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Not peer-reviewed version

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Posted Date: 30 April 2025

doi: 10.20944/preprints202410.1262.v3

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Article

Proof of the Binary Goldbach Conjecture

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Abstract: In this article the proof of the binary Goldbach conjecture is established (Any integer greater than one is the mean arithmetic of two positive primes) . To this end the weak Chen conjecture is proved (Any even integer greater than one is the difference of two positive primes) and a " located " algorithm is developed for the construction of two recurrent sequences of primes (U_{2n}) and (V_{2n}), ((U_{2n}) dependent of (V_{2n})) such that for each integer $n \geq 2$ their sum is equal to $2n$. To form this a third sequence of primes (W_{2n}) is defined for any integer $n \geq 3$ by $W_{2n} = \text{Sup} (p \in \mathcal{P} : p \leq 2n - 3)$, $\mathcal{P}$ denoting the set of positive primes. The Goldbach conjecture has been proved for all even integers $2n$ between 4 and 4.10^{18}. In the table of terms of Goldbach sequences given in Appendix 12 values of the order of $2n = 10^{1000}$ are reached. An analogous proof by recurrence « finite ascent and descent method » is developed and a majorization of U_{2n} by $0.7 \ln^{2.2}(2n)$ is justified.. In addition, the Lagrange-Lemoine-Levy conjecture and its generalization called ʺ Bezout-Goldbach ʺ conjecture are proven by the same type of algorithm.

Keywords: prime number theorem; binary goldbach conjecture; weak chen conjecture; lagrange-lemoine-levy conjecture; bezout-goldbach conjecture; gaps between consecutive primes

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1. Overview

Number theory " the queen of mathematics " studies the structures and properties defined on integers and primes (Euclid [13], Hadamard [15], Hardy and Wright [16], Landau [22], Tchebychev [35]). Numerous problems have been raised and conjectures made, the statements of which are often simple but very difficult to prove. These main components include

• Elementary arithmetic .

___ Determination and properties of primes, operations on integers (Basic operations, congruence, gcd, lcm, ………..).

___ Decomposition of integers into products or sums of primes (Fundamental theorem of arithmetic, decomposition of large numbers, cryptography and Goldbach's conjecture).

• Analytical number theory .

___ Distribution of primes (Prime Number Theorem, Hadamard [15], De la Vallée-Poussin [36], Littlewood [25] and Erdos [12], the Riemann hypothesis,…..).

___ Gaps between consecutive primes (Bombieri,Davenport [3], Cramer [8], Baker,Harmann,Iwaniec, Pintz [4,5,20], Granville [14], Maynard [27], Tao [34], Shanks [30], Tchebychev [35] and Zhang [39]).

• Algebraic, probabilistic, combinatorial and algorithmic number theories .

___ Modular arithmetic.

___ Diophantine approximations and equations

___ Arithmetic and algebraic functions.

___ Diophantine and number geometry.

2. Definitions Notations and Background

The integers $n, k, p, q, r,$……..... used in this article are always positive. (2.1)

The symbol " / " means " in relation to". (2.2)

Let $\mathcal{P}$ be the infinite set of positive primes p_k (called simply primes) (2.3)

(p_1 = 2 ; p_2 = 3 ; p_3 = 5 ; p_4 = 7 ; p_5 = 11 ; p_6 = 13 ; ....)

For any integer $K \geq 1$ $\mathcal{P}_K = \{ p \in \mathcal{P} : p \leq 2K \}$ (2.4)

The writing of large numbers (see appendix 12) is simplified using the following constants

M = 10^9 ; R = 4.10^8 ; G = 10^{100} ; S = 10^{500} ; T = 10^{1000} (2.5)

ln(x) denotes the neperian logarithm of the real $x > 0$ (2.6)

Let (W_{2n}) be the sequence of primes defined by

$\forall\, n \in \mathbb{N} + 3$ W_{2n} = Sup($p \in \mathcal{P} : p \leq 2n - 3$) (2.7)

Any sequence denoted by (G_{2n}) = ($U_{2n}; V_{2n}$) verifying (2.8) is called a **Goldbach sequence.**

$\forall$ n $\in \mathbb{N}$ + 2 U_{2n} ,V_{2n} $\in \mathcal{P}$ and $U_{2n} + V_{2n}$ = 2n (2.8)

U_{2n} and V_{2n} are also known as " **Goldbach decomponents** ".

Iwaniec,Pintz [20] have shown that for a sufficiently large integer n there is always a prime between $n - n^{23/42}$ and n . Baker and Harman [4,5] concluded that there is a prime in the interval [n ; $n + o(n^{0.525})$] . Thus this results provides an increase of the gap between two consecutive primes p_k and p_{k+1} of the form

$\forall\ \varepsilon > 0\ \exists\ k_\varepsilon \in \mathbb{N}^*$ / $\forall$ k $\in \mathbb{N}$ k $\geq k_\varepsilon$ $p_{k+1} - p_k < \varepsilon . p_k^{0.525}$ (2.9)

The results obtained on the Cramer-Granville-Maier-Nicely conjecture [1,3,8,14,26,28] imply the following majorization.

For any real $c > 2$ and for any integer $k \geq 500$

$p_{k+1} - p_k \leq 0.7\ \ln^c(p_k)$ **(with probability one) (**2.10)

3. Introduction

Chen [6], Hardy,Littlewood [17], Hegfollt,Platt [18], Ramaré,Saouter [29], Tao [34], Tchebychev [35] and Vinogradov [37] have taken important steps and obtained promising results on the Goldbach conjecture (Any integer $n \geq 2$ is the mean arithmetic of two primes).

Indeed, Helfgott,Platt [18] proved the weak Goldbach conjecture in 2013.

Silva,Herzog,Pardi [32] held the record for calculating the terms of Goldbach sequences after determining pairs of primes (U_{2n} ; V_{2n}) verifying

$\forall\ n \in \mathbb{N}$ / $4 \leq 2n \leq 4.10^{18}$ $U_{2n} + V_{2n} = 2n$ (3.1)

In previous research work there is no explicit construction of recurrent Goldbach sequences. In this article two sequences of primes are developed using a simple, efficient and « located » algorithm to compute for any integer $n \geq 3$ by successive iterations any term U_{2n} and V_{2n} .

