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# A Pseudo-Riemannian Generalization of Euclidean Subspace Hierarchies

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**Abstract**

We propose a unifying framework that generalizes classical Euclidean subspace hierarchies into a pseudo-Riemannian landscape by introducing a signature-based stratification of manifolds. For a total dimension $D = m + n$, we systematically construct the space of submanifolds $\mathcal{M}_{m,n}$ with $m$ spacelike and $n$ timelike dimensions. These give rise to a structured family of pseudo-Grassmannians $\mathrm{Gr}_{(m,n)}(M, N)$, where signature plays a central geometric and physical role. We extend key constructions—such as Plücker embeddings, local charts, homogeneous space representations, volume forms, and cohomological invariants—to these indefinite-signature settings. Furthermore, we explore implications for symmetry breaking, field theory on signature-changing manifolds, and brane-world cosmologies involving M(1,3), M(2,2), and M(3,1) universes. Applications to twistor theory, supersymmetry, and quantum gravity foams with signature fluctuations are included. The resulting geometric machinery provides new tools for modelling transitions between classical, subtle, and metaphysical layers of spacetime.

**Keywords:** pseudo Riemannian spaces; grassmanian; signature changing cosmologies; branes; twistor physics; supersymmetry

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## 1. Introduction

In differential geometry and general relativity, the structure of spacetime is modeled by a differentiable manifold equipped with a metric tensor of specified signature. Classical physics typically adopts the Riemannian signature $(+, +, +, \ldots, +)$ for Euclidean spaces $\mathbb{R}^N$, while modern physics, particularly general relativity, adopts a Lorentzian signature $(-, +, +, +)$ for spacetime. More exotic theories, such as two-time physics and F-theory, have proposed the use of additional time dimenseions.

This paper seeks to extend the classical notion of nested Euclidean spaces to a comprehensive structure incorporating all pseudo-Riemannian manifolds of a fixed total dimension. We define each element $\mathcal{M}_{m,n}$ in terms of its signature and dimension, and then construct the full hierarchy $\mathcal{S}_N$ of all such manifolds for a given total dimension $N = m + n$. For example, $\mathcal{S}_2$ includes $\mathcal{M}_{2,0}$, $\mathcal{M}_{1,1}$, and $\mathcal{M}_{0,2}$.

This framework provides a new algebraic and geometric method of classifying spacetime manifolds by their signature decomposition, and opens up pathways to studying transitions between such manifolds in cosmological and high-energy contexts. Prior efforts to model variable or multiple time dimensions include Bars' work on two-time physics [1], Hull's T-duality approach [2], and Penrose's conformal cyclic cosmology [3], which incorporates signature dynamics.

The structure of spacetime, as captured in differential geometry and the mathematical foundations of general relativity, is encoded in smooth manifolds endowed with a metric tensor of specified signature. The classical Riemannian picture considers purely positive-definite metrics over manifolds $\mathbb{R}^N$, yielding subspaces $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^N$. This hierarchical framework underlies much of Euclidean geometry and analysis. In contrast, modern physics requires pseudo-Riemannian signatures to model causality and relativistic phenomena, with the standard spacetime modeled as a Lorentzian manifold $\mathcal{M}_{1,3}$ having one temporal and three spatial directions.

Beyond this, various speculative theories, such as F-theory, two-time physics, and twistor theory, extend the possible spacetime signatures to include multiple temporal dimensions. These motivate the construction of a broader framework: a generalization of the classical Euclidean ladder into a family of pseudo-Riemannian subspaces. For any fixed total dimension $D = m + n$, we define a stratified set $\mathcal{S}_D$ containing submanifolds $\mathcal{M}_{m,n}$ with $m$ spacelike and $n$ timelike directions. This gives rise to pseudo-Riemannian generalizations of classical constructs, including Grassmannians, homogeneous spaces, volume forms, and curvature structures.

