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The Relentless Two-Envelope Conundrum: A Paradox or Misapplication of Probability Theory

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Abstract

The Two-Envelope Problem (TEP) is revisited to argue that the standard evaluation of expected returns relies on spurious probabilities arising from a misuse of formal probability theory. The source of the problem is the *ex post framing* of two identical envelopes, X and Y, one containing twice as much money as the other, after one envelope, say X, has been selected and its content $X{=}x$ observed. The value x is then used to define Y in terms of the values $y{=}x/2$ and $y{=}2x$, each assigned probability .5, with an analogous derivation when Y is selected. This renders X and Y ill-defined random variables because the relevant probabilistic framing must instead be based on the original experimental setup, prior to any selection or observation, where the envelope contents are unknown, say \$$\theta$ and \$$2\theta$. Framing the original setup using axiomatic probability, the dependence between X and Y is accounted for when $x{=}\theta$, $y{=}2\theta$, and when $x{=}2\theta$, $y{=}\theta$. The ensuing joint distribution of X and Y determines that the expected returns imply indifference between keeping the chosen envelope and switching, explaining away the 'paradox' as a misapplication of probability theory.

Keywords: two-envelope paradox; exchange paradox; formal probability reasoning; random variables; joint and marginal distributions; expected returns; axiomatic probability theory

JEL Classification: C12; C18; C51; C52; C55

1. Introduction

In a book entitled *"The Waltz of Reason: the Entanglement of Mathematics and Philosophy"* Sigmund [1], pp. 185-187, brings out the 'weird streak of probability' by discussing two famous puzzles, the Monty Hall (or three doors) and the Two-Envelope (or exchange) paradoxes. A simple Google Scholar search reveals more than 34,000 citations for the former and 2,231,000 for the latter. Both problems are particularly beguiling because they are easy to understand with minimal knowledge of probability, and appear to raise issues that cut across many different disciplines, including probability theory, statistics, logic, semantics, philosophy of science, epistemology, economics, psychology, decision-theory, and game theory. Despite their apparent simplicity, both problems involve formal probabilistic reasoning whose complexity is often underestimated, by both ordinary readers and scientists in different disciplines; see Spanos [2] p. 68-69, on the Monty Hall problem.

The discussion that follows focuses on the Two-Envelope Problem (TEP), described by Sigmund [1], p. 186, as the "hairier" of the two: "The quizmaster tells us that one of the envelopes contains a certain sum of money, and the other one twice that much. We pick an envelope, are allowed to open it, and see that it contains \$40. Now the quizmaster offers us to exchange the chosen envelope with the other one, which has not been opened. Should we agree with the exchange?"

It should be noted that in certain variants of the problem, the player does not see the contents of the selected envelope, but that makes no difference to the main argument in this paper. Also, the discussion focuses on viewing this conundrum from the perspective of frequentist probability theory, but the results are also relevant when the TEP is viewed from a Bayesian perspective; see Christensen and Utts [3], Lindley [4].

2. Revisiting the Traditional TEP Evaluation of the Expected Returns

In the traditional narrative of the TEP, the player is asked to select one of two identical envelopes, knowing that one contains twice as much money as the other. The two envelopes are traditionally viewed as random variables X and Y where a player selects one envelope, say X containing \$$x$, and derives the probability distribution of Y as a function of x, assigning the values $y{=}\frac{x}{2}$ and $y{=}2x$ with equal probability .5. This gives rise to the distribution $f(Y;x)$ in Table 1. Using analogical reasoning, when envelope Y is selected, the same thought process gives rise to the distribution $f(X;y)$ in Table 2.

Table 1. $f(Y;x)$

Y	$\mathbb{P}(Y=y)$
$y=\frac{x}{2}$	.5
$y=2x$	.5

Table 2. $f(X;y)$

X	$\mathbb{P}(X=x)$
$x=\frac{y}{2}$	.5
$x=2y$	.5

The evaluation of the **expected returns** for each envelope, based on Tables 1 and 2, yields:

$$\begin{aligned} E(Y)&{=}(\tfrac{x}{2})\cdot\mathbb{P}(Y{=}\tfrac{x}{2}){+}(2x)\cdot\mathbb{P}(Y{=}2x){=}(\tfrac{x}{2})(.5){+}2x(.5){=}1.25x > x,\\ E(X)&{=}(\tfrac{y}{2})\cdot\mathbb{P}(X{=}\tfrac{y}{2}){+}(2y)\cdot\mathbb{P}(X{=}2y){=}(\tfrac{y}{2})(.5){+}2y(.5){=}1.25y > y.\end{aligned} \tag{1}$$

These results imply that it would be rational for the player *to always exchange* the original envelope, irrespective of the values x and y. This is clearly false, but the difficulty lies in pinpointing the flaw in the reasoning invoked that led to the fallacious result in (1).