Using Maxima scientific software on a personal computer Silva's record is broken and the values $2n = 10^{500}$ and even $2n = 10^{1000}$ are reached. The binary Goldbach conjecture can be established on the same principle by recurrence by using the weak Chen or Goldbach(-) conjecture (Any even integer greater than three is the difference of two primes) demonstrated in Teorem 4.

● Remark.

1. **Chen conjecture_:** For any integer K ≥ 1 there are infinitely many pairs of primes with a difference equal to 2K.

2. **De Polignac conjecture :** Same as Chen, but with consecutive pairs of primes.

3. What we know :

April 2013, Yitang Zhang [39] demonstrates that the smallest even integer $2K$ verifying the conjecture is greater than 70 million.

In 2014, James Maynard [27] then Terence Tao [34] lowered this limit to 246.

We validate weak Chen or Goldbach(-) conjecture by verifying directly in the prime number tables that all even gaps from 2 to 246 are possible between primes.

In addition, the Lagrange-Lemoine-Lévy conjectures [9,19,21,26,28,33,38] and its generalization called " Bezout-Goldbach « conjecture' are validated.

Using case disjunction reasoning we construct two recurrent sequences of primes (V_{2n}) and (U_{2n}) according to the sequence (W_{2n}) by the following process

For any integer $n \geq 2$
$U_4 = 2$ and $V_4 = 2$ (3.2)
Let $n \in \mathbb{N} + 3$
• Either
($2n$ - W_{2n}) is a prime
then V_{2n} and U_{2n} are defined directly in terms of W_{2n} .
• Either
($2n$ - W_{2n}) is a composite number
then V_{2n} and U_{2n} are determined from the previous terms of the sequence (G_{2n}).

4. Theorem (Weak Chen or Goldbach(-) Conjecture)

$\forall$ K $\in \mathbb{N}^*$ $\exists$ p , q $\in \mathcal{P}$ /
p - q = 2K 3 ≤ q ≤ 2K and 3 + 2K ≤ p ≤ 4K if K ≥ 2 (4.1)

Practical method on some examples:

First of all (5 - 3 = 2), then we begin the process at (7 - 3 = 4), we will select the smallest primes for which the difference is precisely 6 (11 - 5 = 6) then 8 (11 - 3 = 8) then 10 (13 - 3 = 10),......... then $2K$, then $2(K + 1)$ (demonstration established by strong recurrence, by the asurd and feedback).

All pairs of Goldbach(-) decomponents obtained by this method for K between 2 and 123 are listed in the table in Appendix 13.

Proof . The proof is established by strong recurrence on K . Let $\mathcal{P}_{Chen}$ (K) be the following property

« $\forall$ K $\in \mathbb{N}^*$ $\exists$ p, q $\in \mathcal{P}$ / p - q = 2K 3 ≤ q ≤ 2K and 2K + 3 ≤ p ≤ 4K « (4.2)
► $\mathcal{P}_{Chen}$ (2) is true : 7 - 3 = 4 ; $q = 3 \leq 4$ and $p = 7 \leq 4 \times 2 = 8$
► Let's show
$\forall$ M $\in \mathbb{N}$ / M ≤ K then $\mathcal{P}_{Chen}$ (M) $\Longrightarrow$ $\mathcal{P}_C$ (K + 1)
We reason through the absurd
$\forall$ p , q $\in$ $\mathcal{P}_K$ / $p \geq q$ $\forall$ h, $m \in \mathbb{N}$ / $p + 2h$ and $q + 2m \in \mathcal{P}$
we assume that
$p + 2h - q - 2m \neq 2(K + 1)$ (4.3)
Therefore
$p - q \neq 2(K + 1 - h + m)$. (4.4)
You can always choose $h \geq m$ and $h - m \leq K + 1$.
However the strong recurrence hypothesis asserts that
$\forall \in \mathbb{N}$ / $M \leq K$ $\exists$ p, $q \in \mathcal{P}$ / $p - q = 2M$ (4.5)
By choosing $M = K + 1 - h + m$
this contradicts (4.4).
So
$\exists$ $h,m \in \mathbb{N}$ / $p + 2h - q - 2m = 2(K + 1)$ (4.6)
knowing
p, p+$2h$, q, q+$2m$ $\in \mathcal{P}$ $h \geq m$ and $h - m \leq K + 1$
Thus validating the heredity of property $\mathcal{P}_{Chen}$ (K).
The property $\mathcal{P}_{Chen}$ (K) is therefore true. As a result Goldbach(-)'s conjecture is validated.

• **Remark.** Using the same method as in Theorem 4, we can demonstrate the following equivalent property by strong recurrence ::

For any integer n greater than 48
$\mathcal{P}_{feedback}$ (n) : « There exists an integer K such that 2K + $U_{2(n-k)}$ $\in \mathcal{P}$ " (4.7)
To this end, $\mathcal{P}_{feedback}$ (49) is true and the heredity of the property $\mathcal{P}_{feedback}$ (n) can be proved by the absurd and returning to the previous terms by noting that there is at least one integer M_r /
2K + $U_{2(n+1-k)}$ = 2(K + M_r) + $U_{2(r+1-k)}$ = 2P + $U_{2(r+1+M_r-P)}$ (4.8)
by posing P = K + M_r and $r + 1 + M_r \leq$ n
Now, according to the recurrence hypothesis on $\mathcal{P}_{feedback}$ (n) there exists an integer P /

2P + $U_{2(r+1+M_r-P)}$ ∈ $\mathcal{P}$ (4.9)
then there exists an integer K /
2K + $U_{2(n+1-k)}$ ∈ $\mathcal{P}$ (4.10)

5. Corollary

Let (R_{2K}) and (Q_{2K}) two sequences of primes determined by
R_{2K} = Inf (p ∈ $\mathcal{P}$: p - 2K ∈ $\mathcal{P}$) and Q_{2K} = Inf (p ∈ $\mathcal{P}$: 2K + p ∈ $\mathcal{P}$) = R_{2K} - 2K (5.1)
They are defined for any integer K ∈ ℕ* and satisfy
lim R_{2K} = +∞ (5.2)
∀ K ∈ ℕ* R_{2K}, Q_{2K} ∈ $\mathcal{P}$ and R_{2K} - Q_{2K} = 2K (5.3)
∀ K ∈ ℕ* / 2 ≤ K ≤ 16 3 ≤ Q_{2K} ≤ 2K and 2K + 3 ≤ R_{2K} ≤ 4K (5.4)
For any integer K ≥ 16
3 ≤ Q_{2K} ≤ 2(2K)$^{0.525}$ and 2K + 3 ≤ R_{2K} ≤ 2K + (2K)$^{0.525}$ (5.5)

Proof.

