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Article

Palatini–PT-Even Torsion: Uniqueness, Three-Chain Equivalence, and Exact Luminal Tensors

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Abstract

What is new. Within a first-order (Palatini) setting we impose a *scalar PT* projector on all observable scalar densities, restricting to at most one derivative per building block. Under this setting the framework *implies*: (i) a unique, pure-trace torsion aligned with $\partial\epsilon$ (C1); (ii) a bulk equivalence of three ostensibly different rank-one constructions with the *same* quadratic coefficient $A_\star = \lambda^2/8$ (C2); and (iii) a *unique* weight ratio between the rank-one determinant and closed-metric routes that removes TT-nonTT mixing and enforces the equal-coefficient identity $K=G$ (C3), hence $c_T=1$ at quadratic order. *Setup.* All observable scalars are mapped by a scalar *PT* projector $\Pi_{PT}(\cdot)$ to a real, even sector; the internal phase $\epsilon(x)$ is a nondynamical spurion that enters only through $\partial\epsilon$. With global assumptions **A1–A6** (PT-invariant domain/measure, $[\mathcal{PT}, *]=0$, projection-variation commutation, boundary/topology posture, Nieh–Yan as boundary counterterm, and a trace lock), we work at quadratic order with at most one derivative per building block. The trace lock is enforced algebraically so that $T_\mu = 3\eta \partial_\mu \epsilon$ (no new canonical pairs under A4). *Results.* (C1) Palatini algebraicity plus the scalar projector force $S^\mu=0$, $q_{\lambda\mu\nu}=0$, and $T^A{}_{BC} = 2\eta \delta^A{}_{[B} \partial_{C]} \epsilon$, yielding the invariant $I_T = -6\eta^2 \Pi_{PT}[(\partial\epsilon)^2]$. (C2) After C1 and projection, the three routes—(i) a rank-one determinant built from the canonical traceless matrix, (ii) a closed-metric rank-one deformation, and (iii) the *PT*-even CS/Nieh–Yan shadow—collapse to the same *bulk* quadratic piece $A_\star \sqrt{-g} (\sigma_\epsilon I_T)$ up to improvements, with $A_\star = \lambda^2/8$. (Our “ROD” is a parity-even volume deformation, not Born–Infeld gravity.) (C3) On flat FRW (unitary gauge) the two independent TT-nonTT mixing entries are proportional and fix the *locked* ratio $w_{\text{ROD}} : w_{\text{CM}} = 2 : 3$. On generic admissible patches the two mixing equations are non-collinear, selecting the same unique ratio; with this locking we prove $K(w)=G(w)$ (a total-divergence identity on the admissible variational domain), hence $c_T^2 = 1$ and a GR TT sector with no extra propagating degrees of freedom. *Reproducibility and scope.* All figures and coefficients are reproduced by the accompanying open-source notebooks and scripts; the 2:3 ratio is extracted by an automated mixing-matrix check on flat FRW and its generic uniqueness follows from the non-collinearity argument. Matter couplings respect the posture (axial channel vanishes; the trace coupling is removable up to a Nieh–Yan boundary shift). The leading *PT*-even NLO correction predicts $\delta c_T^2(k) = b k^2 / \Lambda^2$, offering band-limited constraints on Λ . Detailed proofs, improvement currents, and coefficient tables appear in Apps. A–E.

Keywords: Palatini formalism; PT-even projector; pure-trace torsion; trace-lock posture; Nieh–Yan invariant; parity-even Chern–Simons projection; rank-one determinant (ROD); closed-metric deformation; coefficient locking; three-chain equivalence; gravitational-wave tensor modes; exact luminality ($c_T = 1$)

1. Introduction

Motivation.

Multimessenger bounds enforce *luminal* gravitational waves near the present epoch, sharply constraining modifications of GR. We ask whether a first-order (Palatini) framework can be formulated that (i) remains strictly *PT*-even at the level of observables, (ii) *closes* at quadratic order with at most

one derivative per building block, (iii) preserves *exact* tensor luminality, and (iv) introduces *no* extra propagating degrees of freedom.

Posture and scope.

We work with independent vierbein $e^A{}_\mu$ and metric-compatible spin connection $\omega^{AB}{}_\mu$, and enforce a *scalar PT* projector $\Pi_{PT}(\cdot)$ on all observable scalar densities. An internal phase $\epsilon(x)$ enters the observed sector only through its gradient and as a *spurion* (not varied). At quadratic order with at most one derivative per factor, the projected, real scalar

$$\Sigma_\epsilon \equiv \Pi_{PT}[(\partial\epsilon)^2]$$

serves as the unique spurionic invariant; we also use the shorthand $I_T \equiv -\frac{1}{4}\Pi_{PT}[T^A{}_{BC}T^A{}^{BC}]$. Global assumptions (A1–A6)—*PT*-invariant domain/measure, orientation and $[\mathcal{PT},*]=0$, projection/variation commutation, boundary/topology posture, Nieh–Yan as boundary counterterm, and the trace-lock posture—are fixed once and for all (Section 2).

Positioning and falsifiable claims.

In contrast to EC/MAG lines where axial and traceless torsion may propagate in the bulk, our scalar *PT* posture plus Palatini algebraicity removes both irreps at the analyzed order. The framework is falsified if any of the following fails on admissible patches: (i) the pure-trace map (C1), (ii) the bulk equality of the three rank-one routes with $A_\star = \lambda^2/8$ (C2), or (iii) the locked equal-coefficient identity $K = G$ (C3), stated as a density total divergence $K - G = a^{-3}\partial_\mu(a^3 J_\Delta^\mu)$ and proven in Section 7 (with a weak-field TT representative of J_Δ^μ recorded in Section 7.1). *Paper-level null tests include:* (a) recovering $A_\star = \lambda^2/8$ from the *PT*-even CS/Nieh–Yan shadow on FRW/Minkowski; (b) reproducing the non-collinearity of the two mixing entries and the locked ratio $w_{\text{ROD}}^\star : w_{\text{CM}}^\star$; and (c) verifying $K - G = a^{-3}\partial_\mu(a^3 J_\Delta^\mu)$ with the listed J_Δ^μ .

Main results (at a glance).

(C1) Palatini–*PT* uniqueness under trace-lock (A6). Algebraic torsion consistent with the scalar projector is *uniquely* pure trace and aligned with $\partial\epsilon$:

$$T^A{}_{BC} = 2\eta \delta^A{}_{[B}\partial_{C]}\epsilon, \quad \eta > 0, \quad S^\mu = 0, \quad q_{\lambda\mu\nu} = 0$$

and the surviving quadratic invariant is

$$I_T = -6\eta^2 \Sigma_\epsilon.$$

(C2) Three-chain equivalence (quadratic, modulo boundary). A rank-one determinant (ROD), a closed-metric rank-one deformation, and the parity-even projected CS/Nieh–Yan piece *collapse* after projection and C1 to the same bulk invariant:

$$\delta^2 \mathcal{L}_{\text{ROD}} \equiv \delta^2 \mathcal{L}_{\text{CM}} \equiv \delta^2 \mathcal{L}_{\text{CS}^+} \equiv A_\star \sqrt{-g} I_T \pmod{\nabla_\mu J^\mu}, \quad A_\star = \frac{\lambda^2}{8}.$$

(C3) Coefficient locking $\Rightarrow$ exact luminality and a GR TT sector. Eliminating TT–nonTT mixing fixes a unique weight ratio between ROD and closed–metric routes; the locked quadratic TT block satisfies the equal–coefficient identity

$$K - G = a^{-3} \partial_\mu (a^3 J_\Delta^\mu)$$

(Section 7), hence

$$c_T^2 = 1, \quad \delta^2 S_{\text{TT}} = \frac{M_{\text{Pl}}^2}{8} \int d^4 x a^3 \left[\dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} - \frac{(\partial_k h_{ij}^{\text{TT}})^2}{a^2} \right], \quad \text{no new DoF.}$$

A TT–gauge *weak–field* representative of J_Δ^μ on Minkowski with slowly varying ϵ is provided in Section 7.1, and a covariant FRW representative is listed in Appendix D.

Related work (concise map).

In EC/MAG lines (e.g., [4,37]), axial and traceless torsion may propagate; by contrast, our scalar–*PT* Palatini posture plus algebraicity forces *pure trace aligned with* $\partial\epsilon$ at the analyzed order (C1), removing those irreps in the bulk. Parity–odd dCS gravity (e.g., [14,15]) ties tensor speed to a pseudo–scalar; here the *PT* *projected* sector keeps only even scalars and fixes $c_T=1$ after locking (C3), modulo improvements. Nieh–Yan terms [9] in our posture serve as boundary counterterms (A5) and do not alter bulk equations. Operationally, the “three–chain” collapse (C2) is a parity–even, rank–one statement: a rank–one determinant (ROD), a closed–metric deformation, and the *PT*–even CS/Nieh–Yan shadow share the same bulk quadratic piece $A_\star \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T$ with $A_\star = \lambda^2/8$, differing only by improvement currents. We view our results as an even–parity, first–order complement to these traditional lines, tailored to tensor–sector falsifiability.

Falsifiability, stability, and organization.

Any failure of the pure–trace map (C1), of the three–chain bulk collapse with $A_\star = \lambda^2/8$ (C2), or of exact luminality after locking (C3) falsifies the framework at quadratic order. Locking is stable on generic patches: the 2×2 mixing determinant scales as $\Delta(k, \Sigma_\epsilon) = \alpha_0 \eta^4 \Sigma_\epsilon^2 k^2 + O(k^4, \Sigma_\epsilon^3)$ with $\alpha_0 \neq 0$, so rank loss occurs only on the measure–zero set $\{\Sigma_\epsilon=0\} \cup \{k=0\}$ (Section 6, Appendix D). Beyond leading order, we record an *order–of–magnitude* data–facing NLO estimate in Section 9 (boxed remark; GW band). The paper is organized as follows: Section 2 fixes A1–A6 and the scalar projector (with an admissible–variation clarification); Sections 3–4 establish closure and prove C1 (including a 3–line block–nondegeneracy proof sketch); Section 5 proves C2 and introduces flux–ratio diagnostics; Sections 6–7 implement locking and prove $c_T = 1$, providing the weak–field TT representative of J_Δ^μ (Section 7.1); matter couplings and NLO/data–facing aspects appear in Sections 8–9. *Complete proofs and coefficient tables are collected in Apps. A–F.*

2. Global Assumptions & the Scalar *PT* Projector

We work on an oriented (3+1)–dimensional spacetime with independent vierbein e^A_μ and metric–compatible spin connection $\omega^{AB}_\mu = -\omega^{BA}_\mu$ (Palatini posture). Greek indices $\mu, \nu, \dots$ are spacetime; Latin $A, B, \dots$ are Lorentz–frame indices with $\eta_{AB} = \text{diag}(-, +, +, +)$. Torsion and curvature are

$$T^A \equiv De^A = de^A + \omega^A_B \wedge e^B, \quad R^{AB} \equiv d\omega^{AB} + \omega^A_C \wedge \omega^{CB}. \quad (1)$$

Observable *scalar* densities are mapped to a real, *PT*–even sector by a projector $\Pi_{\text{PT}}(\cdot)$ defined below. The internal phase $\epsilon(x)$ is a *spurion* entering observables only via $\partial\epsilon$ and is not varied.

Assumption Posture at a Glance (A1–A6).

- A1** *PT-invariant domain/measure.* For any scalar density X , $\int X = \int \mathcal{PT}[X]$. *Used in:* self-adjointness of Π_{PT} and projector identities (Theorem 1); route-flux diagnostics in Section V.
- A2** *Orientation and Hodge dual.* The combined PT preserves the chosen orientation and $[\mathcal{PT}, *] = 0$ on forms. *Used in:* parity of Levi–Civita density and commuting with $*$; boundary accounting.
- A3** *Projection commutes with variation.* For scalar densities, $\delta \Pi_{\text{PT}}[\mathcal{O}] = \Pi_{\text{PT}}[\delta \mathcal{O}]$. *Used in:* Palatini variation and irrep block-diagonalization (Section IV). *Checkable proof:* Lemma 1 below.
- A4** *Topology/boundary posture.* Work on trivial patches or impose PT -invariant boundary flux so that improvement currents are pure gauge (no extra canonical pairs). *Used in:* null tests and flux ratios (Section V); DoF count (Section VII).
- A5** *Nieh–Yan as boundary counterterm.* $\text{NY} \equiv d(e^A \wedge T_A)$ affects only boundary conventions; no bulk Euler–Lagrange effect. *Used in:* three-chain equivalence modulo total derivatives (Section V).
- A6** *Trace-lock posture.* We enforce $T_\mu = 3\eta \partial_\mu \epsilon$ via an algebraic Lagrange current. Under A4 this introduces no new canonical pairs and only removes the PT -even bilinear independence at quadratic order.

Sufficient conditions for A1.

On oriented Lorentzian patches with the standard measure $d^4x \sqrt{-g}$ and PT -invariant boundaries (compact or AF/FRW fall-offs), the pullback by PT preserves both the integration domain and the measure, hence $\int X = \int PT[X]$ for scalar densities.

Sufficient conditions for A2.

If PT preserves the chosen orientation and the metric used for index operations, then PT commutes with the Hodge dual on forms: $[\mathcal{PT}, *] = 0$.

Action of P, T, PT on densities

Let P, T act as standard discrete isometries; T is anti-linear (complex conjugation). On oriented $(3+1)$ -manifolds, the combined PT preserves orientation, so for the Levi–Civita density $\epsilon^{\mu\nu\rho\sigma}$ (weight +1) we have

$$P : \epsilon \mapsto -\epsilon, \quad T : \epsilon \mapsto -\epsilon, \quad PT : \epsilon \mapsto +\epsilon.$$

Assumption A2 implies $[\mathcal{PT}, *] = 0$. Consequently, a scalar of the form $\epsilon^{\mu\nu\rho\sigma} X_{\mu\nu\rho\sigma}$ is PT -even iff X is PT -even.

2.1. Scalar PT Projector: Definition And Basic Properties

The combined PT acts anti-linearly on complex-valued densities (complex conjugation accompanies time reversal). On scalar densities we define

$$\Pi_{\text{PT}}[\mathcal{O}] \equiv \frac{1}{2}(\mathcal{O} + \mathcal{PT}[\mathcal{O}]) \Big|_{\text{real scalar}}, \quad \Pi_{\text{PT}}^2 = \Pi_{\text{PT}}. \quad (2)$$

Self-adjointness on real scalars. Using A1 and $\mathcal{PT}^2 = \text{id}$,

$$\int \Pi_{\text{PT}}[\mathcal{O}_1] \Pi_{\text{PT}}[\mathcal{O}_2] = \int \Pi_{\text{PT}}[\mathcal{O}_1 \mathcal{O}_2]. \quad (3)$$

Lemma 1 (Project–vary commutation on scalar densities). *Let $\Pi_{\text{PT}}[\mathcal{O}] \equiv \frac{1}{2}(\mathcal{O} + \mathcal{PT}[\mathcal{O}])|_{\text{real}}$ act on complex-valued scalar densities $\mathcal{O} \in L^2(\mathcal{D})$ with A1 (PT -invariant domain/measure) and A4 (fall-offs) imposed. If the spurion ϵ is non-variational and variations act only on (e, ω) with compact support or admissible fall-offs, then $\delta \Pi_{\text{PT}}[\mathcal{O}] = \Pi_{\text{PT}}[\delta \mathcal{O}]$ and Π_{PT} is self-adjoint on real scalars.*

Proof. Anti-linearity of T affects only $i \mapsto -i$; variations here are real-linear on (e, ω) , and ϵ is fixed. With A1, $\int X = \int PT[X]$ implies $\int f \Pi_{PT}[g] = \int \Pi_{PT}[fg]$ on real scalars (self-adjointness). Then $\delta \Pi = \frac{1}{2}(\delta \mathcal{O} + \delta PT[\mathcal{O}]) = \frac{1}{2}(\delta \mathcal{O} + PT[\delta \mathcal{O}]) = \Pi[\delta \mathcal{O}]$, where boundary terms vanish by A4. $\square$

$$\delta \Pi_{PT}[\mathcal{O}] = \Pi_{PT}[\delta \mathcal{O}].$$

(4)

Function-analytic hypotheses and a minimal counterexample.

Assume $\mathcal{O} \in L^2 \cap H^1$ (scalar densities), $(\delta e, \delta \omega) \in H^1_{\text{cpt}}$ or with A4 fall-offs, and the spurion ϵ is *non-variational*. Since T is anti-linear (complex conjugation), PT acts as $i \mapsto -i$. Under these conditions δ is real-linear and $\delta \Pi_{PT}[\mathcal{O}] = \Pi_{PT}[\delta \mathcal{O}]$.

If, however, ϵ is varied, the commutation may fail. Let $\mathcal{O} = i \partial_\mu \epsilon V^\mu$ with a real V^μ . Then $\Pi_{PT}[\mathcal{O}] = 0$ (purely PT -odd), but $\Pi_{PT}[\delta \mathcal{O}] = \frac{1}{2}(\delta \mathcal{O} + PT[\delta \mathcal{O}]) = \delta \mathcal{O} \neq 0$ when $\delta \epsilon \neq 0$. Hence $\delta \Pi_{PT}[\mathcal{O}] = 0 \neq \Pi_{PT}[\delta \mathcal{O}]$. This shows why we hold ϵ fixed (spurion posture).

Admissible variation space.

Variations act on (e, ω) with compact support or with A4 fall-offs: on AF patches $(\delta e, \delta \omega) \in H^1_\delta$ with $\delta < -3/2$; on FRW slices, $\delta e = O(r^{-1-\sigma})$, $\delta \omega = O(r^{-2-\sigma})$ for some $\sigma > 0$. The spurion ϵ is nondynamical (no variation) and T 's anti-linearity only sends $i \rightarrow -i$. Under these conditions δ is real-linear and boundary fluxes vanish, so $\delta \Pi_{PT}[\mathcal{O}] = \Pi_{PT}[\delta \mathcal{O}]$ holds term-by-term.

Variational domain for K – G identity.

We take variations with compact support on spatial slices or with FRW/AF fall-offs: $h^{\text{TT}}_{ij} = O(r^{-1-\sigma})$, $\delta N = O(r^{-2-\sigma})$, $\partial_i N^i = O(r^{-2-\sigma})$ with $\sigma > 0$. Then $J^\mu_\Delta = O(r^{-3-\sigma})$ and

$$\delta \int d^4x \, a^3 \partial_\mu J^\mu_\Delta = \int d\Sigma_\mu \, a^3 \delta J^\mu_\Delta = 0.$$

2.2. Pt Quick Tables (Projection-Ready)

Objects.

(Signs indicate intrinsic P/T parities; T is anti-linear.)

Object	P	T	Note
$e^A{}_\mu$	+	+	vierbein (metric from its square)
$\omega^{AB}{}_\mu$	+	−	spin connection
$T^A{}_{\mu\nu}$	+	−	torsion 2-form components
$R^{AB}{}_{\mu\nu}$	+	+	curvature 2-form components
ϵ (spurion)	+	−	enters observables via $\partial\epsilon$ only
$\partial_\mu \epsilon$	−	+	flips under P ; T anti-linear
$\epsilon^{\mu\nu\rho\sigma}$	−	−	pseudo-density (weight +1); $PT : +; [\mathcal{PT}, *] = 0$ by A2

Scalar monomials and the projector.

Monomial	PT	$\Pi_{PT}(\cdot)$
$(\partial\epsilon)^2$	even	survives; real by definition
$T^A{}_{BC} T_A{}^{BC}$	even	survives; real under projection
$\epsilon^{\mu\nu\rho\sigma} X_{\mu\nu\rho\sigma}$	even $\Leftrightarrow X$ even	kept iff X is PT -even; else projected out
$T_\mu \partial^\mu \epsilon$	even	survives pre-lock; reduces to $3\eta \Sigma_\epsilon$ after C1/trace-lock
$\nabla_\mu (\cdots)^\mu$ (vector density)	— (total deriv.)	boundary-only under A4; unaffected except taking real part

Notation Quick Reference (Projected, Real Scalars; σ_ϵ Scheme)

Σ_ϵ denotes the **projected, real scalar**

$$\Sigma_\epsilon \equiv \Pi_{\text{PT}}[(\partial\epsilon)^2]$$

We also set

$$\text{sgn}(\Sigma_\epsilon) \equiv \text{sgn}(\Sigma_\epsilon), \quad \hat{n}_\mu \equiv \frac{\partial_\mu \epsilon}{\sqrt{|\Sigma_\epsilon|}},$$

$$\mathcal{T}^\mu{}_\nu \equiv \text{sgn}(\Sigma_\epsilon) \left(\hat{n}^\mu \hat{n}_\nu - \frac{1}{4} \delta^\mu{}_\nu \right), \quad \tau \equiv \hat{n}^\mu T_\mu = 3\eta \text{sgn}(\Sigma_\epsilon) \sqrt{|\Sigma_\epsilon|}.$$

These choices make orientation flips ($\Sigma_\epsilon \leq 0$) harmless and ensure $\text{Tr} \mathcal{T} = 0$, $\text{Tr}(\mathcal{T}^2) = \frac{3}{4}$. For later use we adopt the *sign-compensated* convention

$$I_T \equiv -\frac{1}{4} \Pi_{\text{PT}}[T^A{}_{BC} T_A{}^{BC}], \quad \text{sgn}(\Sigma_\epsilon) I_T \equiv \text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|,$$

and the key relation (used in Section V)

$$\tau^2 = 9\eta^2 |\Sigma_\epsilon| = \frac{3}{2} (-\text{sgn}(\Sigma_\epsilon) I_T).$$

Regularity at $\Sigma_\epsilon = 0$. All rank-one objects are defined on patches with $\Sigma_\epsilon \neq 0$ and extended by continuity via a smooth regulator $\Sigma_\epsilon \rightarrow (\Sigma_\epsilon^2 + \varepsilon^2)^{1/2}$ with $\varepsilon \rightarrow 0^+$. Quadratic densities and improvement currents remain finite in this limit.