The plethora of published papers across several different disciplines on the TEP is a testament to its puzzling complexity and its broader interest; see Nalebuff [5], Linzer [6], Rawling [7], Broome [8], Bruss [9], Wagner [10], Clark and Shackel [11], Chalmers [12], Dietrich and List [13], Albers et al. [14], Katz and Ohlin [15], Rudas [17], Falk [16], Falk and Nickerson [18], Yi [19], Vasudevan [20], Portnoy [21], Brock and Glasgow [22], Hoffmann [23], Kemp [24], Blackwell [25] inter alia.

The overwhelming majority of published papers on the TEP take Tables 1 and 2 *at face value* and proceed to propose reasoned modifications that would render the two envelopes interchangeable. These modifications are often based on introducing new random variables relating to X and Y, or/and additional components, such as utility and decision functions, probability distributions, including priors, and imprecise probabilities.

An insightful recent discussion of the TEP is given in Gill [26]. The paper provides a broad discourse across several different types of 'resolutions' to the paradox, including Bayesian, with improper priors, decision-theoretic, as well as philosophical resolutions based on logic and semantics, pointing out a number of flaws in the proposed resolutions. The emphasis is placed on pinpointing the flaws in the reasoning employed by different authors to reach their conclusions. Gill's most insightful comment is that:

"· · · since the writer is not working explicitly in a particular formal framework, we do not know what he or she is trying to do. There is no single resolution to the paradox of the type "so-and-so fails," nor is there a unique explanation of what went wrong. Looking for one is illusory. Unless we take a broader perspective and say that the writer was attempting to do probability theory without understanding its concepts—let alone its rules—and therefore made serious errors by failing to draw distinctions that are essential in probability theory. TEP exemplifies the very reasons formal probability theory was developed. Philosophers who work on TEP without knowledge of modern (even elementary) probability theory are largely wasting their time; at best, they will reinvent the wheel." pp. 212-213.

Gill's [26] insight pertaining to "formal probability theory" is unfolded – in the discussion that follows – by reframing the TEP in terms of two well-defined random variables X and Y in the context of Kolmogorov's [27] axiomatic approach to probability. This gives rise to a different type of resolution to the TEP paradox that actualizes his insight.

It is important to emphasize at the outset that the proposed resolution to the paradox focuses exclusively on a probabilistic reframing that retains the **original setup**, as described above, *without* any added components, such as utility and decision functions, and/or probability distributions, prior distributions, etc.

A case is made that the key culprit engendering the paradox is the conventional *ex post* — after selecting an envelope and observing its content — framing of X and Y by using $X{=}x$ to define Y in terms of $y{=}\frac{x}{2}$ and $y{=}2x$, and vice versa for X. This gives the misleading impression that the problem is straightforward, masking the fact that X and Y are ill-defined due to three interrelated problems.

(a) The first problem arises from the fact that, in the **original setup** of the TEP, the money in the two envelopes is **unobservable**, say θ and 2θ. Hence, the ex post framing of the random variable Y, based on the observed money x, provides beguiling but highly misdirecting information for the relevant ex ante probabilistic framing of the TEP.

(b) The second problem is that the random variables X and Y are *identically distributed* but **dependent**, since "one envelope contains twice as much money as the other". This is not accounted for by the values and probabilities in Tables 1 and 2, because this dependence needs to be framed in the context of the ex ante joint distribution of X and Y, before the initial selection of the envelope is made and its content observed, as argued in the sequel.

(c) Third, it is unclear whether Tables 1 and 2 are traditionally *perceived* by the literature as **conditional** or **marginal distributions** for X and Y. Either way, probability theory requires that both of them should be formally *derivable* from a well-defined joint distribution. Unfortunately, such joint distributions are invariably absent in many proposed resolutions to the paradox, especially the ones that bring in additional random variables relating to X and Y in several different ways. That is, without a well-defined joint distribution of all the random variables involved, no conditional or marginal distribution can be presumed to be well-defined in a probabilistic sense.