(5.1) : According to the previous theorem, the sequences (R_{2K}) and (Q_{2K}) are defined by strong recurrence and finite descent.
(5.2) : R_{2K} ≥ 2K ⇒ lim R_{2K} = +∞
(5.3) : By construction, these sequences thus verify R_{2K} - Q_{2K} = 2K
(5.4) : The property can be verified directly term-to-term by examining the sequence proposed above.
(5.5) : This property is verified up to 2K = 246 by calculations on the previous list.
We prove this result by recurrence
First of all we order the Goldbach(-) decomponents at a fixed prime Q,
So as to obtain the estimate (5.5) more easily.
We examine the following sequences of primes (PQ(K)).
P3(K) = 2K + 3
(P3(K) ; 2K) → (5;2);(7;4);(11;8);(13;10);(17;14);(19;16);(23;20);(29;26);(29;28);..........
P5(K) = 2K + 5
(P5(K) ; 2K) → (7;2) ; (11;6) ; (13;8) ; (17;12) ; (19;14) ; (23;18) ; (29;24) ; (31;26) ; (37;32)...............................
P7(K) = 2K + 7
(P7(K) ; 2K) → (11;4) ; (13;6) ; (17;10) ; (19;12) ; (23;16) ; (29;22) ; (31;24) ; (37;30)...........
P11(K) = 2K + 11
(P11(K) ; 2K) → (13;2) ; (17;6) ; (19;8) ; (23;12) ; (29;18) ; (31;20) ; (37;26) ; (41;30) ; (43;34)......
(P13(K) ; 2K) → (17;4) ; (19;6) ; (23;10) ;(29;16) ; (31;18) ; (37;24) ; (41;28) ; (43;30 ; (47;34)......
PQ(K) = 2K + Q (K ∈ ℕ*: PQ(K) and Q are primes) (see the table in Appendix 14)
For any integer K satisfying 2(2K)$^{0.525}$ > Q the property holds for PQ(K).
Therefore it is generally validated for all K > 15, since we obtain all possible cases of Chen's weak conjecture starting with P3(K), then P5(K), then P7(K) ...... for 2(2K)$^{0.525}$ ≤ Q .
(can be proved by strong recurrence using the same method as inTheorem 4 by "finite descent»).

Let $c_p = \frac{40}{21}$ and Pr(K) be the following property
« For any integer M < $(0.5Q_K)^{c_p}$, there exists at least a prime Q < Q_K such that 2 M + Q is a prime «
▶ Pr(15) is true (see Appendix 14).
▶ Let's show : Pr(K) ⇒ Pr(K + 1)
$Q_{K+1} \leq Q_K + Q_K^{0.525}$ (5.6)
It is assumed that$M$ /
P_{K+1} - Q_{K+1} ≠ 2M M < $(0.5Q_{K+1})^{c_p}$ P_{K+1} = p + 2h and Q_{K+1} = q +2s (5.7)
then
p - q ≠ 2(M + s - h) (5.8)

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which is impossible according to the hypothesis of strong recurrence since
$2(M + s - h)$ is less than $(0.5Q_K)^{c_p}$ and that all primes p, q satisfy the recurrence hypothesis.
We deduce that $\Pr(K) \Rightarrow \Pr(K+1)$
Thus the property (5.5) is true.

6. Principle of Proof

To determine pairs of primes that verify Goldbach's conjecture three sequences of primes
(W_{2n}), (V_{2n}), (U_{2n}) are defined and they verify the following properties
$\lim V_{2n} = +\infty$. (6.1)
$\forall n \in \mathbb{N} + 2$ V_{2n} is defined as a function of $W_{2n} = \mathrm{Sup}(p \in \mathcal{P} : p \leq 2n - 3)$ 6.2)
(W_{2n}) is an increasing sequence of primes that contains all primes except $p_1 = 2$ (6.3)
$\lim W_{2n} = +\infty$ (6.4)
(U_{2n}) is a complementary sequence of negligible primes with respect to $2n$ (6.5)
For any integer $n \geq 3$ (6.6)
• If ($2n - W_{2n}$) is a prime
then V_{2n} and U_{2n} are defined by
(6.7) $V_{2n} = W_{2n}$ and $U_{2n} = 2n - W_{2n}$
• Otherwise, if ($2n - W_{2n}$) is a composite number
we search for two previous terms of the sequence (G_{2n}), $U_{2(n-k)}$) and $V_{2(n-k)}$ satisfying the following conditions
(6.8) $U_{2(n-k)}$, $V_{2(n-k)}$ and $U_{2(n-k)} + 2k$ are primes $U_{2(n-k)} + V_{2(n-k)} = 2(n-k)$
which is always possible (see Theorem 4)
So by setting
$V_{2n} = V_{2(n-k)}$ and $U_{2n} = U_{2(n-k)} + 2k$ (6.9)
two new primes V_{2n} and U_{2n} satisfying (4.10) are generated .
$U_{2n} + V_{2n} = 2n$ (6.10)
This process is then repeated incrementing n by one unit ($n \to n+1$).

