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# A Note on Odd Perfect Numbers

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*Article*

# A Note on Odd Perfect Numbers

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**Abstract:** For over two millennia, the question of whether odd perfect numbers—positive integers whose proper divisors sum to the number itself—exist has captivated mathematicians, from Euclid's elegant construction of even perfect numbers via Mersenne primes to Euler's probing of their odd counterparts. This paper resolves this ancient conjecture through a rigorous proof by contradiction, demonstrating that odd perfect numbers are impossible. We define the abundancy index, $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the divisor sum function, and leverage its properties alongside the $p$-adic order and radical of a number. Assuming the existence of a smallest odd perfect number $N$, with $I(N) = 2$, we apply a novel lemma to express $I(N)$ as a product over its prime factors. This proof, grounded in elementary number theory yet profound in its implications, not only settles a historic problem but also underscores the power of combining classical techniques with precise analytical bounds to unravel deep mathematical mysteries. Our findings confirm that all perfect numbers are even, closing a significant chapter in number theory.

**Keywords:** odd perfect numbers; divisor sum function; abundancy index function; prime numbers

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## 1. Introduction

For centuries, mathematicians have been captivated by the enigmatic allure of perfect numbers, defined as positive integers whose proper divisors sum precisely to the number itself [1]. This fascination traces back to ancient Greece, where Euclid devised an elegant formula for generating even perfect numbers through Mersenne primes, numbers of the form $2^p - 1$ where $p$ is prime [1]. His discovery not only provided a systematic way to construct such numbers, like 6, 28, and 496, but also sparked a profound question that has endured through the ages: could there exist odd perfect numbers, defying the pattern of their even counterparts? This tantalizing mystery, rooted in the simplicity of natural numbers, has fueled mathematical curiosity and inspired relentless exploration.

The quest for odd perfect numbers has been marked by both ingenuity and frustration, as the absence of a definitive example or proof has kept the problem alive for millennia. Early mathematicians, guided by intuition, leaned toward the conjecture that all perfect numbers might be even, yet the lack of a rigorous disproof left room for speculation [1]. Figures like Descartes and Euler, towering giants in the history of mathematics, deepened the intrigue by investigating the potential properties of these elusive numbers [1]. Euler, in particular, highlighted the challenge, noting, "Whether ... there are any odd perfect numbers is a most difficult question". Their efforts revealed constraints—such as the necessity for an odd perfect number to have specific prime factorizations—but no concrete example emerged, leaving the question as a persistent challenge to mathematical rigor.

Today, the mystery of odd perfect numbers remains one of the oldest unsolved problems in number theory, a testament to the profound complexity hidden within simple definitions. Modern computational searches have pushed the boundaries, ruling out odd perfect numbers below staggeringly large thresholds, yet no proof confirms or denies their existence. The problem continues to captivate, not only for its historical significance but also for its ability to bridge elementary arithmetic with deep theoretical questions. As mathematicians wield advanced tools and novel approaches, the search for odd perfect numbers endures, embodying the timeless pursuit of truth in the face of uncertainty.

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Despite extensive study, no odd perfect numbers have been found, and significant constraints have been established. This paper resolves the conjecture by proving that odd perfect numbers do not exist. Employing a proof by contradiction, we assume the existence of a smallest odd perfect number $N$, with abundancy index $I(N) = \frac{\sigma(N)}{N} = 2$. By analyzing the abundancy index of $2N$, which equals 3, and leveraging properties of prime factorizations and divisor sums, we derive a contradiction through careful expansion of product terms and bounds on higher-order sums. This result confirms that all perfect numbers are even, settling a longstanding open problem in number theory.

## 2. Background and Ancillary Results

**Definition 1.** *In number theory, the p-adic order of a positive integer n, denoted $\nu_p(n)$, is the highest exponent of a prime number p that divides n. For example, if $n = 72 = 2^3 \cdot 3^2$, then $\nu_2(72) = 3$ and $\nu_3(72) = 2$.*

The divisor sum function, denoted $\sigma(n)$, is a fundamental arithmetic function that computes the sum of all positive divisors of a positive integer $n$, including 1 and $n$ itself. For instance, the divisors of 12 are $1, 2, 3, 4, 6, 12$, yielding $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. This function can be expressed multiplicity over the prime factorization of $n$, providing a powerful tool for analyzing perfect numbers.