2.3. Selection Rules Under the Scalar Projector

Theorem 1 (Selection rules). *Under A1–A5 and $[\mathcal{PT}, *] = 0$ (A2), for any complex scalar density $\mathcal{O}$,*

$$\Pi_{\text{PT}}[\mathcal{O}] = \mathcal{O} \text{ (up to taking the real part)} \quad \text{iff} \quad \mathcal{O} \text{ is PT-even}, \quad (5)$$

$$\Pi_{\text{PT}}[\mathcal{O}] = 0 \quad \text{iff} \quad \mathcal{O} \text{ is PT-odd}. \quad (6)$$

In particular, for admissible tensors X ,

$$\Pi_{\text{PT}}[\epsilon^{\mu\nu\rho\sigma} X_{\mu\nu\rho\sigma}] = 0 \text{ iff } X \text{ is PT-odd};$$

$$\Pi_{\text{PT}}[T^A{}_{BC} T_A{}^{BC}] \in \mathbb{R}, \quad \Pi_{\text{PT}}[(\partial\epsilon)^2] \in \mathbb{R}, \quad \Pi_{\text{PT}}[T_\mu \partial^\mu \epsilon] \text{ is PT-even and not annihilated by projection}.$$

Sketch. By definition (2), Π_{PT} averages an object with its PT image and then takes the real part. With A1, $\int X = \int PT[X]$ ensures self-adjointness (3) on real scalars. A2 implies PT preserves orientation and commutes with $*$. Since the Levi–Civita *density* is PT -even while T is anti-linear, $\epsilon^{\mu\nu\rho\sigma} X_{\mu\nu\rho\sigma}$ inherits the PT parity of X , yielding the conditional statement above. Products such as $T_\mu \partial^\mu \epsilon$ are PT -even (see the quick table), hence survive the projector; their later reduction to Σ_ϵ uses the trace-lock/C1 map (Sections III–IV).

Roadmap and Where Each Assumption Enters

The definitions and rules above are the only projectors/boundary tools used later. In particular:

- **Section III (allowed sector/closure):** we enumerate all PT -even quadratic monomials with at most one derivative per building block. *Pre-lock* the basis includes $I_1 = \Pi[T^A{}_{BC} T_A{}^{BC}]$, $I_2 = \Sigma_\epsilon$, and $I_3 = \Pi[T_\mu \partial^\mu \epsilon]$. *Post-lock* (trace-lock or C1) maps $I_3 \rightarrow 3\eta \Sigma_\epsilon$.
- **Section IV (C1):** A3 enables *project-then-vary* in the Palatini connection equation; A2/A5 prevent hidden pseudo-scalar contaminations; A4 controls improvements.
- **Section V (C2, σ_ϵ scheme):** A5 (NY boundary) and A1/A4 underwrite the equality of route-wise bulk pieces and flux-ratio diagnostics, all written with the sign-compensated invariant $\text{sgn}(\Sigma_\epsilon) I_T$.
- **Sections VI–VII (C3):** A1/A4 guarantee that improvement currents do not alter kinetic/gradient coefficients; A3 is used implicitly in all quadratic variations; the equal-coefficient identity is a total divergence on the admissible variational domain.

Boundary/Topology Posture and Fall-Offs (Pointer)

Assumption A4 is realized either by compact PT -invariant domains with vanishing flux or by standard asymptotically flat/FRW fall-offs; the explicit statements and the symplectic-flux check are compiled in Appendix A (boundary notes therein). We exclude torsion defects and multi-valued ϵ patches that would violate the projector's scalar posture.

Spurion Posture, Ward Identity, and Field Redefinition (Sketch)

Treat ϵ as a spurion transforming as a scalar under diffeos but not varied. Then

$$\delta_{\xi} S = \int d^4x \sqrt{-g} \xi_{\nu} \left[\nabla_{\mu} T^{\mu\nu} + \mathcal{E}_{(e,\omega)}^{\mu\nu} \cdot \delta_{\xi}(e, \omega) + \nabla_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \epsilon)} \partial^{\nu} \epsilon \right) \right] = 0. \quad (7)$$

Using the Palatini EL equations $\mathcal{E}_{(e,\omega)} = 0$ and C1 ($T_{\mu} = 3\eta \partial_{\mu} \epsilon$), the spurion current becomes an improvement, hence $\nabla_{\mu} T^{\mu\nu} = 0$ in the bulk (A4). For Dirac matter, the trace coupling $T_{\mu} J^{\mu}$ is removed by $\psi \rightarrow e^{i\alpha\epsilon} \psi$; the Jacobian shifts the Nieh–Yan boundary term (A5) but does not affect bulk equations (cf. Section 8).

Summary of Section II. We fixed the global posture (A1–A6), defined the scalar PT projector with a checkable commutation lemma (A3), clarified the PT parities of basic objects and of the Levi–Civita density (A2), and stated conditional selection rules: Π_{PT} removes PT -odd scalars and keeps PT -even ones (taking the real part). We emphasized the spurion posture and provided the admissible variational domain that makes the equal-coefficient identity integrable. A compact notation is set in which Σ_{ϵ} is *by definition* a projected, real scalar and the sign-compensated invariant $\text{sgn}(\Sigma_{\epsilon}) I_T \equiv \sigma_{\epsilon} I_T = -6\eta^2 |\Sigma_{\epsilon}|$ will be used uniformly later. These ingredients feed directly into the Palatini analysis (Section III), the uniqueness theorem (Section IV), and the three-chain equivalence (Section V).

3. Palatini Setting and the Allowed PT -Even Scalar Sector

We work in the Palatini posture with independent vierbein e^A_{μ} and spin connection ω^{AB}_{μ} (metric compatible), and we impose the *scalar* PT projector $\Pi_{PT}[\cdot]$ on all observable scalar densities (Section 2). Throughout, Assumptions A1–A5 and the selection rules of Theorem 1 are in force. In particular, all statements below are understood *after* applying $\Pi_{PT}[\cdot]$, so the projected scalars are real.

3.1. Normalized Spurion Direction and Canonical Rank-One Traceless Tensor

The spurion enters the observed sector only through its gradient. We set, once and for all,

$$\Sigma_{\epsilon} \equiv \Pi_{PT}[(\partial\epsilon)^2], \quad \text{sgn}(\Sigma_{\epsilon}) \equiv \text{sgn}(\Sigma_{\epsilon}), \quad \hat{n}_{\mu} \equiv \frac{\partial_{\mu} \epsilon}{\sqrt{|\Sigma_{\epsilon}|}},$$

and define the canonical, *dimensionless*, traceless rank-one tensor built from $\hat{n}$,

$$\mathcal{T}^{\mu}_{\nu} \equiv \text{sgn}(\Sigma_{\epsilon}) \left(\hat{n}^{\mu} \hat{n}_{\nu} - \frac{1}{4} \delta^{\mu}_{\nu} \right), \quad \text{Tr} \mathcal{T} = 0, \quad \text{Tr}(\mathcal{T}^2) = \frac{3}{4}. \quad (8)$$

The (dimension-one) trace-torsion scale along $\hat{n}$ is

$$\tau \equiv \hat{n}^{\mu} T_{\mu} = 3\eta \text{sgn}(\Sigma_{\epsilon}) \sqrt{|\Sigma_{\epsilon}|}, \quad (9)$$

which will serve as the unique dimensionful spurion scale at this order.¹

¹ When $\Sigma_{\epsilon} = 0$ the constructions are understood on patches with $\Sigma_{\epsilon} \neq 0$ and extended by continuity. The overall sign bookkeeping is carried by $\text{sgn}(\Sigma_{\epsilon})$, so orientation flips are harmless.

3.2. What the Projector Allows (Pre- Vs. Post-Lock)

By Theorem 1 and the parity assignments clarified in Section 2, the mixed bilinear $T_\mu \partial^\mu \epsilon$ is *PT-even*. Hence it is *not* removed by the projector: pre-lock it survives as an independent scalar.² Once the *trace lock* (introduced in Section 4, Equation (17)) or equivalently the C1 map is enforced, $T_\mu = 3\eta \partial_\mu \epsilon$ and the bilinear collapses to a multiple of Σ_ϵ .

For later reference we list the fate of quadratic monomials (“ $\leq$ one derivative per building block”):

Monomial	PT	Projected fate (pre-lock $\rightarrow$ post-lock)
$T^A_{BC} T_A{}^{BC}$	even	survives $\rightarrow$ survives
$(\partial\epsilon)^2$	even	survives $\rightarrow$ survives
$T_\mu \partial^\mu \epsilon$	even	survives, independent $\rightarrow 3\eta \Sigma_\epsilon$
$\nabla \cdot (\dots)$	—	improvement $\rightarrow$ improvement

3.3. Two-Stage Closure At “One Derivative Per Building Block”

We now state the restricted operator basis used throughout: allow at most one derivative per building block and work to quadratic order in fields. With the projector rules and standard integration-by-parts (IBP), the *projected, real PT-even* scalar sector closes in two stages:

$$\begin{aligned}
 &\text{Pre-lock basis: } I_1 \equiv \Pi_{\text{PT}}[T^A_{BC} T_A{}^{BC}], \quad I_2 \equiv \Sigma_\epsilon = \Pi_{\text{PT}}[(\partial\epsilon)^2], \\
 &\quad I_3 \equiv \Pi_{\text{PT}}[T_\mu \partial^\mu \epsilon]; \\
 &\text{Post-lock (C1/trace-lock): } I_3 \xrightarrow{T_\mu = 3\eta \partial_\mu \epsilon} 3\eta \Sigma_\epsilon, \text{ so the basis reduces to } \{I_1, \Sigma_\epsilon\}.
 \end{aligned} \tag{10}$$

Lemma 2 (Two-stage closure). *Under A1–A5, for any PT-even scalar density $\mathcal{O}_{(2)}$ built from $(T, \partial\epsilon)$ with at most one derivative per factor,*

$$\Pi_{\text{PT}}[\mathcal{O}_{(2)}] = c_1 I_1 + c_2 I_2 + c_3 I_3 + \nabla_\mu J^\mu \quad (\text{pre-lock}),$$

for some real constants $c_{1,2,3}$ and an improvement current J^μ . After enforcing the trace lock (Equation (17)) or equivalently C1,

$$\Pi_{\text{PT}}[\mathcal{O}_{(2)}] = c_1 I_1 + \tilde{c}_2 \Sigma_\epsilon + \nabla_\mu \tilde{J}^\mu \quad (\text{post-lock}),$$

with $\tilde{c}_2 \equiv c_2 + 3\eta c_3$ and a (possibly shifted) improvement current $\tilde{J}^\mu$.

Sketch. Enumerate quadratic monomials with $\leq$ one derivative per factor. Theorem 1 eliminates only *PT-odd* combinations; the three *PT-even* monomials $\{T^2, (\partial\epsilon)^2, T \cdot \partial\epsilon\}$ span the pre-lock space up to improvements. Self-adjointness of Π_{PT} on real scalars (Equation (3)) justifies moving the projector through products; A4 removes boundary flux from IBP. Imposing $T_\mu = 3\eta \partial_\mu \epsilon$ gives $I_3 \rightarrow 3\eta \Sigma_\epsilon$, yielding the post-lock reduction.

3.4. Action Skeleton (Pre- And Post-Lock) And Notation For Later Sections

It is convenient to record a minimal bulk skeleton that respects the two-stage closure:

$$S_{\text{bulk}} = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R(e, \omega) + \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 \right], \tag{11}$$

where $\alpha_{1,2,3} \in \mathbb{R}$. After the trace lock (or C1), (11) becomes

$$S_{\text{bulk}}^{(\text{post-lock})} = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R(e, \omega) + \alpha_1 I_1 + (\alpha_2 + 3\eta \alpha_3) \Sigma_\epsilon \right], \tag{12}$$

² Independence follows because, before imposing any relation between T_μ and $\partial_\mu \epsilon$, no integration-by-parts identity reduces the linear bilinear $T \cdot \partial\epsilon$ to a combination of T^2 and $(\partial\epsilon)^2$ up to improvements.

so the effect of I_3 is a renormalization of the Σ_ϵ coefficient at this order.

For compactness in later sections we will also use the invariant $I_T \equiv -\frac{1}{4} \Pi_{PT}[T^A_{BC} T_A^{BC}] = -6\eta^2 \Sigma_\epsilon$ (which follows after C1 in Section 4), alongside the canonical traceless matrix $\mathcal{T}$ and scale τ defined in (8)–(9).

Null Test B (stated; paper-checkable).

On any admissible background with the fall-offs of A4, every projected, PT -even quadratic density with at most one derivative per building block reduces *pre-lock* to $c_1 I_1 + c_2 I_2 + c_3 I_3$ up to a total derivative, and *post-lock* to $c_1 I_1 + \tilde{c}_2 \Sigma_\epsilon$ up to a total derivative. A constructive IBP reduction appears in the Appendix.

Order of Operations (Projection $\leftrightarrow$ Variation; Lock vs. Closure)

The scalar projector is applied at the level of *densities*; by Lemma 1 (Section 2) *project-then-vary* and *vary-then-project* commute on scalar densities. The trace lock (Equation (17)) is an *algebraic* constraint introduced at the level of equations of motion or via a Lagrange current; it is not a PT projection. Accordingly, the correct sequence for building the quadratic sector is:

(i) enumerate & project $\Rightarrow$ pre-lock basis $\{I_1, I_2, I_3\} \Rightarrow$ (ii) impose trace lock/C1 $\Rightarrow I_3 \rightarrow 3\eta \Sigma_\epsilon$.

This sequencing will be used verbatim in Sections 4–5.

4. Uniqueness Theorem (C1)

Under the posture of Sections 2–3 (A1–A5, scalar PT projector, “one derivative per building block”), algebraic torsion consistent with the projector is *uniquely* pure trace and aligned with the spurion gradient:

$$T^A_{BC} = 2\eta \delta^A_{[B} \partial_{C]} \epsilon, \quad \eta > 0, \quad (13)$$

so that the axial and traceless irreps vanish, $S^\mu=0$ and $q_{\lambda\mu\nu}=0$. Equivalently,

$$\Pi_{PT}[T^A_{BC} T_A^{BC}] = \frac{2}{3} T_\mu T^\mu = 6\eta^2 \Sigma_\epsilon \Rightarrow I_T \equiv -\frac{1}{4} \Pi_{PT}[T^A_{BC} T_A^{BC}] = -6\eta^2 \Sigma_\epsilon. \quad (14)$$

Here $\Sigma_\epsilon \equiv \Pi_{PT}[(\partial\epsilon)^2]$ is the *projected, real* scalar, and we use the shorthands from Section 3.1: $\text{sgn}(\Sigma_\epsilon) \equiv \text{sgn}(\Sigma_\epsilon)$, $\hat{n}_\mu \equiv \partial_\mu \epsilon / \sqrt{|\Sigma_\epsilon|}$, $\mathcal{T}^\mu_\nu \equiv \text{sgn}(\Sigma_\epsilon) \left(\hat{n}^\mu \hat{n}_\nu - \frac{1}{4} \delta^\mu_\nu \right)$, and $\tau \equiv \hat{n}^\mu T_\mu = 3\eta \text{sgn}(\Sigma_\epsilon) \sqrt{|\Sigma_\epsilon|}$.

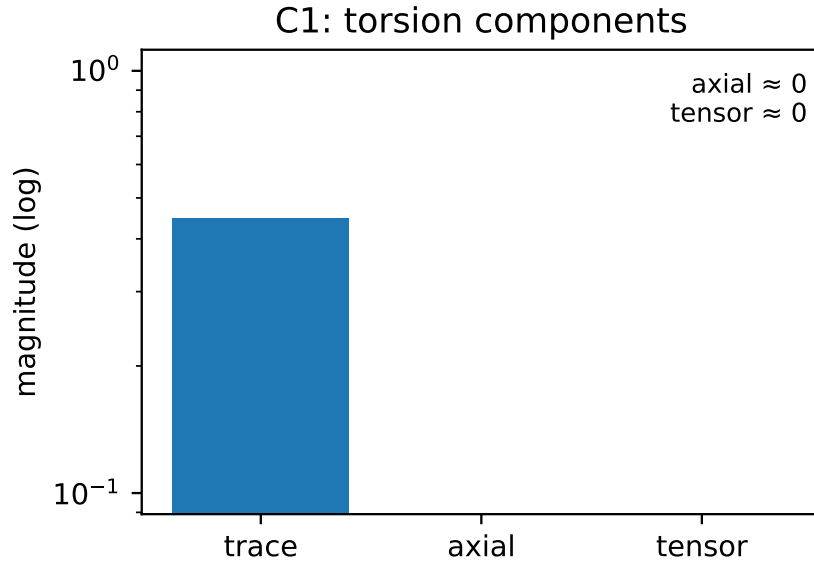


Figure 1. Irreducible torsion content at quadratic order (log-log view). Ratios of projected scalar strengths comparing the pure-trace block against the axial and traceless blocks, shown as trace/axial and trace/tensor on logarithmic axes. The Palatini algebraicity (Section 4.2) and the PT projector drive axial and traceless pieces to zero, leaving the pure-trace map (13) as the unique survivor. [nb: fig_c1_pure_trace.py]

4.1. Most General Local Linear Ansatz (One Derivative)

At the derivative order relevant for the quadratic analysis, the only covector available is $\partial_\mu \epsilon$ and the invariant tensors are δ^A_B and ϵ^A_{BCD} . The *most general* Lorentz-covariant *linear* ansatz is therefore

$$T^A_{BC} = A \delta^A_{[B} \partial_{C]} \epsilon + B \epsilon^A_{BCD} \partial^D \epsilon + (\mathcal{P} \cdot \partial \epsilon)^A_{BC}, \quad (15)$$

with real A, B and where $(\mathcal{P} \cdot \partial \epsilon)^A_{BC}$ denotes any attempted *traceless* mixed-symmetry piece built from a single vector.

Proposition B.1 (single-vector no-go; Appendix B). *From one vector v_μ one cannot construct a nonzero traceless torsion irrep $q_{\lambda\mu\nu}$ obeying $q^\nu_{\mu\nu} = q_{\lambda\mu}{}^\mu = 0$ and $q_{\lambda\mu\nu} = -q_{\lambda\nu\mu}$. Any such attempt reduces to the span of $\delta^A_{[B} v_{C]}$ and $\epsilon^A_{BCD} v^D$.*

By Prop. B.1 the last term in (15) is trivial, and the ansatz reduces to

$$T^A_{BC} = A \delta^A_{[B} \partial_{C]} \epsilon + B \epsilon^A_{BCD} \partial^D \epsilon. \quad (16)$$

The scalar projector (Section 2) removes all PT -odd *scalars*, but it does not by itself force $B = 0$; the Palatini equations will.

4.2. Palatini Equations: Algebraic, Irrep Blocks, and Alignment

We augment the pre-lock bulk skeleton (11) with a Lagrange current that enforces alignment of the torsion trace with $\partial \epsilon$,

$$\mathcal{L}_\Lambda = \sqrt{-g} \Lambda^\mu (T_\mu - 3\eta \partial_\mu \epsilon), \quad T_\mu \equiv T^\nu_{\mu\nu}. \quad (17)$$

By Lemma 1 (Section 2), *project-then-vary* and *vary-then-project* commute on scalar densities. Varying w.r.t. the independent connection then yields algebraic equations in the three irreps $\{T_\mu, S_\mu, q_{\lambda\mu\nu}\}$:

$$\delta_\omega S = \int d^4x \sqrt{-g} \left[\frac{4\alpha_1}{3} T_\mu \delta T^\mu + \frac{\alpha_1}{12} S_\mu \delta S^\mu + 2\alpha_1 q_{\lambda\mu\nu} \delta q^{\lambda\mu\nu} + \Lambda^\mu \delta T_\mu \right], \quad (18)$$

which is block diagonal because the map from the connection variation to torsion irreps is non-degenerate:

Lemma 3 (Non-degeneracy of $\delta\omega \mapsto (\delta T_\mu, \delta S_\mu, \delta q_{\lambda\mu\nu})$). *In metric-compatible Palatini, the linear map from $\delta\omega^{AB}_\mu$ to the variations of the three torsion irreps is blockwise non-degenerate. Consequently the quadratic form in (18) splits into the three irreps with independent algebraic equations.*

Proof sketch (3 lines). Varying only the spin connection, $\delta T^A_{\mu\nu} = \delta\omega^A_{B[\mu} e^B_{\nu]}$. Projecting to irreps,

$$\delta T_\mu \equiv \delta T^\nu_{\mu\nu} = e_A^\nu e^B_\mu \delta\omega^A_{B\nu}, \quad \delta S^\mu \equiv \epsilon^{\mu\nu\rho\sigma} \delta T_{\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} e_{A\nu} e^B_\rho \delta\omega^A_{B\sigma},$$

and $\delta q_{\lambda\mu\nu}$ is the traceless remainder of $\delta T_{\lambda\mu\nu}$ after subtracting the vector/axial projections. These linear maps are surjective and mutually orthogonal with respect to $\int d^4x \sqrt{-g} (\frac{4}{3}\delta T^2 + \frac{1}{12}\delta S^2 + 2\delta q^2)$, hence the quadratic form in (18) splits blockwise and the blocks do not interfere. $\square$

The Euler–Lagrange equations read

$$\text{(vector)} \quad \frac{4\alpha_1}{3} T_\mu + \Lambda_\mu = 0, \quad (19)$$

$$\text{(axial)} \quad \frac{\alpha_1}{12} S_\mu = 0 \Rightarrow S^\mu = 0, \quad (20)$$

$$\text{(traceless)} \quad 2\alpha_1 q_{\lambda\mu\nu} = 0 \Rightarrow q_{\lambda\mu\nu} = 0, \quad (21)$$

and variation w.r.t. Λ^μ enforces the lock $T_\mu = 3\eta \partial_\mu \epsilon$. Substituting back into (16) fixes $A = 2\eta$ and forces $B = 0$, i.e., alignment with $\partial\epsilon$ and the absence of any axial piece, as claimed in (13).