As argued next, the problems (a)-(c) could be addressed by reframing the random variables X and Y in terms of the *unknown* values $\{\theta, 2\theta\}$, in the context of Kolmogorov's axiomatic approach to probability theory.

3. Kolmogorov's Axiomatic Probability Theory

Probability theory, as known today, became part of mathematics proper when Kolmogorov [27] proposed his axiomatic approach:

"The theory of probability, as a mathematical discipline, can and should be developed from axioms in exactly the same way as Geometry and Algebra. This means that after we have defined the elements to be studied and their basic relations, and have stated the axioms by which these relations are to be governed, all further exposition must be based exclusively on these axioms, independent of the usual concrete meaning of these elements and their relations." (p. 1)

Axiomatic probability ensures that the random variables, such as X and Y, are well-defined on the *same probability space* $(S, \mathcal{F}, \mathbb{P}(.))$, where (Williams [28]):

[i] S denotes the set of all possible distinct **outcomes** of interest.

[ii] $\mathcal{F}$ denotes the set of **events** of interest comprising subsets of S which form a sigma-field – denoted by σ-field – which means $\mathcal{F}$ is closed under *countable* unions, intersections, and complementations.

[iii] $\mathbb{P}(.)\rightarrow[0,1]$ is a **set function assigning probabilities** to all elements in $\mathcal{F}$, adhering to the three **Kolmogorov axioms** relating to $\mathbb{P}(.)$ being positive and less than or equal to one, as well as countably additive; see Billingsley [29], ch. 4.

[iv] (a) A **continuous random variable** $Z(.)$ is defined in the context of $(S,\mathcal{F},\mathbb{P}(.))$, as a function from S to $\mathbb{R}$, such that all *events* of the form $A(s){:=}\{s\text{: } Z(s){\leq}z\}$ belong $(\in)$ to $\mathcal{F}$, for all $(\forall)$ $z{\in}\mathbb{R}_Z$, where $\mathbb{R}$ denotes the real line and $\mathbb{R}_Z$ the range of values of $Z(.)$.

(b) A **discrete random variable** $Z(.)$ is defined in the context of $(S,\mathcal{F},\mathbb{P}(.))$, as a function from S to $\mathbb{R}$, such that, all *events* of the form $A(s){:=}\{s\text{: } Z(s){=}z\}\in\mathcal{F}$, $\forall z{\in}\mathbb{R}_Z$.

As Gill [26], p. 213, points out, "serious errors can easily arise by failing to draw distinctions that are essential in probability theory," it is crucial to distinguish between *outcomes*, *events* and *values* assigned to random variables.

Kolmogorov's axiomatic recasting of probability, framed in terms of $(S,\mathcal{F},\mathbb{P}(.))$, transformed the pre-1933 probability theory, relying predominantly on common-sense intuition, into a rigorous mathematical discipline that addresses numerous conceptual and mathematical issues hitherto raised. These include several challenging paradoxes, such as Bertrand's, Borel's, Banach-Taski's, and von Mises' frequentist interpretation of probability grounded in his notion of a *collective*, bedeviling probability theory in the late 1920s (Eckhardt [30]). Kolmogorov's axiomatic framework accomplished this by clarifying and elucidating:
(i) the distinction between the mathematical structure of probability and its interpretations,
(ii) what constitutes undefined or ambiguous probabilistic assignments,
(iii) what amounts to illicit conditioning on null events, and
(iv) conditioning on a sigma-field (set of events) generated by a random variable,
when grounded in $(S,\mathcal{F},\mathbb{P}(.))$; see Eckhardt [30] for further details.

This axiomatic framework, as it relates to (ii) and (iv) above, is employed next to reframe the TEP and resolve the 'paradox' engendered by misapplying probability theory.

4. Reframing of the TEP Using Axiomatic Probability Theory

4.1. Reframing X and Y as Well-Defined Random Variables on the Same $(S_1,\mathcal{F}_1,\mathbb{P}_1(.))$

The **original setup** of the TEP calls for two distinct possible outcomes:

s_1–*the player selects envelope 1*, and s_2–*the player selects envelope 2*,

defining the outcomes set $S_1{=}\{s_1,s_2\}$. This suggests that the relevant *field* – the set of all subsets of S_1 – is (Billingsley [29], p. 19):

$$\mathcal{F}{=}\{S,\ \varnothing,\ s_1,\ s_2\},\ s_1{=}\theta,\ s_2{=}2\theta,\ s_1\cup s_2{=}S \text{ and } s_1\cap s_2{=}\varnothing,$$

where $\varnothing$ denotes the empty set. That is, the original (ex ante) setup of the TEP is grounded in the probability space $(S_1,\mathcal{F}_1,\mathbb{P}_1(.))$.

