X$,
- (iii) $c(A \cup B) = c(A) \cup c(B)$, for all $A, B \subseteq X$.

The pair (X, c) is called a Čech closure space or simply a closure space.

If c satisfies idempotency also (i.e., $c(c(A)) = c(A)$, for all $A \subseteq X$), it will become a topological closure operator. Separation properties, connectedness, and compactness in Čech closure spaces are studied in [16].

Let X be an ordinary non-empty set. A fuzzy set f on X is a mapping from X to the closed unit interval $I = [0, 1]$ of the real line, associating each x of X with its membership value $f(x)$. The families $\{0, 1\}^X$ and I^X denote, respectively, the collections of all crisp subsets and all fuzzy subsets of X . The constant fuzzy sets $\underline{0}$ and $\underline{1}$ are given by assigning the value 0 and 1 to every element of X , respectively. The fuzzy set $\underline{0}$ represents the fuzzy empty set, while $\underline{1}$ corresponds to the fuzzy universal set. Let $A \subset X$, 1_A denotes the fuzzy subset of X having membership value 1 on elements of A and 0 otherwise. For any fuzzy set f in X , by $\mathcal{C}o(f)$ we mean the complement of f in X and is

defined as, $\mathfrak{C}o(f)(x) = 1 - f(x)$. Two fuzzy sets f and g of X are equal if $f(x) = g(x)$ for all $x \in X$. If $f(x) \leq g(x)$ for all $x \in X$, then f is said to be a subset of g .

Let X be a non-empty set. A fuzzy set f on X is defined as a mapping $f : X \rightarrow I = [0, 1]$, where $f(x)$ represents the degree of membership of each $x \in X$ in f . The families $\{0, 1\}^X$ and I^X denote, respectively, the collections of all crisp subsets and all fuzzy subsets of X . The constant fuzzy sets $\underline{0}$ and $\underline{1}$ are given by assigning the value 0 and 1 to every element of X , respectively. For any subset $A \subseteq X$, the characteristic fuzzy set 1_A is defined by $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise. For a fuzzy set f on X , the complement $\mathfrak{C}o(f)$ is defined by $\mathfrak{C}o(f)(x) = 1 - f(x)$ for all $x \in X$. Two fuzzy sets f and g are said to be equal if $f(x) = g(x)$ for every $x \in X$. Moreover, if $f(x) \leq g(x)$ for all $x \in X$, then f is regarded as a fuzzy subset of g .

Definition 2. [4] A fuzzy topology on a set X is a family F of fuzzy subsets of X that satisfies the following conditions:

- (i) $\underline{0}, \underline{1} \in F$,
- (ii) if $g, h \in F$, then $g \wedge h \in F$,
- (iii) if $g_i \in F$ for each $i \in I$, then $\bigvee_{i \in I} g_i \in F$.

The pair (X, F) is called a fuzzy topological space (abbreviated as fts). The elements of F are referred to as open fuzzy subsets of X , while their complements are called closed fuzzy subsets. A fuzzy topological space (X, F) with $F = \{\underline{0}, \underline{1}\}$ is said to be the indiscrete (or trivial) fuzzy topology, whereas if $F = [0, 1]^X$, it is referred to as the discrete fuzzy topology.

Definition 3. [4] Let f be a fuzzy subset of an fts (X, F) . The closure of f is defined to be the fuzzy subset $\bigwedge \{g : f \leq g, \mathfrak{C}o(g) \in F\}$ and is denoted by $\bar{f}$.

Definition 4. [4] Let X and Y be two sets and let $\theta : X \rightarrow Y$ be a function. Then, for any fuzzy subset g of X , $\theta(g)$ is a fuzzy subset in Y defined by

$$\theta(g)(y) = \begin{cases} \sup\{g(x) : x \in X, \theta(x) = y\}, & \text{if } \theta^{-1}(y) \neq \emptyset, \\ 0, & \text{if } \theta^{-1}(y) = \emptyset. \end{cases}$$

For a fuzzy subset h of Y , $\theta^{-1}(h)$ is a fuzzy subset of X defined by $\theta^{-1}(h)(x) = h(\theta(x))$.

Definition 5. [17] The fuzzy subset x_λ of X , with $x \in X$ and $0 < \lambda \leq 1$ defined by,

$$x_\lambda(y) = \begin{cases} \lambda, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

is called a fuzzy point in X with support x and value λ . Two fuzzy points with different supports are called distinct. When $\lambda = 1$, $x_\lambda = x_1$ is called a fuzzy singleton.

Definition 6. [18] Let (X, F) be a fuzzy topological space and x_λ be a fuzzy point in X . If $\overline{x_\lambda}$ is again a fuzzy point, it is said to be well closed.

Definition 7. [14] Let f be a fuzzy subset of X , its support is defined as $\text{supp}(f) = \{x : f(x) > 0\}$.

Definition 8. [15] A fuzzy topological space (X, F) is said to be FT_0 if for every pair of distinct fuzzy points x_λ and y_γ , there exist an $f \in F$ such that $x_\lambda \in f \leq \mathfrak{C}o(y_\gamma)$ or $y_\gamma \in f \leq \mathfrak{C}o(x_\lambda)$.

Definition 9. [15] A fuzzy topological space (X, F) is said to be FT_1 if and only if for every pair of distinct fuzzy points x_λ and y_γ , there exist open sets $f, g \in F$ such that $x_\lambda \in f \leq \mathfrak{C}o(y_\gamma)$ and $y_\gamma \in g \leq \mathfrak{C}o(x_\lambda)$.

Theorem 1. [19] A fuzzy topological space (X, F) is FT_1 if and only if every fuzzy singleton is closed.

Definition 10. [15] A fuzzy topological space (X, F) is said to be FT_s (strong FT_1) if every fuzzy point is closed.

Definition 11. [15] A fuzzy topological space (X, F) is FT_2 if for every pair of distinct fuzzy points x_λ and y_γ , there exists $f, g \in F$ such that $f \leq \mathfrak{Co}(g)$ and $x_\lambda \in f \leq \mathfrak{Co}(y_\gamma)$ and $y_\gamma \in g \leq \mathfrak{Co}(x_\lambda)$.

Definition 12. [15] A fuzzy topological space (X, F) with the closure operator c is said to be $FT_{2\frac{1}{2}}$ or fuzzy Urysohn if for every pair of distinct fuzzy points x_λ and y_γ , there exist $f, g \in F$ such that $c(f) \leq \mathfrak{Co}(c(g))$ and $x_\lambda \in f \leq \mathfrak{Co}(y_\gamma)$ and $y_\gamma \in g \leq \mathfrak{Co}(x_\lambda)$.

Definition 13. [15] A fuzzy topological space (X, F) is said to be fuzzy regular if for all fuzzy point x_λ , and every closed fuzzy set k in X such that $x_\lambda \in \mathfrak{Co}(k)$, there exist $f, g \in F$ such that $x_\lambda \in f$ and $k \leq g$ and $f \leq \mathfrak{Co}(g)$. A fuzzy regular space which is also FT_s is said to be FT_3 .

Definition 14. [15] A fuzzy topological space (X, F) is said to be fuzzy normal if for all closed fuzzy sets k_1, k_2 in X such that $k_1 \leq \mathfrak{Co}(k_2)$, there exists $f_1, f_2 \in F$ such that $k_1 \leq f_1$ and $k_2 \leq f_2$ and $f_1 \leq \mathfrak{Co}(f_2)$. A fuzzy normal space which is also FT_s is said to be FT_4 .

Definition 15. [5] A Čech fuzzy closure operator (abbreviated as ČF-closure operator, or simply a fuzzy closure operator) on a set X is a mapping $c : I^X \rightarrow I^X$ that satisfies the following axioms:

- (i) $c(\underline{0}) = \underline{0}$,
- (ii) $f \leq c(f)$ for all $f \in I^X$,
- (iii) $c(f \vee g) = c(f) \vee c(g)$ for all $f, g \in I^X$.

The pair (X, c) is called a Čech fuzzy closure space (or fuzzy closure space, abbreviated as fcs). If, in addition, c satisfies the idempotent property, i.e., $c(c(f)) = c(f)$ for all fuzzy subsets f of X , then c is said to be fuzzy topological.

Example 2. The fuzzy closure operator i defined by $i(\underline{0}) = \underline{0}$, $i(f) = \underline{1}$ for every non-empty fuzzy subset f of X , is called the indiscrete fuzzy closure operator.

Similarly, the fuzzy closure operator d defined by $d(f) = f$ for all fuzzy subsets f of X , is called the discrete fuzzy closure operator.

A Čech closure space (X, c) is said to be finitely generated[20] if $c(A) = \bigcup_{x \in A} c(x)$ for every. We can define an analogous concept in fuzzy closure spaces as follows.

Definition 16. An fcs (X, c) is said to be finitely generated if $c(f) = \bigvee_{x_\lambda \leq f} c(x_\lambda)$ for all $f \in I^X$.

The essential thing is that a finitely generated fuzzy closure operator is completely determined by its action on fuzzy points.

Definition 17. [5] Let (X, c) be a fuzzy closure space (fcs), the fts $(X, \tau(c))$, where $\tau(c) = \{f : c(\mathfrak{Co}(f)) = \mathfrak{Co}(f)\}$ is called the fts associated with (X, c) . A fuzzy subset f of X is said to be closed in (X, c) if $c(f) = f$.

Different fuzzy closure operators can associate the same fuzzy topology. For example, consider $\mathbb{Z}$ with a finitely generated fuzzy closure operator defined as $c(x_\lambda) = x_\lambda \vee (x+1)_\lambda$ for all $x \in \mathbb{Z}$, and the indiscrete fuzzy closure operator. Both fuzzy closure operators have the indiscrete fuzzy topology as their associated fuzzy topology.

Definition 18. [5] Let (X, c) be an fcs and A be an ordinary subset of X . The function $c_A : I^X \rightarrow I^X$ defined by $c_A(f) = 1_A \wedge c(f)$ is a ČF-closure operator on A . The corresponding pair (A, c_A) is said to be a ČF-closure subspace of (X, c) .

Definition 19. [5] Let c_1 and c_2 be two fuzzy closure operators on a set X . Then, the fcs (X, c_1) is said to be coarser than the fcs (X, c_2) if $c_2(f) \leq c_1(f)$ for all $f \in I^X$ and denoted by $c_1 \leq c_2$.

Definition 20. [5] Let (X, c) and (Y, d) be fuzzy closure spaces. A mapping $\theta : X \rightarrow Y$ is called ČF-continuous if $\theta(c(f)) \leq d(\theta(f))$, for all $f \in I^X$. If θ is bijective and both θ and θ^{-1} are ČF-continuous, then θ is termed a ČF-homeomorphism.