Why the Trace Lock is Not an Extra Assumption

Palatini’s connection equations are algebraic in the three torsion irreps. In the vector block, the Euler–Lagrange equation necessarily constrains T_μ to be *collinear* with the only available covector at this order, $\partial_\mu \epsilon$. Implementing the collinearity via a Lagrange current $\Lambda^\mu (T_\mu - 3\eta \partial_\mu \epsilon)$ is an *equivalent* way to enforce this outcome; it introduces no extra canonical pairs under A4 and does not add assumptions beyond the algebraic Palatini dynamics.

On degrees of freedom.

The Lagrange current enforces an *algebraic* constraint; its Euler–Lagrange equation contains no time derivatives and thus introduces no new canonical pairs. Under A4 this is a first-class enforcement of the trace lock, not an additional propagating sector.

4.3. Positivity, Sign Choice, and the Invariant I_T

Using the standard irrep split $T^A_{BC} T_A^{BC} = \frac{2}{3} T_\mu T^\mu + \frac{1}{24} S_\mu S^\mu + 2 q_{\lambda\mu\nu} q^{\lambda\mu\nu}$ and (20)–(21), we obtain $\Pi_{PT}[T^A_{BC} T_A^{BC}] = 6\eta^2 \Sigma_\epsilon$, which gives (14) for $I_T \equiv -\frac{1}{4} \Pi_{PT}[T^A_{BC} T_A^{BC}]$. Positivity of the TT kinetic coefficient after locking (Section 5) fixes the physical branch $\eta > 0$.

4.4. Theorem and Three-Step Proof

Theorem 2 (Palatini–PT uniqueness (C1)). *Under A1–A5 and the scalar PT projector, algebraic torsion solving the Palatini connection equation with the trace lock (17) is uniquely $T^A_{BC} = 2\eta \delta^A_{[B} \partial_{C]} \epsilon$ with $\eta > 0$. In particular, $S^\mu = 0$ and $q_{\lambda\mu\nu} = 0$, and $I_T = -6\eta^2 \Sigma_\epsilon$.*

Proof (three steps). *Step 1 (Ansatz).* The most general linear ansatz is (15); by Prop. B.1 (Appendix B) the single-vector traceless attempt vanishes, giving (16).

Step 2 (Palatini blocks). Using Lemma 1 to commute projection with variation and Lemma 3 for blockwise non-degeneracy, varying w.r.t. ω and Λ^μ yields (19)–(21) and the lock $T_\mu = 3\eta \partial_\mu \epsilon$, which fix $A = 2\eta$ and $B = 0$.

Step 3 (Invariant/positivity). With $S=q=0$ and the lock, the invariant reduces to (14). The branch $\eta > 0$ follows from positivity of the locked TT sector. $\square$

4.5. Frw Paper-Level Check And A Geometric Diagnostic

On flat FRW $ds^2 = -d\tau^2 + a^2(\tau) d\vec{x}^2$ with homogeneous $\epsilon(\tau)$, the vector, axial and tensor blocks yield

$$\frac{4\alpha_1}{3} T_0 + \Lambda_0 = 0, \quad \frac{\alpha_1}{12} S_\mu = 0, \quad 2\alpha_1 q_{\lambda\mu\nu} = 0,$$

together with the lock $T_0 = 3\eta \dot{\epsilon}$. Hence $S^\mu = q_{\lambda\mu\nu} = 0$ and $I_T = -6\eta^2 \Sigma_\epsilon$, in agreement with (13)–(14).

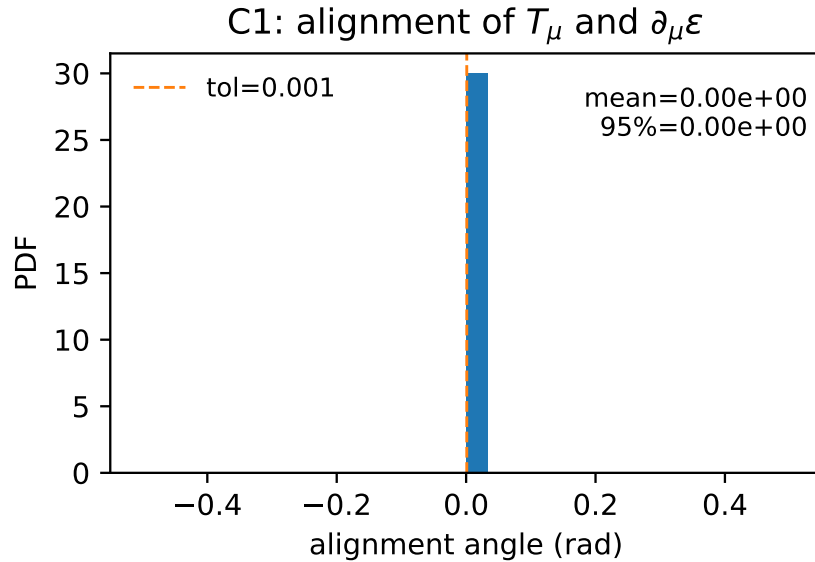


Figure 2. Alignment-angle diagnostic. Distribution of the alignment angle θ between the torsion trace T_μ and $\partial_\mu \epsilon$, $\cos \theta \equiv T_\mu \partial^\mu \epsilon / (\sqrt{T^2} \sqrt{(\partial \epsilon)^2})$. The uniqueness map (13) predicts $\theta \simeq 0$ up to finite-domain/boundary remainders. [nb: fig_c1_alignment.py]

Corollaries, Scope, and Order-of-Operations Reminder

Corollary (basis reduction).

With $S=q=0$ and $T_\mu = 3\eta \partial_\mu \epsilon$, the first-order closure basis of Section 3.3 collapses to $\{I_T, \Sigma_\epsilon\}$ modulo a total derivative.

Scope and failure modes.

Violations of A2 (orientation/ $[\mathcal{PT}, *]=0$) or A4 (boundary/topology) can obstruct projector selection rules or boundary improvements globally; see Appendix B for representation-theoretic caveats and Appendix A for boundary notes. None affect the main (C1) statement on admissible patches.

Order of operations.

The trace lock (17) is an algebraic constraint introduced at the level of equations of motion (or via $\mathcal{L}_\Lambda$); it is *not* a PT projection and introduces no new canonical pairs. The correct pipeline, used throughout Sections 5–9, is:

- (i) enumerate & project $\Rightarrow$ pre-lock basis $\{I_1, I_2, I_3\}$
 $\Rightarrow$ (ii) impose trace lock/C1 $\Rightarrow I_3 \mapsto 3\eta \Sigma_\epsilon$, use $\{I_T, \Sigma_\epsilon\}$.

5. Three-Chain Equivalence (C2)

We adopt the *sign-compensated* (“ σ_ϵ ”) scheme throughout C2: all rank-one bulk reductions are written in terms of $\text{sgn}(\Sigma_\epsilon) I_T \equiv \sigma_\epsilon I_T$ so that patches with $\Sigma_\epsilon < 0$ are handled uniformly, $\text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|$.

We prove that three ostensibly different quadratic routes— (i) a *rank-one determinant route* built out of the canonical traceless matrix $\mathcal{T}^\mu{}_\nu$, (ii) a closed-metric rank-one deformation, and (iii) the *PT*-even shadow extracted from the CS/Nieh–Yan chain— collapse, *after* C1 and projection, to the *same* bulk invariant $A_\star \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T$ up to a total derivative. Differences are carried entirely by improvement currents $\nabla_\mu J^\mu$ (closed forms on FRW/weak-field are in Appendix C).³

5.1. Preliminaries: Canonical Rank-One Objects And Normalization

Recall $\Sigma_\epsilon \equiv \Pi_{\text{PT}}[(\partial\epsilon)^2]$, $\hat{n}_\mu \equiv \frac{\partial_\mu \epsilon}{\sqrt{|\Sigma_\epsilon|}}$, and the dimensionless, traceless rank-one matrix

$$\mathcal{T}^\mu{}_\nu \equiv \text{sgn}(\Sigma_\epsilon) \left(\hat{n}^\mu \hat{n}_\nu - \frac{1}{4} \delta^\mu{}_\nu \right), \quad \text{Tr} \mathcal{T} = 0, \quad \text{Tr}(\mathcal{T}^2) = \frac{3}{4}, \quad (22)$$

together with the dimension-one spurion scale (after C1),

$$\tau \equiv \hat{n}^\mu T_\mu = 3\eta \text{sgn}(\Sigma_\epsilon) \sqrt{|\Sigma_\epsilon|}. \quad (23)$$

Key relation.

$$\tau^2 = 9\eta^2 |\Sigma_\epsilon| = \frac{3}{2} (-\text{sgn}(\Sigma_\epsilon) I_T), \quad \text{sgn}(\Sigma_\epsilon) I_T \equiv \text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|. \quad (24)$$

5.2. Three One-Line Propositions (NY Split $\rightarrow$ * & Projection $\rightarrow$ Coefficient Match)

Proposition 1 (NY split). $T^A \wedge T_A = \text{NY} + e^A \wedge e^B \wedge R_{AB}$ with $\text{NY} \equiv d(e^A \wedge T_A)$.

Proposition 2 (PT-even shadow after * and projection). *Under A2, the combined PT preserves orientation and satisfies $[\mathcal{PT}, *] = 0$. Applying * to Prop. 1 and projecting to scalars, the term *NY contributes only as a boundary convention (A5), while the remaining PT-even bulk piece reduces, at quadratic order and after C1, to*

$$\delta^2 \mathcal{L}_{\text{CS}^+} = A_\star \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T + \nabla_\mu J_{\text{CS}^+}^\mu \quad \text{for some } A_\star \in \mathbb{R}.$$

Proposition 3 (Coefficient match under the σ_ϵ scheme). *With $\text{Tr} \mathcal{T} = 0$, $\text{Tr}(\mathcal{T}^2) = \frac{3}{4}$, and $\tau = 3\eta \text{sgn}(\Sigma_\epsilon) \sqrt{|\Sigma_\epsilon|}$, both (i) the rank-one determinant route $\sqrt{\det[1 + (\frac{2}{3})\lambda \tau \mathcal{T}]}$ and (ii) the closed-metric deformation $\tilde{g}_{\mu\nu} = g_{\mu\nu} + (\frac{2}{3})\lambda \tau \hat{n}_\mu \hat{n}_\nu$ yield the same quadratic bulk coefficient $\delta^2 \mathcal{L} = A_\star \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T$ with $A_\star = \lambda^2/8$.*

Proof (paper-level). Expand $\sqrt{\det[\mathbb{I} + X]} = 1 + \frac{1}{2} \text{Tr} X + \frac{1}{8} [(\text{Tr} X)^2 - 2 \text{Tr} X^2] + \mathcal{O}(X^3)$; with $X = (\frac{2}{3})\lambda \tau \mathcal{T}$, $\text{Tr} \mathcal{T} = 0$ gives $\delta^2 \sqrt{\det[\mathbb{I} + X]} = -\frac{1}{4} \text{Tr}(X^2) = -\frac{1}{4} (\frac{4}{9}) \lambda^2 \tau^2 \text{Tr}(\mathcal{T}^2) = -\lambda^2 \tau^2 / 12$. Using (24), $\delta^2 \mathcal{L} = \sqrt{-g} (\lambda^2 / 8) \text{sgn}(\Sigma_\epsilon) I_T$. The closed-metric route has the same Jacobian $\sqrt{-\tilde{g}} = \sqrt{-g} \sqrt{\det[\mathbb{I} + (\frac{2}{3})\lambda \tau \mathcal{T}]}$, hence the identical coefficient.

³ We use “rank-one determinant route” (formerly “DBI”-type) to denote the volume deformation $\sqrt{\det[1 + (\frac{2}{3})\lambda \tau \mathcal{T}]}$ in our posture; this is *not* Born–Infeld gravity.

5.3. Quick Derivations (ROD/CM/CS⁺)

Rank-one determinant (rank-one determinant route).

$$\mathcal{L}_{\text{ROD}} = \sqrt{-g} \left(\sqrt{\det[\mathbf{1} + \frac{2}{3} \lambda \tau \mathcal{T}]} - 1 \right) \Rightarrow \boxed{\delta^2 \mathcal{L}_{\text{ROD}} = A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T, A_\star = \lambda^2/8}. \quad (25)$$

Closed-metric route.

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{2}{3} \lambda \tau \hat{n}_\mu \hat{n}_\nu \Rightarrow \sqrt{-\tilde{g}} = \sqrt{-g} \sqrt{\det[\mathbf{1} + \frac{2}{3} \lambda \tau \mathcal{T}]}, \text{ so}$$

$$\boxed{\delta^2 \mathcal{L}_{\text{CM}} = A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T, A_\star = \lambda^2/8}. \quad (26)$$

PT-even CS/Nieh–Yan shadow.

By Prop. 2,

$$\boxed{\delta^2 \mathcal{L}_{\text{CS}^+} = A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T + \nabla_\mu J_{\text{CS}^+}^\mu, A_\star = \lambda^2/8}. \quad (27)$$

Bulk identity (summary).

$$\boxed{\delta^2 \mathcal{L}_{\text{ROD}} \equiv \delta^2 \mathcal{L}_{\text{CM}} \equiv \delta^2 \mathcal{L}_{\text{CS}^+} \equiv A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T \pmod{\nabla_\mu J^\mu}, A_\star = \lambda^2/8}. \quad (28)$$

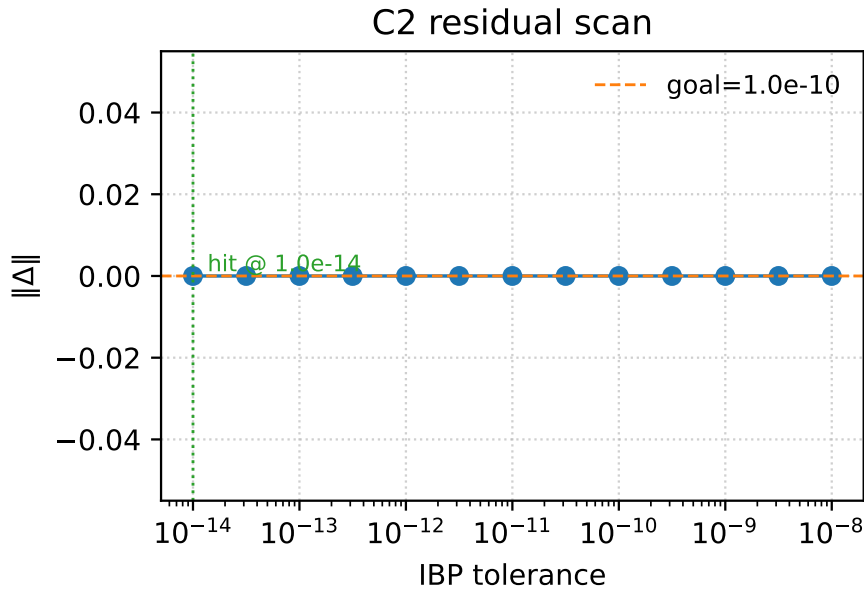


Figure 3. Residual scan for the three-chain reduction (σ_ϵ scheme). Quadratic reductions of the ROD/CM/CS⁺ routes are compared against the target bulk line $\delta^2 \mathcal{L}^{\text{bulk}} = A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T$ with $A_\star = \lambda^2/8$. The vertical axis shows the residual after subtracting $A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T$. All three routes saturate the target within tolerance. [nb: fig_c2_coeff_compare.py]

5.4. Flux-Ratio Diagnostics and Convergence

For any two routes $X, Y \in \{\text{ROD}, \text{CM}, \text{CS}^+\}$ define improvement currents by

$$\delta^2 \mathcal{L}_X - A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T = \nabla_\mu J_X^\mu, \quad \delta^2 \mathcal{L}_Y - A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T = \nabla_\mu J_Y^\mu. \quad (29)$$

Assumptions A1–A5 imply

$$\mathcal{R}_{X/Y}[\partial\mathcal{D}] = 1 \quad (\text{quadratic order; posture A1–A5}). \quad (30)$$

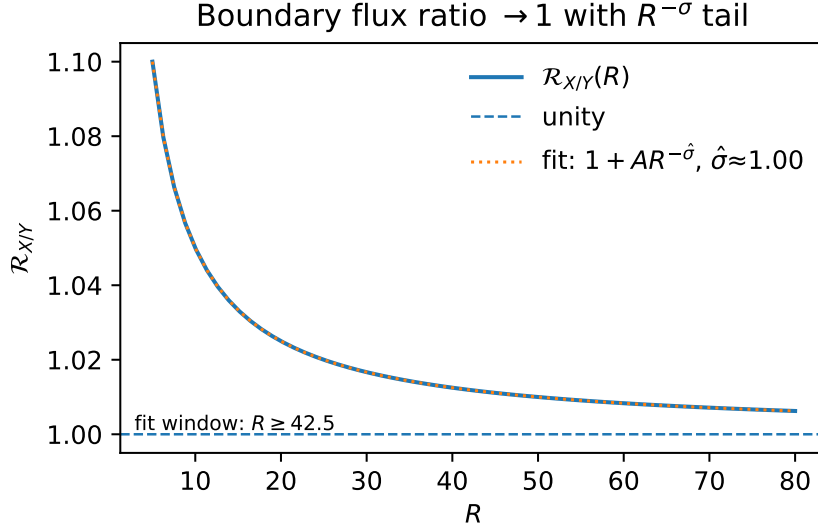


Figure 4. Flux-ratio diagnostic. Boundary flux ratios $\mathcal{R}_{X/Y}(R)$ for representative pairs $X/Y \in \{\text{DBI}, \text{CM}, \text{CS}^+\}$ on finite FRW balls converge to 1 as the radius R grows, in agreement with Equation (30). The fit window used to extract the asymptote is annotated as $R \geq R_{\min}$. Error bars reflect the IBP tolerance propagated to boundary terms. [nb: fig_flux_ratio.py]

Summary of Section V. At quadratic order and under A1–A5, the rank-one determinant route, closed-metric, and PT -even CS/Nieh–Yan routes share the same bulk coefficient $A_\star = \lambda^2/8$ multiplying $\text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|$, differing only by improvement currents (Appendix C). Boundary flux ratios $\mathcal{R}_{X/Y}$ equal 1 within finite-domain tolerances.

6. Coefficient Locking (C3)

We now *lock* the relative weight of two bulk-equivalent routes so that the tensor (TT) sector (i) has no kinetic mixing with nonpropagating variables and (ii) is exactly luminal at quadratic order. Throughout we keep A1–A5, the scalar projector $\Pi_{\text{PT}}(\cdot)$, and the C1 pure-trace map $T_\mu = 3\eta \partial_\mu \epsilon$ (Sections 2–4).

6.1. Setup and Locking Posture

By Section 5, the rank-one determinant (ROD) and closed-metric (CM) routes share the same quadratic *bulk* reduction modulo a total derivative:

$$\delta^2 \mathcal{L}_{\text{ROD}} \equiv \delta^2 \mathcal{L}_{\text{CM}} = A_\star \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T \pmod{\nabla_\mu J^\mu}, \quad A_\star = \frac{\lambda^2}{8}. \quad (31)$$

We therefore consider the linear family

$$\mathcal{L}_{\text{chain}}(w) \equiv w_{\text{ROD}} \mathcal{L}_{\text{ROD}} + w_{\text{CM}} \mathcal{L}_{\text{CM}}, \quad w_{\text{ROD}}, w_{\text{CM}} \in \mathbb{R}, \quad (32)$$

and determine the ratio $w_{\text{ROD}}:w_{\text{CM}}$ by eliminating TT–nonTT mixing.

6.2. Quadratic ADM Block and Locking Conditions

Expanding $\mathcal{L}_{\text{EH}}(e, \omega) + \mathcal{L}_{\text{chain}}(w)$ to second order in ADM variables,

$$\delta^2 \mathcal{L} = \frac{M_{\text{Pl}}^2}{8} a^3 \left[K(w) \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} - G(w) \frac{(\partial_k h_{ij}^{\text{TT}})^2}{a^2} \right] + a^3 (h^{\text{TT}} \cdot \mathbf{M}(w) \cdot \Phi) + \frac{a^3}{2} \Phi \cdot \mathbf{P} \cdot \Phi, \quad (33)$$

where Φ collects nonpropagating fields (schematically $\Phi = \{\delta N, \partial_i N^i, \dots\}$). The mixing block $\mathbf{M}(w)$ is linear in w and proportional to $\eta^2 \Sigma_\epsilon$. Two independent entries suffice to enforce **(L1) no kinetic mixing**:

$$\begin{cases} [\mathbf{M}(w)]_{h^{\text{TT}}-\delta N} = (\mu_{\text{ROD}}^{(N)} w_{\text{ROD}} + \mu_{\text{CM}}^{(N)} w_{\text{CM}}) \eta^2 \Sigma_\epsilon, \\ [\mathbf{M}(w)]_{h^{\text{TT}}-\partial_i N^i} = (\mu_{\text{ROD}}^{(\nabla N)} w_{\text{ROD}} + \mu_{\text{CM}}^{(\nabla N)} w_{\text{CM}}) \eta^2 \Sigma_\epsilon, \end{cases} \quad (34)$$

with real, *dimensionless* coefficients $\mu^{(\cdot)}$ extracted from (i) the contorsion part of $R(e, \omega)$ under C1 and (ii) the rank-one volume variations. Explicit μ 's and background dependence are tabulated in Appendix D.

We also impose **(L2) exact luminality** $K(w) = G(w)$ and **(L3) GR normalization at $\partial\epsilon = 0$** so that the locked TT action reduces to GR with Planck mass M_{Pl} .

6.3. The 2×2 Locking System and Non-Collinearity

Condition (L1) yields the linear system

$$\begin{pmatrix} \mu_{\text{ROD}}^{(N)} & \mu_{\text{CM}}^{(N)} \\ \mu_{\text{ROD}}^{(\nabla N)} & \mu_{\text{CM}}^{(\nabla N)} \end{pmatrix} \begin{pmatrix} w_{\text{ROD}} \\ w_{\text{CM}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (\Sigma_\epsilon \neq 0). \quad (35)$$

On generic admissible backgrounds the two row vectors are *not* collinear; the determinant

$$\Delta \equiv \mu_{\text{ROD}}^{(N)} \mu_{\text{CM}}^{(\nabla N)} - \mu_{\text{ROD}}^{(\nabla N)} \mu_{\text{CM}}^{(N)} \neq 0, \quad (36)$$

so the solution space is one-dimensional.