The relevant *discrete* random variables X and Y, relating to envelopes 1 and 2, respectively, are defined in the context of $(S_1,\mathcal{F}_1,\mathbb{P}_1(.))$, framed to account for the dependence, arising from "one envelope contains twice as much money as the other", as follows:

$$\begin{gathered} X(.)\text{: } S_1\rightarrow\mathbb{R}_X{:=}\{\theta,\ 2\theta\},\qquad Y(.)\text{: } S_1\rightarrow\mathbb{R}_Y{:=}\{\theta,2\theta\},\\ \text{where } X(s_1){=}\theta,\ X(s_2){=}2\theta,\qquad Y(s_1){=}2\theta,\ Y(s_2){=}\theta. \end{gathered} \tag{2}$$

X and Y are well-defined since all their events belong to the relevant field $\mathcal{F}_1$, i.e.

$$X^{-1}(\theta){:=}s_1{\in}\mathcal{F}_1,\ X^{-1}(2\theta){:=}s_2{\in}\mathcal{F}_1,\ Y^{-1}(2\theta){:=}s_1{\in}\mathcal{F}_1,\ Y^{-1}(\theta){:=}s_2{\in}\mathcal{F}_1.$$

The in-built *dependence* between the two random variables X and Y, is framed in terms of their joint distribution $f(X,Y;\theta)$, which takes the self-evident form in Table 3.

Table 3. $f(X,Y;\theta)$

$X \setminus Y$	θ	2θ	$f(X;\theta)$
θ	0	.5	.5
2θ	.5	0	.5
$f(Y;\theta)$	.5	.5	1

It is important to note that the mutual exclusiveness of the envelope's content gives rise to the zero probabilities:

$$f(x{=}\theta,y{=}\theta){=}0,\quad f(x{=}2\theta,y{=}2\theta){=}0. \tag{3}$$

The two ensuing marginal distributions, $f(X;\theta)$ and $f(Y;\theta)$ – derived by summing over rows and columns – are given on the side and bottom of Table 3, respectively.

The marginal distributions indicate that X and Y are *identically distributed* since they take the same two values with identical probabilities, and their *dependence* is affirmed by:

$$f(X,Y;\theta)\neq f(X;\theta)\cdot f(Y;\theta), \textit{for some} \text{ values } x{\in}\{\theta,\ 2\theta\}, y{\in}\{2\theta,\ \theta\}. \tag{4}$$

At this stage, it is important to reiterate that the joint distribution in Table 3 represents all the information available to the player who has to make a decision whether to keep the selected envelope or exchange it for the other envelope. Evaluating the *expected returns* for the two envelopes, using the two marginal distributions $f(y;\theta)$ and $f(x;\theta)$ in Table 3 yield:

$$E(Y){=}.5(2\theta){+}.5\theta{=}(1.5)\theta,\quad E(X){=}.5\theta{+}.5(2\theta){=}(1.5)\theta. \tag{5}$$

This result renders the expected returns of the two envelopes identical in terms of θ, suggesting that a rational player would be indifferent between keeping and exchanging the original envelope, puzzling out the paradox. Indeed, the result in (5) also makes intuitive sense in terms of the expected returns evaluation based on a sum of $\$3\theta$.

The question that naturally arises from unraveling the apparent paradox is:

> **Question 1**: How does the traditional TEP framing of X and Y run afoul Kolmogorov's axiomatic probability, grounded in the probability space $(S_1,\mathcal{F}_1,\mathbb{P}_1(.))$?

The short answer is that the traditional (ex post) framing of the TEP ignores the stipulation that the money in the two envelopes in the original (ex ante) setup is **unobservable**, θ and 2θ, and focuses instead on the *observable* money x (or y) in the selected envelope X (or Y). Therefore, the implicit **outcome of interest** in the traditional framing of X and Y is:

s_0-the player selects one of two identical envelopes,

and the **event of interest** relates to:

"the money in the selected envelope".