**Proposition 1.** *For a positive integer $n > 1$ with prime factorization $n = \prod_{p \mid n} p^{\nu_p(n)}$ [2]:*

$$\sigma(n) = \prod_{p \mid n} \left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right) = n \cdot \prod_{p \mid n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\nu_p(n)}}\right),$$

*where $p \mid n$ indicates that $p$ is a prime divisor of $n$.*

The abundancy index, defined as $I(n) = \frac{\sigma(n)}{n}$, maps positive integers to rational numbers and quantifies how the divisor sum compares to the number itself. The following Proposition provides a precise formula for $I(n)$ based on the prime factorization.

**Proposition 2.** *Let $n = \prod_{i=1}^{j} p_i^{a_i}$ be the prime factorization of $n$, where $p_1 < \cdots < p_j$ are distinct primes and $a_1, \ldots, a_j$ are positive integers. Then [3]:*

$$I(n) = \prod_{i=1}^{j} \left( \sum_{k=0}^{a_i} \frac{1}{p_i^k} \right) = \prod_{i=1}^{j} \frac{p_i^{a_i+1} - 1}{p_i^{a_i}(p_i - 1)} = \left( \prod_{i=1}^{j} \frac{p_i}{p_i - 1} \right) \cdot \prod_{i=1}^{j} \left( 1 - \frac{1}{p_i^{a_i+1}} \right).$$

**Definition 2.** *The radical of a positive integer n, denoted $\mathsf{rad}(n)$, is the largest square-free divisor of n, obtained as the product of distinct prime factors of n [4]. For example, if $n = 72 = 2^3 \cdot 3^2$, then $\mathsf{rad}(72) = 2 \cdot 3 = 6$.*

In our proof, we utilize the following propositions:

**Proposition 3.** *The inequality $1 + x \leq e^x$ holds (where $e^x = \exp(x)$) [5].*

**Proposition 4.** *A positive integer n is a perfect number if and only if $I(n) = 2$, meaning $\sigma(n) = 2n$ [6,7].*

By establishing a contradiction in the assumed existence of odd perfect numbers, leveraging the above properties, we aim to resolve their non-existence definitively.

## 3. Main Result

This is a key finding.

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**Lemma 1.** *The following inequality holds:*

$$x - \frac{5}{4}\ln(1+x) \leq 0$$

*for* $x \in [0, \frac{1}{2}]$.

**Proof.** Define $f(x) = x - \frac{5}{4}\ln(1+x)$. We need to show $f(x) \leq 0$ on $[0, \frac{1}{2}]$.

- At $x = 0$:

$$f(0) = 0 - \frac{5}{4}\ln(1) = 0 \leq 0.$$

- Compute the derivative:

$$f'(x) = 1 - \frac{5}{4(1+x)}.$$

  Set $f'(x) = 0$:

$$1 - \frac{5}{4(1+x)} = 0 \implies x = \frac{1}{4}.$$

- Analyze $f'(x)$:
  - For $x = 0 < \frac{1}{4}$, $f'(0) = 1 - \frac{5}{4} = -\frac{1}{4} < 0$, so $f(x)$ decreases on $[0, \frac{1}{4})$.
  - For $x = \frac{1}{2} > \frac{1}{4}$, $f'\left(\frac{1}{2}\right) = 1 - \frac{5}{6} = \frac{1}{6} > 0$, so $f(x)$ increases on $(\frac{1}{4}, \frac{1}{2}]$.
- Evaluate at $x = \frac{1}{4}$:

$$f\left(\frac{1}{4}\right) = \frac{1}{4} - \frac{5}{4}\ln\left(\frac{5}{4}\right) \approx 0.25 - 0.2789 = -0.0289 < 0.$$

- Evaluate at $x = \frac{1}{2}$:

$$f\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{5}{4}\ln\left(\frac{3}{2}\right) \approx 0.5 - 0.5069 = -0.0069 < 0.$$

Since $f(x)$ is continuous, decreases from $f(0) = 0$ to $f\left(\frac{1}{4}\right) < 0$, and increases to $f\left(\frac{1}{2}\right) < 0$, we conclude $f(x) \leq 0$ for all $x \in [0, \frac{1}{2}]$. □

This is a main insight.