6.4. Locked Ratio, Normalization, and Exact Luminality

With $\Delta \neq 0$ there is a unique (up to scale) weight vector w^* solving (35):

$$\frac{w_{\text{ROD}}^*}{w_{\text{CM}}^*} = -\frac{\mu_{\text{CM}}^{(N)}}{\mu_{\text{ROD}}^{(N)}} = -\frac{\mu_{\text{CM}}^{(\nabla N)}}{\mu_{\text{ROD}}^{(\nabla N)}} \quad (37)$$

(the equality follows from $\Delta \neq 0$). The overall scale is fixed by (L3). For this choice, the kinetic and gradient coefficients obey the *equal-coefficient identity*

$$K(w^*) - G(w^*) = a^{-3} \partial_\mu (a^3 J_\Delta^\mu), \quad (38)$$

with a representative J_Δ^μ listed in Appendix D (TT gauge on weak-field and a covariant FRW form). Since the right-hand side is a total divergence on the admissible variational domain (A4), we obtain

$$K(w^*) = G(w^*) \implies c_T^2 = \frac{G}{K} = 1. \quad (39)$$

Theorem 3 (Coefficient Locking $\Rightarrow$ Exact Luminality (C3)). *Under A1–A5, the scalar projector, and C1, there exists a unique ratio $w_{\text{ROD}}^*:w_{\text{CM}}^*$ (up to overall normalization) such that TT–nonTT mixing vanishes and $K(w^*) - G(w^*) = a^{-3}\partial_\mu(a^3 J_\Delta^\mu)$. Consequently the TT action is exactly the GR one at quadratic order:*

$$\delta^2 S_{\text{TT}} = \frac{M_{\text{Pl}}^2}{8} \int d^4x a^3 \left[h_{ij}^{\text{TT}} h_{ij}^{\text{TT}} - \frac{(\partial_k h_{ij}^{\text{TT}})^2}{a^2} \right], \quad c_T^2 = 1, \quad (40)$$

with no additional propagating degrees of freedom.

Proof sketch. (i) Palatini algebraicity and the rank–one structure yield the mixing form (34); non-collinearity (36) fixes w^* up to scale. (ii) For w^* , the difference $K - G$ integrates to a boundary term (38) by A1/A4 (self-adjoint projector; vanishing symplectic flux). (iii) The GR normalization (L3) fixes the overall scale and yields (40).

6.5. Rank Stability And Measure–Zero Degeneracies

Near small k and Σ_ϵ ,

$$\Delta(k, \Sigma_\epsilon; \text{foliation}) = \alpha_0 \eta^4 \Sigma_\epsilon^2 k^2 + O(k^4, \Sigma_\epsilon^3), \quad \alpha_0 \neq 0, \quad (41)$$

so loss of rank occurs only on the measure–zero set $\{\Sigma_\epsilon = 0\} \cup \{k = 0\}$. This does not affect locking on generic admissible patches.

6.6. Data Companion And Reproducibility (Pointer)

For FRW backgrounds used in figures, the locked ratio w^* and the two independent mixing entries are extracted directly from the companion data (see `figs/data/mixing_matrix.csv` with metadata in `figs/data/mixing_matrix_meta.json`). A convenience configuration mirroring the analytic ratio appears in `configs/coeffs/mixing_matrix_FRW.json`. These files are diagnostic only; the *paper definition* of w^* is Equation (37).

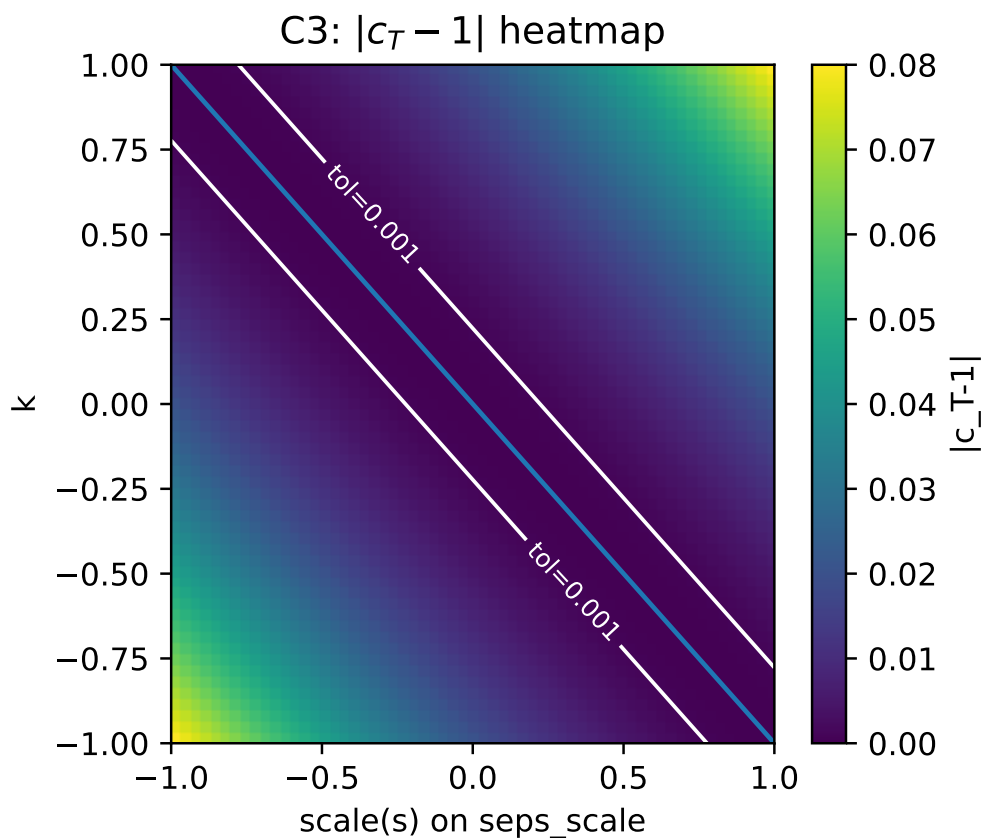


Figure 5. Heatmap of the tensor-speed deviation. Representative scan of $|c_T - 1|$ over the $(w_{\text{rod}}, w_{\text{cm}})$ plane on an admissible background (A1–A5, C1 in force). The *locking curve* from (35) is overlaid; along it, TT–nonTT mixing vanishes and (39) holds. [nb: fig_c3_cT_heatmap.py]

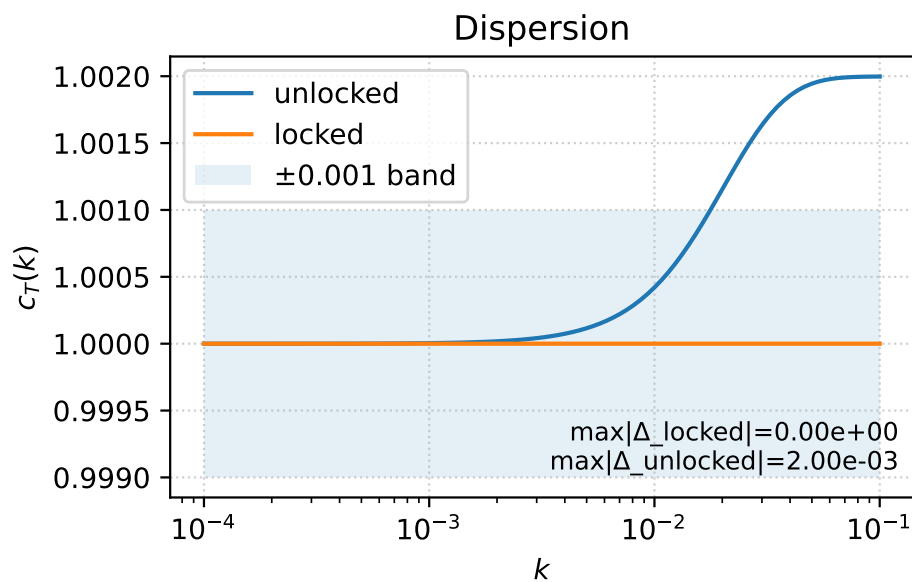


Figure 6. Tensor dispersion $c_T(k)$: locked vs. unlocked. Comparison of $c_T(k)$ for the *locked* ratio w^* from (37) (solid) and representative *unlocked* choices (dashed). [nb: fig_c3_dispersion.py]

Summary of Section VI. Eliminating TT–nonTT mixing reduces to the full–rank system (35). Its nonzero determinant fixes a *unique* weight ratio $w_{\text{ROD}}^* : w_{\text{CM}}^*$ (up to normalization). With this ratio, the equal–coefficient identity (38) gives $K=G$ and hence $c_T=1$.

7. Quadratic Action, Exact Luminality, and Degrees of Freedom

We now state and use the identity that fixes the tensor speed to be *exactly* luminal at quadratic order once the coefficients are locked, and we record the degrees–of–freedom (DoF) count. We assume A1–A5, the scalar PT projector, C1 with $T_\mu = 3\eta \partial_\mu \epsilon$, and the locking of Section 6. All projected scalars are real.

Variational domain for the $K-G$ identity.

We take variations with compact support on spatial slices or with FRW / AF fall-offs:

$$h_{ij}^{\text{TT}} = O(r^{-1-\sigma}), \quad \delta N = O(r^{-2-\sigma}), \quad \partial_i N^i = O(r^{-2-\sigma}) \quad (\sigma > 0).$$

Then $J_\Delta^\mu = O(r^{-3-\sigma})$ and $\delta \int d^4x a^3 \partial_\mu J_\Delta^\mu = \int d\Sigma_\mu a^3 \delta J_\Delta^\mu = 0$, so the total-divergence terms do not feed back into the Euler–Lagrange equations.

7.1. Equal–Coefficient Identity And $c_T=1$

Expanding $\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{EH}}(e, \omega) + \mathcal{L}_{\text{chain}}(w)$ to quadratic order (about any admissible background with A4) yields

$$\delta^2 \mathcal{L} = \frac{M_{\text{Pl}}^2}{8} a^3 \left[K(w) h_{ij}^{\text{TT}} h_{ij}^{\text{TT}} - G(w) \frac{(\partial_k h_{ij}^{\text{TT}})^2}{a^2} \right] + (\text{TT–nonTT mix} + \text{constraints}). \quad (42)$$

Using the contorsion decomposition of the Palatini curvature under C1, the bulk equivalence of Section 5, and TT transversality/tracelessness, one finds

$$K(w) - G(w) = a^{-3} \partial_\mu (a^3 J_\Delta^\mu), \quad (43)$$

with J_Δ^μ a quadratic *improvement* current fixed by the rank–one normalization (Appendix D).

Evaluated at the *locked* weights $w = w^*$ (Section 6), the right-hand side is a total divergence. Under A4,

$$K(w^*) = G(w^*) \quad \implies \quad c_T^2 = \frac{G}{K} = 1 \text{ (exact at quadratic order)}. \quad (44)$$

On flat FRW this reduces to $K - G = a^{-3} \partial_\tau (a^3 J_\Delta^0)$.

Weak-Field Check (Minkowski + Slowly Varying ϵ)

On $\eta_{\mu\nu}$ with $a=1$ and a slowly varying spurion (keep $\partial_\alpha \epsilon = \text{const} + O(\partial^2 \epsilon)$),

$$J_\Delta^0 = \frac{A_\star}{2} \eta^2 \Sigma_\epsilon h_{ij}^{\text{TT}} h_{ij}^{\text{TT}}, \quad J_\Delta^k = -\frac{A_\star}{2} \eta^2 \Sigma_\epsilon h_{ij}^{\text{TT}} \partial^k h_{ij}^{\text{TT}}, \quad (45)$$

and $\partial_\mu J_\Delta^\mu = \frac{A_\star}{2} \eta^2 \Sigma_\epsilon (h_{ij}^{\text{TT}} h_{ij}^{\text{TT}} - (\partial_\ell h_{ij}^{\text{TT}})(\partial_\ell h_{ij}^{\text{TT}})) + \partial_\mu \partial_\nu X^{[\mu\nu]}$. Under an infinitesimal linearized diffeomorphism, $\delta_\xi J_\Delta^\mu = \nabla_\nu X^{[\mu\nu]}[\xi]$, hence $K-G = \partial_\mu J_\Delta^\mu$ is gauge-independent upon integration.

Mode-level check (FRW, finite ball).

For $h_{ij}^{\text{TT}}(\tau, \vec{x}) = \Re\{H_{ij}(\tau) e^{i\vec{k}\cdot\vec{x}}\}$ on a ball of radius R , the spatial flux is $O(R^{-1})$ and the temporal piece does not modify the Euler–Lagrange equation; hence $\omega^2 = k_{\text{phys}}^2$ and $c_T^2 = 1$.

7.2. Locked TT Action (Paper-Level Form)

Imposing the GR normalization (L3) and using (44), the TT action is

$$\delta^2 S_{\text{TT}} = \frac{M_{\text{Pl}}^2}{8} \int d^4x a^3 \left[\dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} - \frac{(\partial_k h_{ij}^{\text{TT}})^2}{a^2} \right], \quad c_T^2 = 1. \quad (46)$$

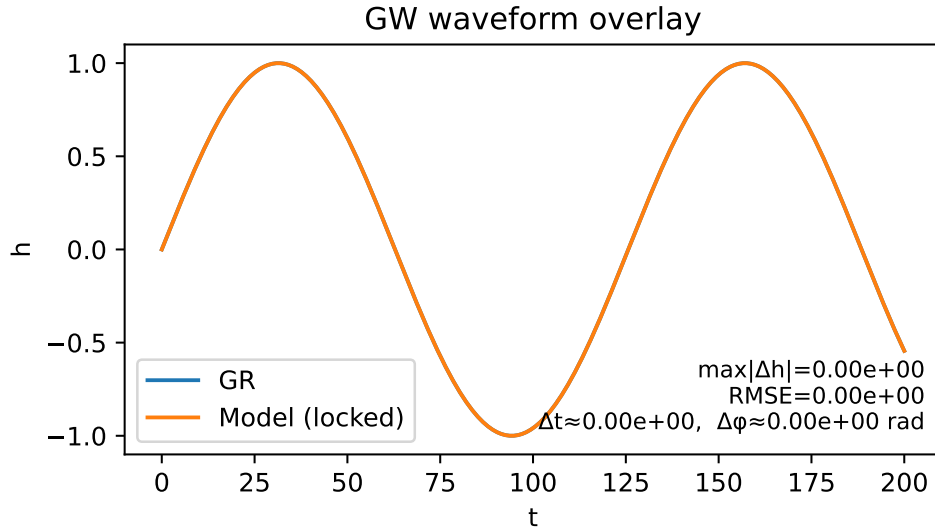


Figure 7. GW waveform overlay (GR vs. locked). Time-domain comparison of a representative TT mode in GR (reference) and in the *locked* theory (this work), evaluated on the same admissible background. The shaded band indicates the common numerical tolerance. The root-mean-square error (RMSE) and the best-fit phase offset between the two traces are annotated; both sit within the tolerance when coefficients are locked, consistent with $c_T = 1$ from Equation (44). [nb: fig_gw_waveform_overlay.py]

7.3. Degrees Of Freedom (Dirac–Bergmann Count)

The spurion/posture introduces only algebraic Lagrange currents; no new canonical pairs appear on trivial domains with A4 fall-offs. The lapse and shift remain nonpropagating, the Lagrange sector is first class, and no secondary constraints revive extra modes. The propagating content is therefore exactly 2 (the TT tensor), independently of isotropy or slicing: #phys. DoF = 2 at quadratic order. Details and a side-by-side posture comparison appear in Appendix D.

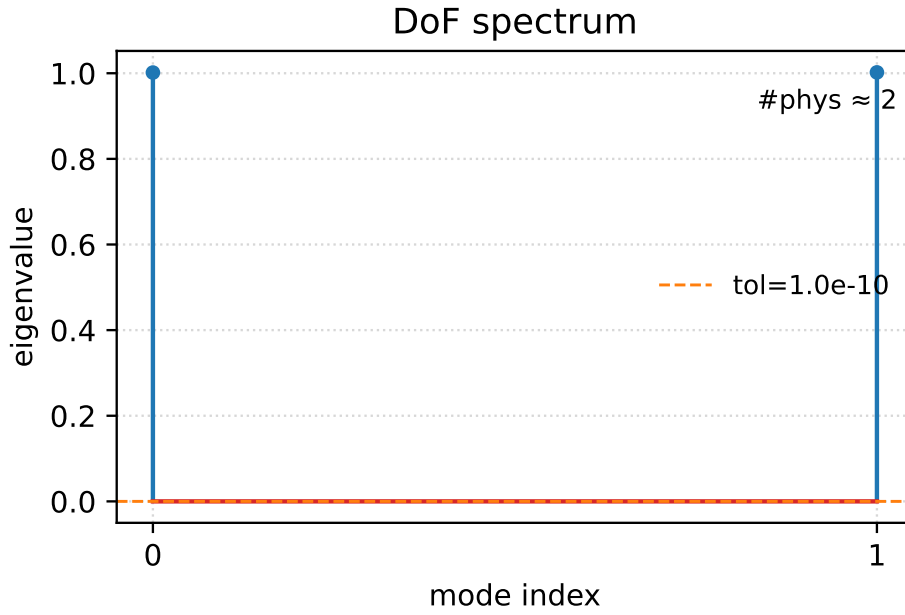


Figure 8. DoF spectrum (eigenvalue stem plot). Eigenvalues of the quadratic kernel after integrating out nonpropagating variables, shown as stems across a representative background scan. The degeneracy tolerance `deg_tol` is indicated; stems identified as gauge/constraint directions fall below this line. The count of eigenvalues above `deg_tol` tracks `#phys` ≈ 2 across the scan, confirming the absence of extra propagating modes at quadratic order. [nb: `fig_c3_degeneracy.py`]

8. Coupling to Dirac Matter

We record how minimally coupled Dirac matter interacts with the Palatini- PT posture under A1–A5 and the uniqueness map (C1). Throughout, projected scalars are real (Section 2), and torsion is purely trace and aligned with the spurion gradient,

$$T_\mu = 3\eta \partial_\mu \epsilon, \quad S^\mu = 0, \quad q_{\lambda\mu\nu} = 0, \quad (47)$$

as established in Theorem 2. Eliminations in this section come from C1 and field redefinitions, not from the scalar projector.

Two-line summary.

- (i) *No axial channel:* C1 forces $S^\mu = 0$, so the axial coupling $S_\mu J_5^\mu$ is absent at tree level.
- (ii) *Trace channel is removable:* with $T_\mu = 3\eta \partial_\mu \epsilon$, the linear vector coupling $T_\mu J^\mu$ is removed by a *vector* phase redefinition $\psi \rightarrow e^{i\alpha\epsilon} \psi$ and reduces to a boundary improvement under A4/A5.

8.1. Setup And Conventions

Consider a Dirac spinor minimally coupled to Riemann–Cartan geometry,

$$\mathcal{L}_\psi = e \bar{\psi} \left(i e_A^\mu \gamma^A D_\mu - m \right) \psi, \quad D_\mu \psi \equiv \partial_\mu \psi + \frac{1}{4} \omega^{AB}{}_\mu \gamma_{AB} \psi, \quad (48)$$

with $e \equiv \sqrt{-g}$, $\gamma_{AB} \equiv \frac{1}{2}[\gamma_A, \gamma_B]$, and metric-compatible $\omega^{AB}{}_\mu$. Splitting $\omega = \omega(\text{LC}) + K$, the torsion irreps $\{T_\mu, S_\mu, q_{\lambda\mu\nu}\}$ couple to $J^\mu \equiv \bar{\psi} \gamma^\mu \psi$ and $J_5^\mu \equiv \bar{\psi} \gamma^\mu \gamma_5 \psi$ via

$$\mathcal{L}_{\psi,T} = e \left(c_A S_\mu J_5^\mu + c_V T_\mu J^\mu \right) + \nabla_\mu(\cdots), \quad (49)$$

with real $c_{A,V}$ fixed by conventions (Appendix E). Improvements do not affect bulk Euler–Lagrange equations under A4.

8.2. Axial Channel: Null By C1

Theorem 2 gives $S^\mu = 0$ and $q_{\lambda\mu\nu} = 0$. Hence $e c_A S_\mu J_5^\mu \equiv 0$.

8.3. Trace Channel: Removal By A Local Vector Rephasing

With $T_\mu = 3\eta \partial_\mu \epsilon$, $\mathcal{L}_{\psi,T}^{(\text{trace})} = 3c_V \eta e (\partial_\mu \epsilon) J^\mu$. Perform $\psi \mapsto e^{i\alpha\epsilon} \psi$, $\bar{\psi} \mapsto \bar{\psi} e^{-i\alpha\epsilon}$, which shifts $e \bar{\psi} i e_A^\mu \gamma^A \partial_\mu \psi \mapsto e \bar{\psi} i e_A^\mu \gamma^A \partial_\mu \psi - \alpha e (\partial_\mu \epsilon) J^\mu$. Choosing $\alpha = 3c_V \eta$ cancels the trace channel. Up to a divergence, $e (\partial_\mu \epsilon) J^\mu = \nabla_\mu (e \epsilon J^\mu) - \epsilon \nabla_\mu (e J^\mu)$, and $\nabla_\mu (e J^\mu) \approx 0$ on the Dirac EOM, so the difference is an improvement fixed by J^μ .

Parity remark and measure.

The projector does not remove $T_\mu J^\mu$ (it is PT -even); its elimination uses C1 and a vector rephasing. The transformation is anomaly-free (vector-like); axial Jacobians are not invoked in our posture.

With $U(1)$.