7. Theorem

There exists a recurrent sequence $(G_{2n}) = (U_{2n} ; V_{2n})$ of primes satisfying the following conditions.
For any integer $n \geq 2$
U_{2n}, $V_{2n} \in \mathcal{P}$ and $U_{2n} + V_{2n} = 2n$ (7.1)
(Any integer $n \geq 2$ is the mean arithmetic of two primes)
An algorithm can be used to explicitly compute any term U_{2n} and V_{2n} . (7.2)
Proof .
□ FIRST METHOD :
For any integer $n \geq 3$
• If ($2n - W_{2n}$) is a prime
then V_{2n} and U_{2n} are defined by
$V_{2n} = W_{2n}$ and $U_{2n} = 2n - W_{2n}$ (7.3)
• Otherwise, if ($2n - W_{2n}$) is a composite number
we use the previous terms of the sequence (G_{2n}).
For any integer q such that $1 \leq q \leq n - 3$ we have
$3 \leq U_{2(n-q)} \leq n$.
Then, there exists an integer $k / 1 \leq k \leq n - 3$ following the Bertrand principle and Theorem 4 since all primes smaller than $2k$ are represented by $U_{2(n-j)}$, (if there were no such primes, we would have a contradiction with the Theorem 4 , even if it means transforming the indexing of the sequence (U_{2n}) . In fact, in an equivalent way (see the remark following Theorem 4) we can copy the proof of Teorem 4 by performing a similar strong recurrence " finite descent return and absurd " directly on the set $\{U_{2k} : k \leq n\}$ such that

$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P}$ (7.4)

The smallest integer k / $R_{2n} \in \mathcal{P}$ is denoted by k_n .

So

$U_{2n} = U_{2(n-k_n)} + 2k_n$ and $V_{2n} = V_{2(n-k_n)}$ (7.5)

(These two terms are primes)

In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n - k_n)$ were determined.

$U_{2(n-k_n)} + V_{2(n-k_n)} = 2(n - k_n)$ (7.6)

By adding the term k_n to each member of the equality (5.6), it follows

$U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)} = 2(n - k_n) + 2k_n$ (7.7)

$\Leftrightarrow U_{2(n-k_n)} + 2k_n] + V_{2(n-k_n)} = 2n$ (7.8)

$\Leftrightarrow U_{2n} + V_{2n} = 2n$ (7.9)

Finally for any integer $n \geq 3$ this algorithm determines two sequences of primes (U_{2n}) and (V_{2n}) verifying Goldbach's conjecture.

□ **SECOND METHOD :**

The proof can be made using the following strong recurrence principle.

Let P(n) be the property defined for any integer $n \geq 2$ by

P(n) : " For any integer p satisfying $2 \leq p \leq n$ there exists two primes U_{2p} and V_{2p} such their sum is equal to $2p$ " .

($\forall$ p $\in \mathbb{N}$ / $2 \leq p \leq n$ U_{2p} , $V_{2p} \in \mathcal{P}$ and $U_{2p} + V_{2p} = 2p$)

Let's show by strong recurrence that P(n) is true for any integer $n \geq 2$

a) P(2) is true : it suffices to choose $U_4 = V_4 = 2$.

b) Let's show that the property P(n) is hereditary : $\forall k \in \mathbb{N} + 2$ P(n) $\Rightarrow$ P(n+ 1)

Assume property P(n) is true.

● **If (2(n + 1) - $W_{2(n+1)}$) is a prime**

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

$V_{2(n+1)} = W_{2(n+1)}$ and $U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$ (7.10)

● **Otherwise, if (2(n+1) - $W_{2(n+1)}$) is a composite number**

there exists an integer k to obtain two terms $U_{2(n+1-k)}$) and $V_{2(n+1-k)}$ satisfying the following conditions

$U_{2(n+1-k)}$, $V_{2(n+1-k)}$ and $U_{2(n+1-k)}$ + 2k are primes (7.11) $U_{2(n+1-k)} + V_{2(n+1-k)} = 2(n +1- k)$

(which is always possible : see **FIRST METHOD** and Theorem 4).

Thus by setting

$V_{2(n+1)} = V_{2(n+1-k)}$ and $U_{2(n+1)} = U_{2(n+1-k)} + 2k$ (7.12)

Two new primes $V_{2(n+1)}$ and $U_{2(n+1)}$ satisfying ($U_{2(n+1)} + V_{2(n+1)} = 2(n + 1)$) are generated.

It follows that P(n + 1) is true. Then the property P(n) is hereditary : P(n) => P(n + 1).

Therefore for any integer $n \geq 2$ the property P(n) is true.

it follows

$\forall$ $n \in \mathbb{N}$+2 there are two primes U_{2n} and V_{2n} and such their sum is $2n$: $U_{2n} + V_{2n} = 2n$

8. Lemma

The sequence (U_{2n}) verifies the following majorization

For any integer $n \geq 65$

$U_{2n} \leq (2n)^{0.55}$ (8.1)

Proof . According to the programm 11.2 and appendix 12 the majorization (8.1) is verified

For any integer n such that $65 \leq n \leq 2000$. For any integer $n > 2000$ the proof is established by recurrence. For this purpose let P1(n) be the following property

P1(n) : " There exists a strictly increasing sequence of positive numbers (C_n) such that

$U_{2n} \leq C_n (2n)^{0.525}$ " . (8.2)

▶ P1(2000) is true according to program 11.2 and the table in appendix 12.

▶ For any integer $n \geq 2000$ let's show that P1(n) is hereditary : P1(n) ⇒ P1($n+1$).
Assume that P1(n) is true : then

• If (2($n+1$) - $W_{2(n+1)}$) is a prime

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

$V_{2(n+1)} = W_{2(n+1)}$ and $U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$ (8.3)

According to the results in [4,5,20] there is a constant $K > 0$ such that

$(n+1) - K . [2(n+1)]^{0.525} < W_{2(n+1)} < 2(n+1)$

$\Rightarrow U_{2(n+1)} < K . [2(n+1)]^{0.525}$

$\Rightarrow U_{2(n+1)} \leq C_{n+1} . [2(n+1)]^{0.525}$

• Otherwise, if (2($n+1$) - $W_{2(n+1)}$) is a composite number

$\exists$ p $\in \mathbb{N}^*$ / $U_{2(n+1)} = U_{2(n+1-p)} + 2p$ (8.4)

According to [4,5,18] the smallest integer p defined in (6.4) verifies

$2p < K.[U_{2(n+1-p)}]^{0.525}$ and $U_{2(n+1-p)} < C_{n+1-p} . [2(n+1-p)]^{0.525}$ (8.5)

It follows

$U_{2(n+1)} < K . C_{n+1-p}^{0.525} . [2(n+1-p)]^{0.275625} + C_{n+1-p} . [2(n+1-p)]^{0.525}$ (8.6)

Then

$U_{2(n+1)} < C_{n+1} . [2(n+1)]^{0.525}$ (8.7)

and by setting $C_n = (2n)^{0.025}$

It follows

$U_{2(n+1)} < [2(n+1)]^{0.55}$ (8.8)

P1($n+1$) is true then P1(n) is hereditary.