Our approach aims to develop a systematic theory of pseudo-Grassmannians $\mathrm{Gr}_{(m,n)}(M, N)$, extending classical Grassmannians $\mathrm{Gr}(d, D) \subset \mathbb{R}^D$ which parametrize $d$-dimensional linear subspaces. The pseudo-Grassmannian instead parametrizes subspaces with a fixed signature embedded in a larger space $\mathbb{R}^{M,N}$. Each point in $\mathrm{Gr}_{(m,n)}(M, N)$ corresponds to a linear subspace with $m$ spacelike and $n$ timelike directions, with a preserved inner product signature.

This unifying language has profound implications in theoretical physics. In particular, by organizing the families of submanifolds and their interactions via signature transitions, we construct models for universe transitions, symmetry breaking phenomena, and higher-dimensional dualities. Our framework naturally connects to the Trilok model of three interleaved universes: the physical $\mathcal{M}_{1,3}$, the subtle $\mathcal{M}_{2,2}$, and the metaphysical $\mathcal{M}_{3,1}$, each admitting distinct causal and geometric structures, and each embedded into an extended pseudo-Riemannian stratification.

This work develops the necessary mathematical machinery to study metrics, curvature, volume, cohomology, representation theory, and quantum field theory across these generalized manifolds. In particular, we define coordinate charts, metric invariants, and curvature tensors within pseudo-Grassmannians, formulate quantum actions on signature-varying branes, and derive transition rules via moduli of metric deformations. The ultimate goal is to classify and geometrize the full landscape of possible pseudo-Riemannian universes, and apply these results to geometry, particle physics, and cosmology.

## 2. Pseudo-Riemannian Subspace Hierarchies

Let $\mathcal{M}_{m,n}$ denote a pseudo-Riemannian manifold with $m$ spacelike and $n$ timelike dimensions. The total dimension is then defined as

$$D = m + n. \tag{1}$$

The metric tensor $g_{ab}$ on $\mathcal{M}_{m,n}$ has signature $(m, n)$, which means the diagonalized form of the metric includes $m$ positive and $n$ negative entries:

$$g_{ab}^{(m,n)} = \mathrm{diag}(\underbrace{+1, \ldots, +1}_{m}, \underbrace{-1, \ldots, -1}_{n}). \tag{2}$$

For each integer $N = m + n$, we define the hierarchy $\mathcal{S}_N$ of signature classes as:

$$\mathcal{S}_N = \{\mathcal{M}_{k,N-k} \mid k = 0, 1, \ldots, N\}. \tag{3}$$

The cardinality of $\mathcal{S}_N$ is clearly:

$$|\mathcal{S}_N| = N + 1. \tag{4}$$

Each member of $\mathcal{S}_N$ corresponds to a different causal and geometric structure. For instance, $\mathcal{M}_{N,0}$ corresponds to a Riemannian space $\mathbb{R}^N$, while $\mathcal{M}_{N-1,1}$ corresponds to a Lorentzian spacetime.

## 3. Causal Structure and Metric Properties

The presence of multiple time dimensions fundamentally alters the causal structure of the manifold. In standard Lorentzian manifolds with $n = 1$, causality is determined by the future and past

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light cones. For manifolds with $n > 1$, multiple null directions exist, and the light cone becomes a higher-dimensional manifold itself, potentially leading to causal ambiguity.

For any $\mathcal{M}_{m,n}$, define the tangent space at a point $p$ as $T_p\mathcal{M}_{m,n}$. The causal structure is derived from the sign of the squared norm of a vector $v \in T_p\mathcal{M}$:

$$g(v,v) > 0 \Rightarrow \text{spacelike}, \quad g(v,v) < 0 \Rightarrow \text{timelike}, \quad g(v,v) = 0 \Rightarrow \text{null}. \tag{5}$$

In $\mathcal{M}_{1,1}$, for instance, the null cone is two-dimensional, including vectors of the form $(v^1, v^2)$ with $(v^1)^2 - (v^2)^2 = 0$. The cone structure in $\mathcal{M}_{n,n}$ becomes more intricate, as the bilinear form admits multiple null directions.