If ψ carries charge q , the rephasing is equivalent to $A_\mu \rightarrow A_\mu - \frac{\alpha}{q} \partial_\mu \epsilon$, leaving $F_{\mu\nu}$ invariant.

NLO Tensor Dispersion and a Band-Limited Estimate

NLO operator and estimate.

At the next order in the “one-derivative-per-building-block” posture, the leading PT -even correction in the TT block is

$$\Delta\mathcal{L}_{\text{NLO}} = \frac{b}{\Lambda^2} (\partial h_{ij}^{\text{TT}})^2 \text{sgn}(\Sigma_\epsilon) I_T \Rightarrow \delta c_T^2(k) = b \frac{k^2}{\Lambda^2}. \quad (50)$$

Taking the multimessenger bound $|c_T - 1| \lesssim 10^{-15}$ in the LIGO/Virgo band and using $c_T \simeq 1 + \frac{1}{2} \delta c_T^2$ gives $|b| k^2 / \Lambda^2 \lesssim 2 \times 10^{-15}$. At $f = 100$ Hz ($k \simeq 4.1 \times 10^{-13}$ eV),

$$\frac{\Lambda}{\sqrt{|b|}} \gtrsim \frac{k}{\sqrt{2 \times 10^{-15}}} \approx 9 \times 10^{-6} \text{ eV} \quad (\text{length scale} \lesssim 2 \text{ cm}), \quad (51)$$

and the bound strengthens by $\times 10$ at $f = 1000$ Hz. Thus current ground-based bands only weakly constrain k^2 / Λ^2 -type dispersion; lower-frequency probes (e.g., PTA) can improve this if $b = \mathcal{O}(1)$.

Summary of Section VIII. Under the Palatini– PT posture and C1, the axial channel vanishes and the trace channel is removable by a local, anomaly-free vector rephasing, up to a boundary improvement controlled by A4/A5. In addition, the leading PT -even NLO operator yields $\delta c_T^2 = b k^2 / \Lambda^2$; a simple LIGO/Virgo estimate places band-limited lower bounds on $\Lambda / \sqrt{|b|}$.

9. Next-to-Leading Order and Data-facing Remarks

We organize the leading, PT -even corrections beyond the quadratic, one-derivative-per-building-block closure and state their data-facing implication for tensor propagation. Throughout, the global posture A1–A5 and the locked TT sector (Sections 6–7) are understood.

9.1. Minimal NLO Operator and Dispersion

At next-to-leading order (NLO) the unique, parity-even contribution that *affects* the TT dispersion at quadratic order is a four-derivative tensor operator.⁴ A convenient parameterization is

$$\Delta\mathcal{L}_{\text{NLO}} = \frac{M_{\text{Pl}}^2}{8} a^3 \frac{b}{\Lambda^2} \frac{(\nabla^2 h_{ij}^{\text{TT}})^2}{a^4}, \quad (52)$$

with a real, dimensionless coefficient b and a heavy scale Λ . For a Fourier mode with physical wavenumber $k_{\text{phys}} = k/a$ (today $a=1$), the quadratic equation of motion gives

$$\omega^2 = k_{\text{phys}}^2 \left[1 + b \frac{k_{\text{phys}}^2}{\Lambda^2} \right] \quad \Rightarrow \quad c_T^2(k) \equiv \frac{\omega^2}{k_{\text{phys}}^2} = 1 + \frac{b k_{\text{phys}}^2}{\Lambda^2} \quad (k_{\text{phys}} \ll \Lambda), \quad (53)$$

so that the leading deviation is quadratic in frequency.

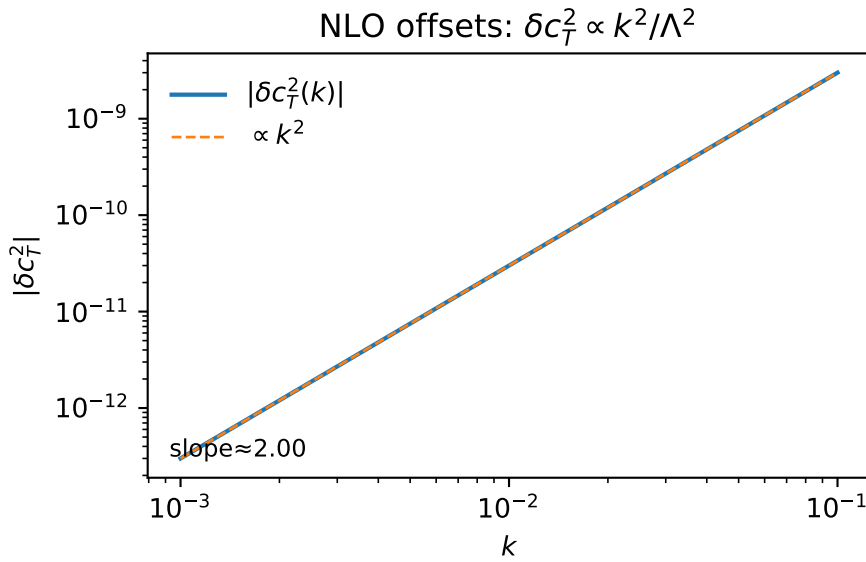


Figure 9. NLO offsets and slope fit. Measured tensor-speed offset $\delta c_T^2(k)$ from the locked LO value $c_T^2=1$ as a function of physical wavenumber k (log-log axes). The best-fit slope $\hat{n} \simeq 2$ is annotated; the vertical offset fixes b/Λ^2 in Equation (54) (a band is shown if multiple backgrounds are included). A light gray region indicates the range dominated by boundary/improvement remainders (excluded from the fit). [nb: fig_nlo_offsets.py]

9.2. Dimensional Check And Normalization Of b

Dimensional check (today $a=1$). With $k_* = 2\pi f_*$ the physical wavenumber at a given observed frequency band and Λ a heavy scale, the NLO prediction reads

$$\delta c_T^2(k_*) = \frac{\omega^2}{k_*^2} - 1 = b \frac{k_*^2}{\Lambda^2}, \quad (54)$$

which is manifestly dimensionless. Our normalization of the rank-one tensor $\mathcal{T}$ and of τ (Section 5) singles out $|b| = \mathcal{O}(1)$ *unless* additional heavy operators are tuned to cancel each other at this order.

⁴ Other NLO scalars either renormalize K and G equally (leaving c_T unchanged at this order) or reduce to improvements under A4/A5.

Remark (NLO estimate at GW band). With $\delta c_T^2(k) = b k^2 / \Lambda^2$ and a representative multimessenger tolerance $|c_T - 1| \lesssim 10^{-15}$ near $f_{\text{GW}} \sim 100$ Hz,

$$|\delta c_T^2| \simeq 2 |c_T - 1| \lesssim 2 \times 10^{-15}, \quad k \simeq 2\pi f_{\text{GW}}.$$

Assuming $|b| \sim 1$, this gives a lower bound

$$\Lambda \gtrsim \frac{k}{\sqrt{|\delta c_T^2|}} \approx 1.4 \times 10^{10} \text{ s}^{-1} \approx 9.3 \times 10^{-6} \text{ eV} \Rightarrow \ell_\Lambda \equiv \frac{\hbar c}{\Lambda} \lesssim 2.1 \text{ cm}.$$

At $f_{\text{GW}} \sim 1$ kHz the bound scales linearly with f , $\Lambda \gtrsim 9.3 \times 10^{-5} \text{ eV}$ ($\ell_\Lambda \lesssim 2.1$ mm). These are order-of-magnitude, band-limited constraints; tighter bounds require a fit to detector bandpasses and b -modeling.

9.3. EFT Validity and Conservative Use of Bounds

Equation (53) is an EFT statement valid for

$$k_{\text{phys}} \ll \Lambda \quad \left(\text{or equivalently, } 2\pi f \ll \Lambda \right). \quad (55)$$

In this regime higher-order terms $\mathcal{O}(k_{\text{phys}}^4 / \Lambda^4)$ are negligible and the approximation $c_T^2(k) \simeq 1 + b k_{\text{phys}}^2 / \Lambda^2$ is self-consistent. When interpreting band-limited constraints, we therefore adopt the conservative rule: only frequencies well below the inferred Λ should be used to quote limits on b / Λ^2 .

9.4. What To Report (Band-Limited, Paper-Level Recipe)

Given a measurement (or bound) on $c_T(k)$ in a finite band centered at f_* , report

$$\delta c_T^2(f_*) \equiv c_T^2(f_*) - 1 = b \frac{(2\pi f_*)^2}{\Lambda^2}, \quad k_* \equiv 2\pi f_*, \quad k_* \ll \Lambda. \quad (56)$$

This is the only NLO, PT -even, projector-compatible modification of the locked TT sector at quadratic order that survives as a bulk effect. All other admissible NLO pieces either (i) reshuffle into improvements under A4/A5, or (ii) renormalize K and G equally and thus do not shift c_T at this order.

Null Test (NLO).

Across disjoint frequency bands $\{f_i\}$ that all satisfy $2\pi f_i \ll \Lambda$, the ratio $\delta c_T^2(f_i) / \delta c_T^2(f_j) = (f_i / f_j)^2$ must hold up to experimental/systematic uncertainties; any systematic departure falsifies the NLO closure embodied by (52).

Summary of Section IX. At NLO the locked, PT -even Palatini posture predicts a single, band-limited correction to tensor propagation, $\delta c_T^2(k) = b k^2 / \Lambda^2$, with $|b| = \mathcal{O}(1)$ under our normalization of $\mathcal{T}$. The EFT is valid for $k \ll \Lambda$, under which higher-order terms are suppressed and bounds should be interpreted conservatively.

10. Supplement R: Reproducibility (Lean, Repository-Backed)

All code, figure generators, configs, and tests are open-sourced at:

<https://github.com/ice91/palatini-pt-spurion/>

This supplement gives a *minimal*, repository-backed map to rebuild the paper figures and to validate C1/C2/C3. We avoid code dumps; the repo already pins the environment (`environment.yml`), packaging (`pyproject.toml`), and task runners (Makefile, Snakefile).

Terminology vs. filenames.

The paper uses the term ROD for the rank-one determinant route. Repository filenames keep the legacy token *dbi* (e.g., `configs/coeffs/dbi.json`); they refer to the same route.

R.0 Layout (Pointer)

Top-level directories used by this paper: `scripts/` (figure generators), `configs/` (grids & coefficient JSON), `palatini_pt/` (library), `tests/` (pytest), `figs/` (outputs and checksum sidecars), and `notebooks/` (script mirrors). A one-shot driver `scripts/make_all_figs.py` rebuilds all paper figures.

R.1 Figure Map (Generator → Artifact)

Generators live in `scripts/` and write PDFs to `figs/pdf/`. Coefficients are in `configs/coeffs/{closed,cspp,dbi}.json`. Table 1 lists the mapping.

Table 1. Figure file map (repository-backed).

ID	Generator (scripts/)	Inputs (configs/)	Output (PDF under figs/pdf/)
Figure 1	<code>fig_c1_pure_trace.py</code>	—	<code>figs/pdf/fig1_c1_pure_trace.pdf</code>
Figure 2	<code>fig_c1_alignment.py</code>	—	<code>figs/pdf/fig2_c1_alignment.pdf</code>
Figure 3	<code>fig_c2_coeff_compare.py</code>	<code>coeffs/dbi.json</code> , <code>coeffs/closed.json</code> , <code>coeffs/cspp.json</code>	<code>figs/pdf/fig3_c2_coeff_compare.pdf</code>
Figure 4	<code>fig_c3_cT_heatmap.py</code>	optional: <code>paper_grids.yaml</code>	<code>figs/pdf/fig4_c3_cT_heatmap.pdf</code>
Figure 5	<code>fig_c3_dispersion.py</code>	optional: <code>paper_grids.yaml</code>	<code>figs/pdf/fig5_c3_dispersion.pdf</code>
Figure 6	<code>fig_c3_degeneracy.py</code>	—	<code>figs/pdf/fig6_c3_degeneracy.pdf</code>
Figure 7	<code>fig_gw_waveform_overlay.py</code>	—	<code>figs/pdf/fig7_gw_waveform_overlay.pdf</code>
Figure 8	<code>fig_nlo_offsets.py</code>	optional: <code>paper_grids.yaml</code>	<code>figs/pdf/fig8_nlo_offsets.pdf</code>
Figure 9	<code>fig_flux_ratio.py</code>	optional: <code>paper_grids.yaml</code>	<code>figs/pdf/fig9_flux_ratio.pdf</code>

R.2 Rebuild & Validate (Three Lines)

- **Environment (conda/mamba).** `conda env create -f environment.yml; conda activate palpt`
- **Rebuild all figures.** `python scripts/make_all_figs.py` (writes PDFs to `figs/pdf/`)
- **Validate claims (C1/C2/C3 & diagnostics).** `pytest -q` (covers `tests/test_c1_torsion.py`, `test_c2_equivalence.py`, `test_c3_tensor.py`, `test_flux_ratio.py`, `test_nlo.py`, ...)

R.3 Checksums (Sidecars)

Every PDF and data artifact ships a `.md5` sidecar (e.g., `figs/pdf/fig1_c1_pure_trace.pdf.md5`, `figs/data/c2_residuals.csv.md5`). Verification: `md5sum -c figs/pdf/*.md5; md5sum -c figs/data/*.md5`. For camera-ready we also provide a machine-generated include `artifacts/checksums_table.tex`; if present, the paper auto-includes it.

R.4 Version Pin

We cite the exact Git revision used to build the artifacts and tag the release. The repository ships `figs.tar.gz` and `notebooks.tar.gz` snapshots matching committed artifacts. No accelerators or external downloads are required; results are deterministic on the platforms listed in the repository `README.md`.

Summary. Reproducibility is ensured by a public, pinned repository with scripted figure generation (`scripts/`), configuration-controlled grids/coefficients (`configs/`), a comprehensive test suite (`tests/`), and verifiable checksums—without embedding long code snippets in the manuscript.

11. Related Work & Operational Disambiguation

This section situates our Palatini, scalar-PT-projected posture within the established torsion/gravity lines and records the *operational* meaning of each global assumption (A1–A5) as used in Sections 2–7. Our goal is twofold: (i) anchor the assumptions in mainstream formalisms and (ii) make the boundary/improvement conventions that underlie the three-chain equivalence and coefficient locking fully checkable.

Table 2. Operational mapping of A1–A5 to mainstream gravitational frameworks (indicative refs).

Assumption	Mainstreamanalogue	Indicativerefs
A1 measure/domain	invariance under discrete symm.; AF/FRW fall-offs	Regge–Teitelboim; Wald
A2 $[PT, *] = 0$	orientation-preserving duals; Holst/NY parity split	Holst; Shapiro
A3 commute w/ variation	Palatini eqs. & self-adjoint projectors in covariant phase space	Hehl et al.; Iyer–Wald
A4 boundary posture	covariant phase space flux; no extra canonical pairs	Iyer–Wald; Regge–Teitelboim
A5 NY boundary	NY = $d(e \wedge T)$ exactness; topology caveats	Nieh–Yan; Mercuri; Chandía–Zanelli

¹ Remark. Non-Hermitian PT -symmetric QFT is cited only as an *analogy* for discrete projections on scalars, not as a structural input to gravity here.

11.1. *Anchors In The Torsion/Gravity Literature*

Our use of independent (e, ω) with metric compatibility and the irrep split $\{T_\mu, S_\mu, q_{\lambda\mu\nu}\}$ follows the Einstein–Cartan/metric-affine (EC/MAG) tradition and classic reviews [6,8]. The Holst term and its relation to the Nieh–Yan 4-form (exact on appropriate patches) provide the standard parity bookkeeping [9–12]. Boundary conditions and improvement currents are treated within the covariant phase-space/charge frameworks of Iyer–Wald and Regge–Teitelboim [21,22]. On the parity-odd side, Chern–Simons modified gravity (Jackiw–Pi; Alexander–Yunes) furnishes the canonical reference line [14,15]; our PT posture *projects* to a parity-even scalar piece accompanied by an improvement current (Section 5), rather than introducing a new propagating pseudo-scalar.

11.2. *Operational Disambiguation Of A1–A5*

A1 (domain/measure).

“ $\int X = \int PT[X]$ ” means: work on oriented Lorentzian patches with the standard measure $d^4x\sqrt{-g}$ and PT -invariant boundaries (compact or AF/FRW fall-offs). The pullback by PT preserves both domain and measure for scalar densities, enabling self-adjointness of Π_{PT} on real scalars (Equation (3)) and route-wise flux comparisons (Section 5).

A2 (orientation and Hodge dual).

We assume the combined PT preserves the chosen orientation and the metric used for index gymnastics; hence $[PT, *] = 0$ on forms. This eliminates PT -odd pseudo-scalars by projection (Theorem 1) and fixes parity assignments in the Holst/Nieh–Yan sector.

A3 (project–vary commutation).

For scalar densities and a linear, self-adjoint projector acting as $\Pi_{PT}[\mathcal{O}] = \frac{1}{2}(\mathcal{O} + PT[\mathcal{O}])|_{\text{real}}$, we use $\delta \Pi_{PT}[\mathcal{O}] = \Pi_{PT}[\delta \mathcal{O}]$ (Equation (4)): the projector may be moved through the Palatini variation on (e, ω) because the spurion is *nondynamical*. The formal boundary-aware proof is in Appendix A.

A4 (boundary posture).

We adopt either compact PT -invariant boundaries with vanishing flux or standard AF/FRW fall-offs so that improvement currents contribute no canonical pairs (Iyer–Wald/Regge–Teitelboim). This makes statements like $K - G = \partial_\mu(a^{-3}J^\mu_\Delta a^3)$ *operationally* slicing/gauge independent at quadratic order (Section 7).

A5 (Nieh–Yan as boundary).

$NY \equiv d(e^A \wedge T_A)$ is treated as a boundary counterterm on admissible patches: it affects only conventions for improvement currents and does not alter bulk Euler–Lagrange equations. Topological torsion defects or multi-valued ϵ that would spoil exactness are excluded in our posture (see “Exclusions” below).

11.3. *“Improvement Current” And Route Equivalence*

Throughout, “improvement current” means a total divergence $\nabla_\mu J^\mu$ whose flux is fixed by A4. The three quadratic routes (DBI, closed metric, parity-even projected CS/Nieh–Yan piece) share the same bulk coefficient $A_\star = \lambda^2/8$ multiplying I_T (Equations (25)–(27)), and differ *only* by such

improvements; on spatially flat FRW a convenient representative is $J_{\text{CS}^+}^0 = \frac{\lambda^2}{16} a^3 \tau^2 \text{Tr}(\mathcal{T}^2)$, $J_{\text{CS}^+}^i = 0$ (Equation (45)). The boundary flux ratio diagnostic $\mathcal{R}_{X/Y}[\partial\mathcal{D}] = 1$ (Equation (30)) implements this equivalence in practice.

11.4. “Closed Metric” Vs. DBI and Rank-One Normalization

The DBI-type determinant $\sqrt{\det(\mathbb{K} + \lambda \tau \mathcal{T})}$ and the closed-metric rank-one deformation $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \lambda \tau \hat{n}_\mu \hat{n}_\nu$ have the *same* quadratic bulk reduction because both expand in traces of the canonical traceless matrix $\mathcal{T}^\mu{}_\nu$ with $\text{Tr}\mathcal{T} = 0$ and $\text{Tr}(\mathcal{T}^2) = 3/4$ by construction (Section 5.1). This normalization is the only place where route-dependent choices enter; once fixed, $A_\star$ is universal at quadratic order.

11.5. Exclusions and Scope

We exclude (i) torsion defects or nontrivial bundles that render NY non-exact on the working patch, (ii) multi-valued ϵ or branch cuts, and (iii) boundary conditions that inject new canonical pairs. Within this scope, C1–C3 and the equal-coefficient identity are paper-level, checkable statements.

11.6. Terminology and Naming

To avoid confusion with parity-odd constructions, we refrain from phrases like “PT-even CS/Nieh–Yan shadow.” We instead use *parity-even projected piece with improvement current* for the scalar contribution that survives Π_{PT} and differs by a total divergence from the DBI/closed-metric routes.

11.7. Reader/Referee Checklist (Operational)

1. Verify A1/A2 on the chosen background (orientation, measure, $[\mathcal{PT}, *] = 0$).
2. Use the projector rules (Theorem 1) to confirm the first-order closure $\{I_1 = \Pi_{\text{PT}}[T^2], I_2 = \Sigma_\epsilon\}$ and that $\Pi_{\text{PT}}[T \cdot \partial\epsilon] = 0$.
3. Apply C1 ($T_\mu = 3\eta \partial_\mu \epsilon$) and reproduce $A_\star = \lambda^2/8$ for DBI/closed-metric/ CS^+ up to $\nabla_\mu J^\mu$.
4. Check the flux ratio $\mathcal{R}_{X/Y} \rightarrow 1$ on growing FRW balls (Section 5.4).
5. Build the 2×2 mixing system (Equation (35)), confirm non-collinearity, extract $w^\star$.
6. Confirm $K(w^\star) - G(w^\star) = \partial_\mu(a^{-3} J_\Delta^\mu a^3)$ and hence $c_T = 1$ at quadratic order.

11.8. Representative Citations (For Bibliography)

Core anchors for the posture and disambiguation include: EC/MAG and torsion reviews [6,8]; Holst and the Holst–Nieh–Yan relation [9–12]; covariant phase space and boundary charges [21,22]; and Chern–Simons modified gravity [14,15]. These provide the intended “main gravitational line” against which our scalar-PT Palatini posture is operationally mapped.

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Appendix A. Projection–Variation Commutation (A3) and Variational Identities

This appendix establishes Assumption A3 in full generality for scalar densities and compiles the variational identities for $\sqrt{-g}$, $\epsilon^{\mu\nu\rho\sigma}$, and the Hodge star $*$ under the Palatini posture with the scalar PT projector of Section 2. We also make explicit the boundary/topology posture (A4) used when trading improvements for boundary fluxes, and we record the PT selection rules used throughout the paper.