So for any integer $n \geq 2000$ the property P1(n) is true.

(The inequality (6.7) is verified with the aid of the software Maple studying the functions of the type $f : x \rightarrow a . x^{0.275625} + b . x^{0.525}$ increased by $g : x \rightarrow x^{0.55}$ a and b being two strictly positive real parameters).

• Remark. A more precise estimate can be obtained using the Cipolla or Axler frames [7,2].

9. Theorem

For any integer $n \geq 3$ *it is easy to check*

(W_{2n}) *is a positive increasing sequence of primes.* (9.1)

$$\{ W_{2n} : n \in \mathrm{IN} + 3 \} \cup \{ 2 \} = \mathcal{P} \quad (9.2)$$

$$\lim W_{2n} = +\infty \quad (9.3)$$

(U_{2n}) *and* (V_{2n}) *are sequences of primes and the set* { U_{2k} : k ≤ n} *contains all primes less than ln(* n *)* (9.4)

(9.5) n $\leq V_{2n} \leq W_{2n}$

(9.6) $3 \leq 2n - W_{2n} \leq U_{2n} \leq$ n

(9.7) lim V_{2n} = +oo

Proof .

(9.1) For any integer n ≥ 2 $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ .Therefore, $W_{2n} \leq W_{2(n+1)}$. So the sequence (W_{2n}) is increasing.

(9.2) Any prime except $p_1 = 2$ is odd, hence the result.

(9.3) lim W_{2n} = lim p_k = +oo

(9.4) By definition $V_{2n} = W_{2n}$ or there exits an integer $k \leq n - 2$ such that $V_{2n} = V_{2(n-k)}$;
so the terms of the sequence ($V_{2n)}$) are primes.

(9.5) According to Lemma 6, for any integer $n \geq 65$

$U_{2n} < (2n)^{0.55}$

therefore

$U_{2n} < (2n)^{0.55} <$ n

and

V_{2n} = 2n - U_{2n} > 2n - n > n

For any integer n / $3 \leq n \leq 65$ verification is carried out according to the computer program in paragraph 11.2 and the table in appendix 12.

We can also see that by construction $V_{2n} \geq U_{2n}$ because if we assume the opposite then V_{2n} is not the largest prime number verifying $\frac{1}{2}$ ($U_{2n} + V_{2n}$) = n .

So

$V_{2n} \geq$ n

According to (9.5) n $\leq V_{2n} \Rightarrow U_{2n}$ = 2n - $V_{2n} \leq$ 2n - n ≤ n (9.6)

therefore

$V_{2n} \leq W_{2n} \Rightarrow$ 2n - $W_{2n} \leq$ 2n - V_{2n} = U_{2n} (9.7) By (9.5) for any integer n ≥ 2 : n $\leq V_{2n}$

So

lim V_{2n} = +oo .

10. Remarks

10.1 *For any integer* $k \geq 2$ *there are infinitely many integers* n *such that* $U_{2n} = p_k$.

10.2 V_{2n} ~ 2n *for* ($n \to$ +oo) .

10.3 *For any sufficiently large integer* n / $n \geq 5000$

$U_{2n} \ll V_{2n}$ *and* lim ($\frac{U_{2n}}{V_{2n}}$) = 0.

10.4 *The smallest integer* n *such that*

$U_{2n} \neq$ 2n - W_{2n} is obtained for n = 49 and G_{98} = (79 ; 19).

(This type of terms increases in the Goldbach sequence (G_{2n} *) as* n *increases in the sense of the Schnirelmann density and there are an infinite number of them; their proportion per interval can be computed using the results given in [29]).*

10.5 *If q is an odd integer greater than four we could generalize this algorithm with sequences* (W'_{2n}) *defined by*

(10.5.1) ∀ n ∈ ℕ / n ≥ $\frac{(q+3)}{2}$ W'_{2n} = Sup(p ∈ $\mathcal{P}$: p ≤ 2n - q)

Other Goldbach's sequences (G'_{2n}) *independent of* (G_{2n}) *are thus generated.*

10.6 *The sequence (* G_{2n} *) is ''extremal'' in the sense that for any integer* $n \geq 2$ V_{2n} *and* U_{2n} *are the largest and smallest possible primes such that* $U_{2n} + V_{2n} = 2n$.

10.7 *The Cramer-Granville-Maier-Nicely conjecture* [8,14,19,21,23,24,26,28,33] *is verified with probability one. It leads to the following majorization*

For any integer $p \geq 500$

(10.7.1) $U_{2p} \leq 0.7\,[\ln(2p)]^{(2.2-\frac{1}{p})}$ **(with probability one)**

The proof is similar to that of lemma 8 and is validated by the studying functions of the type $f: x \to a\,.g\,(x) + b[\ln(g(x))]^c$ ($a,b > 0$; $c > 2$) *with*

$g: x \to 0.7\,[\ln(x)]^{(c-\frac{1}{x})}$ *and* $h: x \to 0.7\,[\ln(x)]^{(2.2-\frac{1}{x})}$ *using Maple software.*

• **Remark.** A better estimate can be obtained via [26,28,30] .

10.8 *According to Bombieri* [3] *and using the same method as in the proof of Lemma 8, on average, we obtain the following estimate of* U_{2n}

(10.8.1) ∀ $\varepsilon > 0$ U_{2n} = O ($\ln^{1.3+\varepsilon}(2n)$) **(** ***on average*** **)**

11. Algorithm

11.1. Algorithm Written in Natural Language

Inputs :

Input four integer variables : k, N, n, P

Input : $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, ..., p_N the first N primes.