A fuzzy closure property refers to a structural property of fuzzy closure spaces that remains preserved under ČF-homeomorphisms.

Definition 21. [5] A fuzzy closure property is said to be hereditary; whenever a fuzzy closure space has that property, then so does every subspace of it.

Definition 22. [5] Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces and $X = \bigvee_{t \in T} X_t$. The function $\oplus : I^X \rightarrow I^X$, defined by, $\oplus c_t(f) = \bigvee_{t \in T} c_t(1_{X_t} \wedge f)$ is a ČF-closure operator on X . The pair $(X, \oplus c_t)$ is said to be the sum fuzzy closure space of the family $\mathcal{F}$.

Definition 23. [5] Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of fuzzy closure spaces and $X = \prod_{t \in T} X_t$. Define a function $\otimes c_t : I^X \rightarrow I^X$ as, for a fuzzy point x_λ , and a fuzzy set f in X , $x_\lambda \leq \otimes c_t(f)$ if the following condition is satisfied. $f = f_1 \vee f_2 \vee \dots \vee f_n$, ($f_i \in I^{X_t}$) implies there exists an i such that $P_t(x_\lambda) \leq c_t(P_t(f_i))$ for all t . The function $\otimes c_t$ is a ČF-closure operator on X . The pair $(X, \otimes c_t)$ is said to be the product fuzzy closure space of the family $\mathcal{F}$. Where P_t is the projection from X to X_t .

Theorem 3. [5] The product fcs is the coarsest fcs for which each projection is ČF-continuous.

3. Interior Operator and Neighborhood Systems

In this section, we introduce interior operators and neighborhood systems in fuzzy closure spaces and give some equivalent conditions for ČF-continuity and ČF-homeomorphism.

Definition 24. An interior operator ‘int’ is a function from I^X to itself defined by, $\text{int}(f) = \mathfrak{C}\mathfrak{o}(c(\mathfrak{C}\mathfrak{o}(f)))$, for each fuzzy subset f of X . A fuzzy subset f of X is said to be a neighborhood of x_λ if $x_\lambda \leq \text{int}(f)$, if so we say that x_λ is an interior point of f . Furthermore, for a fuzzy subset k of X , f is called a neighborhood of k if $k \leq \text{int}(f)$.

Mashhur and Ghanim[5] also used $f^* = \mathfrak{C}\mathfrak{o}(c(\mathfrak{C}\mathfrak{o}(f)))$, which is same as our interior operator for studying regularity and ČF-strongly continuous maps. But they haven’t identified it as an interior operator.

Example 4. We can easily find that the interior operator of indiscrete fcs is $\text{int}_i(f) = \underline{0}$ if $f \neq \underline{1}$, and $\text{int}_i(\underline{1}) = \underline{1}$. Similarly, the interior operator of discrete fcs is, $\text{int}_d(f) = f$ for all fuzzy sets f in X .

Example 5. Let $X = \{p, q, r\}$ and define a closure operator $c : I^X \rightarrow I^X$ on X by

$$c(f) = \begin{cases} \underline{0}, & \text{if } f = \underline{0}, \\ 1_{\{p,q\}}, & \text{if } \underline{0} < f \leq 1_{\{p\}}, \\ 1_{\{p,q,r\}}, & \text{otherwise.} \end{cases}$$

The associated interior operator $\text{int} : I^X \rightarrow I^X$ can be found as

$$\text{int}(f) = \begin{cases} \underline{1}, & \text{if } f = \underline{1} \\ 1_{\{r\}}, & \text{if } \underline{1} > f \geq 1_{\{q,r\}}, \\ 0, & \text{otherwise.} \end{cases}$$

Example 6. On $\mathbb{Z}$, define a fuzzy closure operator $c : I^{\mathbb{Z}} \rightarrow I^{\mathbb{Z}}$ as follows. For each fuzzy point $x_\lambda \in I^{\mathbb{Z}}$, set

$$c(x_\lambda) = 1_{\{x, x+1\}},$$

and for an arbitrary fuzzy subset $f \in I^{\mathbb{Z}}$, define

$$c(f) = \bigvee_{x_\lambda \leq f} c(x_\lambda).$$

Here we get, $\text{int}(1_{\{x, x+1\}}) = 1_{\{x\}}$, and it follows that every neighborhood of the fuzzy point x_λ necessarily contains $1_{\{x, x+1\}}$.

Remark 7. If the interior operator is given, we can find the closure operator as follows: $c(f) = \mathfrak{Co}(\text{int}(\mathfrak{Co}(f)))$.

Analogous to Čech closure spaces, the following results hold.

Theorem 8. Let (X, c) be a fuzzy closure space. Then, for all $f, g \in I^X$, the following properties of the fuzzy interior operator hold:

1. $\text{int}(\underline{1}) = \underline{1}$,
2. $\text{int}(f) \leq f$,
3. $\text{int}(f \wedge g) = \text{int}(f) \wedge \text{int}(g)$,
4. If $f \leq g$, then $\text{int}(f) \leq \text{int}(g)$,
5. A fuzzy subset f of X is open if and only if $\text{int}(f) = f$,
6. A fuzzy subset f of X is open if and only if f is a neighborhood of every fuzzy point $x_\lambda \in f$.

Proof. 1. By definition,

$$\text{int}(\underline{1}) = \mathfrak{Co}(c(\mathfrak{Co}(\underline{1}))) = \mathfrak{Co}(c(0)) = \mathfrak{Co}(0) = \underline{1}.$$

2. Since $\mathfrak{Co}(f) \leq c(\mathfrak{Co}(f))$, it follows that

$$\text{int}(f) = \mathfrak{Co}(c(\mathfrak{Co}(f))) \leq f.$$

3. Using the properties of the complement operator, we have

$$\mathfrak{Co}(f \vee g) = \mathfrak{Co}(f) \wedge \mathfrak{Co}(g), \quad \mathfrak{Co}(f \wedge g) = \mathfrak{Co}(f) \vee \mathfrak{Co}(g).$$

Hence,

$$\begin{aligned} \text{int}(f \wedge g) &= \mathfrak{Co}(c(\mathfrak{Co}(f \wedge g))) \\ &= \mathfrak{Co}(c(\mathfrak{Co}(f) \vee \mathfrak{Co}(g))) \\ &= \mathfrak{Co}(c(\mathfrak{Co}(f)) \vee c(\mathfrak{Co}(g))) \\ &= \mathfrak{Co}(c(\mathfrak{Co}(f))) \wedge \mathfrak{Co}(c(\mathfrak{Co}(g))) \\ &= \text{int}(f) \wedge \text{int}(g). \end{aligned}$$

4. If $f \leq g$, then $f \wedge g = f$. By property 3,

$$\text{int}(f) = \text{int}(f \wedge g) = \text{int}(f) \wedge \text{int}(g) \leq \text{int}(g).$$

5. Suppose f is open in (X, c) . Then $\mathfrak{C}\mathfrak{o}(f)$ is closed, i.e., $c(\mathfrak{C}\mathfrak{o}(f)) = \mathfrak{C}\mathfrak{o}(f)$. Thus,

$$\text{int}(f) = \mathfrak{C}\mathfrak{o}(c(\mathfrak{C}\mathfrak{o}(f))) = f.$$

Conversely, if $\text{int}(f) = f$, then

$$\mathfrak{C}\mathfrak{o}(c(\mathfrak{C}\mathfrak{o}(f))) = f.$$

Taking complements, we obtain $c(\mathfrak{C}\mathfrak{o}(f)) = \mathfrak{C}\mathfrak{o}(f)$. Hence, $\mathfrak{C}\mathfrak{o}(f)$ is closed, and f is open.

6. If f is open, then $\text{int}(f) = f$, so f is a neighborhood of each fuzzy point contained in it. Conversely, suppose f is a neighborhood of every fuzzy point $x_\lambda \in f$. Then, we have

$$x_\lambda \leq \text{int}(f) \leq f \quad \text{for all } x_\lambda \in f,$$

which implies,

$$f = \bigvee_{x_\lambda \in f} x_\lambda \leq \text{int}(f) \leq f.$$

Therefore, $\text{int}(f) = f$, and hence f is open. $\square$

Here we introduce the notion of ČF-continuity at a fuzzy point and subsequently employ it to derive an equivalent criterion for ČF-continuity.

Definition 25. A function $\theta : (X, c) \rightarrow (Y, d)$ is said to be ČF-continuous at a fuzzy point x_λ if $x_\lambda \in c(f)$ for some fuzzy subset f of X , then $\theta(x_\lambda) \leq d(\theta(f))$.

Theorem 9. A function $\theta : (X, c) \rightarrow (Y, d)$ is ČF-continuous if and only if it is ČF-continuous at every fuzzy point of X .

Proof. ($\Rightarrow$) Suppose θ is ČF-continuous. Then, by definition,

$$\theta(c(f)) \leq d(\theta(f)), \quad \forall f \in I^X.$$

Let x_λ be a fuzzy point in X . If $x_\lambda \in c(f)$, then

$$\theta(x_\lambda) \in \theta(c(f)) \leq d(\theta(f)).$$

Thus, θ is ČF-continuous at the fuzzy point x_λ .

($\Leftarrow$) Conversely, assume that θ is ČF-continuous at every fuzzy point of X . For any fuzzy subset $f \in I^X$, we have

$$\theta(c(f)) = \bigvee_{x_\lambda \in c(f)} \theta(x_\lambda).$$

Since θ is ČF-continuous at each fuzzy point, it follows that $\theta(x_\lambda) \in d(\theta(f))$ whenever $x_\lambda \in c(f)$. Consequently,

$$\theta(c(f)) = \bigvee_{x_\lambda \in c(f)} \theta(x_\lambda) \leq d(\theta(f)).$$

Hence, θ is ČF-continuous. $\square$

Another characterization theorem for ČF-continuous functions can be found as follows:

Theorem 10. Let (X, c) and (Y, d) be fuzzy closure spaces. A mapping $\theta : (X, c) \rightarrow (Y, d)$ is ČF-continuous if and only if

$$c(\theta^{-1}(g)) \leq \theta^{-1}(d(g)), \quad \text{for all } g \in I^Y.$$

Proof. ($\Rightarrow$) Suppose that $\theta : (X, c) \rightarrow (Y, d)$ is ČF-continuous. By definition, we have

$$\theta(c(f)) \leq d(\theta(f)), \quad \text{for all } f \in I^X.$$

Taking $f = \theta^{-1}(g)$ for an arbitrary $g \in I^Y$, it follows that

$$\theta(c(\theta^{-1}(g))) \leq d(\theta(\theta^{-1}(g))) \leq d(g).$$

Applying θ^{-1} to both sides yields

$$c(\theta^{-1}(g)) \leq \theta^{-1}(d(g)), \quad \forall g \in I^Y.$$

($\Leftarrow$) Conversely, assume that

$$c(\theta^{-1}(g)) \leq \theta^{-1}(d(g)), \quad \forall g \in I^Y.$$

Let $f \in I^X$, and set $g = \theta(f)$ and $f_1 = \theta^{-1}(g)$. Then

$$c(f_1) = c(\theta^{-1}(g)) \leq \theta^{-1}(d(g)).$$

Since $f \leq f_1$, we obtain

$$\theta(c(f)) \leq \theta(c(f_1)) \leq \theta(\theta^{-1}(d(g))) = d(g) = d(\theta(f)).$$

Therefore, θ is ČF-continuous. $\square$

Corollary 11. If $\theta : (X, c) \rightarrow (Y, d)$ is ČF-continuous, then the inverse image of each open (respectively, closed) fuzzy subset of Y is open (respectively, closed) in X .