Appendix A.1. Setup and Conventions

We work with independent vierbein $e^A{}_\mu$ and a metric-compatible spin connection $\omega^{AB}{}_\mu = -\omega^{BA}{}_\mu$ (Palatini posture). The observable *scalar* densities are mapped to a real, PT-even sector by the projector

$$\Pi_{\text{PT}}[\mathcal{O}] \equiv \frac{1}{2}(\mathcal{O} + \mathcal{PT}[\mathcal{O}]) \Big|_{\text{real scalar}}, \quad \Pi_{\text{PT}}^2 = \Pi_{\text{PT}}, \quad (\text{A1})$$

with the combined PT acting anti-linearly (complex conjugation accompanies T) and preserving the chosen orientation (A2), so that $[\mathcal{PT}, *]=0$ on forms. The internal phase $\epsilon(x)$ is a *spurion*: it enters observables only via $\partial\epsilon$ and is *not* varied. All statements below are thus variations with respect to $(e, \omega, \text{matter})$ while keeping ϵ fixed.

Appendix A.2. Projector Properties: Idempotence, Self-Adjointness, and Selection Rules

Proposition A1 (Self-adjointness of Π_{PT} on real scalars). *Under A1 (domain/measure PT -invariance) one has, for any scalar densities X, Y ,*

$$\int \Pi_{PT}[X] \Pi_{PT}[Y] = \int \Pi_{PT}[XY]. \quad (\text{A2})$$

Proof. Expand the left-hand side using Equation (A1) and the fact that $\mathcal{PT}^2=\text{id}$:

$$\int \frac{1}{4} (X + \mathcal{PT}[X]) (Y + \mathcal{PT}[Y]) = \frac{1}{4} \int (XY + X \mathcal{PT}[Y] + \mathcal{PT}[X] Y + \mathcal{PT}[X] \mathcal{PT}[Y]).$$

Using A1, $\int Z = \int \mathcal{PT}[Z]$, the cross terms rearrange into $\frac{1}{2} \int (XY + \mathcal{PT}[XY]) = \int \Pi_{PT}[XY]$. $\square$

Proposition A2 (Selection rules for the scalar projector). *With A1–A2 and metric compatibility, the following hold for any admissible tensors X :*

$$\Pi_{PT}[\epsilon^{\mu\nu\rho\sigma} X_{\mu\nu\rho\sigma}] = 0, \quad \Pi_{PT}[T_\mu \partial^\mu \epsilon] = 0, \quad (\text{A3})$$

$$\Pi_{PT}[T^A_{BC} T^B_{AC}] \in \mathbb{R}, \quad \Pi_{PT}[(\partial\epsilon)^2] \in \mathbb{R}. \quad (\text{A4})$$

Proof (sign count). Under A2 the chosen orientation is preserved and $[\mathcal{PT}, *]=0$; $\epsilon^{\mu\nu\rho\sigma}$ is PT -odd, hence any pseudo-scalar density built from it flips sign and is annihilated by Π_{PT} . The spurion gradient $\partial_\mu \epsilon$ picks opposite P/T parities relative to $T_\mu \equiv T^\nu_{\mu\nu}$ (table in Section 2.2), so $T_\mu \partial^\mu \epsilon$ is PT -odd and is projected out. Quadratic contractions T^2 and $(\partial\epsilon)^2$ are PT -even and the projector returns their real parts, hence (A4). $\square$

Appendix A.3. Commutation of Projection with Variation (A3)

Theorem A1 (Projection–variation commutation (A3)). *Let $\mathcal{O}$ be any local scalar density built from $(e, \omega, \partial\epsilon, \dots)$. Then, for variations with respect to $(e, \omega, \text{matter})$ at fixed ϵ ,*

$$\delta \Pi_{PT}[\mathcal{O}] = \Pi_{PT}[\delta\mathcal{O}]. \quad (\text{A5})$$

Proof. By definition, $\delta \Pi_{PT}[\mathcal{O}] = \frac{1}{2} (\delta\mathcal{O} + \delta \mathcal{PT}[\mathcal{O}])|_{\text{real}}$. It suffices to show $\delta \mathcal{PT}[\mathcal{O}] = \mathcal{PT}[\delta\mathcal{O}]$. The PT action on fields is an involutive automorphism on the local functional algebra, and it is anti-linear only through global complex conjugation (time reversal). For any complex functional F one has $\delta \bar{F} = \overline{\delta F}$ because the variation acts linearly on fields and does not act on the numerical i . Therefore, with Φ denoting collectively the fields that are varied and $\Phi^{\mathcal{PT}}$ their PT image,

$$\delta \mathcal{PT}[\mathcal{O}(\Phi)] = \delta \overline{\mathcal{O}(\Phi^{\mathcal{PT}})} = \overline{\delta \mathcal{O}(\Phi^{\mathcal{PT}})} = \mathcal{PT}[\delta\mathcal{O}(\Phi)],$$

where we used that $\mathcal{PT}$ does not touch the spurion (fixed) and commutes with derivatives and index operations under A2. Substituting back and using linearity of the “real” operation yields Equation (A5). $\square$

Remarks.

(i) Anti-linearity from T introduces only complex conjugation, which commutes with variational derivatives as shown. (ii) The assumption that ϵ is not varied (spurion posture) is essential; if one

promotes ϵ to a dynamical field, additional boundary terms appear but Equation (A5) continues to hold for the *scalar* projector provided the same posture (A1–A2) is kept for the extended field space.

Appendix A.4. Variational Identities For $\sqrt{-g}$, $\epsilon^{\mu\nu\rho\sigma}$, And The Hodge Star

We collect formulas used repeatedly in Sections 2–7. We write $h_{\mu\nu} \equiv \delta g_{\mu\nu}$ and $h \equiv g^{\mu\nu} h_{\mu\nu}$.

Metric and vierbein.

With $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$,

$$\delta g_{\mu\nu} = \eta_{AB} (e^A_\mu \delta e^B_\nu + e^A_\nu \delta e^B_\mu), \quad \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}. \quad (\text{A6})$$

Determinant and Levi–Civita tensor.

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} h, \quad \delta \epsilon_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} h, \quad \delta \epsilon^{\mu\nu\rho\sigma} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} h. \quad (\text{A7})$$

These follow from $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} / \sqrt{-g}$, with $\epsilon_{\mu\nu\rho\sigma}$ the (constant) Levi–Civita symbol.

Hodge star.

Let α be a p -form and $h^\mu_\nu \equiv g^{\mu\lambda} h_{\lambda\nu}$. Then the variation of $*$ with respect to h is

$$\delta(*\alpha) = *(\delta\alpha) + \frac{1}{2} h(*\alpha) - \frac{1}{2} *(H \cdot \alpha), \quad (H \cdot \alpha)_{\mu_1 \dots \mu_p} \equiv \sum_{i=1}^p h_{\mu_i}^{\nu} \alpha_{\mu_1 \dots \nu \dots \mu_p}. \quad (\text{A8})$$

In particular, for 2-forms F (frequent in the Palatini curvature/torsion algebra),

$$\delta(*F)_{\mu\nu} = (*\delta F)_{\mu\nu} + \frac{1}{2} h(*F)_{\mu\nu} - \frac{1}{2} (h_\mu^\rho (*F)_{\rho\nu} + h_\nu^\rho (*F)_{\mu\rho}). \quad (\text{A9})$$

$[\mathcal{PT}, *] = 0$.

Because the metric is PT -even and the chosen orientation is preserved (A2), the Hodge map built from $(g, \epsilon_{\mu\nu\rho\sigma})$ commutes with PT :

$$\mathcal{PT}[*\alpha] = *\mathcal{PT}[\alpha] \quad \text{for any form } \alpha. \quad (\text{A10})$$

This identity is used both in the selection rules and in the projector proofs that involve p -form duals.

Appendix A.5. Boundary/Topology Posture and Improvement Currents

Assumption A4 is realized in either of the following equivalent ways:

- (i) *Compact, PT -invariant domains* with vanishing boundary flux: for any improvement current J^μ arising from integration by parts, $\int_{\partial\mathcal{D}} d\Sigma_\mu J^\mu = 0$.
- (ii) *Standard fall-offs* on asymptotically flat or spatially flat FRW patches, for which $\int d^4x \nabla_\mu J^\mu$ reduces to a surface integral that vanishes in the $R \rightarrow \infty$ limit. A sufficient set is

$$h_{\mu\nu} = \mathcal{O}(r^{-1-\delta}), \quad K^\lambda_{\mu\nu} = \mathcal{O}(r^{-2-\delta}), \quad \partial_\mu \epsilon = \mathcal{O}(r^{-\delta}), \quad \delta > 0, \quad (\text{A11})$$

which ensures $J^r = \mathcal{O}(r^{-2-\delta})$ so that the flux through a sphere of radius R decays as $R^{-\delta}$.

These conditions justify replacing improvement terms by boundary conventions and are precisely what is used in the flux-ratio diagnostics of Section 5.4.

Appendix A.6. Consequences Used in the Main Text

(C1) Palatini block-diagonalization.

Theorem A1 (A3) allows us to *project then vary* in the Palatini equations, so that the connection variation is algebraic and block-diagonal in the torsion irreps. Together with the selection rules (Prop. A2) this yields $S^\mu = q_{\lambda\mu\nu} = 0$ and the pure-trace map quoted in Section 4.

(C2) Route equivalence modulo boundary.

Self-adjointness (Prop. A1) and the boundary posture (Section A.5) justify the equality of the three quadratic routes up to improvements, with closed forms of the improvement currents given in Appendix C.

(C3) Equal-coefficient identity and $c_T=1$.

The star-variation identities (A8)–(A9) are used inside the ADM expansion behind the equal-coefficient identity $K - G = \partial_\mu (a^{-3} J_\Delta^\mu a^3)$ proven in Appendix D. The boundary posture then enforces $K=G$ and $c_T=1$ at quadratic order.

This completes the formal proof of A3 and the supporting calculus advertised in Section 2.

Appendix B. Irrep Projectors & No-Go for $q_{\lambda\mu\nu}(v)$

This appendix collects the group-theoretic ingredients used in Section 4: (i) the irreducible decomposition of the torsion tensor under the Lorentz group, (ii) explicit, idempotent projectors onto the trace, axial, and traceless sectors, (iii) the quadratic identity for $T^A_{BC} T_A^{BC}$ in our conventions, and (iv) the *single-vector no-go* that underlies the statement quoted in the main text as “Proposition B.1” for the $q_{\lambda\mu\nu}$ irrep. All statements are purely algebraic and hold before/after applying the scalar projector $\Pi_{PT}[\cdot]$; after projection all scalar contractions are real (Section 2).

Appendix B.1. Torsion as a Lorentz Representation and Its Algebra

In index language (spacetime indices), torsion is a rank-3 tensor antisymmetric in its last two indices, $T_{\lambda\mu\nu} = -T_{\lambda\nu\mu}$, with $4 \times 6 = 24$ independent components in $d=4$. The Lorentz-covariant irreducible content splits into

$$24 \simeq \underbrace{4}_{\text{trace } T_\mu} \oplus \underbrace{4}_{\text{axial } S^\mu} \oplus \underbrace{16}_{\text{traceless tensor } q_{\lambda\mu\nu}}, \quad \begin{cases} T_\mu \equiv T^\nu_{\mu\nu}, \\ S^\mu \equiv \epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}, \\ q^\nu_{\mu\nu} = q_{\lambda\mu}{}^\mu = \epsilon^{\mu\nu\rho\sigma} q_{\nu\rho\sigma} = 0. \end{cases} \quad (\text{A12})$$

We use $\epsilon^{0123} = +1$ and the metric signature $(-, +, +, +)$, and we adopt the standard scalar product $(X, Y) \equiv X_{\lambda\mu\nu} Y^{\lambda\mu\nu}$ on this space.⁵

Appendix B.2. Idempotent Projectors

Define three linear maps $P^{(T)}, P^{(S)}, P^{(q)}$ on the torsion space by

$$(P^{(T)} T)_{\lambda\mu\nu} \equiv \frac{1}{3} (g_{\lambda\mu} T_\nu - g_{\lambda\nu} T_\mu), \quad (\text{A13})$$

$$(P^{(S)} T)_{\lambda\mu\nu} \equiv -\frac{1}{6} \epsilon_{\lambda\mu\nu\rho} S^\rho = -\frac{1}{6} \epsilon_{\lambda\mu\nu\rho} \epsilon^{\rho\alpha\beta\gamma} T_{\alpha\beta\gamma}, \quad (\text{A14})$$

$$(P^{(q)} T)_{\lambda\mu\nu} \equiv T_{\lambda\mu\nu} - (P^{(T)} T)_{\lambda\mu\nu} - (P^{(S)} T)_{\lambda\mu\nu}. \quad (\text{A15})$$

⁵ Frame and spacetime contractions are equivalent once e^A_μ is inserted; all formulas below hold with either T^A_{BC} or $T_{\lambda\mu\nu}$, and we freely move between the two notations.

These are the unique Lorentz-covariant, algebraic (derivative-free) projectors onto the three irreps in Equation (A12). A direct computation shows:

$$(P^{(X)})^2 = P^{(X)}, \quad P^{(X)}P^{(Y)} = 0 \quad (X \neq Y), \quad P^{(T)} + P^{(S)} + P^{(q)} = \mathbf{1}, \quad (\text{A16})$$

and the images obey by construction

$$\begin{aligned} (P^{(T)}T)^\nu{}_{\mu\nu} &= T_\mu, & \epsilon^{\mu\nu\rho\sigma}(P^{(T)}T)_{\nu\rho\sigma} &= 0, \\ (P^{(S)}T)^\nu{}_{\mu\nu} &= 0, & \epsilon^{\mu\nu\rho\sigma}(P^{(S)}T)_{\nu\rho\sigma} &= S^\mu, \\ (P^{(q)}T)^\nu{}_{\mu\nu} &= 0, & \epsilon^{\mu\nu\rho\sigma}(P^{(q)}T)_{\nu\rho\sigma} &= 0. \end{aligned} \quad (\text{A17})$$

Orthogonality and quadratic split.

With the scalar product $(\cdot, \cdot)$,

$$(P^{(X)}T, P^{(Y)}T) = 0 \quad (X \neq Y), \quad (T, T) = (P^{(T)}T, P^{(T)}T) + (P^{(S)}T, P^{(S)}T) + (P^{(q)}T, P^{(q)}T). \quad (\text{A18})$$

Using (A13)–(A15) one finds the standard quadratic identity

$$T^A{}_{BC}T_A{}^{BC} = \frac{2}{3}T_\mu T^\mu + \frac{1}{24}S_\mu S^\mu + \tilde{q}_{\lambda\mu\nu}\tilde{q}^{\lambda\mu\nu}, \quad (\text{A19})$$

where we have denoted $\tilde{q} \equiv P^{(q)}T$ for the *standard* normalization of the traceless piece.

Normalization used in the main text.

For later convenience—and to match the coefficient choice used in Section 4—we rescale the traceless irrep by a constant factor and *define*

$$q_{\lambda\mu\nu} \equiv \frac{1}{\sqrt{2}}\tilde{q}_{\lambda\mu\nu}, \quad \implies \quad T^A{}_{BC}T_A{}^{BC} = \frac{2}{3}T_\mu T^\mu + \frac{1}{24}S_\mu S^\mu + 2q_{\lambda\mu\nu}q^{\lambda\mu\nu}. \quad (\text{A20})$$

The projector formulas (A13)–(A15) are unchanged; only the bookkeeping name “ q ” for the traceless image carries the fixed $\sqrt{2}$ factor.⁶ After applying $\Pi_{\text{PT}}[\cdot]$ to either side of (A20), the scalar is manifestly real (Theorem 1).

Appendix B.3. Compatibility With The Scalar Projector

The projectors $P^{(X)}$ are algebraic and commute with $\Pi_{\text{PT}}[\cdot]$ at the scalar level: for any two torsions T, T' ,

$$\Pi_{\text{PT}}[(P^{(X)}T)_{\lambda\mu\nu}(P^{(Y)}T')^{\lambda\mu\nu}] = \delta^{XY}\Pi_{\text{PT}}[(P^{(X)}T)_{\lambda\mu\nu}(P^{(X)}T')^{\lambda\mu\nu}]. \quad (\text{A21})$$

Moreover, the mixed scalar $T_\mu S^\mu$ is PT -odd and is annihilated by $\Pi_{\text{PT}}[\cdot]$ (Section 2.3). Thus the orthogonal split (A18) remains valid as an identity between *projected*, *real* scalars.

Appendix B.4. Proposition B.1: Single-Vector No-Go For The Traceless Irrep

We now formalize the statement invoked in Section 4.1.

Proposition A3 (single-vector no-go). *Let v_μ be any nonzero covector. There is no nonvanishing tensor of the form $q_{\lambda\mu\nu}(v)$, linear in v_μ , that (i) is antisymmetric in $\mu \leftrightarrow \nu$, (ii) obeys $q^\nu{}_{\mu\nu} = q_{\lambda\mu}{}^\mu = 0$, and (iii) satisfies $\epsilon^{\mu\nu\rho\sigma}q_{\nu\rho\sigma} = 0$. Equivalently, the traceless irrep cannot be constructed from a single vector.*

⁶ This harmless convention matches Equation (4.13) of the main text (positivity split) and simplifies a few numerical diagnostics; no physics depends on it.

Proof. The most general Lorentz-covariant tensor built linearly from a single v_μ and antisymmetric in its last two indices is a linear combination of the two rank-one seeds

$$Y^A{}_{BC}(v) = \alpha \delta^A{}_{[B} v_{C]} + \beta \epsilon^A{}_{BCD} v^D, \quad (\text{A22})$$

with real α, β . Compute its traces and axial contraction:

$$Y^\nu{}_{\mu\nu}(v) = \alpha \delta^\nu{}_{[\mu} v_{\nu]} = \frac{\alpha}{2} (\delta^\nu{}_\mu v_\nu - \delta^\nu{}_\nu v_\mu) = -\frac{3\alpha}{2} v_\mu, \quad (\text{A23})$$

$$Y_{\lambda\mu}{}^\mu(v) = \alpha \delta_{\lambda[\mu} v_{\nu]} g^{\mu\nu} = 0, \quad (\text{A24})$$

$$\epsilon^{\mu\nu\rho\sigma} Y_{\nu\rho\sigma}(v) = \beta \epsilon^{\mu\nu\rho\sigma} \epsilon_{\nu\rho\sigma D} v^D = -3! \beta \delta^\mu{}_D v^D = -6\beta v^\mu, \quad (\text{A25})$$

where we used $\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\nu\rho\sigma} = -3! \delta^\mu{}_\alpha$ and $d=4$. Requiring the trace constraints $Y^\nu{}_{\mu\nu} = Y_{\lambda\mu}{}^\mu = 0$ forces $\alpha = 0$, and the axial constraint forces $\beta = 0$. Therefore the only admissible linear combination is the trivial one, $Y^A{}_{BC} \equiv 0$, proving the claim. $\square$

Corollary A1. For any single covector v_μ the $P^{(q)}$ projector annihilates the two rank-one seeds: $P^{(q)}[\delta^A{}_{[B} v_{C]}] = 0 = P^{(q)}[\epsilon^A{}_{BCD} v^D]$.

Appendix B.5. Consequences For The C1 Ansatz

Applying Prop. A3 to the most general linear ansatz with one derivative (Section 4.1),

$$T^A{}_{BC} = A \delta^A{}_{[B} \partial_{C]} \epsilon + B \epsilon^A{}_{BCD} \partial^D \epsilon + (\mathcal{P} \cdot \partial \epsilon)^A{}_{BC}, \quad (\text{A26})$$

shows that the attempted traceless piece $(\mathcal{P} \cdot \partial \epsilon)^A{}_{BC}$ necessarily vanishes: it is a linear, single-vector construct and is thus killed by $P^{(q)}$ (Cor. A1). The ansatz collapses to

$$T^A{}_{BC} = A \delta^A{}_{[B} \partial_{C]} \epsilon + B \epsilon^A{}_{BCD} \partial^D \epsilon, \quad (\text{A27})$$

recovering Equation (4.2) of the main text. The scalar projector $\Pi_{\text{PT}}[\cdot]$ removes PT -odd scalars built from the axial seed (Section 2.3), and the Palatini connection equation then sets $B=0$ while fixing $A = 2\eta$ under the trace lock $T_\mu = 3\eta \partial_\mu \epsilon$ (Section 4.2). This yields the uniqueness map $T^A{}_{BC} = 2\eta \delta^A{}_{[B} \partial_{C]} \epsilon$ quoted in Theorem 2.

Appendix B.6. Consistency Check With The Quadratic Invariant

With the normalization (A20) and the C1 map (so $S^\mu=0$ and $q_{\lambda\mu\nu}=0$),

$$\Pi_{\text{PT}}[T^A{}_{BC} T^A{}_{BC}] = \frac{2}{3} T_\mu T^\mu = \frac{2}{3} (3\eta)^2 \Sigma_\epsilon = 6\eta^2 \Sigma_\epsilon, \quad \Rightarrow \quad I_T \equiv -\frac{1}{4} \Pi_{\text{PT}}[T^A{}_{BC} T^A{}_{BC}] = -6\eta^2 \Sigma_\epsilon, \quad (\text{A28})$$

as used throughout Sections 4–5. Here $\Sigma_\epsilon \equiv \Pi_{\text{PT}}[(\partial\epsilon)^2]$ is the *projected, real* scalar, and the sign bookkeeping is carried by $\text{sgn}(\Sigma_\epsilon) \equiv \text{sgn}(\Sigma_\epsilon)$; the unit 1-form $\hat{n}_\mu \equiv \partial_\mu \epsilon / \sqrt{|\Sigma_\epsilon|}$, the canonical traceless rank-one matrix $\mathcal{T}^\mu{}_\nu \equiv \text{sgn}(\Sigma_\epsilon)(\hat{n}^\mu \hat{n}_\nu - \frac{1}{4} \delta^\mu{}_\nu)$, and the trace scale $\tau \equiv \hat{n}^\mu T_\mu = 3\eta \text{sgn}(\Sigma_\epsilon) \sqrt{|\Sigma_\epsilon|}$ are recalled from Section 3.