## 4. Enumeration and Moduli of Signature Spaces

We can represent the totality of all pseudo-Riemannian subspaces up to a dimension $D$ by the union:

$$\mathcal{H}_{\leq D} = \bigcup_{N=1}^{D} \mathcal{S}_N. \tag{6}$$

The total number of signature classes up to dimension $D$ is then given by the triangular sum:

$$|\mathcal{H}_{\leq D}| = \sum_{k=1}^{D}(k+1) = \frac{(D+1)(D+2)}{2} - 1. \tag{7}$$

This result is easily verified by direct computation. For example, for $D = 3$, we have $\mathcal{S}_1 = 2$, $\mathcal{S}_2 = 3$, $\mathcal{S}_3 = 4$, so:

$$|\mathcal{H}_{\leq 3}| = 2 + 3 + 4 = 9. \tag{8}$$

The moduli space of such manifolds can be considered as a parameter space over signature tuples $(m,n)$, with possible constraints on curvature, topology, or symmetry. A natural candidate is:

$$\mathcal{M}_D^{\text{sig}} = \{g_{ab} \in \text{Sym}^2(T^*\mathcal{M}) \mid \text{sig}(g_{ab}) = (m,n), m+n = D\}. \tag{9}$$

## 5. Physical Interpretation and Applications

Several theoretical frameworks suggest or allow the existence of multiple time dimensions. In Bars' two-time physics [1], an additional temporal dimension is introduced to derive physical symmetries more naturally. F-theory, a geometric approach to Type IIB string theory, employs a 12-dimensional spacetime with signature (2,10), offering a compelling example of $\mathcal{M}_{2,10}$ [4].

Conformal cyclic cosmology as proposed by Penrose [3] implies signature transitions at the end and beginning of cosmic epochs. Furthermore, in quantum cosmology and certain path-integral approaches, spacetime metrics may transition dynamically between Riemannian and Lorentzian regimes [5].

Understanding the full hierarchy $\mathcal{S}_N$ may provide a geometrically consistent framework for analyzing such transitions, offering insights into symmetry breaking, causality, and cosmological boundary conditions. It also naturally supports formulations of quantum gravity where spacetime signature is not fixed a priori but emerges dynamically.

## 6. Structural Foundations of Pseudo-Riemannian Grassmannians

The classical Grassmannian $\text{Gr}(d, D)$, defined as the manifold of $d$-dimensional subspaces of $\mathbb{R}^D$, possesses a rich set of geometric, algebraic, and topological structures. These include a homogeneous space representation, Plücker embedding, Schubert calculus, cohomological invariants, and applications across geometry and physics. In the pseudo-Riemannian setting, one considers subspaces of fixed indefinite signature $(m,n)$ inside an ambient pseudo-Euclidean framework.

Let $V \subset \mathbb{R}^{M,N}$ be a $d = m + n$-dimensional subspace with induced signature $(m, n)$. The set of such subspaces forms the pseudo-Riemannian Grassmannian:

$$\mathrm{Gr}_{(m,n)}(M,N) = \{V \subset \mathbb{R}^{M,N} \mid \dim V = m + n,\ \mathrm{sig}(g|_V) = (m,n)\}. \tag{10}$$

We now outline and define the structural elements necessary to generalize Grassmannians to the pseudo-Riemannian category.

### 6.1. Manifold and Topological Structure

In the Riemannian case, the Grassmannian is a compact manifold of dimension $d(D-d)$. The pseudo-Riemannian case, however, is non-compact due to the indefinite signature and has disconnected components depending on the signature class. The pseudo-Grassmannian $\mathrm{Gr}_{(m,n)}(M,N)$ is a real smooth manifold realized as a homogeneous space under the pseudo-orthogonal group:

$$\mathrm{Gr}_{(m,n)}(M,N) = \frac{O(M,N)}{O(m,n) \times O(M-m, N-n)}. \tag{11}$$

The smooth structure is inherited from this quotient, as $O(M,N)$ acts transitively on non-degenerate subspaces of given signature.