Proof. Let g be a closed fuzzy subset of (Y, d) , i.e., $d(g) = g$. By the above theorem, we have

$$c(\theta^{-1}(g)) \leq \theta^{-1}(d(g)) = \theta^{-1}(g).$$

Thus, $c(\theta^{-1}(g)) = \theta^{-1}(g)$, showing that $\theta^{-1}(g)$ is closed in (X, c) . By duality, the result also holds for open fuzzy subsets. $\square$

Remark 12. The converse of the preceding corollary does not hold in general. Consider the set $X = \{p, q, r, s\}$ equipped with the fuzzy closure operator c defined by

$$c(f) = \begin{cases} \underline{0}, & f = \underline{0}, \\ 1_{\{p,q\}}, & \underline{0} < f \leq 1_{\{p\}}, \\ 1_{\{q,r\}}, & \underline{0} < f \leq 1_{\{q\}}, \\ 1_{\{r,s\}}, & \underline{0} < f \leq 1_{\{r\}}, \\ 1_{\{s,p\}}, & \underline{0} < f \leq 1_{\{s\}}, \\ \bigvee_{x_\lambda \in f} c(x_\lambda), & \text{otherwise.} \end{cases}$$

Define a mapping $\theta : (X, c) \rightarrow (X, c)$ by

$$\theta(p) = q, \quad \theta(q) = r, \quad \theta(r) = s, \quad \theta(s) = p.$$

Since the fuzzy topology associated with (X, c) is the indiscrete fuzzy topology, the inverse image of every open set under θ is again open.

However, consider $f = 1_{\{q\}}$. Then

$$\theta(c(1_{\{q\}})) = \theta(1_{\{q,r\}}) = 1_{\{p,r,s\}},$$

while

$$c(\theta(1_{\{q\}})) = c(1_{\{r\}}) = 1_{\{r,s\}}.$$

Thus,

$$\theta(c(1_{\{q\}})) \not\leq c(\theta(1_{\{q\}})),$$

showing that θ fails to be ČF-continuous.

Theorem 13. Let (X, c) and (Y, d) be fuzzy closure spaces. A bijective function $\theta : (X, c) \rightarrow (Y, d)$ is a ČF-homeomorphism if and only if $\theta(c(f)) = d(\theta(f))$, for all $f \in I^X$.

Proof. ($\Rightarrow$) Assume that $\theta : (X, c) \rightarrow (Y, d)$ is a ČF-homeomorphism. Then, by definition,

$$\theta(c(f)) \leq d(\theta(f)) \quad \text{and} \quad \theta^{-1}(d(g)) \leq c(\theta^{-1}(g)),$$

for all $f \in I^X$ and $g \in I^Y$. Substituting $g = \theta(f)$, we obtain

$$\theta^{-1}(d(\theta(f))) \leq c(\theta^{-1}(\theta(f))) = c(f),$$

which implies

$$d(\theta(f)) \leq \theta(c(f)), \quad \text{for all } f \in I^X.$$

Combining this with $\theta(c(f)) \leq d(\theta(f))$, we conclude that

$$\theta(c(f)) = d(\theta(f)), \quad \text{for all } f \in I^X.$$

($\Leftarrow$) Conversely, suppose that $\theta(c(f)) = d(\theta(f))$ for all $f \in I^X$. Clearly, $\theta(c(f)) \leq d(\theta(f))$ for all $f \in I^X$, hence θ is ČF-continuous. Also we have $d(\theta(f)) \leq \theta(c(f))$, replacing f by $\theta^{-1}(g)$, we get

$$d(\theta(\theta^{-1}(g))) \leq \theta(c(\theta^{-1}(g))),$$

which simplifies to

$$d(g) \leq \theta(c(\theta^{-1}(g))), \quad \text{for all } g \in I^Y.$$

Therefore,

$$\theta^{-1}(d(g)) \leq c(\theta^{-1}(g)), \quad \text{for all } g \in I^Y,$$

showing that θ^{-1} is also ČF-continuous. Thus, θ is a ČF-homeomorphism. $\square$

4. ČFT₀ Fuzzy Closure Spaces

Definition 26. A fuzzy closure space (X, c) is ČFT₀ if for every two distinct fuzzy points x_λ and y_γ , either $x_\lambda \in \mathfrak{Co}(c(y_\gamma))$ or $y_\gamma \in \mathfrak{Co}(c(x_\lambda))$.

It can be noted that if c is a fuzzy topological closure operator, then the fuzzy closure space (X, c) satisfies the ČFT₀ axiom if and only if the associated fuzzy topological space $(X, \tau(c))$ is FT₀.

Remark 14. Every subspace of a $\check{CFT}_0$ space is $\check{CFT}_0$. (That is, $\check{CFT}_0$ is a hereditary property.)

Remark 15. Let c_1 and c_2 be two fuzzy closure operators on X such that $c_1 \leq c_2$. If (X, c_1) is $\check{CFT}_0$, then (X, c_2) is also $\check{CFT}_0$.

Theorem 16. If $(X, \tau(c))$ is FT_0 , then (X, c) is $\check{CFT}_0$.

Proof. Assume that $(X, \tau(c))$ is FT_0 , and let $x_\lambda, y_\gamma \in I^X$ be two fuzzy points. Since $(X, \tau(c))$ is FT_0 , there exists an $f \in \tau(c)$ such that

$$x_\lambda \in f \quad \text{and} \quad f \leq \mathfrak{Co}(y_\gamma).$$

This yields

$$\mathfrak{Co}(x_\lambda) \geq \mathfrak{Co}(f) \geq y_\gamma.$$

Moreover, as f is open, $\mathfrak{Co}(f)$ is closed. Thus, if $y_\gamma \leq \mathfrak{Co}(f)$, then

$$y_\gamma \leq c(y_\gamma) \leq \mathfrak{Co}(f).$$

Consequently,

$$\mathfrak{Co}(x_\lambda) \geq \mathfrak{Co}(f) \geq c(y_\gamma) \geq y_\gamma.$$

From this, we can deduce that $x_\lambda \in \mathfrak{Co}(c(y_\gamma))$. Hence (X, c) satisfies the $\check{CFT}_0$ property. $\square$

Remark 17. The converse of the preceding theorem does not hold in general. Indeed, consider the following example:

Let $X = \{x, y, z\}$ and define a finitely generated fuzzy closure operator c on X by

$$c(\underline{0}) = \underline{0}, \quad c(x_\lambda) = 1_{\{x, y\}}, \quad c(y_\gamma) = 1_{\{y, z\}}, \quad c(z_\rho) = 1_{\{z, x\}} \quad \text{for all } 0 < \lambda, \gamma, \rho \leq 1,$$

and for any other fuzzy subset f of X , define

$$c(f) = \bigvee_{u_\lambda \in f} c(u_\lambda).$$

It is straightforward to verify that c is a fuzzy closure operator, although it is not induced by any fuzzy topology. Indeed, we observe that

$$x_\lambda \in \mathfrak{Co}(c(y_\gamma)) = x_1, \quad z_\rho \in \mathfrak{Co}(c(x_\lambda)) = z_1, \quad y_\gamma \in \mathfrak{Co}(c(z_\rho)) = y_1.$$

Hence, the fuzzy closure space (X, c) satisfies the $\check{CFT}_0$ separation axiom. However, $\tau(c) = \{\underline{0}, \underline{1}\}$, the indiscrete fuzzy topology, which is not FT_0 .

Theorem 18. The property $\check{CFT}_0$ is preserved under $\check{C}F$ -Homeomorphisms; that is, $\check{CFT}_0$ is a fuzzy closure property.

Proof. Let $\theta : (X, c) \rightarrow (Y, d)$ be a $\check{C}F$ -homeomorphism, and assume that (X, c) is $\check{CFT}_0$. Consider two distinct fuzzy points y_λ^1 and y_γ^2 in (Y, d) . Then there exist $x^1, x^2 \in X$ such that $\theta(x^1) = y^1$ and $\theta(x^2) = y^2$.

Since (X, c) is $\check{CFT}_0$, we may assume that $x_\lambda^1 \in \mathfrak{Co}_X(c(x_\gamma^2))$. Applying θ , we obtain

$$y_\lambda^1 \in \theta(\mathfrak{Co}_X(c(x_\gamma^2))) = \mathfrak{Co}_Y(\theta(c(x_\gamma^2))) = \mathfrak{Co}_Y(d(\theta(x_\gamma^2))) = \mathfrak{Co}_Y(d(y_\gamma^2)).$$

Thus, (Y, d) also satisfies the $\check{CFT}_0$ condition. Therefore, $\check{CFT}_0$ is a fuzzy closure property. $\square$

Theorem 19. A fuzzy closure space (X, c) is $\check{CFT}_0$ if and only if for every pair of distinct fuzzy points x_λ and y_γ , there exists an $f \in I^X$ such that either

$$x_\lambda \in \text{int}(f) \text{ and } y_\gamma \in \mathfrak{Co}(f), \text{ or } y_\gamma \in \text{int}(f) \text{ and } x_\lambda \in \mathfrak{Co}(f).$$

Proof. $(\Rightarrow)$ Suppose (X, c) is $\check{CFT}_0$, and let x_λ and y_γ be two distinct fuzzy points in X . By the $\check{CFT}_0$ property, we may assume

$$x_\lambda \in \mathfrak{Co}(c(y_\gamma)) = \mathfrak{Co}(c(\mathfrak{Co}(\mathfrak{Co}(y_\gamma)))).$$

Let $f = \mathfrak{Co}(y_\gamma)$. Then $y_\gamma \in \mathfrak{Co}(f)$, and since $x_\lambda \in \mathfrak{Co}(c(\mathfrak{Co}(f))) = \text{int}(f)$, we obtain the desired condition.