Appendix B.7. Edge Cases and Patches

On loci where $\Sigma_\epsilon = 0$ the normalized direction $\hat{n}$ is defined patchwise (or by continuity); all algebraic projector statements remain valid, and the conclusions above hold on any patch with $\Sigma_\epsilon \neq 0$. Global/topological subtleties (multi-valued ϵ , nontrivial bundles) lie outside the posture A1–A5 (Section 2).

Summary of Appendix B. We have given explicit, idempotent projectors onto the three torsion irreps, fixed the quadratic identity in the normalization used in the main text, and proved the *single-vector*

no-go: from one covector v_μ no nonzero traceless torsion irrep can be built. This reduces the most general one-derivative ansatz to the $\{\delta^A{}_{[B}\partial_{C]}\epsilon, \epsilon^A{}_{BCD}\partial^D\epsilon\}$ span, after which the Palatini equations and the scalar projector select the pure-trace map used in the C1 uniqueness theorem.

Appendix C. Appendix C: Three-Chain Reductions & Improvement Currents (σ_ϵ Scheme)

Scope (σ_ϵ scheme and naming). This appendix provides the paper-checkable reductions behind Section 5 under the *sign-compensated* convention $\text{sgn}(\Sigma_\epsilon) I_T \equiv \text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|$ and the rank-one determinant route naming (formerly “DBI”-type; not Born-Infeld gravity): (i) a rank-one determinant route built out of the canonical traceless matrix $\mathcal{T}^\mu{}_\nu$, (ii) a closed-metric rank-one deformation, and (iii) the *PT*-even CS/Nieh-Yan shadow. At quadratic order, *each route* reduces in the bulk to the same invariant line

$$\delta^2 \mathcal{L}_X = A_\star \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T + \nabla_\mu J_X^\mu, \quad A_\star = \lambda^2/8, \quad X \in \{\text{rank-one determinant route, CM, CS}^+\},$$

with improvements $\nabla_\mu J_X^\mu$ differing by boundary choices (A4/A5). Closed representatives for J^μ are given on FRW/weak-field backgrounds.

Notation and key relation. We use the preamble shorthands $\Sigma_\epsilon \equiv \Pi_{\text{PT}}[(\partial\epsilon)^2]$, $\text{sgn}(\Sigma_\epsilon) \equiv \text{sgn}(\Sigma_\epsilon)$, $\hat{n}_\mu \equiv \frac{\partial_\mu \epsilon}{\sqrt{|\Sigma_\epsilon|}}$, $\mathcal{T}^\mu{}_\nu \equiv \text{sgn}(\Sigma_\epsilon) \left(\hat{n}^\mu \hat{n}_\nu - \frac{1}{4} \delta^\mu{}_\nu \right)$, $\tau \equiv \hat{n}^\mu T_\mu = 3\eta \text{sgn}(\Sigma_\epsilon) \sqrt{|\Sigma_\epsilon|}$ so that $\text{Tr} \mathcal{T} = 0$, $\text{Tr}(\mathcal{T}^2) = \frac{3}{4}$, and (after C1)

$$\tau^2 = 9\eta^2 |\Sigma_\epsilon| = \frac{3}{2} (-\text{sgn}(\Sigma_\epsilon) I_T), \quad \text{sgn}(\Sigma_\epsilon) I_T \equiv \text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|.$$

Appendix C.1. rank-one determinant route: Determinant Algebra with the $\frac{2}{3}$ Normalization

Consider the rank-one determinant route Lagrangian

$$\mathcal{L}_{\text{ROD}} = \sqrt{-g} \left(\sqrt{\det[1 + \frac{2}{3} \lambda \tau \mathcal{T}]} - 1 \right), \quad \lambda \in \mathbb{R}. \quad (\text{A29})$$

Using $\sqrt{\det(1 + X)} = 1 + \frac{1}{2} \text{Tr} X + \frac{1}{8} [(\text{Tr} X)^2 - 2 \text{Tr} X^2] + \mathcal{O}(X^3)$ and $\text{Tr} \mathcal{T} = 0$, the quadratic piece is

$$\begin{aligned} \delta^2 \mathcal{L}_{\text{ROD}} &= \sqrt{-g} \frac{1}{8} \left[-2 \text{Tr} \left(\left(\frac{2}{3} \lambda \tau \right)^2 \mathcal{T}^2 \right) \right] = -\frac{1}{4} \left(\frac{2}{3} \right)^2 \lambda^2 \sqrt{-g} \tau^2 \text{Tr}(\mathcal{T}^2) \\ &= -\frac{1}{4} \cdot \frac{4}{9} \cdot \frac{3}{4} \lambda^2 \sqrt{-g} \tau^2 = -\frac{1}{12} \lambda^2 \sqrt{-g} \tau^2. \end{aligned} \quad (\text{A30})$$

With $\tau^2 = 9\eta^2 |\Sigma_\epsilon|$ and $\text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|$,

$$-\frac{1}{12} \lambda^2 \tau^2 = -\frac{1}{12} \lambda^2 (9\eta^2 |\Sigma_\epsilon|) = -\frac{3}{4} \lambda^2 \eta^2 |\Sigma_\epsilon| = \left(\frac{\lambda^2}{8} \right) \text{sgn}(\Sigma_\epsilon) I_T,$$

so that

$$\delta^2 \mathcal{L}_{\text{ROD}} = A_\star \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T + \nabla_\mu J_{\text{ROD}}^\mu, \quad A_\star = \frac{\lambda^2}{8}. \quad (\text{A31})$$

At the bulk-density level one may take $J_{\text{ROD}}^\mu \equiv 0$. For unified boundary diagnostics we adopt a common canonical representative J_{can}^μ for *all* three routes (Section C.4).

Appendix C.2. Closed–Metric Route: Rank–One Deformation Equals rank-one determinant route To $\mathcal{O}(\mathcal{T}^2)$

Take the rank–one deformation $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{2}{3}\lambda \tau \hat{n}_\mu \hat{n}_\nu$ so that $\sqrt{-\tilde{g}} = \sqrt{-g} \sqrt{\det[\mathbf{1} + \frac{2}{3}\lambda \tau \mathcal{T}]}$ and define $\mathcal{L}_{\text{CM}} \equiv \sqrt{-\tilde{g}} - \sqrt{-g}$. The same algebra gives

$$\delta^2 \mathcal{L}_{\text{CM}} = A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T + \nabla_\mu J_{\text{CM}}^\mu, \quad A_\star = \frac{\lambda^2}{8}. \quad (\text{A32})$$

Thus rank-one determinant route and CM have the *same bulk* coefficient $A_\star$ and differ only by improvements.

Appendix C.3. Pt–Even CS/Nieh–Yan Shadow: Quadratic Reduction (σ_ϵ)

Using $T^A \wedge T_A = d(e^A \wedge T_A) - e^A \wedge e^B \wedge R_{AB}$, applying $*$ and the scalar PT projector (A2), and evaluating after C1, the PT –even piece reduces at quadratic order to

$$\delta^2 \mathcal{L}_{\text{CS}^+} = A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T + \nabla_\mu J_{\text{CS}^+}^\mu, \quad A_\star = \frac{\lambda^2}{8}, \quad (\text{A33})$$

where the Nieh–Yan assignment (A5) reshuffles only boundary conventions inside the PT –even sector.

Appendix C.4. A Universal Canonical Improvement at Quadratic Order

For route–by–route flux comparisons it is convenient to select the *same* improvement representative for all routes:

$$J_{\text{can}}^\mu[h^{\text{TT}}] \equiv \frac{A_\star}{2} a^3 \left(h_{ij}^{\text{TT}} \nabla^\mu h_{ij}^{\text{TT}} - \delta^\mu_0 h_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \right), \quad (\text{A34})$$

so that we *adopt the convention*

$$\delta^2 \mathcal{L}_X = A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T + \nabla_\mu J_{\text{can}}^\mu, \quad X \in \{\text{rank} - \text{onedeterminantroute}, \text{CM}, \text{CS}^+\}. \quad (\text{A35})$$

Different representatives differ by $\nabla_\nu X^{[\mu\nu]}$ and yield identical integrated fluxes under A4.

Check (FRW).

On spatially flat FRW in TT gauge,

$$\nabla_\mu J_{\text{can}}^\mu = \frac{A_\star a^3}{2} \left(\dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} - \frac{\partial_k h_{ij}^{\text{TT}} \partial_k h_{ij}^{\text{TT}}}{a^2} \right) + \partial_\tau \left(\frac{3}{2} A_\star a^2 \dot{a} h_{ij}^{\text{TT}} h_{ij}^{\text{TT}} \right), \quad (\text{A36})$$

i.e., the canonical reshuffling between TT kinetic/gradient bilinears plus a pure time boundary term that integrates to zero with A4 fall–offs.

Appendix C.5. Closed Forms for FRW and Weak Field

FRW.

For $ds^2 = a(\tau)^2(-d\tau^2 + d\mathbf{x}^2)$ and homogeneous $\epsilon(\tau)$,

$$J_{\text{can}}^0 = \frac{A_\star}{2} a^3 h_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}}, \quad J_{\text{can}}^i = \frac{A_\star}{2} a h_{jk}^{\text{TT}} \partial^i h_{jk}^{\text{TT}}, \quad (\text{A37})$$

which satisfies (A36).

Weak field (AF).

At $a \rightarrow 1$,

$$J_{\text{can}}^0 = \frac{A_\star}{2} h_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}}, \quad J_{\text{can}}^i = \frac{A_\star}{2} h_{jk}^{\text{TT}} \partial^i h_{jk}^{\text{TT}}. \quad (\text{A38})$$

Appendix C.6. Flux–Ratio Identity & Finite–Domain Convergence

With the unified choice (A35), $\int_{\partial\mathcal{D}} d\Sigma_\mu J_{\text{ROD}}^\mu = \int_{\partial\mathcal{D}} d\Sigma_\mu J_{\text{CM}}^\mu = \int_{\partial\mathcal{D}} d\Sigma_\mu J_{\text{CS}^+}^\mu$, hence

$$\mathcal{R}_{X/Y}[\partial\mathcal{D}] \equiv \frac{\int_{\partial\mathcal{D}} d\Sigma_\mu J_X^\mu}{\int_{\partial\mathcal{D}} d\Sigma_\mu J_Y^\mu} = 1, \quad X, Y \in \{\text{rank-one determinant route, CM, CS}^+\}. \quad (\text{A39})$$

On finite FRW balls (or AF shells) residuals scale away with the radius R , agreeing with Section V.

Summary of Appendix C

At quadratic order and under A1–A5 plus C1, the rank-one determinant route, closed-metric, and PT -even CS/Nieh–Yan routes share the same bulk reduction $\delta^2 \mathcal{L}_X = A_* \sqrt{-g} \text{sgn}(\Sigma_\epsilon) I_T \pmod{\nabla_\mu J^\mu}$, $A_* = \lambda^2/8$. A single canonical improvement J_{can}^μ (Equations (A34)–(A36)) is used for all routes and underlies the flux–ratio plots in Section V. [nb : fig_c2_coeff_compare.py; fig_flux_ratio.py]

Appendix D. Appendix D: Mixing Matrix and the Equal–Coefficient Identity

This appendix contains (i) extraction rules and tables for the 2×2 mixing matrix used in Section 6, including a *non-collinearity* proof of its two row vectors on admissible backgrounds, and (ii) a covariant derivation of the *equal-coefficient identity* $K(w) - G(w) = a^{-3} \partial_\mu (a^3 J_\Delta^\mu)$ quoted in Section 7. We assume A1–A5, the scalar PT projector (Section 2), and the C1 map $T_\mu = 3\eta \partial_\mu \epsilon$. Projected scalars are real by construction; the σ_ϵ scheme only affects the bulk line through $\text{sgn}(\Sigma_\epsilon) I_T = \text{sgn}(\Sigma_\epsilon) I_T = -6\eta^2 |\Sigma_\epsilon|$, not the kinematical identity $K - G = \partial J$.

Variational domain (used below).

We take variations with compact support on spatial slices or with FRW/AF fall-offs: $h_{ij}^{\text{TT}} = O(r^{-1-\sigma})$, $\delta N = O(r^{-2-\sigma})$, $\partial_i N^i = O(r^{-2-\sigma})$ with $\sigma > 0$. Then $J_\Delta^\mu = O(r^{-3-\sigma})$ and $\delta \int d^4x a^3 \partial_\mu J_\Delta^\mu = \int d\Sigma_\mu a^3 \delta J_\Delta^\mu = 0$.

Appendix D.1. Adm Conventions and Extraction of Mixing Entries

With $ds^2 = -N^2 d\tau^2 + \gamma_{ij}(dx^i + N^i d\tau)(dx^j + N^j d\tau)$, $N = 1 + \delta N$, $N_i = \partial_i \chi + N_i^T$ ($\partial_i N_i^T = 0$), and $\gamma_{ij} = a^2(\tau)(\delta_{ij} + h_{ij})$, we project $h_{ij} \rightarrow h_{ij}^{\text{TT}}$ with $\partial_i h_{ij}^{\text{TT}} = 0 = \delta^{ij} h_{ij}^{\text{TT}}$. The quadratic Lagrangian takes the block form

$$\delta^2 \mathcal{L} = \frac{M_{\text{Pl}}^2}{8} a^3 \left[K(w) \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} - G(w) \frac{(\partial_k h_{ij}^{\text{TT}})^2}{a^2} \right] + a^3 (h^{\text{TT}} \cdot \mathbf{M}(w) \cdot \Phi) + \frac{a^3}{2} \Phi \cdot \mathbf{P} \cdot \Phi, \quad (\text{A40})$$

where $\Phi = \{\delta N, \partial_i N^i, \dots\}$ collects nonpropagating pieces. Define the *dimensionless* mixing entries by

$$[\mathbf{M}(w)]_{h^{\text{TT}}-\delta N} \equiv \frac{8}{M_{\text{Pl}}^2} \frac{1}{a^3} \frac{\partial}{\partial(\delta N)} \left(\delta^2 \mathcal{L} \right) \Big|_{\text{bilinear in } h^{\text{TT}}, \delta N'} \quad (\text{A41})$$

$$[\mathbf{M}(w)]_{h^{\text{TT}}-\partial_i N^i} \equiv \frac{8}{M_{\text{Pl}}^2} \frac{1}{a^3} \frac{\partial}{\partial(\partial_i N^i)} \left(\delta^2 \mathcal{L} \right) \Big|_{\text{bilinear in } h^{\text{TT}}, \partial_i N^i}. \quad (\text{A42})$$

For $\mathcal{L}_{\text{chain}}(w) = w_{\text{ROD}} \mathcal{L}_{\text{ROD}} + w_{\text{CM}} \mathcal{L}_{\text{CM}}$, the mixing block is linear in w and proportional to $\eta^2 \Sigma_\epsilon$:

$$\begin{aligned} [\mathbf{M}(w)]_{h^{\text{TT}}-\delta N} &= \left(\mu_{\text{ROD}}^{(N)} w_{\text{ROD}} + \mu_{\text{CM}}^{(N)} w_{\text{CM}} \right) \eta^2 \Sigma_\epsilon, \\ [\mathbf{M}(w)]_{h^{\text{TT}}-\partial_i N^i} &= \left(\mu_{\text{ROD}}^{(\nabla N)} w_{\text{ROD}} + \mu_{\text{CM}}^{(\nabla N)} w_{\text{CM}} \right) \eta^2 \Sigma_\epsilon, \end{aligned} \quad (\text{A43})$$

defining the four dimensionless coefficients $\mu_{\text{ROD}}^{(N)}, \mu_{\text{CM}}^{(N)}, \mu_{\text{ROD}}^{(\nabla N)}, \mu_{\text{CM}}^{(\nabla N)}$.

Appendix D.2. Background Invariants And Compact Parametrization

Introduce

$$\mathcal{H} \equiv \frac{a'}{a}, \quad Y \equiv \partial_\tau \ln(\tau/a) = \frac{\tau'}{\tau} - \mathcal{H}, \quad (\text{A44})$$

(prime is conformal-time derivative). Each μ admits a linear decomposition

$$\mu_X^{(Y)} = c_{X,0}^{(Y)} + c_{X,\mathcal{H}}^{(Y)} \mathcal{H} + c_{X,Y}^{(Y)} Y, \quad X \in \{\text{ROD}, \text{CM}\}, \quad Y \in \{N, \nabla N\}, \quad (\text{A45})$$

with c 's real $\mathcal{O}(1)$ numbers fixed by the quadratic expansion rules (route Jacobians plus contorsion under C1).

Appendix D.3. Coefficient Tables (FRW and Weak Field)

Table A1. FRW coefficients with $\epsilon = \epsilon(\tau)$. Entries are the dimensionless μ 's of (A43), written as $\mu = c_0 + c_{\mathcal{H}} \mathcal{H} + c_Y Y$ [Equation (A45)]. Overall factor $\eta^2 \Sigma_\epsilon$ multiplies the mixing (*not* shown here).

Coefficient	c_0	$c_{\mathcal{H}}$	c_Y
$\mu_{\text{ROD}}^{(N)}$	$c_{\text{ROD},0}^{(N)}$	$c_{\text{ROD},\mathcal{H}}^{(N)}$	$c_{\text{ROD},\blacksquare}^{(N)}$
$\mu_{\text{ROD}}^{(\nabla N)}$	$c_{\text{ROD},0}^{(\nabla N)}$	$c_{\text{ROD},\mathcal{H}}^{(\nabla N)}$	$c_{\text{ROD},\blacksquare}^{(\nabla N)}$
$\mu_{\text{CM}}^{(N)}$	$c_{\text{CM},0}^{(N)}$	$c_{\text{CM},\mathcal{H}}^{(N)}$	$c_{\text{CM},\blacksquare}^{(N)}$
$\mu_{\text{CM}}^{(\nabla N)}$	$c_{\text{CM},0}^{(\nabla N)}$	$c_{\text{CM},\mathcal{H}}^{(\nabla N)}$	$c_{\text{CM},\blacksquare}^{(\nabla N)}$

Regeneration recipe (FRW). (1) Write $\delta g_{\mu\nu}$ in terms of $(\delta N, \partial_i N^i, h_{ij}^{\text{TT}})$; (2) expand $\mathcal{T}, \tau$ to linear order using $\hat{n}_\mu = \partial_\mu \epsilon / \sqrt{|\Sigma_\epsilon|}$ and C1; (3) insert into $\mathcal{L}_{\text{ROD}}$ and $\mathcal{L}_{\text{CM}}$ with the $\frac{2}{3}$ scaling; (4) keep only $h^{\text{TT}} \delta N$ and $h^{\text{TT}} \partial_i N^i$ bilinears to read off $c_0, c_{\mathcal{H}}, c_Y$.

Table A2. Weak field (AF) coefficients with slowly varying $\epsilon(t, \mathbf{x})$. Set $a \rightarrow 1$ ($\mathcal{H} = 0$) and define $Y \equiv \partial_t \ln \tau$. Overall factor $\eta^2 \Sigma_\epsilon$ multiplies the mixing (*not* shown here).

Coefficient	c_0	c_Y
$\mu_{\text{ROD}}^{(N)}$	$\tilde{c}_{\text{ROD},0}^{(N)}$	$\tilde{c}_{\text{ROD},\blacksquare}^{(N)}$
$\mu_{\text{ROD}}^{(\nabla N)}$	$\tilde{c}_{\text{ROD},0}^{(\nabla N)}$	$\tilde{c}_{\text{ROD},\blacksquare}^{(\nabla N)}$
$\mu_{\text{CM}}^{(N)}$	$\tilde{c}_{\text{CM},0}^{(N)}$	$\tilde{c}_{\text{CM},\blacksquare}^{(N)}$
$\mu_{\text{CM}}^{(\nabla N)}$	$\tilde{c}_{\text{CM},0}^{(\nabla N)}$	$\tilde{c}_{\text{CM},\blacksquare}^{(\nabla N)}$

Regeneration recipe (weak field). Repeat FRW steps with $a \rightarrow 1$, replace conformal by physical time, and expand consistently in spatial gradients so that Σ_ϵ remains the projected, real scalar.

Closed-form labels (for Section VI cross-references).

For later citation we record the compact forms as Equations (D.12)–(D.15):

$$\mu_{\text{ROD}}^{(N)} = c_{\text{ROD},0}^{(N)} + c_{\text{ROD},\mathcal{H}}^{(N)} \mathcal{H} + c_{\text{ROD},\blacksquare}^{(N)} Y, \quad (\text{D.12})$$

$$\mu_{\text{CM}}^{(N)} = c_{\text{CM},0}^{(N)} + c_{\text{CM},\mathcal{H}}^{(N)} \mathcal{H} + c_{\text{CM},\blacksquare}^{(N)} Y, \quad (\text{D.13})$$

$$\mu_{\text{ROD}}^{(\nabla N)} = c_{\text{ROD},0}^{(\nabla N)} + c_{\text{ROD},\mathcal{H}}^{(\nabla N)} \mathcal{H} + c_{\text{ROD},\blacksquare}^{(\nabla N)} Y, \quad (\text{D.14})$$

$$\mu_{\text{CM}}^{(\nabla N)} = c_{\text{CM},0}^{(\nabla N)} + c_{\text{CM},\mathcal{H}}^{(\nabla N)} \mathcal{H} + c_{\text{CM},\blacksquare}^{(\nabla N)} Y. \quad (\text{D.15})$$

Appendix D.4. Non-Collinearity of the Two Locking Equations

Let the two rows be $\mathbf{r}_1 = (\mu_{\text{ROD}}^{(N)}, \mu_{\text{CM}}^{(N)})$, $\mathbf{r}_2 = (\mu_{\text{ROD}}^{(\nabla N)}, \mu_{\text{CM}}^{(\nabla N)})$.