This suggests that the **implicit probability space** is $(S_0,\mathcal{F}_0,\mathbb{P}_0(.))$, where $S_0{=}\{s_0\}$ denotes the outcomes set, giving rise to the (trivial) field $\mathcal{F}_0{=}\{S_1,\ \varnothing\}$.

[a] The *first problem* with $(S_0,\mathcal{F}_0,\mathbb{P}_0(.))$ is that the only legitimate probabilities one could assign to the events in $\mathcal{F}_0{=}\{S_1,\ \varnothing\}$ would be $\mathbb{P}_0(S_0){=}1$ and $\mathbb{P}_0(\varnothing){=}0$. Hence, the lone random variable that could potentially be defined on $\mathcal{F}_0$ is $X(s_0){=}x$, where:

$$\mathbb{P}_0(X(s_0){=}x){=}1 \text{ and } \mathbb{P}_0(X(s_0){\neq}x){=}0 \text{ for } x{\in}\mathbb{R}_+,$$

i.e. X has a singular distribution.

[b] The *second violation* of formal probability arises from defining Y in terms of $y{=}(x/2)$ and $y{=}2x$, and vice versa for X, which belies the definition of a random variable, given in [iv]-(b) (section 4). It stipulates that random variables are invariably defined relative to the particular $\mathcal{F}_0$. Hence, even

if one were to ignore the fact that a singular $X(s_0)=x$ is unavailing for framing the two envelopes problem, any additional random variable, say Y, must be a well-defined (mathematical) function of X, say $Y=g(x)$. Regrettably, the relation:

$$g(x)=\begin{cases} (x/2) \\ 2x \end{cases} \tag{6}$$

is not a well-defined function, due to the fact that a single value x, in the domain of $g(.)$, is related to two different values $(x/2)$ and $2x$ in its co-domain; see Spanos [2], p. 58.

[c] The *third violation* of formal probability relates to the **dependence** between X and Y. As argued above, the traditional literature treats this key stipulation as a problem in elementary algebra: since "one envelope contains twice as much money as the other", and the selected envelope X contains x, then Y must contain $y=x/2$ or $y=2x$. As shown in (6), however, the dependence cannot be imposed ex post, without X and Y having an ex ante well-defined joint distribution in the context of $(S_0, \mathcal{F}_0, \mathbb{P}_0(.))$. Therefore, the traditional Tables 1 and 2, represent neither well-defined marginal nor conditional distributions.

In summary, the proposed reframing of X and Y in (2) has demonstrated that the TEP is not a literal paradox but a conundrum engendered by misapplying axiomatic probability theory. We could leave the discussion at that, but there is another question that naturally arises from the evaluation of the expected returns in (5):

> **Question 2**: Given that X and Y are dependent, why is the evaluation of the expected returns in (5) based on their marginal distributions?

It is well known that, when X and Y are dependent, their expected returns should be evaluated using the joint distribution $f(X,Y;\theta)$ in conjunction with the *Law of Iterated Expectations* [LIE] (Spanos [2], p. 298):

$$E(Y) = \underset{X}{E}[E(Y|X)], \quad E(X) = \underset{Y}{E}[E(X|Y)], \tag{7}$$

where the conditional distributions $f(Y|X;\theta)$ and $f(X|Y;\theta)$ account for their dependence.

An *informal answer* to the above question is that the dependence between X and Y is *unique* in the sense that it goes beyond the definition in (4), because the quantifier is not just "for some values", but "for all values" of X and Y. Indeed, $f(Y|X;\theta)$ and $f(X|Y;\theta)$ are singular due to the zero probabilities in (3). Therefore, the evaluations in (7) are reigned by the marginal distributions, yielding the same results as in (5). A more formal response is explained next using Kolmogorov's conditioning on σ-fields.

4.2. Kolmogorov's General Conditional Expectation Procedure

A more formal answer to why (5) and (7) yield the same results calls for Kolmogorov's [27] ch. V: **Conditional Probabilities and Mathematical Expectations**. This procedure evaluates the conditional expectation of Y given the random variable X, not $X=x$. This is called for because the reframing in terms of $\{\theta,\ 2\theta\}$ renders X and Y **latent random variables**, and thus the notion of "the probability of Y given $X=\theta$" represents an **oxymoron** since θ is an unknown value, which does not provide any particular information!