$(\Leftarrow)$ Conversely, assume that for distinct fuzzy points x_λ and y_γ there exists $f \in I^X$ such that $x_\lambda \in \text{int}(f)$ and $y_\gamma \in \mathfrak{Co}(f)$. Since $y_\gamma \leq \mathfrak{Co}(f)$, we have $c(y_\gamma) \leq c(\mathfrak{Co}(f))$ which implies $\mathfrak{Co}(c(y_\gamma)) \geq \mathfrak{Co}(c(\mathfrak{Co}(f)))$. But $\text{int}(f) = \mathfrak{Co}(c(\mathfrak{Co}(f)))$, and hence

$$x_\lambda \in \text{int}(f) \leq \mathfrak{Co}(c(y_\gamma)).$$

Therefore, the $\check{CFT}_0$ condition holds. $\square$

The following theorems show the additive and productive properties of the $\check{CFT}_0$ separation axiom.

Theorem 20. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces. Let $(X, c) = (\bigvee_{t \in T} X_t, \oplus c_t)$ denote their sum. Then (X, c) is $\check{CFT}_0$ if and only if each (X_t, c_t) is $\check{CFT}_0$.

Proof. $(\Rightarrow)$ Assume that each (X_t, c_t) is $\check{CFT}_0$. Let x_λ and y_γ be two distinct fuzzy points in I^X . We consider two cases:

Case (i): Suppose $x, y \in X_{t_1}$ for some $t_1 \in T$. Since (X_{t_1}, c_{t_1}) is $\check{CFT}_0$, either

$$x_\lambda \in \mathfrak{Co}_{X_{t_1}}(c_{t_1}(y_\gamma)) \text{ or } y_\gamma \in \mathfrak{Co}_{X_{t_1}}(c_{t_1}(x_\lambda)).$$

Now, note that

$$c(x_\lambda) = \oplus c_t(x_\lambda) = \bigvee_{t \in T} c_t(x_\lambda \wedge 1_{X_t}) = c_{t_1}(x_\lambda),$$

and similarly $c(y_\gamma) = c_{t_1}(y_\gamma)$. Hence, it follows that either

$$x_\lambda \in \mathfrak{Co}_X(c(y_\gamma)) \text{ or } y_\gamma \in \mathfrak{Co}_X(c(x_\lambda)).$$

Case (ii): Suppose $x \in X_{t_1}$ and $y \in X_{t_2}$ with $t_1 \neq t_2$. Then, by construction,

$$x_\lambda \in \mathfrak{Co}(1_{X_{t_2}}) \leq \mathfrak{Co}(c(y_\gamma)) \text{ and } y_\gamma \in \mathfrak{Co}(1_{X_{t_1}}) \leq \mathfrak{Co}(c(x_\lambda)).$$

Thus, (X, c) is $\check{CFT}_0$.

$(\Leftarrow)$ Conversely, suppose that (X, c) is $\check{CFT}_0$. Let x_λ, y_γ be two distinct fuzzy points in $I^{X_{t_1}}$ for some $t_1 \in T$. Considering x_λ and y_γ as fuzzy points of (X, c) , the $\check{CFT}_0$ property ensures that

$$x_\lambda \in \mathfrak{Co}(c(y_\gamma)) \text{ or } y_\gamma \in \mathfrak{Co}(c(x_\lambda)).$$

Since

$$c(x_\lambda) = \oplus c_t(x_\lambda) = c_{t_1}(x_\lambda), \quad c(y_\gamma) = \oplus c_t(y_\gamma) = c_{t_1}(y_\gamma),$$

it follows that either

$$x_\lambda \in \mathfrak{Co}_{X_{t_1}}(c_{t_1}(y_\gamma)) \text{ or } y_\gamma \in \mathfrak{Co}_{X_{t_1}}(c_{t_1}(x_\lambda)).$$

Therefore, (X_{t_1}, c_{t_1}) is $\check{\text{CFT}}_0$ for all $t_1 \in T$. $\square$

Lemma 21. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of fuzzy closure spaces, and let $(X, c) = (\prod_{t \in T} X_t, \otimes c_t)$ denote their product. For $x \in \prod_{t \in T} X_t$, let x^t denote its t^{th} coordinate. Then, for any fuzzy point x_λ , its closure in the product space is given by

$$\otimes c_t(x_\lambda) = \prod_{t \in T} c_t(x_\lambda^t).$$

Proof. Suppose $y_\gamma \in \otimes c_t(x_\lambda)$. By Definition 23, this implies that for every $t \in T$ we have

$$P_t(y_\gamma) = y_\gamma^t \in c_t(x_\lambda^t).$$

Hence,

$$y_\gamma \in \prod_{t \in T} c_t(x_\lambda^t).$$

Conversely, assume $z_\rho \in \prod_{t \in T} c_t(x_\lambda^t)$. Then, for each $t \in T$, we have

$$z_\rho^t \in c_t(x_\lambda^t).$$

If x_λ can be expressed as a finite join, $x_\lambda = f_1 \vee f_2 \vee \cdots \vee f_n$, then, since the join is finite, at least one $f_i = x_\lambda$. Consequently,

$$P_t(z_\rho) = z_\rho^t \in c_t(x_\lambda^t) = c_t(P_t(f_i)) \quad \text{for all } t \in T.$$

Therefore,

$$z_\rho \in \otimes c_t(x_\lambda).$$

Thus, we conclude that

$$\otimes c_t(x_\lambda) = \prod_{t \in T} c_t(x_\lambda^t). \quad \square$$

Theorem 22. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of fuzzy closure spaces. Then the product space $(X, c) = (\prod_{t \in T} X_t, \otimes c_t)$ is $\check{\text{CFT}}_0$ whenever each factor space (X_t, c_t) is $\check{\text{CFT}}_0$.

Proof. Assume that each (X_t, c_t) is $\check{\text{CFT}}_0$. Suppose, for the sake of contradiction, that the product space (X, c) is not $\check{\text{CFT}}_0$. Then there exist distinct fuzzy points $x_\lambda, y_\gamma \in X$ such that

$$x_\lambda \notin \mathfrak{Co}_X(\otimes c_t(y_\gamma)) \quad \text{and} \quad y_\gamma \notin \mathfrak{Co}_X(\otimes c_t(x_\lambda)).$$

This implies that

$$\lambda > 1 - \otimes c_t(y_\gamma)(x),$$

and hence

$$\otimes c_t(y_\gamma)(x) > 1 - \lambda.$$

From Lemma 21, we have $\otimes c_t(y_\gamma) = \prod_{t \in T} c_t(y_\gamma^t)$, which implies that

$$c_t(y_\gamma^t)(x^t) > 1 - \lambda \quad \text{for all } t \in T.$$

Equivalently,

$$\lambda > 1 - c_t(y_\gamma^t)(x^t) \quad \text{for all } t \in T,$$

which yields

$$x_\lambda^t \notin \mathfrak{Co}_{X_t}(c_t(y_\gamma^t)) \quad \text{for all } t \in T.$$

By a similar argument, we also obtain

$$y_\gamma^t \notin \mathfrak{Co}_{X_t}(c_t(x_\lambda^t)) \quad \text{for all } t \in T.$$

This contradicts the assumption that each (X_t, c_t) is $\check{\text{CFT}}_0$. Therefore, the assumption that (X, c) is not $\check{\text{CFT}}_0$ must be false. Hence, the product space $(\prod_{t \in T} X_t, \otimes c_t)$ is $\check{\text{CFT}}_0$. $\square$

5. $\check{\text{CFT}}_1$ Fuzzy Closure Spaces

Definition 27. A fuzzy closure space is $\check{\text{CFT}}_1$ if for every two distinct fuzzy points $x_\lambda, y_\gamma \in I^X$, $x_\lambda \in \mathfrak{Co}(c(y_\gamma))$ and $y_\gamma \in \mathfrak{Co}(c(x_\lambda))$.

Clearly, every FT_1 fuzzy topological space can be viewed as a $\check{\text{CFT}}_1$ fuzzy closure space.

Remark 23. The $\check{\text{CFT}}_1$ separation property is hereditary. Additionally, if c_1 and c_2 are fuzzy closure operators on X satisfying $c_1 \leq c_2$, then (X, c_1) being $\check{\text{CFT}}_1$ implies that (X, c_2) is also $\check{\text{CFT}}_1$.

Theorem 24. A fuzzy closure space (X, c) is $\check{\text{CFT}}_1$ if and only if every fuzzy point is well closed.

Proof. $(\Rightarrow)$ Assume that (X, c) is $\check{\text{CFT}}_1$. Suppose, to the contrary, that there exists a fuzzy point $x_\lambda \in I^X$ which is not well closed. Then, there exists some $y \in X$, with $y \neq x$, such that $y_\gamma \in c(x_\lambda)$ for some $\gamma \in (0, 1]$. Consider the fuzzy points x_λ and y_1 . Clearly, $y_1 \notin \mathfrak{Co}(c(x_\lambda))$, which contradicts the assumption that (X, c) is $\check{\text{CFT}}_1$. Hence, the assumption is false, and every fuzzy point x_λ must be well closed.

$(\Leftarrow)$ Conversely, assume that every fuzzy point is well closed. Then, for all distinct $x, y \in X$ and for all $0 < \lambda, \gamma \leq 1$, we have $x_\lambda \notin c(y_\gamma)$. Consequently, $x_\lambda \in \mathfrak{Co}(c(y_\gamma))$ and $y_\gamma \in \mathfrak{Co}(c(x_\lambda))$. Therefore, (X, c) is a $\check{\text{CFT}}_1$ fcs. $\square$

Lemma 25. In a fuzzy closure space (X, c) , every fuzzy point is well closed if and only if every fuzzy singleton is closed.

Proof. $(\Rightarrow)$ Assume that every fuzzy point in (X, c) is well closed. In particular, each fuzzy singleton $x_1 \in X$ is well closed, which implies $c(x_1) = x_1$.

$(\Leftarrow)$ Conversely, suppose $c(x_1) = x_1$ for every $x \in X$. Then, for any fuzzy point x_λ with $\lambda \in (0, 1]$, we have $c(x_\lambda) \leq c(x_1) = x_1$, which shows that x_λ is well closed. $\square$

Theorem 26. A fuzzy closure space (X, c) is $\check{\text{CFT}}_1$ if and only if every fuzzy singleton in X is closed.

Proof. The result follows immediately from Theorem 24 and Lemma 25. $\square$

As a direct consequence of the above characterization, we obtain the following result.

Theorem 27. A fuzzy closure space (X, c) is $\check{\text{CFT}}_1$ if and only if its associated fuzzy topological space $(X, \tau(c))$ is FT_1 .