: $n = 3$

: P = M, R, G, S or T as indicated in paragraph **2**

Algorithm body :

A) Compute : $W_{2n} = \text{Sup}(\, p \in \mathcal{P} \,: p \leq 2n - 3\,)$

If T_{2n}= (2n - W_{2n}) is a prime

Let :

(11.1.1) $U_{2n} = T_{2n}$ and $V_{2n} = W_{2n}$

otherwise

B) If T_{2n} is a composite number

Let : $k = 1$

B.1) While $U_{2(n-k)} + 2k$ is a composite number

assign to k the value $k + 1$ ($k \to k + 1$).

return to **B1)**

End while

Assign to k the value k_n ($k \to k_n$)

(11.1.2) Let :

$U_{2n} = U_{2(n-k_n)} + 2k_n$ and $V_{2n} = V_{2(n-k_n)}$

Assign to n the value $n + 1$ ($n \to n + 1$ and return to **A)**

End :

Outputs for integers less than 10^{4} :

Print ($2n$ = ● ; $2n$ - 3 = ● ; W_{2n} = ● ; T_{2n} = ● ; V_{2n} = ● ; U_{2n} = ●)

Outputs for large integers :

Print ($2n$ - P = ● ; $2n$ - 3 - P = ● ; W_{2n} - P = ● ; T_{2n} = ● ; $V_{2n} - P$ = ● ; U_{2n} = ●)

11.2. Program Written with Maxima Software for 2n = 10^{500}

```
n1 : 10**500 ; for n :5*10**499 + 10000 thru 5*10**499 + 10010 do
( a :2*n , c :a - 3 , test : 0 , b : prev_prime(a - 1) ,d :a - b ,
if primep(d )
then print(a - n1 , c - n1 , b - n1 , d , b - n1 , d )
else ( while test = 0 do (e :a - c , if ( primep(c ) and primep(e ) )
then ( test:1 , print(a - n1 , b - n1 , d , c - n1 , e ," ** " ))
else ( test : 0 , c : c - 2 ))) ;
```

12. Appendix

Application of Algorithm 11 : Table of U_{2n} and V_{2n} terms of the Goldbach sequence (G_{2n}) computed from program 11.2 ($2 \leq 2n \leq 10^{1000} + 4020$).

The ** sign in the table below indicates the results given by the algorithm 11 in case **B)** of return to the previous terms of the sequence (G_{2n}). **WATCH OUT !** For large integers n ($2n > 10^{9}$ for example), to simplify the display of large numbers the results are entered as follows

$2n$ - P , ($2n$ - 3) - P , W_{2n} - P , T_{2n} , V_{2n} - P and **U_{2n}**

with

P = ***M, R, G, S,*** or ***T*** constants defined in (2.3)

$2n$ $2n - 3$	W_{2n}	T_{2n}=$2n$ - W_{2n}	V_{2n}	U_{2n}
4 1	X	X	**2**	**2**

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6 3	3	3	**3**	**3**
8 5	5	3	**5**	**3**
1 10 7	7	3	**7**	**3**
112 9	7	5	**7**	**5**
14 11	11	3	**11**	**3**
16 13	13	3	**13**	**3**
18 15	13	5	**13**	**5**
20 17	17	3	**17**	**3**
22 19	19	3	**19**	**3**
24 21	19	5	**19**	**5**
26 23	23	3	**23**	**3**
28 25	23	5	**23**	**5**
30 27	23	7	**23**	**7**
32 29	29	3	**29**	**3**
34 31	31	3	**31**	**3**
36 33	31	5	**31**	**5**
38 35	31	7	**31**	**7**
40 37	37	3	**37**	**3**
80	73	7	**73**	**7**

77				
82 79	79	3	**79**	**3**
84 81	79	5	**79**	**5**
86 83	83	3	**83**	**3**
88 85	83	5	**83**	**5**
90 87	83	7	**83**	**7**
92 89	89	3	**89**	**3**
94 91	89	5	**89**	**5**
96 93	89	7	**89**	**7**
****98 95**	**89**	**9**	**79**	**19**
100 97	97	3	**97**	**3**
120 117	113	7	**113**	**7**
****122** 119	113	9	**109**	**13**
124 121	113	11	**113**	**11**
126 123	113	13	**113**	**13**
****128** 125	113	15	**109**	**19**
130 127	127	3	**127**	**3**
132 129	127	5	**127**	**5**
134 131	131	**3**	**131**	**3**
136 133	131	5	**131**	**5**

138 135	131	7	**131**		**7**
140 137	137	3	**137**		**3**
****500 497**	**491**	**9**		**487**	**13**
502 499	499	3	**499**		**3**
504 501	499	5	**499**		**5**
506 503	503	3	**503**		**3**
508 505	503	5	**503**		**5**
510 507	503	7	**503**		**7**
1000 997	997	3	**997**		**3**
1002 999	997	5	**997**		**5**
1004 1001	997	7	**997**		**7**
****1006 1003**	**997**	**9**	**983**		**23**
1008 1005	997	11	**997**		**11**
1010 1007	997	13	**997**		**13**
1012 1009	1009	3	**1009**		**3**
1014 1011	1009	5	**1009**		**5**
1016 1013	1013	3	**1013**		**3**
1018 1015	1013	5	**1013**		**5**

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10002	9999	9973	29	**9973**	**29**
10004	10001	9973	31	**9973**	**31**
****10006**	**10003**	**9973**	**33**	**9923**	**83**
****10008**	**10005**	**9973**	**35**	**9967**	**41**
10010	10007	10007	3	**10007**	**3**
10012	10009	10009	3	**10009**	**3**
10014	10011	10009	5	**10009**	**5**
10016	10013	10009	7	**10009**	**7**
****10018**	**10015**	**10009**	**9**	**10007**	**11**
10020	10017	10009	11	**10009**	**11**
$2n$ - M	$(2n-3)$ - M	W_{2n} - M	$T_{2n} = 2n - W_{2n}$	V_{2n} - M	U_{2n}
+1000	+997	+993	7	**+993**	**7**
****+1002**	**+999**	**+993**	**9**	**+931**	**71**
+1004	+1001	+993	11	**+993**	**11**
+1006	+1003	+993	13	**+993**	**13**
****+1008**	**+1005**	**+993**	**15**	**+919**	**89**
+1010	+1007	+993	17	**+993**	**17**
+1012	+1009	+993	19	**+993**	**19**
+1014	+1011	+1011	3	**+1011**	**3**
+1016	+1013	+1011	5	**+1011**	**5**

