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Metallic Ratios are Defined by an Argument of a Normalized Complex Number

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Abstract: We show that metallic ratios for $k \in \mathbb{R}$ are defined by an argument of a normalized complex number, while for rational $k \neq \{0, \pm 2\}$, they are defined by Pythagorean triples.

Keywords: metallic ratios; Pythagorean triples; emergent dimensionality

1. Introduction

Metallic ratios of $k \in \mathbb{N}_0$ are defined as

$$M_{k\pm} := \frac{k \pm \sqrt{k^2 + 4}}{2}, \tag{1}$$

being the roots of the quadratic equation

$$M_{k\pm}^2 - kM_{k\pm} - 1 = 0, \tag{2}$$

with the property $M_{k-} = -1/M_{k+} = k - M_{k+}$. They are shown in Figure 1.

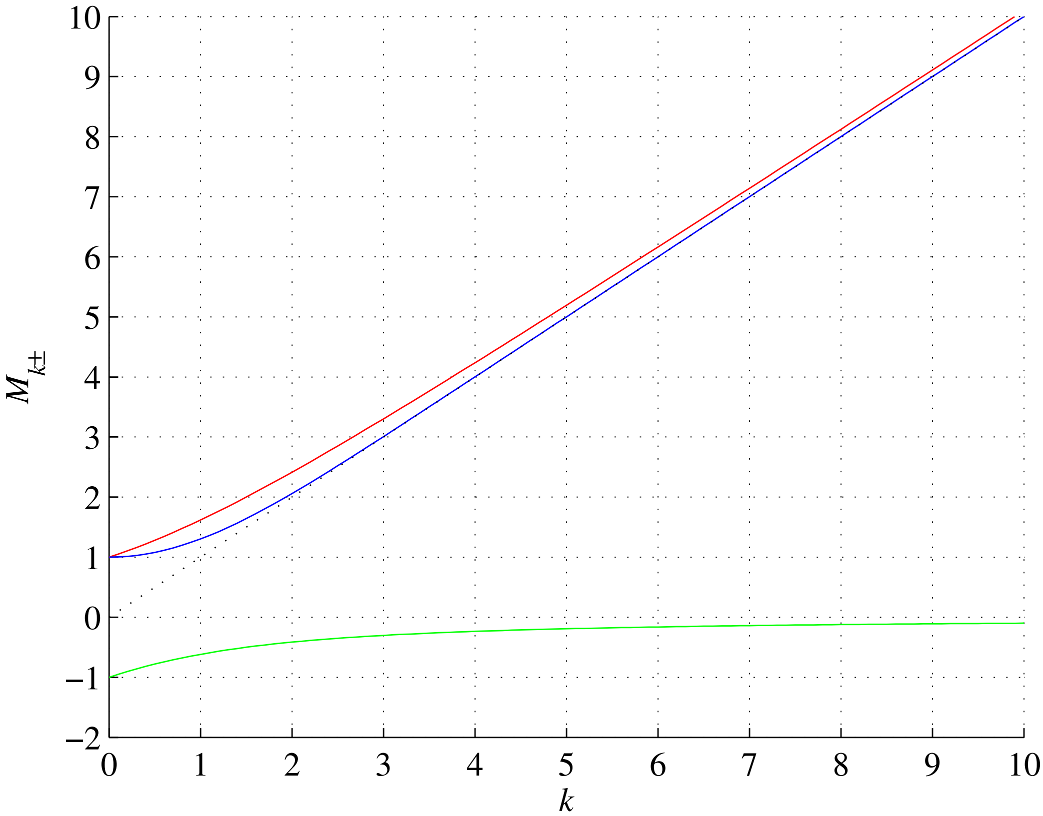

Figure 1. Metallic ratios: positive M_{k+} (red), negative M_{k-} (green) as continuous functions of $0 \leq k \leq 10$ and Ł-K metric $D_{NN}(k, s)$ for $s = \sqrt{\pi}/2$ (blue).

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The positive metallic ratio M_{k+} as a continuous function of $k \in \mathbb{R}$ has the same property as the Łukaszyk-Karmowski (Ł-K) metric [1] between two independent continuous random variables: It becomes asymptotic to $M_{k+} = k$ as k goes to ifinity, as the $+4$ factor in the square root becomes negligible and $M_{k\pm} \approx \{k, 0\}$ for large k. Because the ratios (1) are usually visualized as ratios of the edge lengths of a rectangle and these are assumed to be nonnegative, usually only the positive principal square root M_{k+} of (2) is considered, where for $k = 1$ the golden ratio is obtained, for $k = 2$ the silver ratio, for $k = 3$ the bronze ratio, etc. However, distance nonnegativity does not hold for the Ł–K metric [1], for example; such axiomatization may be misleading [2].