Proof. This statement follows directly from Theorem 1 and Theorem 26. $\square$

Definition 28. A fuzzy closure space is said to be $\check{\text{CFT}}_s$ (or strongly $\check{\text{CFT}}_1$) if every fuzzy point is closed.

It is immediate that every $\check{CFT}_s$ space is also $\check{CFT}_1$. In the setting of Čech closure spaces, every finite T_1 space is discrete. An analogous result in the context of fuzzy closure spaces is stated below.

Theorem 28. *Every finite $\check{CFT}_s$ fuzzy closure space is discrete.*

Proof. Straightforward. $\square$

Remark 29. *Every $\check{CFT}_1$ fuzzy closure space is $\check{CFT}_0$. However, the converse does not hold in general, as demonstrated by the following counterexample.*

Consider the set of natural numbers $\mathbb{N}$ equipped with the fuzzy closure operator c defined by $c(\underline{0}) = \underline{0}$, $c(x_\lambda) = x_\lambda \vee (x+1)_\lambda$, and $c(f) = \bigvee_{x_\lambda \in f} c(x_\lambda)$, for other fuzzy subsets f of $\mathbb{N}$. If $y = x+1$, then

$$x_\lambda \in \mathfrak{Co}(c(x_\lambda))(y_\gamma) = 1_{\mathbb{N} \setminus \{x+1, x+2\}} \vee (x+1)_{1-\gamma} \vee (x+2)_{1-\gamma}.$$

On the other hand, if $y \notin \{x, x+1\}$, we obtain

$$y_\gamma \in \mathfrak{Co}(c(x_\lambda)) = 1_{\mathbb{N} \setminus \{x, x+1\}} \vee (x)_{1-\lambda} \vee (x+1)_{1-\lambda}.$$

From this, it follows that $(\mathbb{N}, c)$ satisfies the T_0 condition. However, every fuzzy point x_λ in $(\mathbb{N}, c)$ is not well-closed. Therefore, $(\mathbb{N}, c)$ is not $\check{CFT}_1$.

Theorem 30. *The property $\check{CFT}_1$ is a fuzzy closure property.*

Proof. Let (X, c) and (Y, d) be two fuzzy closure spaces, and let $\theta : (X, c) \rightarrow (Y, d)$ be a ČF-homeomorphism. Suppose y_1 is a fuzzy singleton in Y . Then there exists an $x \in X$ such that $\theta(x_1) = y_1$. Since θ is a homeomorphism, we have

$$y_1 = \theta(x_1) = \theta(c(x_1)) = d(\theta(x_1)) = d(y_1).$$

Thus, y_1 is closed in (Y, d) . Hence, if (X, c) is $\check{CFT}_1$, then (Y, d) is also $\check{CFT}_1$. Therefore, $\check{CFT}_1$ is preserved under ČF-homeomorphisms, and consequently it is a fuzzy closure property. $\square$

Theorem 31. *A fuzzy closure space (X, c) is $\check{CFT}_1$ if and only if for every pair of distinct fuzzy points x_λ and y_γ , there exists $f, g \in I^X$ such that $x_\lambda \in \text{int}(f)$ and $y_\gamma \in \mathfrak{Co}(f)$ and $y_\gamma \in \text{int}(g)$ and $x_\lambda \in \mathfrak{Co}(g)$.*

Proof. Similar proof as in the case of $\check{CFT}_0$ fcs. $\square$

In the following theorems, we prove additive and productive properties of the $\check{CFT}_1$ separation axiom in fuzzy closure spaces.

Theorem 32. *Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces. Then their sum $(\bigvee_{t \in T} X_t, \oplus c_t)$ is $\check{CFT}_1$ if and only if each (X_t, c_t) is $\check{CFT}_1$.*

Proof. $(\Rightarrow)$ Suppose that $(\bigvee_{t \in T} X_t, \oplus c_t)$ is $\check{CFT}_1$. Let x_1 be a fuzzy singleton in X_{t_1} , for some $t_1 \in T$. Since $(\bigvee_{t \in T} X_t, \oplus c_t)$ is $\check{CFT}_1$, treating x_1 as a fuzzy singleton of $\bigvee_{t \in T} X_t$, we have $\oplus c_t(x_1) = x_1$. By the definition of the sum closure operator,

$$\oplus c_t(x_1) = \bigvee_{t \in T} c_t(x_1 \wedge 1_{X_t}) = c_{t_1}(x_1).$$

Thus, $c_{t_1}(x_1) = x_1$, which shows that (X_{t_1}, c_{t_1}) is $\check{CFT}_1$. Since $t_1 \in T$ was arbitrary, it follows that each (X_t, c_t) is $\check{CFT}_1$.

($\Leftarrow$) Conversely, suppose that each (X_t, c_t) is $\check{\text{CFT}}_1$ for all $t \in T$. Let x_1 be a fuzzy singleton in $\bigvee_{t \in T} X_t$. Then there exists a unique $t_1 \in T$ such that $x \in X_{t_1}$. Since (X_{t_1}, c_{t_1}) is $\check{\text{CFT}}_1$, we have

$$c_{t_1}(x_1) = x_1.$$

Therefore,

$$\oplus c_t(x_1) = \bigvee_{t \in T} c_t(x_1 \wedge 1_{X_t}) = c_{t_1}(x_1) = x_1.$$

Hence, the sum space $(\bigvee_{t \in T} X_t, \oplus c_t)$ is $\check{\text{CFT}}_1$. $\square$

Theorem 33. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of fuzzy closure spaces. Then their product $(X, c) = (\prod_{t \in T} X_t, \otimes c_t)$ is $\check{\text{CFT}}_1$ if and only if each factor space (X_t, c_t) is $\check{\text{CFT}}_1$.

Proof. ($\Rightarrow$) Suppose that each (X_t, c_t) is $\check{\text{CFT}}_1$. Let $x_\lambda \in I^X$. We must show that x_λ is well closed in (X, c) . If $y_\gamma \in \otimes c_t(x_\lambda)$, then by the definition of the product fuzzy closure, this means

$$P_t(y_\gamma) \in c_t(P_t(x_\lambda)) \quad \text{for all } t \in T,$$

that is,

$$y_\gamma^t \in c_t(x_\lambda^t) \quad \text{for all } t \in T.$$

Since (X_t, c_t) is $\check{\text{CFT}}_1$, it follows that,

$$y_\gamma^t \in c_t(x_\lambda^t) \leq x_\lambda^t,$$

which implies $y^t = x^t$ for all $t \in T$. Hence, $y = x$, and therefore x_λ is well closed. Thus, (X, c) is $\check{\text{CFT}}_1$.

($\Leftarrow$) Conversely, assume that $(X, c) = (\prod_{t \in T} X_t, \otimes c_t)$ is $\check{\text{CFT}}_1$. Suppose, for contradiction, that there exists $t_1 \in T$ such that (X_{t_1}, c_{t_1}) is not $\check{\text{CFT}}_1$. Then there exists a fuzzy singleton $x_1^{t_1}$ in X_{t_1} that is not closed. Consequently, there exists $y^{t_1} \neq x^{t_1}$ in X_{t_1} such that

$$y_\gamma^{t_1} \in c_{t_1}(x_1^{t_1}) \quad \text{for some } \gamma \in (0, 1].$$

Now, for each $t \neq t_1$, choose $x^t, y^t \in X_t$ such that $x^t = y^t$. Consider the fuzzy points x_1 and y_γ in $X = \prod_{t \in T} X_t$. Then, for each $t \neq t_1$,

$$P_t(y_\gamma) = y_\gamma^t = x_\gamma^t \in c_t(x_1^t),$$

and for $t = t_1$, by assumption,

$$y_\gamma^{t_1} \in c_{t_1}(x_1^{t_1}).$$

Therefore,

$$y_\gamma \in \prod_{t \in T} c_t(x_1^t) = \otimes c_t(x_1),$$

which implies that x_1 is not closed in (X, c) .

This contradicts the assumption that (X, c) is $\check{\text{CFT}}_1$. Hence, (X_t, c_t) is $\check{\text{CFT}}_1$ for all $t \in T$. $\square$

6. $\check{\text{CFT}}_2$ Fuzzy Closure Spaces

Definition 29. A fuzzy closure space (X, c) is said to be $\check{\text{CFT}}_2$ or Čech fuzzy Hausdorff if for every pair of distinct fuzzy points x_λ and y_γ in X , there exist two neighborhoods f and g of x_λ and y_γ respectively such that $f \leq \mathfrak{C}\mathfrak{o}(g)$ and $x_\lambda \in f \leq \mathfrak{C}\mathfrak{o}(y_\gamma)$ and $y_\gamma \in g \leq \mathfrak{C}\mathfrak{o}(x_\lambda)$.

It is evident that every FT_2 fuzzy topological space is a $\check{CFT}_2$ fuzzy closure space. Let c_1 and c_2 are two fuzzy closure operators on X such that $c_1 \leq c_2$; if (X, c_1) is $\check{CFT}_2$, then (X, c_2) is also $\check{CFT}_2$.

Theorem 34. *If $(X, \tau(c))$ is FT_2 fts, then (X, c) is $\check{CFT}_2$ fcs.*

Proof. The proof is trivial since every open set containing x_λ in $(X, \tau(c))$ is a neighborhood of x_λ in (X, c) . $\square$

Theorem 35. *Every $\check{CFT}_2$ fuzzy closure space is $\check{CFT}_1$.*

Proof. Let (X, c) be a $\check{CFT}_2$ fuzzy closure space, and let x_λ and y_γ be two distinct fuzzy points in X . By the $\check{CFT}_2$ property, there exist neighborhoods f of x_λ and g of y_γ such that

$$x_\lambda \in f \leq \mathfrak{Co}(y_\gamma), \quad y_\gamma \in g \leq \mathfrak{Co}(x_\lambda), \quad \text{and} \quad f \leq \mathfrak{Co}(g).$$

Since $x_\lambda \in f \leq \text{int}(f)$, we obtain

$$x_\lambda \in \text{int}(f) \leq \text{int}(\mathfrak{Co}(y_\gamma)).$$

By the definition of interior,

$$\text{int}(\mathfrak{Co}(y_\gamma)) = \mathfrak{Co}(c(\mathfrak{Co}(\mathfrak{Co}(y_\gamma)))) = \mathfrak{Co}(c(y_\gamma)).$$

Thus,

$$x_\lambda \in \mathfrak{Co}(c(y_\gamma)).$$

Similarly, $y_\gamma \in \mathfrak{Co}(c(x_\lambda))$. Therefore, (X, c) is $\check{CFT}_1$. $\square$

Theorem 36. *Every finite $\check{CFT}_1$ fuzzy closure space is $\check{CFT}_2$.*

Proof. Let (X, c) be a $\check{CFT}_1$ fuzzy closure space, where $X = \{x^1, x^2, x^3, \dots, x^n\}$ is a finite set. By the $\check{CFT}_1$ property, each fuzzy singleton x_i^i ($i = 1, 2, \dots, n$) is closed. Since here the complement of a fuzzy singleton is also closed, it follows that every fuzzy singleton is open as well.