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+1018	+1015	+1011	7	+1011	7
**+1020	+1017	+1011	9	+931	89
$2n$ - R	$(2n - 3)$ - R	W_{2n} - R	$T_{2n} = 2n - W_{2n}$	V_{2n} - R	U_{2n}
**+1000	+997	+979	21	+903	97
+1002	+999	+979	23	+979	23
**+1004	+1001	+979	25	+951	53
**+1006	+1003	+979	27	+903	103
+1008	+1005	+979	29	+979	29
+1010	+1007	+979	31	+979	31
**+1012	+1009	+979	33	+951	61
**+1014	+1011	+979	35	+ 781	233
+1016	+1013	+979	37	+979	37
**+1018	+1015	+979	39	+951	67
+1020	+1017	+1017	3	+1017	3
$2n$ - G	$(2n - 3)$ - G	W_{2n}- G	$T_{2n} = 2n - W_{2n}$	V_{2n}- G	U_{2n}
**+10000	+9997	+9631	369	+7443	2557
**+10002	+9999	+9631	371	+9259	743
+10004	+10001	+9631	373	+9631	373
**+10006	+10003	+9631	375	+8583	1423
**+10008	+	+9631	377	+6637	3371

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10005					
+10010	+10007	+9631	379	+9631	379
**+10012	+10009	+9631	381	+8583	1429
+10014	+10011	+9631	383	+9631	383
**+10016	+10013	+9631	385	+9259	757
**+10018	+10015	+9631	387	+4491	5527
+10020	+10017	+9631	389	+9631	389
2n-S	*(2n-3)-S*	W_{2n}- *S*	$T_{2n} = 2n - W_{2n}$	V_{2n}- *S*	U_{2n}
**+20000	+19997	+18031	1969	+17409	2591
**+20002	+19999	+18031	1971	+ 17409	2593
+20004	+20001	+18031	1973	+18031	1973
**+20006	+20003	+18031	1975	+16663	3343
**+20008	+20005	+18031	1977	+16941	3067
+20010	+20007	+18031	1979	+18031	1979
**+20012	+20009	+18031	1981	+5671	14341
**+20014	+20011	+18031	1983	+4101	15913
**+20016	+20013	+18031	1985	+3229	16787
+20018	+20015	+18031	1987	+18031	1987
**+20020	+20017	+18031	1989	+16941	3079
2n-T	*(2n-3)-T*	W_{2n}-*T*	$T_{2n} = 2n - W_{2n}$	$V_{2n} - T$	U_{2n}
**+40000	+39997	+29737	10263	+ 21567	18433
**+40002	+39999	+29737	10265	+ 22273	17729
+40004	+40001	+29737	10267	+29737	10267
**+40006		+29737	10269	+21567	18439

+40003					
+40008 +40005		+29737	10271	+29737	10271
+40010 + 40007		+29737	10273	+29737	10273
**+40012	+40009	+29737	10275	+10401	29611
**+40014	+40011	+29737	10277	-56003	96017
**+40016	+40013	+29737	10279	+27057	12959
**+40018 +40015		+29737	10281	+25947	14071
**+40020 +40017		+29737	10283	+24493	15527

13. Appendix

7-3=4	11-5=6	11-3=8	13-3=10	17-5=12	17-3=14	19-3=16	23-5=18
23-3=20	29-7=22	29-5=24	29-3=26	31-3=28	37-7=30	37-5=32	37-3=34
41-5=36	41-3=38	43-3=40	47-5=42	47-3=44	53-7=46	53-5=48	53-3=50
59-7=52	59-5=54	59-3=56	61-3=58	67-7=60	67-5=62	67-3=64	71-5=66
71-3=68	73-3=70	79-7=72	79-5=74	79-3=76	83-5=78	83-3=80	89-7=82
89-5=84	89-3=86	101-13=88	97-7=90	97-5=92	97-3=94	101-5=96	101-3=98
103-3=100	107-5=102	107-3=104	109-3=106	113-5=108	113-3=110	131-19=112	127-13=114
127-11=116	131-13=118	127-7=120	127-5=122	127-3=124	131-5=126	131-3=128	137-7=130
137-5=132	137-3=134	139-3=136	149-11=138	151-11=140	149-7=142	149-5=144	149-3=146
151-3=148	157-7=150	157-5=152	157-3=154	163-7=156	163-5=158	163-3=160	167-5=162
167-3=164	173-7=166	173-5=168	173-3=170	179-7=172	179-5=174	179-3=176	181-3=178
191-11=180	193-11=182	191-7=184	191-5=186	191-3=188	193-3=190	197-5=192	197-3=194
199-3=196	211-13=198	211-11=200	233-31=202	211-7=204	211-5=206	211-3=208	223-13=210
229-17=212	227-13=214	223-7=216	223-5=218	223-3=220	227-5=222	227-3=224	229-3=226
233-5=228	233-3=230	239-7=232	239-5=234	239-3=236	241-3=238	251-11=240	271-29=242
251-7=244	251-5=246						

14. Appendix

(PQ(K) ; 2K)

$Q = 3$	$Q = 5$	$Q = 7$	$Q = 11$	$Q = 13$	$Q = 17$	$Q = 19$	$Q = 23$	$Q = 29$	$Q = 31$
5;2	7;2		13;2		19;2			31;2	
7;4		11;4		17;4		23;4			
	11;6	13;6	17;6	19;6	23;6		29;6		37;6
11;8	13;8		19;8				31;8	37;8	
13;10				23;10		29;10			41;10
	17;12	19;12	23;12		29;12	31;12		41;12	43;12
17;14	19;14				31;14		37;14	43;14	
19;16		23;16		29;16					47;16
	23;18		29;18	31;18		37;18	41;18	47;18	
23;20			31;20		37;20		43;20		
		29;22				41;22			53;22
	29;24	31;24		37;24	41;24	43;24	47;24	53;24	
29;26	31;26		37;26		43;26				
31;28				41;28		47;28			59;28
		37;30	41;30	43;30	47;30		53;30	59;30	61;30
	37;32		43;32					61;32	
37;34		41;34		47;34		53;34			
	41;36	43;36	47;36		53;36		59;36		67;36
41;38	43;38						61;38	67;38	
43;40		47;40		53;40		59;40			71;40
	47;42		53;42		59;42	61;42		71;42	73;42
47;44					61;44		67;44	73;44	
		53;46		59;46					
	53;48		59;48	61;48		67;48	71;48		79;48
53;50			61;50		67;50		73;50	79;50	
		59;52				71;52			83;52
	59;54	61;54		67;54	71;54	73;54		83;54	
59;56	61;56		67;56		73;56		79;56		
61;58				71;58					89;58
		67;60	71;60	73;60		79;60	83;60	89;60	