### 6.2. Dimension Formula and Group Representation

The dimension of $\mathrm{Gr}_{(m,n)}(M,N)$ is obtained by subtracting the stabilizer subgroup dimensions:

$$\dim \mathrm{Gr}_{(m,n)}(M,N) = \frac{(M+N)(M+N-1)}{2} - \frac{(m+n)(m+n-1)}{2} - \frac{(M-m+N-n)(M-m+N-n-1)}{2}. \tag{12}$$

This expression reduces to the classical $d(D-d)$ when $N = 0$ and $n = 0$, recovering the compact Riemannian Grassmannian.

### 6.3. Coordinate Charts and Local Structure

The classical Grassmannian is covered by coordinate charts parameterized by matrices with rank $d$, typically represented in Stiefel coordinates. For pseudo-Grassmannians, local charts must respect the non-degeneracy and signature conditions. A possible construction involves embedding $\mathrm{Gr}_{(m,n)}(M,N)$ into the manifold of $(M+N) \times (m+n)$ matrices with fixed signature Gram matrix:

$$G = X^\top \eta X = \eta_{(m,n)}, \tag{13}$$

where $\eta$ is the ambient metric of signature $(M,N)$, and $\eta_{(m,n)}$ is the standard form of the induced metric. The subset of such matrices defines a local trivialization, and the quotient by $O(m,n)$ gives the chart.

### 6.4. Plücker Embedding and Exterior Algebra

The classical Plücker embedding maps $\mathrm{Gr}(d,D)$ into the projective space $\mathbb{P}(\wedge^d \mathbb{R}^D)$. In the pseudo-Riemannian case, one considers an embedding:

$$\mathrm{Gr}_{(m,n)}(M,N) \hookrightarrow \mathbb{P}(\wedge^{m+n} \mathbb{R}^{M,N}), \tag{14}$$

where the wedge product must preserve the signature constraint. This embedding maps each $(m+n)$-dimensional subspace with signature $(m,n)$ to its exterior volume form, constrained by the indefinite inner product.

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*6.5. Tangent Bundle and Hom Spaces*

In the Riemannian Grassmannian, the tangent space at a point $V \in \mathrm{Gr}(d, D)$ is:

$$T_V \mathrm{Gr}(d, D) \cong \mathrm{Hom}(V, V^{\perp}). \tag{15}$$

In the pseudo-Riemannian case, the orthogonal complement $V^{\perp}$ is also indefinite. Thus,

$$T_V \mathrm{Gr}_{(m,n)}(M, N) \cong \mathrm{Hom}(V, V^{\perp}), \tag{16}$$

with additional structure due to the indefinite metrics on both domains.

*6.6. Schubert Stratification and Incidence Geometry*

Schubert cells decompose the classical Grassmannian into strata indexed by combinatorial data associated with flags. In the pseudo-case, one may define strata via incidence conditions respecting the signature, such as:

$$\dim(V \cap F_i) = r_i, \tag{17}$$

where $F_i$ is a flag subspace and $r_i$ accounts for the number of positive and negative signature directions intersected. These stratifications will in general be non-compact and disconnected.

*6.7. Metric and Volume Structures*

The Grassmannian admits a canonical invariant Riemannian metric derived from the ambient orthogonal group. For pseudo-Grassmannians, an indefinite metric can be induced from the bilinear form of $O(M, N)$. The associated volume form must be constructed carefully to respect orientation and signature. The Killing form on the Lie algebra $\mathfrak{o}(M, N)$ provides a candidate for constructing such forms.

*6.8. Cohomology and Topological Invariants*

The cohomology ring of the classical Grassmannian is generated by Schubert classes. In the pseudo-case, non-compactness complicates the standard cohomological picture. Nonetheless, one may define relative cohomology or use equivariant cohomology with respect to $O(M, N)$ to define invariants. Intersection theory remains challenging due to the lack of compactness, but stratified Morse theory or signature-dependent characteristic classes may be employed.

*6.9. Complexification and Holomorphic Structure*

The complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(d, D)$ admits a Kähler structure. Complexifying the pseudo-Grassmannian leads to spaces such as $\mathrm{Gr}_{\mathbb{C}}(m + n, M + N)$, where real signature data is encoded in complex conjugation invariants. These complex pseudo-Grassmannians appear in twistor theory and in conformal compactifications of spacetime.