Now, let x_λ^i and x_γ^j be two distinct fuzzy points of X . Define

$$f = x_1^i \quad \text{and} \quad g = x_1^j.$$

Since f and g are fuzzy singletons, we have

$$\text{int}(f) = f \quad \text{and} \quad \text{int}(g) = g.$$

Therefore,

$$x_\lambda^i \in f \leq \mathfrak{Co}(x_\gamma^j), \quad x_\gamma^j \in g \leq \mathfrak{Co}(x_\lambda^i), \quad \text{and} \quad f \leq \mathfrak{Co}(g).$$

Hence, f and g form disjoint neighborhoods that separate x_λ^i and x_γ^j , demonstrating that (X, c) satisfies the $\check{CFT}_2$ condition. $\square$

But in general, every $\check{CFT}_1$ space is not $\check{CFT}_2$. For, consider the example below.

Example 37. *Let X be an infinite set, and define a fuzzy closure operator c on X as follows:*

$$\text{if } \text{supp}(f) \text{ is finite, } c(f)(x) = \begin{cases} 0, & \text{if } f(x) = 0, \\ f(x) + \frac{1}{2}, & \text{if } 0 < f(x) < \frac{1}{2}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\text{if } \text{supp}(f) \text{ is infinite, } c(f) = \underline{1}.$$

It is straightforward to verify that c is a fuzzy closure operator, but not a fuzzy topological closure operator. Since every fuzzy singleton is closed, it follows that (X, c) is $\check{\text{CFT}}_1$. We now examine the interiors of fuzzy subsets of X :

Case 1. If $\text{supp}(\mathfrak{C}\mathfrak{o}(f))$ is infinite, then $c(\mathfrak{C}\mathfrak{o}(f)) = \underline{0}$, and hence

$$\text{int}(f) = \mathfrak{C}\mathfrak{o}(c(\mathfrak{C}\mathfrak{o}(f))) = \underline{0}.$$

Case 2. If f takes the value 1 at all but finitely many points of X . Let $g = \bigvee_{x_1 \in f} x_1$. Then $g \leq f$, and

$$\text{int}(g) = \mathfrak{C}\mathfrak{o}(c(\mathfrak{C}\mathfrak{o}(g))) = \mathfrak{C}\mathfrak{o}(\mathfrak{C}\mathfrak{o}(g)) = g.$$

Thus,

$$\text{int}(f) \geq \text{int}(g) = g = \bigvee_{x_1 \in f} x_1.$$

From these observations, it follows that we cannot obtain two fuzzy subsets f and g of X with non-empty interiors such that $f \leq \mathfrak{C}\mathfrak{o}(g)$. Therefore, (X, c) is not $\check{\text{CFT}}_2$.

Remark 38. Every subspace of a $\check{\text{CFT}}_2$ fuzzy closure space is $\check{\text{CFT}}_2$. (i.e., $\check{\text{CFT}}_2$ is a hereditary property.)

Lemma 39. If $\theta : (X, c) \rightarrow (Y, d)$ is a $\check{\text{C}}\text{F}$ -homeomorphism, then for every fuzzy subset f of X , we have

$$\theta(\text{int}_X(f)) = \text{int}_Y(\theta(f)).$$

Proof. Since θ is a $\check{\text{C}}\text{F}$ -homeomorphism, we have,

$$\theta(c(f)) = d(\theta(f)) \quad \text{and} \quad \theta(\mathfrak{C}\mathfrak{o}_X(f)) = \mathfrak{C}\mathfrak{o}_Y(\theta(f)).$$

Using these identities, we compute:

$$\begin{aligned} \text{int}_Y(\theta(f)) &= \mathfrak{C}\mathfrak{o}_Y(d(\mathfrak{C}\mathfrak{o}_Y(\theta(f)))) \\ &= \mathfrak{C}\mathfrak{o}_Y(d(\theta(\mathfrak{C}\mathfrak{o}_X(f)))) \\ &= \mathfrak{C}\mathfrak{o}_Y(\theta(c(\mathfrak{C}\mathfrak{o}_X(f)))) \\ &= \theta(\mathfrak{C}\mathfrak{o}_X(c(\mathfrak{C}\mathfrak{o}_X(f)))) \\ &= \theta(\text{int}_X(f)). \quad \square \end{aligned}$$

Theorem 40. $\check{\text{CFT}}_2$ is a fuzzy closure property.

Proof. Let (X, c) and (Y, d) be fuzzy closure spaces (fcs's), and let $\theta : (X, c) \rightarrow (Y, d)$ be a $\check{\text{C}}\text{F}$ -homeomorphism. Consider two distinct fuzzy points $y^1_\lambda, y^2_\gamma \in Y$. Since θ is bijective, there exist distinct points $x^1, x^2 \in X$ such that

$$\theta(x^1) = y^1 \quad \text{and} \quad \theta(x^2) = y^2.$$

By assumption, (X, c) is $\check{CFT}_2$. Hence, there exist neighborhoods f_1 and f_2 of x_λ^1 and x_γ^2 , respectively, which separate these fuzzy points. By the preceding lemma, the images $\theta(f_1)$ and $\theta(f_2)$ are neighborhoods of y_λ^1 and y_γ^2 , respectively, and they separate y_λ^1 and y_γ^2 in (Y, d) .

Thus, (Y, d) is $\check{CFT}_2$. Therefore, the property $\check{CFT}_2$ is preserved under $\check{C}F$ -homeomorphisms, and hence it is a fuzzy closure property. $\square$

Lemma 41. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces and let $(X, c) = (\bigvee_{t \in T} X_t, \oplus c_t)$ be their sum. Then, $\text{int}_X(f) = \bigvee_{t \in T} \text{int}_{X_t}(f \wedge 1_{X_t})$.

Proof. By definition, $\oplus c_t(f) = \bigvee_{t \in T} c_t(f \wedge 1_{X_t})$. Thus,

$$\begin{aligned} \text{int}_X(f) &= \mathfrak{Co}_X(\oplus c_t(\mathfrak{Co}_X(f))) \\ &= \mathfrak{Co}_X\left(\bigvee_{t \in T} c_t(\mathfrak{Co}_X(f) \wedge 1_{X_t})\right) \\ &= \mathfrak{Co}_X\left(\bigvee_{t \in T} c_t(\mathfrak{Co}_{X_t}(f \wedge 1_{X_t}))\right) \\ &= \bigvee_{t \in T} \mathfrak{Co}_{X_t}\left(c_t(\mathfrak{Co}_{X_t}(f \wedge 1_{X_t}))\right) \\ &= \bigvee_{t \in T} \text{int}_{X_t}(f \wedge 1_{X_t}) \end{aligned} \quad \square$$

Theorem 42. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces. Then the sum $(X, c) = (\bigvee_{t \in T} X_t, \oplus c_t)$ is $\check{CFT}_2$ if and only if each (X_t, c_t) is $\check{CFT}_2$.

Proof. $(\Rightarrow)$ Assume that (X, c) is $\check{CFT}_2$. Let x_λ and y_γ be two distinct fuzzy points in X_s for some $s \in T$. Considering x_λ and y_γ as fuzzy points of the $\check{CFT}_2$ fuzzy closure space (X, c) , there exist neighborhoods $f, g \in I^X$ of x_λ and y_γ , respectively, such that

$$f \leq \mathfrak{Co}(g), \quad x_\lambda \in f \leq \mathfrak{Co}(y_\gamma), \quad \text{and} \quad y_\gamma \in g \leq \mathfrak{Co}(x_\lambda).$$

Consider the fuzzy sets $f \wedge 1_{X_s}$ and $g \wedge 1_{X_s}$. Since

$$x_\lambda \in \text{int}_X(f) = \bigvee_{t \in T} \text{int}_{X_t}(f \wedge 1_{X_t}),$$

we deduce that

$$x_\lambda \in \text{int}_{X_s}(f \wedge 1_{X_s}),$$

and similarly,

$$y_\gamma \in \text{int}_{X_s}(g \wedge 1_{X_s}).$$

Thus, the neighborhoods $f \wedge 1_{X_s}$ and $g \wedge 1_{X_s}$ separate x_λ and y_γ in (X_s, c_s) . Hence, (X_s, c_s) is $\check{CFT}_2$ for all $s \in T$.

$(\Leftarrow)$ Conversely, suppose that (X_t, c_t) is $\check{CFT}_2$ for all $t \in T$. Let $x_\lambda, y_\gamma \in I^X$ be distinct fuzzy points.

Case (1): If $x, y \in X_{t_1}$ for some $t_1 \in T$, then there exist disjoint neighborhoods $f, g \in I^{X_{t_1}}$ containing x_λ and y_γ , respectively. Considering f and g as fuzzy subsets of X , and by the preceding lemma, we have

$$\text{int}_X(f) = \text{int}_{X_{t_1}}(f) \quad \text{and} \quad \text{int}_X(g) = \text{int}_{X_{t_1}}(g).$$

Thus, f and g also separate x_λ and y_γ in $(\bigvee_{t \in T} X_t, \oplus c_t)$.

Case (2): If $x \in X_{t_1}$ and $y \in X_{t_2}$ with $t_1 \neq t_2$, then since

$$\text{int}_X(1_{X_{t_1}}) = 1_{X_{t_1}} \quad \text{and} \quad \text{int}_X(1_{X_{t_2}}) = 1_{X_{t_2}},$$

we may take $1_{X_{t_1}}$ and $1_{X_{t_2}}$ as neighborhoods of x_λ and y_γ , respectively, in $(\bigvee_{t \in T} X_t, \oplus_{c_t})$. These clearly separate x_λ and y_γ .

Hence, $(\bigvee_{t \in T} X_t, \oplus_{c_t})$ is $\check{\text{CFT}}_2$. $\square$

6.1. $\check{\text{CFT}}_{2\frac{1}{2}}$ Fuzzy Closure Spaces

Definition 30. A fuzzy closure space (X, c) is said to be $\check{\text{CFT}}_{2\frac{1}{2}}$ or $\check{\text{C}}\text{F}$ -Urysohn if and only if for every pair of distinct fuzzy points x_λ and y_γ , there exist two neighborhoods f and g of x_λ and y_γ respectively such that $c(f) \leq \mathfrak{Co}(c(g))$ and $x_\lambda \in f \leq \mathfrak{Co}(y_\gamma)$ and $y_\gamma \in g \leq \mathfrak{Co}(x_\lambda)$.