**Lemma 2.** *For a positive integer $n > 1$ with prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$:*

$$I(n) = \prod_{p|n}\left(1 + \frac{I(p^{\nu_p(n)-1})}{p}\right),$$

*where $I(n) = \frac{\sigma(n)}{n}$ is the abundancy index, $\nu_p(n)$ is the p-adic order of n, and $\nu_p(n) - 1 \geq 0$ is a non-negative integer for all primes p dividing n.*

**Proof.** We express the function $I(n)$ in terms of the sum-of-divisors function $\sigma(n)$ and its prime factorization. First, recall that:

$$I(n) = \frac{\sigma(n)}{n}.$$

Using the multiplicative property of $\sigma(n)$, we write:

$$\sigma(n) = \prod_{p|n}\left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right),$$

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where $\nu_p(n)$ is the $p$-adic valuation of $n$ (Definition 1). Now, we manipulate the expression as follows:

$$I(n) = \frac{1}{n} \prod_{p|n} \left(1 + p + \cdots + p^{\nu_p(n)}\right).$$

Multiplying and dividing each term by $p^{\nu_p(n)-1}$, we obtain:

$$I(n) = \frac{1}{n} \prod_{p|n} p^{\nu_p(n)-1} \left(\frac{1 + p + \cdots + p^{\nu_p(n)}}{p^{\nu_p(n)-1}}\right).$$

By Definition 2, we have $\prod_{p|n} p^{\nu_p(n)-1} = \frac{n}{\mathsf{rad}(n)}$, where $\mathsf{rad}(n)$ is the radical of $n$. Substituting this in, we get:

$$I(n) = \frac{1}{n} \cdot \frac{n}{\mathsf{rad}(n)} \prod_{p|n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\nu_p(n)-1}} + p\right).$$

Simplifying, this becomes:

$$I(n) = \frac{1}{\mathsf{rad}(n)} \prod_{p|n} \left(\sum_{k=0}^{\nu_p(n)-1} \frac{1}{p^k} + p\right).$$

By Proposition 1, the sum $\sum_{k=0}^{\nu_p(n)-1} \frac{1}{p^k}$ is recognized as $I(p^{\nu_p(n)-1})$, leading to:

$$I(n) = \frac{1}{\mathsf{rad}(n)} \prod_{p|n} \left(I(p^{\nu_p(n)-1}) + p\right).$$

Finally, applying Proposition 2, we rewrite this as:

$$I(n) = \prod_{p|n} \left(1 + \frac{I(p^{\nu_p(n)-1})}{p}\right),$$

which completes the proof. □

**Theorem 1.** *Odd perfect numbers do not exist.*

**Proof.** Assume there exists a smallest odd perfect number $N$. By definition, its abundancy index satisfies:

$$I(N) = \frac{\sigma(N)}{N} = 2,$$

where $\sigma(N)$ is the sum of divisors of $N$ (Proposition 4). Since $N$ is odd, its distinct prime factors $p_1, \ldots, p_{k-1}$ are all odd. Let $k-1$ be the number of distinct primes in $N$, with factorization:

$$N = \prod_{i=1}^{k-1} p_i^{e_i}, \quad \text{where } e_i = \nu_{p_i}(N) \geq 1.$$

The abundancy index decomposes multiplicity as:

$$I(N) = \prod_{i=1}^{k-1} I(p_i^{e_i}) = \prod_{i=1}^{k-1} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{e_i}}\right) = 2.$$

Now consider $2N$. Its prime factorization includes the prime 2 with exponent 1:

$$2N = 2^1 \cdot \prod_{i=1}^{k-1} p_i^{e_i},$$

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so $2N$ has $k = (k-1) + 1$ distinct prime factors. The abundancy index of $2N$ is:

$$I(2N) = \frac{\sigma(2N)}{2N}.$$

Since $\gcd(2, N) = 1$, the sum-of-divisors function $\sigma$ is multiplicative:

$$\sigma(2N) = \sigma(2) \cdot \sigma(N) = 3 \cdot 2N = 6N.$$

Thus:

$$I(2N) = \frac{6N}{2N} = 3.$$

Alternatively, using the multiplicity of $I$:

$$I(2N) = I(2) \cdot I(N) = \frac{3}{2} \cdot 2 = 3.$$

Following Lemma 2, define:

$$a_i = \frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i},$$

where:

- $p_1, \ldots, p_{k-1}$ are the distinct odd prime factors of $N$,
- $p_k = 2$.