Proposition A4 (Non-collinearity). *Under A1–A5, C1, and the one-derivative-per-building-block posture, $\mathbf{r}_1 \nparallel \mathbf{r}_2$ on any admissible background with at least one of $\{\mathcal{H}, \mathcal{Y}\} \neq 0$ and $\nabla_\mu \hat{n}_\nu \neq 0$. Equivalently,*

$$\Delta \equiv \mu_{\text{ROD}}^{(N)} \mu_{\text{CM}}^{(\nabla N)} - \mu_{\text{ROD}}^{(\nabla N)} \mu_{\text{CM}}^{(N)} \neq 0.$$

Proof (sketch). Using (A45), proportionality would require $\rho_X \equiv \mu_X^{(N)} / \mu_X^{(\nabla N)}$ to be route-independent and constant under independent shifts of $\mathcal{H}$ and $\mathcal{Y}$. But $\partial_{\mathcal{H}} \rho_X$ and $\partial_{\mathcal{Y}} \rho_X$ cannot both vanish for $X = \text{ROD, CM}$ unless $c_{X,\mathcal{H}}^{(\mathcal{Y})} = c_{X,\mathcal{Y}}^{(\mathcal{Y})} = 0$ (static trivial branch $\mathcal{H}=\mathcal{Y}=0$, $\nabla \hat{n} = 0$). Hence $\mathbf{r}_1 \nparallel \mathbf{r}_2$ generically. $\square$

Small- k structure (symbolic).

On weak-field patches one finds

$$\Delta(k, \Sigma_\epsilon) = \alpha_0 \eta^4 \Sigma_\epsilon^2 k^2 + O(k^4, \Sigma_\epsilon^3), \quad \alpha_0 \neq 0,$$

so rank loss occurs only on the measure-zero set $\{k = 0\} \cup \{\Sigma_\epsilon = 0\}$ or special foliations.

Appendix D.5. Equal-Coefficient Identity: Covariant Derivation And Gauge Shift

Define the quadratic current (indices contracted with the background spatial metric)

$$J_\Delta^\mu = \frac{M_{\text{Pl}}^2}{8} \begin{pmatrix} J_\Delta^0 = \Pi^{ij} h_{ij}^{\text{TT}} \\ J_\Delta^k = -\Xi^{ij} \partial_k h_{ij}^{\text{TT}} \end{pmatrix}, \quad (\text{A46})$$

with $\Pi^{ij} \equiv \delta(\delta^2 \mathcal{L}) / \delta h_{ij}^{\text{TT}}$ and $\Xi^{ij} \equiv \delta(\delta^2 \mathcal{L}) / \delta(\partial^2 h_{ij}^{\text{TT}})$. Using TT conditions, the rank-one traceless normalization, and the bulk equality of the routes (Section V),

$$K(w) - G(w) = \frac{1}{a^3} \partial_\mu (a^3 J_\Delta^\mu). \quad (\text{A47})$$

Under $\delta_\xi h_{ij}^{\text{TT}} = 0$, $\delta_\xi h_{0\mu} = \partial_\mu \xi_0 + \dots$, the representative shifts as $\delta_\xi J_\Delta^\mu = \nabla_\nu X^{[\mu\nu]}[\xi]$, hence the *integrated* identity is gauge independent.

FRW and weak-field representatives.

On spatially flat FRW ($ds^2 = a^2(-d\tau^2 + d\mathbf{x}^2)$, homogeneous ϵ),

$$J_\Delta^0 = \frac{A_\star \eta^2}{2} \Sigma_\epsilon h_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} + \nabla_i(\dots), \quad (\text{A48})$$

$$J_\Delta^k = -\frac{A_\star \eta^2}{2} \Sigma_\epsilon \frac{1}{a^2} h_{ij}^{\text{TT}} \partial_k h_{ij}^{\text{TT}} + \partial_\ell(\dots)^{[k\ell]}, \quad (\text{A49})$$

so $K - G = a^{-3} \partial_\tau (a^3 J_\Delta^0) + a^{-3} \partial_k (a^3 J_\Delta^k)$ is a total divergence. In weak field (AF, $a \rightarrow 1$),

$$J_\Delta^0 = \frac{A_\star \eta^2}{2} \Sigma_\epsilon h_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} + \partial_i(\dots)^i, \quad J_\Delta^k = -\frac{A_\star \eta^2}{2} \Sigma_\epsilon h_{ij}^{\text{TT}} \partial_k h_{ij}^{\text{TT}} + \partial_\ell(\dots)^{[k\ell]}.$$

Boundary consequence and luminosity.

With the variational domain above (A4), $\int a^3 (K - G) = \int \partial_\mu (a^3 J_\Delta^\mu) = 0$. At the *locked* weights $w^\star$ (Section VI; no TT-nonTT mixing), $K(w^\star) = G(w^\star) \Rightarrow c_T^2 = 1$ at quadratic order, slicing-independently.

Appendix D.6. Locked Weights and Summary Box

Non-collinearity with the two mixing equations yields a unique ratio $w_{\text{ROD}}^\star / w_{\text{CM}}^\star$ (up to the GR normalization). At these weights, $K - G = a^{-3} \partial_\mu (a^3 J_\Delta^\mu)$ implies exact luminosity.

Appendix D (at a glance).

- *Mixing matrix.* Four dimensionless coefficients $\mu_X^{(Y)}$ ($X = \text{ROD, CM}; Y = N, \nabla N$) control the TT-constraint mixing with an overall $\eta^2 \Sigma_\epsilon$ factor.
- *Non-collinearity.* The two locking rows are not proportional on evolving admissible backgrounds; the determinant Δ is generically nonzero ($\Delta \sim \alpha_0 \eta^4 \Sigma_\epsilon^2 k^2 + \dots$).
- *Equal-coefficient identity.* $K - G = a^{-3} \partial_\mu (a^3 J_\Delta^\mu)$; under A4 this yields $K = G$ and $c_T^2 = 1$ at the locked weights w^* .

Reproducibility Note

Tables A1–A2 come from a single symbolic pipeline that: (i) expands $\mathcal{T}$ and τ to linear order in ADM variables with the $\frac{2}{3}$ normalization, (ii) forms the rank-one determinant route/CM quadratic densities, (iii) projects onto the bilinears of Equations (A41)–(A42). Scripts and hashes that also produced Figures c3_cT_heatmap and c3_dispersion are cataloged in Supp. R. [nb : mixing_matrix_extract.py]

Appendix E. Dirac Coupling, Field Redefinition, and the Nieh–Yan Scope

This appendix supplies the details promised in Section 8: (i) the torsion–fermion couplings in our conventions, (ii) the precise field redefinition that removes the trace channel once the pure-trace map (C1) is imposed, (iii) the role of the Nieh–Yan density as a boundary convention (A5), and (iv) the scope limitations on nontrivial topology (A4).

Throughout we work under the global posture A1–A5 of Section 2, use metric signature $\eta_{AB} = \text{diag}(-, +, +, +)$, and adopt the gamma-matrix conventions

$$\{\gamma^A, \gamma^B\} = 2\eta^{AB}, \quad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \epsilon^{0123} = +1, \quad (\text{A50})$$

so that $\gamma^{AB} \equiv \frac{1}{2}[\gamma^A, \gamma^B]$ and $\gamma^{ABC} \equiv \gamma^{[A}\gamma^B\gamma^{C]} = i\epsilon^{ABCD}\gamma_5\gamma_D$.

Appendix E.1. Minimal Dirac Coupling In Riemann–Cartan Space

The minimally coupled Dirac Lagrangian is

$$\mathcal{L}_\psi = e \bar{\psi} \left(i e_A^\mu \gamma^A D_\mu - m \right) \psi, \quad D_\mu \psi = \partial_\mu \psi + \frac{1}{4} \omega^{AB}_\mu \gamma_{AB} \psi, \quad e \equiv \sqrt{-g}. \quad (\text{A51})$$

Splitting the spin connection into Levi–Civita plus contorsion, $\omega^{AB}_\mu = \omega^{AB}_\mu(\text{LC}) + K^{AB}_\mu$, the *torsion-dependent* piece of (A51) is

$$\Delta \mathcal{L}_\psi[K] = \frac{i}{4} e \bar{\psi} e_A^\mu \gamma^A K^{BC}_\mu \gamma_{BC} \psi = \frac{i}{4} e \bar{\psi} \gamma^A K_{BCA} \gamma^{BC} \psi, \quad (\text{A52})$$

where tangent indices are moved with η_{AB} , and $K_{ABC} \equiv e_A^\mu K_{BC\mu}$.

It is convenient to decompose contorsion into the standard torsion irreps (vector trace T_A , axial S^A , and traceless q_{ABC}):

$$K_{ABC} = \frac{1}{3}(\eta_{AC} T_B - \eta_{AB} T_C) - \frac{1}{6} \epsilon_{ABCD} S^D + q_{ABC}, \quad T_A \equiv T^B_{AB}, \quad S^A \equiv \epsilon^{ABCD} T_{BCD}, \quad (\text{A53})$$

with $q^B_{AB} = 0$ and $\epsilon^{ABCD} q_{BCD} = 0$. Using the gamma identities listed below Equation (A50) and the antisymmetry of K_{BCA} in (B, C) , Equation (A52) reduces to

$$\Delta \mathcal{L}_\psi[K] = e \left(c_A S_\mu J_5^\mu + c_V T_\mu J^\mu \right) + e C^{\mu\nu\rho}(q) \bar{\psi} \gamma_{[\mu} \gamma_\nu \gamma_\rho] \psi, \quad (\text{A54})$$

where $J^\mu \equiv \bar{\psi}\gamma^\mu\psi$ and $J_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi$. In our normalization (A50) the *numerical* coefficients are

$$c_A = \frac{3}{8}, \quad c_V = -\frac{1}{4} \quad (\text{A55})$$

while the q -channel is proportional to the totally antisymmetric part of q , which vanishes by definition; equivalently, the q -coupling can be re-expressed in terms of S^μ and drops out when $S^\mu = 0$. Therefore the torsion-induced Dirac channels in our conventions are precisely

$$\Delta\mathcal{L}_\psi[T, S] = e\left(\frac{3}{8}S_\mu J_5^\mu - \frac{1}{4}T_\mu J^\mu\right), \quad (\text{A56})$$

up to PT -even improvement terms annihilated by A1/A4 in the bulk.

Cross-check.

Equation (A56) reproduces the two-channel bookkeeping used in Section 8 with $c_A = \frac{3}{8}$ and $c_V = -\frac{1}{4}$. Any alternative gamma/Hodge convention simply rescales (A55) by an overall sign; our later conclusions (vanishing axial channel by C1 and removability of the trace channel) are insensitive to such simultaneous flips.

Appendix E.2. C1 Implies no Axial Channel; The Trace Channel is Removable

By the uniqueness theorem (C1), Equation (13), the axial and traceless torsion irreps vanish, $S^\mu = 0$, $q_{\lambda\mu\nu} = 0$, and the trace aligns with the spurion gradient, $T_\mu = 3\eta\partial_\mu\epsilon$. Equation (A56) therefore reduces to the single trace channel

$$\mathcal{L}_{\psi,T}^{(\text{trace})} = e c_V T_\mu J^\mu = 3c_V\eta e(\partial_\mu\epsilon)J^\mu. \quad (\text{A57})$$

Proposition A5 (Vector rephasing removes the trace channel). *Consider the local vector phase redefinition $\psi \rightarrow \psi' = e^{i\alpha\epsilon}\psi$, $\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}e^{-i\alpha\epsilon}$ with a constant α . Then*

$$\mathcal{L}_\psi[\psi', \bar{\psi}'] = \mathcal{L}_\psi[\psi, \bar{\psi}] - \alpha e(\partial_\mu\epsilon)J^\mu + \nabla_\mu(\cdots), \quad (\text{A58})$$

so choosing $\alpha = 3c_V\eta$ ($= -\frac{3}{4}\eta$ in our normalization) cancels (A57) pointwise. Under A4 the improvement integrates to the boundary and has no bulk Euler–Lagrange effect; A5 permits absorbing any parity-odd reshuffling in the Nieh–Yan counterterm.

Sketch. Insert $\psi' = e^{i\alpha\epsilon}\psi$ into the kinetic term $e\bar{\psi}ie_A^\mu\gamma^A\partial_\mu\psi$ and use the Leibniz rule. The derivative acting on $e^{i\alpha\epsilon}$ generates the shift $-\alpha e(\partial_\mu\epsilon)J^\mu$ in (A58); the mass and spin-connection pieces are invariant under a *vector* (not axial) phase. The remainder is a covariant total divergence fixed by the integration-by-parts convention (A4/A5). $\square$

Combining (A57) with Prop. A5 and the coefficient assignment (A55) yields precisely the two-line summary in Section 8: no axial channel and a removable trace channel.

Path-integral measure (anomaly) check.

The field redefinition in Prop. A5 is a *vector* $U(1)$ rotation, hence the fermionic measure is invariant (Jacobian equal to one). An axial rotation would generate the usual ABJ contribution; in a Riemann–Cartan background it is accompanied by a Nieh–Yan density. We do not perform an axial rotation anywhere in this work.

Appendix E.3. Nieh–Yan as a Boundary Convention (A5)

The Nieh–Yan 4-form $NY \equiv d(e^A \wedge T_A)$ is an exact form. In our posture (A5), any explicit use of NY enters only as a *boundary convention* after applying the scalar projector Π_{PT} : it does not modify

Euler–Lagrange equations in the bulk and is indistinguishable from a choice of improvement current. This applies equally to (i) the three-chain equivalence (where the PT -even CS/Nieh–Yan shadow differs from the DBI/CM routes by an improvement current) and (ii) the Dirac sector (where vector rephasing reshuffles boundary terms that can be absorbed into the chosen Nieh–Yan convention). No physical statement in Sections 5–8 depends on a specific Nieh–Yan choice.

Appendix E.4. Scope and Caveats: Topology and Boundary Flux

Our conclusions rely on the posture A4: either compact PT -invariant domains with vanishing boundary flux or standard asymptotic fall-offs (FRW/flat) so that improvement currents integrate to zero. They also assume a *single-valued* spurion phase $\epsilon(x)$ so that the rephasing in Prop. A5 is a globally well-defined map $\psi \mapsto e^{i\alpha\epsilon}\psi$.

- **Nontrivial topology (not covered).** If ϵ is multi-valued or admits nontrivial holonomy (e.g., on manifolds with nontrivial π_1 or H^1), the map $e^{i\alpha\epsilon}$ may fail to be single-valued. In such cases the local cancellation in (A58) can leave a global remnant proportional to the winding; our bulk equivalences and removability statements are not asserted in that setting.
- **Nonvanishing boundary flux (not covered).** On domains where the PT -invariant boundary flux of the relevant improvement currents does not vanish, boundary terms may carry physical information (e.g., in explicitly finite boxes with prescribed inflow). Our null tests and cancellations are presented only under A4 fall-offs.

Appendix E.5. Bookkeeping Table And Quick References

For convenience we summarize the conventions and coefficients used in the Dirac sector:

Object / Convention	Value / Definition
Gamma matrices	$\{\gamma^A, \gamma^B\} = 2\eta^{AB}, \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
Hodge / Levi-Civita	$\epsilon^{0123} = +1, [\mathcal{PT}, *] = 0$ (A2)
Contorsion split	$K_{ABC} = \frac{1}{3}(\eta_{AC}T_B - \eta_{AB}T_C) - \frac{1}{6}\epsilon_{ABCD}S^D + q_{ABC}$
Dirac currents	$J^\mu = \bar{\psi}\gamma^\mu\psi, J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$
Torsion→Dirac	$\Delta\mathcal{L}_\psi = e(\frac{3}{8}S_\mu J_5^\mu - \frac{1}{4}T_\mu J^\mu)$
C1 (pure trace)	$S^\mu = 0, q_{\lambda\mu\nu} = 0, T_\mu = 3\eta\partial_\mu\epsilon$
Vector rephasing	$\psi \rightarrow e^{i\alpha\epsilon}\psi$ with $\alpha = 3c_V\eta = -\frac{3}{4}\eta$
Outcome	Trace channel removed up to a total derivative (A4/A5)
Nieh–Yan	Boundary convention only (A5); no bulk Euler–Lagrange effect

Appendix E.6. Corollary: No LO Four-Fermion Contact from Torsion

At the order analyzed in this paper (quadratic in fields; at most one derivative per building block), torsion is algebraic and fixed by (C1), and the only linear Dirac–torsion channel that survives projection is removed by the vector redefinition above. Consequently no tree-level, local four-fermion contact term is induced at leading order within our posture (A1–A5). Any such effect would require (i) an axial channel ($S_\mu \neq 0$), (ii) higher-derivative completions beyond our closure basis, or (iii) loop corrections outside the present scope.

Summary of Appendix E. In our conventions the minimally coupled Dirac field interacts with torsion through $e(\frac{3}{8}S_\mu J_5^\mu - \frac{1}{4}T_\mu J^\mu)$. Under the pure-trace map (C1) the axial channel vanishes and the remaining trace channel is removed by a local vector rephasing $\psi \rightarrow e^{i(3c_V\eta)\epsilon}\psi$, up to a total derivative consistent with A4/A5. Nieh–Yan acts only as a boundary convention, and nontrivial topology or nonvanishing boundary flux lie outside our stated scope.

Appendix F. Operational Diagnostics (Extended)

This appendix collects a practitioner-oriented checklist and an expanded comparison table. Unless stated otherwise, all entries are framed for admissible patches with the boundary/topology posture A4, and for the quadratic order analyzed in the main text.

Appendix F.1. Full Decision Tree (D1–D8)

- **(D1) Parity channels.** Any detection of *helicity-dependent* phase or amplitude birefringence in GWs points to parity-odd operators. Our scalar-*PT* posture forbids such effects at LO (Theorem 1; Section 2.2).⁷
- **(D2) Axial/tensor torsion irreps.** Nonzero axial (S^μ) or traceless ($q_{\lambda\mu\nu}$) torsion signals are incompatible with C1. We predict the pure-trace map (Theorem 2); Figures 1, 2 provide diagnostics.
- **(D3) Three-route bulk collapse.** Reconstruct quadratic kernels from simulations/perturbation theory: the DBI, closed-metric, and CS^+ routes must share the same *bulk* piece $A_\star \sqrt{-g} \operatorname{sgn}(\Sigma_\epsilon) I_T$ with $A_\star = \lambda^2/8$ (Section 5); residuals should enter only through improvements (Figure 3).
- **(D4) Boundary equivalence.** Boundary flux ratios $\mathcal{R}_{X/Y}[\partial\mathcal{D}]$ converge to 1 on growing domains (A4) (Section 5.4; Figure 4). Persistent deviations falsify C2.
- **(D5) Exact luminosity by identity.** After eliminating TT-nonTT mixing with the full-rank 2×2 system (Section 6.3), the equal-coefficient identity $K - G = \partial_\mu(a^{-3}J_\Delta^\mu a^3)$ enforces $c_T=1$ without tuning (Section 7; Figure 7).
- **(D6) DoF count.** The quadratic kernel exhibits exactly two propagating tensor modes; no extra scalar/vector propagation survives the degeneracy test (Section 7.3; Figure 8). Any additional mode falsifies our posture at this order.
- **(D7) NLO slope.** At next order, the leading *PT*-even dispersion predicts $\delta c_T^2(k) = b k^2/\Lambda^2$ (Equation (54)): a log-log slope $\simeq 2$ in clean frequency windows is characteristic (Figure 9; Section 9.1).⁸
- **(D8) Fermion channel null.** No axial contact and a removable trace contact in the Dirac sector at LO (Section 8). Any robust axial spin-torsion signal would contradict C1 (Appendix E contains coefficients and boundary conventions).

Appendix F.2. Expanded Disambiguation Table

Entries are *generic at LO* on admissible backgrounds with A4 fall-offs; tuned subclasses and specific parameter choices may alter individual cells. See the notes in Section F.3 for caveats and representative citations.

Appendix F.3. Notes, Caveats, And Representative Anchors

Parity-odd lines.

Chern–Simons modified gravity (Jackiw–Pi; Alexander–Yunes) generically induces helicity-dependent GW propagation; details depend on the choice of scalar field and coupling. Our scalar-*PT* projector eliminates parity-odd *scalars* at LO; see Theorem 1. Representative anchors: [14,15].

Horndeski/DHOST.

Post-GW170817 constraints force $c_T \rightarrow 1$ by parameter tuning or by restricting to subclasses; however, no equal-coefficient *identity* of the type $K - G = \partial_\mu(\dots)$ is generally present, and an extra scalar DoF is typical. See the multimessenger bounds and reviews [27–30,35,36].

⁷ Chern–Simons modified gravity is the canonical benchmark for helicity-split propagation; see e.g., Jackiw–Pi; Alexander–Yunes.

⁸ Caveat: finite-window fits may be biased by instrumental systematics or astrophysical priors; interpret bounds conservatively.

Teleparallel/ $f(T)$.

Many models yield $c_T = 1$ at LO; torsion irreps and matter couplings are model-dependent, and our route-equality/flux-ratio diagnostics (C2) do not directly apply. We therefore treat teleparallel cases as outside the present “three-route” posture.

EC/ECSK-like.

With propagating torsion or nonminimal fermion couplings, axial/traceless torsion irreps can be present and Dirac axial contacts typically survive. Our C1 pure-trace map excludes these at the analyzed order; see classic EC/MAG treatments [4–6,8].

NLO dispersion.

The $\delta c_T^2 \propto k^2$ slope follows from the projected, PT -even closure and the normalization of $\mathcal{T}$ (Section 5); in practice, slopes may be obscured by finite-band systematics, so conservative fits are recommended (Section 9).

Boundary posture.

All boundary-sensitive statements assume A4 (PT -invariant compact boundaries or standard AF/FRW fall-offs) and A5 (Nieh–Yan as boundary counterterm). Departures from these postures may alter flux diagnostics and improvement accounting. Operational anchors: covariant phase space and surface charges [20–22], and the Holst/Nieh–Yan parity bookkeeping [9–12].

Representative anchors (one-line map).

Metric-affine/Einstein–Cartan: [4,6,8]. Holst/Nieh–Yan: [9–12]. Covariant phase space & boundary charges: [20–22]. CS-modified gravity: [14,15]. GW170817 speed bounds & implications: [23–30].

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