It was shown [3] that for $k \neq \{0, 2\}$ the metallic ratios (1) can be expressed by primitive Pythagorean triples, as

$$M_{k+} = \cot\left(\frac{\theta}{4}\right), \tag{3}$$

and for $k \geq 3$

$$k = 2\sqrt{\frac{c+b}{c-b}}, \tag{4}$$

where θ is the angle between a longer cathetus b and hypotenuse c of a right triangle defined by a Pythagorean triple, as shown in Figure 2, whereas for $k = \{3, 4\}$ it is the angle between a hypotenuse and a shorter cathetus a ($\{M_{3+}, M_{10+}\}$ and $\{M_{4+}, M_{6+}\}$ are defined by the same Pythagorean triples, respectively, $(5, 12, 13)$ and $(3, 4, 5)$), and

$$M_{1+} = \cot\left(\frac{\pi - \theta_{(3,5)}}{4}\right). \tag{5}$$

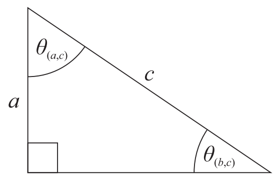

Figure 2. Right triangle showing a longer (b), shorter (a) hypotenuse, catheti (c) and angles $\theta = \theta_{(b,c)}$ and $\theta_{(a,c)}$.

For example the Pythagorean triple $(20, 21, 29)$ defines M_{5+}, the Pythagorean triple $(3, 4, 5)$ defines M_{6+}, the Pythagorean triple $(28, 45, 53)$ defines M_{7+}, and so on.

2. Results

Theorem 1. *The metallic ratio of $k \in \mathbb{R}$ is defined by an acute angle of a right triangle $0 < \theta < \pi/2$.*

Proof. We express the RHS of the equation (3) using half-angle formulae for sine and cosine, and substituting $\varphi := \theta/2$

$$\begin{aligned} \cot\left(\frac{\theta}{4}\right) = \cot\left(\frac{\varphi}{2}\right) = \frac{1+\cos\varphi}{\sin\varphi} = \frac{1+\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \\ = \frac{1+\sqrt{\frac{1+\cos\theta}{2}}}{\sqrt{\frac{1-\cos\theta}{2}}} = M_k, \end{aligned} \tag{6}$$

since $0 < \theta < \pi/2$ (we exclude degenerated triangles), so $\mathrm{sgn}(\sin(\theta/2) = \mathrm{sgn}(\cos(\theta/2) = 1$.

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Multiplying the numerator and denominator of (6) by $\sqrt{(1+\cos\theta)/2}$ and performing some basic algebraic manipulations, we arrive at the quadratic equation for M_k

$$\sin(\theta)^2 M_k^2 - 2\sin(\theta)\left[1+\cos(\theta)\right] M_k - \sin(\theta)^2 = 0, \tag{7}$$

having roots

$$M_{\theta\pm} = \frac{1+\cos(\theta) \pm \sqrt{2\left(1+\cos(\theta)\right)}}{\sin(\theta)}, \tag{8}$$

corresponding to the metallic ratios (1) for $0 < \theta < \pi/2$. □

Theorem 2. *The metallic ratio of $k \in \mathbb{R}$ is defined by an angle $0 \le \theta < 2\pi$.*

Proof. Equating relations (1) and (8) and solving for k gives

$$k = \frac{2\left(\cos(\theta)+1\right)}{\sin(\theta)}, \tag{9}$$

valid for $0 < \theta < \pi/2$. Solving the relation (9) for θ extends its validity to $0 \le \theta < 2\pi$ by analytic continuation, giving

$$\frac{k+2i}{k-2i} = e^{i\theta_k} = \cos\theta_k + i\sin\theta_k = \frac{a}{c} + \frac{b}{c}i := z_k, \tag{10}$$

that relates $k \in \mathbb{R}$ with a normalized complex number z_k. □

The angle $\theta_k = \arg(z_k)$ is shown in Figure 3 for z_k given by the relation (10). Figure 4 shows metallic ratios (8) as a function of this angle for $-\pi \le \theta_k \le \pi$ and extrapolated to $-2\pi \le \theta \le 2\pi$. $z_{k<0}$ and $z_{k>0}$ are complex conjugates of each other.