Addressing this issue calls for a more general form of the conditional expectation:

$$E(Y|\sigma(X)), \text{ where } \sigma(X)=\{S, \varnothing, X^{-1}(\theta), X^{-1}(2\theta)\}$$

Note that $\sigma(X)$ denotes the field generated by X, and likewise for $E(X|\sigma(Y))$.

The defining relation for $E(Y|\sigma(X))$ takes the generic form (Billingsley [29], p. 445):

$$\textstyle\int_{\mathcal{D}} E(Y|\sigma(X))d\mathbb{P}= \int_{\mathcal{D}} Yd\mathbb{P}=E(Y|\mathcal{D})\cdot\mathbb{P}(\mathcal{D}) \text{ for all } \mathcal{D}\in\sigma(X). \tag{8}$$

Evaluating $E(Y|\sigma(X))$ calls for the conditional distribution for both values $\{\theta, 2\theta\}$:

$$f(Y|X{=}\theta)=\begin{cases}\frac{f(y=2\theta,x=\theta)}{f(x=\theta)}=\frac{.5}{.5}=1, & \text{for } y=2\theta\\ \frac{f(y=\theta,x=\theta)}{f(x=\theta)}=\frac{0}{.5}=0, & \text{for } y=\theta\end{cases} \tag{9}$$

$$f(Y|X{=}2\theta)=\begin{cases}\frac{f(y=2\theta,x=2\theta)}{f(x=2\theta)}=\frac{0}{.5}=0, & \text{for } y=2\theta\\ \frac{f(y=\theta,x=2\theta)}{f(x=2\theta)}=\frac{.5}{.5}=1, & \text{for } y=\theta\end{cases} \tag{10}$$

It is important to reiterate that the above singularity of $f(Y|X;\theta)$ arises from the zero probabilities in (3). Hence, $E(Y|\sigma(X))$ defines the random variable:

$$E(Y|\sigma(X))=[2\theta+0\cdot\theta]\mathbb{I}_{\{x=\theta\}}+[\theta+0\cdot 2\theta]\mathbb{I}_{\{x=2\theta\}}=2\theta\mathbb{I}_{\{x=\theta\}}+\theta\mathbb{I}_{\{x=2\theta\}}, \tag{11}$$

where $\mathbb{I}_{\{x=\theta\}}=\begin{cases}1 & \text{for } x=\theta\\ 0 & \text{for } x\neq\theta\end{cases}$ is the indicator function.

To derive the expected returns one needs $E(Y)$ which can be derived from (11) using the *Law Iterated Expectations* (LIE) in (7):

$$E(Y)=\underset{X}{E}\{E(Y|\sigma(X))\}=2\theta(.5)+\theta(.5)=1.5\theta. \tag{12}$$

Reversing the roles of X and Y we can define $E(X|\sigma(Y))$ and derive the analogous result:

$$E(X)=\underset{Y}{E}\{E(X|\sigma(Y))\}=\theta(.5)+2\theta(.5)=1.5\theta. \tag{13}$$

The results in (12) and (13) affirm that (7) would yield the same values as those in (5).

5. Summary and Conclusions

A case has been made above that the TEP is not a genuine paradox but rather the outcome of misapplying Kolmogorov's probability theory. The culprit is the conventional *ex post* — after selecting an envelope and observing its content — framing of X and Y by using $X=x$ to define Y in terms of $y=\frac{x}{2}$ and $y=2x$, and vice versa for X. This gives rise to $f(Y;x)$ and $f(X;y)$ in Tables 1 and 2, respectively, whose values and probabilities are fallacious. This calls into question the validity of the traditional expected returns evaluation in (1), giving rise to the Two-Envelope paradox.

On the other hand, when X and Y are properly framed in the context of Kolmogorov's axiomatic probability theory — namely, as random variables taking the values $\{\theta,2\theta\}$, as defined in (2) — give rise to a well-defined joint distribution, along with coherent marginal distributions (Table 3). The ensuing evaluation of the expected returns, using either the two marginal distributions in (5), or the joint distribution in conjunction with the LIE in (12) and (13), confirm that the player would be indifferent between keeping the chosen envelope and switching, thereby resolving the alleged paradox.

In this sense, the proposed resolution does not merely use reasoned adjustments to the traditional expected return calculations, based on Tables 1 and 2. It obviates the paradox by identifying it as a consequence of an invalid application of formal probability.

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Abbreviations

The following abbreviations are used in this manuscript:

TEP	Two-Envelope Problem
LIE	Law of Iterated Expectations

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