*6.10. Representation Theory and Symmetry Actions*

The action of $O(D)$ on $\mathrm{Gr}(d, D)$ decomposes into irreducible representations. In the pseudo-case, $O(M, N)$ acts transitively on $\mathrm{Gr}_{(m,n)}(M, N)$, and its unitary representations are infinite-dimensional. The structure of orbits, isotropy subgroups, and induced representations becomes important for physical models with signature-changing solutions or higher-dimensional embeddings.

*6.11. Applications in Physics and Geometry*

Pseudo-Grassmannians appear naturally in string theory, M-theory, and models with extra time dimensions. In F-theory [4], the ambient space is 12-dimensional with signature (2,10). In two-time physics [1], the configuration space projects from a phase space structured over a $\mathrm{Gr}_{(m,n)}$ manifold. Signature-changing cosmologies, such as those explored by Hartle and Hawking [5], suggest that the dynamics of the early universe may involve traversals through them.

## 7. Topological and Smooth Manifold Structure of Pseudo-Riemannian Grassmannians

In classical Riemannian geometry, the Grassmannian manifold $\mathrm{Gr}(d, D)$ is defined as the space of all $d$-dimensional linear subspaces of $\mathbb{R}^D$. This space is compact, smooth, and connected. It is a homogeneous space under the orthogonal group $O(D)$, given explicitly as the quotient

$$\mathrm{Gr}(d, D) = \frac{O(D)}{O(d) \times O(D-d)}. \tag{18}$$

The dimension of this manifold is computed as

$$\dim \mathrm{Gr}(d, D) = d(D-d). \tag{19}$$

This reflects the number of degrees of freedom required to specify a $d$-dimensional subspace in $D$-dimensional Euclidean space, up to orthogonal transformations.

We aim to generalize this concept to the setting of pseudo-Riemannian geometry, where the ambient space $\mathbb{R}^{M,N}$ carries a non-degenerate bilinear form of signature $(M, N)$, indicating $M$ spacelike and $N$ timelike directions. Within such a space, we define the pseudo-Riemannian Grassmannian $\mathrm{Gr}_{(m,n)}(M, N)$ as the manifold of all $(m+n)$-dimensional subspaces of signature $(m, n)$. That is, each such subspace $V \subset \mathbb{R}^{M,N}$ satisfies

$$\dim V = m + n, \quad \text{and} \quad g|_V \text{ has signature } (m, n), \tag{20}$$

where $g$ is the pseudo-Riemannian metric of the ambient space.

This manifold is non-compact and potentially disconnected. The topology of $\mathrm{Gr}_{(m,n)}(M, N)$ is richer and more subtle than its Riemannian counterpart. Each connected component of this space corresponds to a fixed signature type $(m, n)$, and varying this pair leads to a stratification of the ambient Grassmannian into disconnected pseudo-Riemannian leaves.

The generalization of the homogeneous space construction becomes

$$\mathrm{Gr}_{(m,n)}(M, N) = \frac{O(M, N)}{O(m, n) \times O(M-m, N-n)}, \tag{21}$$

where $O(M, N)$ denotes the indefinite orthogonal group preserving the bilinear form of signature $(M, N)$. The groups $O(m, n)$ and $O(M-m, N-n)$ preserve the induced metrics on the subspace and its orthogonal complement, respectively. Since $O(M, N)$ is non-compact, the resulting quotient manifold $\mathrm{Gr}_{(m,n)}(M, N)$ is also non-compact. The total dimension of this manifold is given by

$$\begin{aligned} \dim \mathrm{Gr}_{(m,n)}(M, N) &= \dim O(M, N) - \dim O(m, n) - \dim O(M-m, N-n) \\ &= \frac{(M+N)(M+N-1)}{2} - \frac{(m+n)(m+n-1)}{2} \\ &\quad - \frac{(M-m+N-n)(M-m+N-n-1)}{2}. \end{aligned} \tag{22}$$