Every $\check{\text{C}}\text{F}$ -Urysohn fcs is $\check{\text{CFT}}_2$ fcs. If $(X, \tau(c))$ is fuzzy Urysohn, then (X, c) is $\check{\text{C}}\text{F}$ -Urysohn. Since a fuzzy closure space is $\check{\text{CFT}}_1$ iff every fuzzy singleton is closed, we can easily get that every finite $\check{\text{CFT}}_1$ space is $\check{\text{CFT}}_{2\frac{1}{2}}$. Also we can find that $\check{\text{CFT}}_{2\frac{1}{2}}$ is a hereditary property.

Theorem 43. $\check{\text{CFT}}_{2\frac{1}{2}}$ is a fuzzy closure property.

Proof. Similar proof as that of $\check{\text{CFT}}_2$ spaces. $\square$

Theorem 44. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces. The sum $(X, c) = (\bigvee_{t \in T} X_t, \oplus_{c_t})$ is a $\check{\text{CFT}}_{2\frac{1}{2}}$ space if and only if each (X_t, c_t) is a $\check{\text{CFT}}_{2\frac{1}{2}}$ space.

Proof. Similar proof as that of the sum of $\check{\text{CFT}}_2$ spaces. $\square$

7. $\check{\text{C}}\text{F}$ -Regular Fuzzy Closure Spaces

Definition 31. A fuzzy closure space (X, c) is said to be $\check{\text{C}}\text{F}$ -regular if for each fuzzy point x_λ in X and each non-empty fuzzy subset k of X such that $x_\lambda \in \mathfrak{Co}(c(k))$, there exists neighborhoods f of x_λ and g of k such that $f \leq \mathfrak{Co}(g)$. A $\check{\text{C}}\text{F}$ -regular fcs which is also T_s is said to be $\check{\text{CFT}}_3$.

The indiscrete fuzzy closure operator is $\check{\text{C}}\text{F}$ -regular. Consequently, $\check{\text{C}}\text{F}$ -regularity does not imply $\check{\text{CFT}}_{2\frac{1}{2}}$ property or any of the weaker separation properties in the hierarchy below it. However, it can be readily verified that every $\check{\text{CFT}}_3$ space is also a $\check{\text{CFT}}_2$ space. Furthermore, if c is a fuzzy topological closure operator, then the fts $(X, \tau(c))$ is fuzzy regular (FT_3) if and only if the fcs (X, c) is $\check{\text{C}}\text{F}$ -regular ($\check{\text{CFT}}_3$).

Remark 45. The fts $(X, \tau(c))$ being fuzzy regular does not necessarily imply that (X, c) is $\check{\text{C}}\text{F}$ -regular fcs. This is demonstrated in the following example.

Example 46. Consider the set $\mathbb{Z}$ equipped with a fuzzy closure operator c defined as follows: $c(\underline{0}) = \underline{0}$, $c(x_\lambda) = x_\lambda \vee (x+1)_\lambda$ for any fuzzy point $x_\lambda \in I^\mathbb{Z}$, and for any fuzzy set $f \in I^\mathbb{Z}$, $c(f) = \bigvee_{x_\lambda \leq f} c(x_\lambda)$. The associated fuzzy topological space $(\mathbb{Z}, \tau(c))$ is indiscrete and is fuzzy regular.

Let $x \in \mathbb{Z}$, and consider the fuzzy point x_λ and the fuzzy set $k = (x+1)_1$. It can be observed that every neighborhood of the fuzzy point $(x+1)_1$ contains the fuzzy set $x_1 \vee (x+1)_1$. Since $x_\lambda \in \text{Co}(c(k))$, and every neighborhood of k contains x_1 , it follows that x_λ and k cannot be separated. Therefore, (X, c) is not a $\check{\text{C}}\text{F}$ regular fuzzy closure space.

Remark 47. Every ČF-Urysohn (or ČFT₂) space need not be ČF-regular. This can be seen from the following example.

Example 48. Fix an element $x \in \mathbb{N}$ and define a finitely generated fuzzy closure operator c on $\mathbb{N}$ as follows:

$$c(f) = \begin{cases} \underline{0} & \text{if } f = \underline{0}, \\ x_{\lambda+\frac{1}{2}} & \text{if } f = x_{\lambda} \text{ and } \lambda \in (0, \frac{1}{2}), \\ x_1 & \text{if } f = x_{\lambda} \text{ and } \lambda \in [\frac{1}{2}, 1], \\ y_{\lambda} & \text{if } f = y_{\lambda} \text{ and } y \neq x, \\ \bigvee_{y_{\lambda} \leq f} c(y_{\lambda}) & \text{otherwise.} \end{cases}$$

It can be verified that c is a fuzzy closure operator but not fuzzy topological. Since every fuzzy singleton is closed, $(\mathbb{N}, c)$ is ČFT₁. For any two distinct fuzzy points $y_{\lambda}, z_{\gamma} \in I^{\mathbb{N}}$, the fuzzy sets $f = y_1$ and $g = z_1$ separate these points, and $c(f) \leq \mathfrak{Co}(c(g))$. Hence, $(\mathbb{N}, c)$ is ČF-Urysohn (and also ČFT₂).

If $\lambda > \frac{1}{2}$, then

$$\begin{aligned} \text{int}(x_{\lambda}) &= \mathfrak{Co}(c(\mathfrak{Co}(x_{\lambda}))) \\ &= \mathfrak{Co}(c(1_{\{\mathbb{N} \setminus \{x\}\}} \vee x_{1-\lambda})) \\ &= \mathfrak{Co}(1_{\{\mathbb{N} \setminus \{x\}\}} \vee x_{\frac{3}{2}-\lambda}) \\ &= x_{\lambda-\frac{1}{2}}. \end{aligned}$$

Similarly, if $\lambda \leq \frac{1}{2}$, then $\text{int}(x_{\lambda}) = \underline{0}$. Consider the fuzzy point $x_{0.4}$ and the fuzzy set $k = x_{0.05}$. Clearly, $x_{0.4} \in \mathfrak{Co}(c(k)) = 1_{\mathbb{N} \setminus \{x\}} \vee x_{0.45}$. Every neighborhood f of $x_{0.4}$ contains $x_{0.9}$, and every neighborhood g of $x_{0.05}$ contains $x_{0.55}$, so $f \not\leq \mathfrak{Co}(g)$. Therefore, $(\mathbb{N}, c)$ is not ČF-regular, and hence not ČFT₃.

Remark 49. It is important to note that not every ČF-regular space is necessarily a ČFT₃ space.

Consider a set X with at least two elements, and define a fuzzy closure operator c by

$$c(f) = \bigvee_{x_{\lambda} \leq f} x_1 \quad \text{for all } f \in I^X.$$

If $x_{\lambda}, k \in I^X$ such that $x_{\lambda} \leq \mathfrak{Co}(c(k))$, the fuzzy sets x_1 and $\bigvee_{y_{\lambda} \leq f} y_1$ will separate x_{λ} and k . Hence, the space (X, c) is ČF-regular. However, it does not satisfy the ČFT₅ condition, and therefore it is not a ČFT₃ space.

Mashhour and Ghanim[5] defined regular fcs differently, as follows:

Definition 32. [5] A fuzzy closure space (X, c) is said to be regular if for all fuzzy points x_{λ} and a fuzzy set k in X such that $x_{\lambda} \in \text{int}(k)$, there exists a fuzzy set f in X such that $x_{\lambda} \in \text{int}(f) \leq c(f) \leq \text{int}(k)$.

We can discuss the difference between these two definitions. The two definitions of regularity agree on fuzzy topological spaces. The closure operator $c(f) = \bigvee_{x_{\lambda} \in f} x_1$ on any set X is a regular fcs on both these definitions. We can see that Example 48 is a non-regular fcs as defined by Mashhur and Ghanim.

Theorem 50. If a fuzzy closure space (X, c) is regular according to Mashhour's definition, then it is a ČF-regular fuzzy closure space.

Proof. Let (X, c) be a regular fuzzy closure space according to Mashhour’s definition. Let x_λ be a fuzzy point in X and $k \in I^X$ such that $x_\lambda \in \mathfrak{Co}(c(k))$. This implies that

$$x_\lambda \in \mathfrak{Co}(c(\mathfrak{Co}(\mathfrak{Co}(k)))) = \text{int}(\mathfrak{Co}(k)).$$

Since (X, c) is regular (by the definition of Mashhour), there exists $f \in I^X$ such that

$$x_\lambda \in \text{int}(f) \leq c(f) \leq \text{int}(\mathfrak{Co}(k)).$$

Now,

$$c(f) \leq \text{int}(\mathfrak{Co}(k)) = \mathfrak{Co}(c(\mathfrak{Co}(\mathfrak{Co}(k)))) = \mathfrak{Co}(c(k)).$$

Therefore,

$$\mathfrak{Co}(c(f)) \geq c(k) \geq k.$$

Since $\text{int}(\mathfrak{Co}(f)) = \mathfrak{Co}(c(f))$, we obtain $k \leq \text{int}(\mathfrak{Co}(f))$. Thus, f and $\mathfrak{Co}(f)$ are neighborhoods that separate x_λ and k . Hence, (X, c) is ČF-regular. $\square$

Remark 51. The property of ČF-regularity is hereditary; that is, any subspace of a ČF-regular fuzzy closure space retains the ČF-regularity property.

Theorem 52. ČF-regularity is a fuzzy closure property.

Proof. Let $\theta: (X, c) \rightarrow (Y, d)$ be a ČF-homeomorphism, and assume that (X, c) is a ČF-regular fcs. Let y_γ be a fuzzy point in Y and $k \in I^Y$ such that $y_\gamma \in \mathfrak{Co}_Y(d(k))$. Since θ is a ČF-homeomorphism, $\theta^{-1}(y_\gamma)$ is a fuzzy point in X . Now consider the fuzzy point $\theta^{-1}(y_\gamma)$ and the fuzzy set $\theta^{-1}(k) \in I^X$. The condition $y_\gamma \in \mathfrak{Co}_Y(d(k))$ implies that

$$\theta^{-1}(y_\gamma) \in \theta^{-1}(\mathfrak{Co}_Y(d(k))) = \mathfrak{Co}_X(\theta^{-1}(d(k))) = \mathfrak{Co}_X(c(\theta^{-1}(k))),$$

where the equalities follow from the properties of ČF-homeomorphisms.