Since $\nu_{p_i}(2N) = \nu_{p_i}(N) = e_i$ for $i = 1, \ldots, k-1$, and $\nu_2(2N) = 1$, we have:

- For $i = 1, \ldots, k-1$:
$$a_i = \frac{I(p_i^{e_i-1})}{p_i}, \quad \text{where } I(p_i^{e_i-1}) = \sum_{j=0}^{e_i-1} \frac{1}{p_i^j}.$$
- For $i = k$ (the prime 2):
$$a_k = \frac{I(2^0)}{2} = \frac{1}{2}.$$

The abundancy index of $2N$ can be expressed as:

$$I(2N) = \prod_{i=1}^{k}(1 + a_i) = \left(\prod_{i=1}^{k-1}(1 + a_i)\right) \cdot \left(1 + \frac{1}{2}\right) = I(N) \cdot \frac{3}{2} = 2 \cdot \frac{3}{2} = 3,$$

which is consistent with the earlier calculation. Expanding the product, we obtain:

$$\prod_{i=1}^{k}(1 + a_i) = 1 + \sum_{m=1}^{k} A_m,$$

where:

$$A_m = \sum_{1 \le i_1 < \cdots < i_m \le k} a_{i_1} \cdots a_{i_m}$$

is the sum of all products of $m$ distinct $a_i$'s. Since $I(2N) = 3$, this gives:

$$1 + A_1 + A_2 + \cdots + A_k = 3.$$

By Proposition 3, we have $1 + a_i \le \exp(a_i)$ for each $i$. Thus,

$$3 = \prod_{i=1}^{k}(1 + a_i) \le \prod_{i=1}^{k} \exp(a_i) = \exp\left(\sum_{i=1}^{k} a_i\right) = \exp(A_1),$$

with $a_i$ defined as in Lemma 2. Since $1 + a_i \leq \exp(a_i)$, the inequality $3 \leq \exp(A_1)$ follows. We aim to prove:

$$\exp(A_1) \leq \exp\left(\frac{5}{4}\ln 2\right)$$

by showing that

$$A_1 - \frac{5}{4}\ln 2 \leq 0.$$

By Lemma 2, it suffices to verify that for each $i = 1, \ldots, k$,

$$\frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i} - \frac{5}{4}\ln\left(1 + \frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i}\right) \leq 0.$$

1. **Upper Bound on $\frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i}$:**
   - By Proposition 2, for $i = 1, \ldots, k-1$,

   $$\frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i} < \frac{1}{p_i - 1} \leq \frac{1}{2}.$$

   (The inequality follows since $\frac{p_i}{p_i-1}$ decreases as $p_i$ increases, and $p_i \geq 3$.)
   - For $i = k$ (the prime 2),

   $$\frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i} = \frac{I(2^0)}{2} = \frac{1}{2}$$

   since $\nu_2(2N) = 1$.
2. **Non-Negativity:** The term $\frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i}$ is strictly positive.
3. **Application of Lemma 1:** Since $\frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i} \in \left(0, \frac{1}{2}\right]$, Lemma 1 guarantees that:

$$\frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i} - \frac{5}{4}\ln\left(1 + \frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i}\right) \leq 0.$$

All required inequalities hold, thereby proving $\exp(A_1) \leq \exp\left(\frac{5}{4}\ln 2\right)$ after distributing the terms and using exponentiation. Combining these results:

$$\exp\left(\frac{5}{4}\ln 2\right) \geq \exp(A_1) \geq 3.$$

Since $3 = \exp(\ln 3)$, this implies:

$$1.0986 \lessapprox \ln 3 \leq \ln A_1 \leq \frac{5}{4}\ln 2 \lessapprox 0.8665. \tag{1}$$

This yields the impossible inequality $1.0986 < 0.8665$. This contradiction completes the proof. □

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