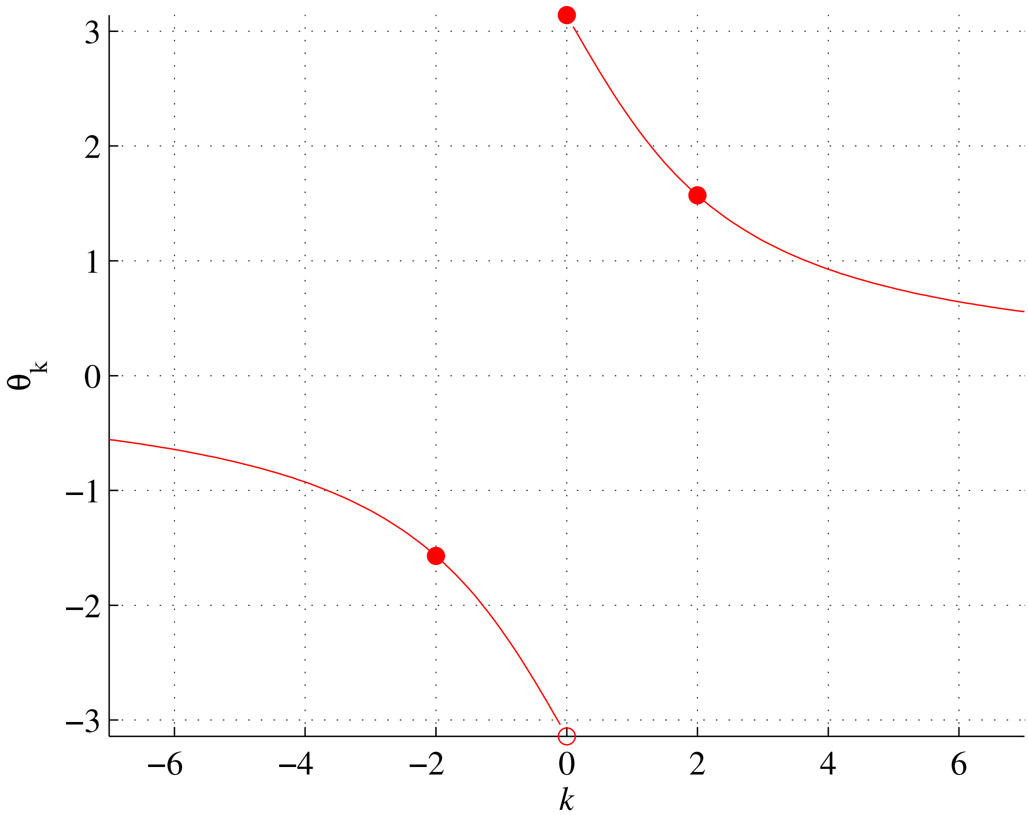

Figure 3. θ_k for $-7 \le k \le 7$. $\theta_{\pm 2} = \pm\pi/2$, $\theta_0 = \pi$.

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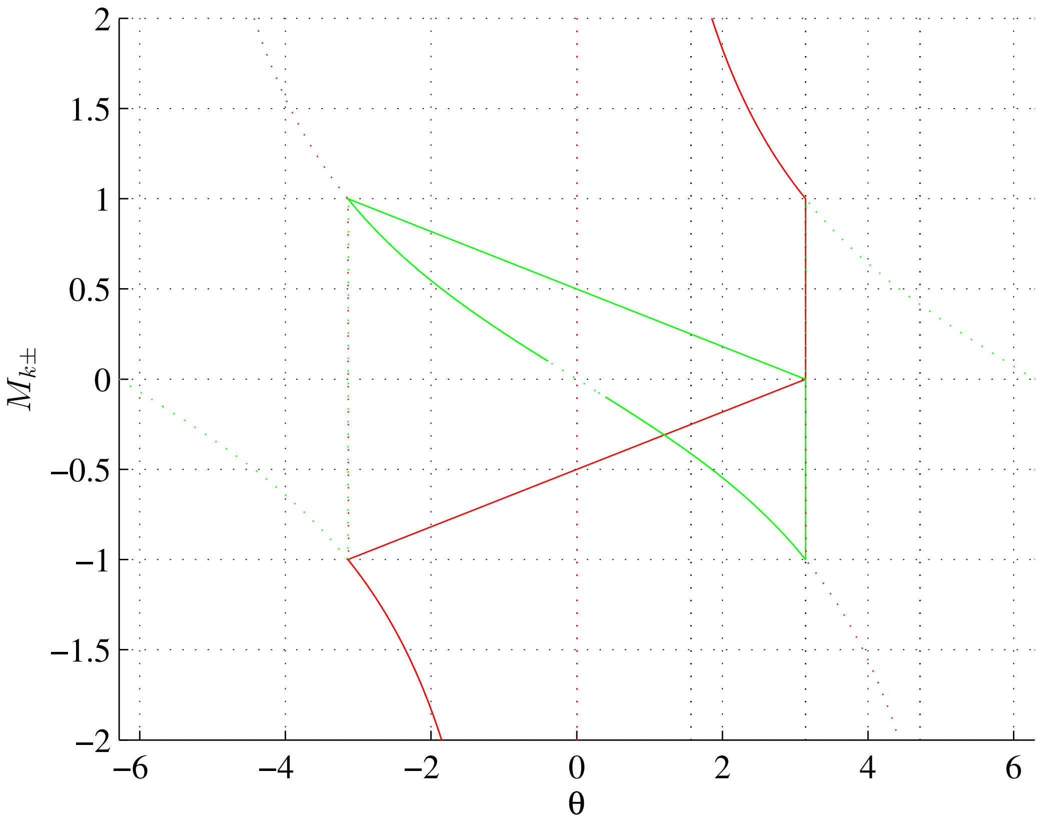