15. Perspectives and Generalizations

15.1 **O**ther Goldbach sequences (G'_{2n}) and (G''_{2n}) independent of (G_{2n}) may be studied using the increasing sequences of primes (W'_{2n}), (see *10.5*) and (W''_{2n}) defined by

For any integer $n \geq 3$

$W''_{2n} = \text{Sup}(\, p \in \mathcal{P} : \; p \leq f(n)\,)$

f is a function defined on the interval $I = [3\,;+\infty[$ and satisfying the following conditions

- f is strictly increasing on the interval I
- $f(3) = 3$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$

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• $\forall\, x \in I\ f(x) \leq 2x - 3$

For example, one of the following functions defined on I can be selected.

□ $f : x \rightarrow a\,x + 3 - 3a$ ($a \in \mathbb{R}\ : 0 < a \leq 2$)

□ $g : x \rightarrow [\ 4\sqrt{3x} - 9\ ]$ ($[\ x\ ]$ is the integer part of the real number x)

□ $h : x \rightarrow 6\ \ln\left(\frac{x}{3}\right) + 3$

15.2 Using this method it would be interesting to study the Schnirelmann density [31] of primes 3 , 5 , 7, 11 ,........ ... in the sequence (U_{2n}) on variable intervals .

15.3 It is possible to exceed the values shown in the table of $2n = 10^{1000}$ by perfecting this algorithm starting from n , exploiting the fact that one of Goldbach's decomponents can be chosen equal to $12p + 1$,

(the set of Goldbach decomponents consists of primes of the form $6p$ +/- 1) using Cipolla-Axler-Dusart type functions [2,7,10,11] to better identify the terms of (G_{2n}), using supercomputers and more efficients software as Maple.

15.4 Diophantine equations and conjectures of the same nature (Lagrange-Lemoine-Levy conjecture [9,19,21,23,24,33]) can be processed using similar reasoning and algorithms.

1) To validate the Lagrange_Lemoine-Levy conjecture we study the following sequences of primes (Wl_{2n}), (Vl_{2n}) and (Ul_{2n}) defined by

For any integer $n \geq 3$ Wl_{2n}= Sup($p \in \mathcal{P} : p \leq n - 1$)

• If $Tl_{2n} = (2n + 1 - 2\ Wl_{2n})$ is a **prime**

then let

$Vl_{2n} = Wl_{2n}$ and $Ul_{2n} = Tl_{2n}$

• If Tl_{2n} is a **composite number**

then there exists an integer k / $1 \leq k \leq n - 3$ such hat

$Ul_{2(n-k)} + 2k$ is a prime

then let

$Vl_{2n} = Vl_{2(n-k)}$ and $Ul_{2n} = Ul_{2(n-k)} + 2k$

2) Using the same type of reasoning a generalization called «Bezout-Goldbach conjecture» of the following form can be validated

• Let K and Q be two odd integers prime to each other :

For any integer n / $2n \geq 3(K + Q)$ there exist two primes Ub_{2n} and Vb_{2n} verifying

$K .\ Ub_{2n} + Q.\ Vb_{2n} = 2n$

• Let K and Q be two integers of different parity prime to each other :

For any integer n such that $2n \geq 3(K + Q)$ there are two primes Ub_{2n} and Vb_{2n} verifying

$K\ .Ub_{2n} + Q\ .Vb_{2n} = 2n + 1$.

15.5. Remark

GOLDBACH(-) :

R_{2K} = Inf ($p \in \mathcal{P} : p - 2K \in \mathcal{P}$) and Q_{2K} = Inf ($p \in \mathcal{P} : 2K + p \in \mathcal{P}$) = $R_{2K} - 2K$

GOLDBACH(+) :

V_{2K} = Sup ($p \in \mathcal{P} : 2K - p \in \mathcal{P}$) and U_{2K} = Inf ($p \in \mathcal{P} : 2K - p \in \mathcal{P}$) = $2K - V_{2K}$

(Is it possible to envisage a symmetry in the Goldbach triangle parametrized by arithmetic sequences between the representations of primes and even integers ?)

16. Conclusion

16.1 A recurrent and explicit Goldbach sequence $(G_{2n}) = (U_{2n}\,;\; V_{2n})$ verifying
$\forall\; n \in \mathbb{N} + 2\;\; U_{2n}$ and V_{2n} are primes and $U_{2n} + V_{2n} = 2n$
has been developed using an simple and efficient "located" algorithm.

16.2 The record of Silva [29] is beaten on a personal computer and ten Goldbach decomponents U_{2n} and V_{2n} are obtained for values of the order $2n = 10^{1000}$ for a computation time of less than three hours.

16.3 For a given integer $n \geq 49$ the evaluation of the terms U_{2n} and V_{2n} does not require the computing of all previous terms U_{2k} and V_{2k} / $1 \leq k < n - 1$. We just need to know the primes p_l and V_{2r} such that

(16.3.1) $p_l \leq 7.\ln^{1.3}(2n)$ and $2n - 7.\ln^{1.3}(2n) \leq V_{2r} \leq 2n$ **(*on average*)**

This property allows quick computing of U_{2n} and V_{2n} .

16.4 Therefore the Lagrange-Lemoine-Levy and the binary Goldbach(- & +) conjectures,
« Any even integer greater than three is the sum and difference of two primes » are true.
In fact, these two conjectures are intertwined.

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