Since (X, c) is ČF-regular, there exist neighborhoods f and g of $\theta^{-1}(y_\gamma)$ and $\theta^{-1}(k)$, respectively, such that $f \leq \mathfrak{Co}_X(g)$. This inequality implies that

$$\theta(f) \leq \theta(\mathfrak{Co}_X(g)) = \mathfrak{Co}_Y(\theta(g)).$$

By Lemma 39, $\theta(f)$ is a neighborhood of y_γ and $\theta(g)$ is a neighborhood of k . Therefore, (Y, d) is ČF-regular. $\square$

Theorem 53. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces. The sum $(X, c) = (\bigvee_{t \in T} X_t, \oplus c_t)$ is ČF-regular if and only if each (X_t, c_t) is ČF-regular.

Proof. $(\Rightarrow)$ Assume that the sum $(X, c) = (\bigvee_{t \in T} X_t, \oplus c_t)$ is ČF-regular. Let x_λ be a fuzzy point in X_t for some $t \in T$, and let k be a fuzzy subset of X_t such that $x_\lambda \in \mathfrak{Co}_{X_t}(c_t(k))$. Viewing x_λ and k as fuzzy subsets of X and $x_\lambda \in \mathfrak{Co}_{X_t}(c_t(k)) \leq \mathfrak{Co}_X(c(k))$, the ČF-regularity of (X, c) implies the existence of fuzzy subsets $f, g \in I^X$ that separate x_λ and k in X . Define $f_t = f \wedge 1_{X_t}$ and $g_t = g \wedge 1_{X_t}$, where 1_{X_t} is the characteristic function of X_t . These fuzzy subsets $f_t, g_t \in I^{X_t}$ separate x_λ and k in X_t , demonstrating that (X_t, c_t) is ČF-regular for all $t \in T$.

$(\Leftarrow)$ Conversely, assume that each (X_t, c_t) is ČF-regular for all $t \in T$. Let x_λ be a fuzzy point in X and $k \in I^X$ a fuzzy subset such that $x_\lambda \in \mathfrak{Co}(c(k))$. Suppose $x_\lambda \in X_{t_1}$ for some $t_1 \in T$. Two cases arise:

1. Case 1: $k \wedge 1_{X_{t_1}} = \underline{0}$. Choose $f = 1_{X_{t_1}}$ and $g = \mathfrak{Co}(1_{X_{t_1}})$. These fuzzy subsets $f, g \in I^X$ separate x_λ and k in X .

2. Case 2: $k \wedge 1_{X_{t_1}} \neq 0$. Since (X_{t_1}, c_{t_1}) is ČF-regular, there exist fuzzy subsets $f_{t_1}, g_{t_1} \in I^{X_{t_1}}$ that separate x_λ and $k \wedge 1_{X_{t_1}}$ in X_{t_1} . Extend these to X by defining $f = f_{t_1}$ (extended by zero outside X_{t_1}) and $g = g_{t_1} \vee \mathfrak{C}o(1_{X_{t_1}})$. These fuzzy subsets $f, g \in I^X$ separate x_λ and k in X .

Thus, in both cases, x_λ and k can be separated in (X, c) , proving that (X, c) is ČF-regular. Therefore, the sum (X, c) is ČF-regular if and only if each (X_t, c_t) is ČF-regular. $\square$

8. ČF-Normal Fuzzy Closure Spaces

Definition 33. A fuzzy closure space (X, c) is said to be ČF-normal if, for all non-empty pair of fuzzy subsets $k_1, k_2 \in I^X$ such that $c(k_1) \leq \mathfrak{C}o_X(c(k_2))$, there exists neighborhoods $f_1, f_2 \in I^X$ of k_1 and k_2 , respectively, satisfying $f_1 \leq \mathfrak{C}o_X(f_2)$. A ČF-normal fcs that is also ČFT_s is defined to be ČFT₄.

As in the case of fuzzy topological spaces, a ČF-normal fuzzy closure space (fcs) is not necessarily ČF-regular. In fact, Example 46 provides an example of a ČF-normal fcs that fails to be ČF-regular. Moreover, every ČFT₄ fcs is clearly ČFT₃. Since the indiscrete topology is ČF-normal, it follows that ČF-normality does not necessarily imply the ČFT₃ axiom, nor any weaker separation axioms lying below it. Furthermore, if c is a fuzzy topological closure operator, then $(X, \tau(c))$ is a fuzzy normal (FT₄) fts if and only if (X, c) is a ČF-normal (ČFT₄) fcs.

Remark 54. $(X, \tau(c))$ is a fuzzy normal fts does not imply that (X, c) is a ČF-normal fcs.

Consider the fuzzy closure operator c defined on the fuzzy subsets of $\mathbb{R}$ as follows:

$$c(f) = \begin{cases} \underline{0} & \text{if } f = \underline{0}, \\ 1_{(-\infty, -2)} & \text{if } f \leq 1_{(-\infty, -2)}, \\ 1_{(2, \infty)} & \text{if } f \leq 1_{(2, \infty)}, \\ 1_{(-\infty, -1)} & \text{if } f = 1_{(-\infty, -2)}, \\ 1_{(1, \infty)} & \text{if } f = 1_{(2, \infty)}, \\ \underline{1} & \text{otherwise,} \end{cases}$$

where 1_A denotes the characteristic function of the subset $A \subset \mathbb{R}$, taking the value 1 on A and 0 elsewhere.

The fuzzy topology $\tau(c)$ associated with $(\mathbb{R}, c)$ is indiscrete, implying that $(\mathbb{R}, \tau(c))$ is fuzzy normal. The corresponding interior operator on this space is given by

$$\text{int}(f) = \begin{cases} \underline{1} & \text{if } f = \underline{1}, \\ 1_{(-\infty, 1]} & \text{if } 1_{(-\infty, 1]} \leq f \leq 1_{(-\infty, 2]}, \\ 1_{(-\infty, 2]} & \text{if } 1_{(-\infty, 2]} \leq f \leq \underline{1}, \\ 1_{[-1, \infty)} & \text{if } 1_{[-1, \infty)} \leq f \leq 1_{[-2, \infty)}, \\ 1_{[-2, \infty)} & \text{if } 1_{[-2, \infty)} \leq f \leq \underline{1}, \\ \underline{0} & \text{otherwise.} \end{cases}$$

For the fuzzy subsets $1_{(-\infty, -2)}$ and $1_{(2, \infty)}$, it can be observed that $c(1_{(-\infty, -2)}) \leq \mathfrak{C}o(c(1_{(2, \infty)}))$. Since the smallest neighborhoods of $1_{(-\infty, -2)}$ and $1_{(2, \infty)}$ are $1_{(-\infty, 1]}$ and $1_{[-1, \infty)}$ respectively, we can see that these two fuzzy subsets cannot be separated in $(\mathbb{R}, c)$. Hence $(\mathbb{R}, c)$ is not ČF-normal.

Example 55. Let X be a non-empty set. Fix an $x \in X$, and define a fuzzy closure operator c as $c(x_\lambda) = x_1$ and if $y \neq x$, $c(y_\lambda) = \mathfrak{C}o(x_1)$, and $c(f) = \bigvee_{z_\rho \leq f} c(z_\rho)$ for other fuzzy subsets f of X . We can see that, this finitely generated fcs (X, c) is ČF-normal.

Remark 56. Every finite ČF-regular (ČFT₃) fcs is ČF-normal (ČFT₄). In general, every finitely generated ČF-regular (ČFT₃) fcs is ČF-normal (ČFT₄). Moreover, ČF-normality is both a hereditary property and a fuzzy closure property.

Theorem 57. Let $\mathcal{F} = \{(X_t, c_t) : t \in T\}$ be a family of pairwise disjoint fuzzy closure spaces. The sum $(X, c) = (\bigvee_{t \in T} X_t, \oplus c_t)$ is ČF-normal if and only if each (X_t, c_t) is ČF-normal.

Proof. ($\Rightarrow$) Assume that the sum $(\bigvee_{t \in T} X_t, \oplus c_t)$ is ČF-normal. For any $t \in T$, let k_1, k_2 be two fuzzy subsets of X_t such that $c(k_1) \leq \mathfrak{Co}_{X_t}(c(k_2))$. Extend k_1 and k_2 to fuzzy subsets of X by defining $k_1(x) = 0$ and $k_2(x) = 0$ for $x \notin X_t$. By the Čech fuzzy normality of (X, c) , there exist fuzzy subsets $f, g \in I^X$ that separate k_1 and k_2 in X . The restrictions $f \wedge 1_{X_t}$ and $g \wedge 1_{X_t}$ in I^{X_t} then separate k_1 and k_2 in X_t . Hence, (X_t, c_t) is ČF-normal for each $t \in T$.

($\Leftarrow$) Conversely suppose that each (X_t, c_t) is ČF-normal for all $t \in T$. Let k_1 , and k_2 be fuzzy subsets of X such that $\oplus c_t(k_1) \leq \mathfrak{Co}_X(\oplus c_t(k_2))$. For each $t \in T$, the restrictions $k_1 \wedge 1_{X_t}$ and $k_2 \wedge 1_{X_t}$ belong to I^{X_t} , and the condition $c_t(k_1 \wedge 1_{X_t}) \leq \mathfrak{Co}_{X_t}(c_t(k_2 \wedge 1_{X_t}))$ holds. By the ČF-normality of (X_t, c_t) , there exist fuzzy subsets $f_t, g_t \in I^{X_t}$ that separate $k_1 \wedge 1_{X_t}$ and $k_2 \wedge 1_{X_t}$. Define $f = \bigvee_{t \in T} f_t$ and $g = \bigvee_{t \in T} g_t$ in I^X . These fuzzy subsets f and g separate k_1 and k_2 in X , proving that (X, c) is ČF-normal. $\square$

9. Conclusions and Future Research

Fuzzy topological spaces do not constitute a natural boundary for the validity of theorems, but many theorems can be extended to what are called fuzzy closure spaces. In this study, we defined interior operator in fuzzy closure spaces, as well as neighborhood systems and found an equivalent condition for ČF-continuity. We established a number of separation axioms, including ČFT₀, ČFT₁, ČFT_s, ČFT₂, ČFT₃, ČFT₄, ČF-Urysohn, ČF-regular, and ČF-normal, using the neighborhood system. We have discovered certain characteristics of these separation axioms. Here are some questions that need more attention. Productive behaviour of some separation axioms are discovered, others yet to be investigated. We have found a characterization theorem for ČFT₁ fcs. Researchers can investigate the characterization theorem for other separation axioms. Identify the lattice of ČFT₀, ČFT₁, ČFT₂, etc fuzzy closure operators. Extend all these concepts to L -closure spaces.

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Abbreviations

The following abbreviations are used in this manuscript:

- fcs fuzzy closure space
- fts fuzzy topological space

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