Figure 4. Metallic ratios: positive $M_{\theta+}$ (red), negative $M_{\theta-}$ (green) as a function of $0 \leq \theta < \pi/2$ and extrapolated to $-2\pi \leq \theta \leq 2\pi$ (dotted), and $\theta = \arg(z_k)$ for $-\pi \leq \theta < \pi$ (solid).

Theorem 3. *For $k \neq \{0, \pm 2\}, k \in \mathbb{Q}$, the triple $\{a, b, c\}$ corresponding to the angle θ_k (10) is a Pythagorean triple.*

Proof. Plugging rational $k := l/m, m \neq 0, l, m \in \mathbb{Z}$ into the relation (10) gives

$$\frac{l^2 - 4m^2}{l^2 + 4m^2} + \frac{4lm}{l^2 + 4m^2} i = \frac{a}{c} + \frac{b}{c} i, \tag{11}$$

and $a = l^2 - 4m^2, b = 4lm, c = l^2 + 4m^2, a, b, c \in \mathbb{Z}$ is a possible solution. It is easy to see that $a^2 + b^2 = c^2$. $k = 0$ implies $l = 0$ and $a = -4m^2, b = 0, c = 4m^2$ valid $\forall m \in \{\mathbb{R}, \mathbb{I}\}$. $k = \pm 2$ implies $l = \pm 2m$ and $a = 0, b = \pm 8m^2, c = 8m^2$ also valid $\forall m \in \{\mathbb{R}, \mathbb{I}\}$. $\square$

Table 1 shows the generalized Pythagorean triples that define the metallic ratios for $k = \{0.1, 0.2, \ldots, 7\}$.

Table 1. Triples.

k	a	b	c	k	a	b	c
0.1	-399	40	401	3.6	28	45	53
0.2	-99	20	101	3.7	969	1480	1769
0.3	-391	120	409	3.8	261	380	461
0.4	-12	5	13	3.9	1121	1560	1921
0.5	-15	8	17	4	3	4	5
0.6	-91	60	109	4.1	1281	1640	2081
0.7	-351	280	449	4.2	341	420	541
0.8	-21	20	29	4.3	1449	1720	2249
0.9	-319	360	481	4.4	48	55	73
1	-3	4	5	4.5	65	72	97
1.1	-279	440	521	4.6	429	460	629
1.2	-8	15	17	4.7	1809	1880	2609
1.3	-231	520	569	4.8	119	120	169
1.4	-51	140	149	4.9	2001	1960	2801
1.5	-7	24	25	5	21	20	29
1.6	-9	40	41	5.1	2201	2040	3001
1.7	-111	680	689	5.2	72	65	97
1.8	-19	180	181	5.3	2409	2120	3209
1.9	-39	760	761	5.4	629	540	829
2				5.5	105	88	137
2.1	41	840	841	5.6	171	140	221
2.2	21	220	221	5.7	2849	2280	3649
2.3	129	920	929	5.8	741	580	941
2.4	11	60	61	5.9	3081	2360	3881
2.5	9	40	41	6	4	3	5
2.6	69	260	269	6.1	3321	2440	4121
2.7	329	1080	1129	6.2	861	620	1061
2.8	12	35	37	6.3	3569	2520	4369
2.9	441	1160	1241	6.4	231	160	281
3	5	12	13	6.5	153	104	185
3.1	561	1240	1361	6.6	989	660	1189
3.2	39	80	89	6.7	4089	2680	4889
3.3	689	1320	1489	6.8	132	85	157
3.4	189	340	389	6.9	4361	2760	5161
3.5	33	56	65	7	45	28	53

For $k = \{-7, -6.9, \ldots, -0.1\}$ set $b \leftrightarrow -b$.
E.g. for $k = -7$, $\{45, 28, 53\} \leftrightarrow \{45, -28, 53\}$.

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