This expression reduces to the classical case when $N = n = 0$. For instance, let us consider the case of $\mathrm{Gr}_{(1,1)}(3, 1)$. Here, $O(3, 1)$ has dimension 6, while $O(1, 1)$ and $O(2, 0)$ both have dimension 1. Hence,

$$\dim \mathrm{Gr}_{(1,1)}(3, 1) = 6 - 1 - 1 = 4. \tag{23}$$

This example represents all 2D Lorentzian planes in Minkowski spacetime. The presence of a time direction makes these subspaces Lorentzian, and their enumeration gives rise to a 4-dimensional non-compact manifold.

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To define a smooth structure on $\mathrm{Gr}_{(m,n)}(M,N)$, we consider the set of all injective linear maps

$$\mathrm{St}_{(m,n)}(M,N) = \{X \in \mathbb{R}^{(M+N)\times(m+n)} \mid X^\top \eta X = \eta_{(m,n)}\}, \tag{24}$$

where $\eta$ is the ambient pseudo-Riemannian metric and $\eta_{(m,n)}$ is the canonical form of the induced metric on the subspace. The pseudo-Grassmannian is then obtained as the quotient

$$\mathrm{Gr}_{(m,n)}(M,N) = \mathrm{St}_{(m,n)}(M,N)/O(m,n). \tag{25}$$

This construction mirrors the classical realization of the Grassmannian as a quotient of the Stiefel manifold, generalized now to include signature constraints.

The topology of $\mathrm{Gr}_{(m,n)}(M,N)$ is heavily influenced by the action of $O(M,N)$. Since $O(M,N)$ is non-compact, the topology is nontrivial. Moreover, the space may have multiple connected components depending on whether or not $O(M,N)$ acts transitively on the space of subspaces with given signature $(m,n)$. It is known that for certain values of $M,N,m,n$, the space of such subspaces breaks into multiple disjoint orbits.

The smooth structure is inherited from the action of the Lie group $O(M,N)$, and as such, $\mathrm{Gr}_{(m,n)}(M,N)$ is a smooth manifold. Its transition functions between charts are given by smooth transformations induced by the group action. The non-compactness manifests in the fact that no global coordinate chart can cover the entire manifold, and compact open subsets are typically needed.

Physically, such manifolds arise in generalized geometries where the local or global structure of spacetime is allowed to vary. Theories like two-time physics [1], F-theory [4], and certain approaches to quantum cosmology [5] use spaces that are naturally constructed from pseudo-Riemannian subspaces. In particular, the configuration space of a physical system might be modeled not by a fixed Lorentzian manifold but by a family of manifolds varying over the signatures.

In summary, the pseudo-Riemannian Grassmannian $\mathrm{Gr}_{(m,n)}(M,N)$ forms a smooth, non-compact manifold with a rich topological and geometric structure. It generalizes the classical Grassmannian by introducing signature as a parameter, leading to stratified, possibly disconnected manifolds. These spaces serve as a geometric foundation for numerous applications in mathematical physics and beyond.

## 8. Homogeneous Space Representation of Pseudo-Riemannian Grassmannians

In differential geometry, the classical Grassmannian manifold $\mathrm{Gr}(d,D)$, representing the space of all $d$-dimensional subspaces of $\mathbb{R}^D$, is constructed as a homogeneous space under the orthogonal group $O(D)$. Specifically, this manifold can be expressed as a quotient

$$\mathrm{Gr}(d,D) = \frac{O(D)}{O(d)\times O(D-d)}, \tag{26}$$

where $O(d)$ is the orthogonal group preserving the standard Euclidean structure on $\mathbb{R}^d$, and $O(D-d)$ similarly acts on the orthogonal complement. This identification reflects the fact that $O(D)$ acts transitively on the space of $d$-dimensional subspaces, and the stabilizer of any fixed $d$-plane is isomorphic to $O(d)\times O(D-d)$. The resulting Grassmannian is compact, connected, and smooth.

To generalize this construction to pseudo-Riemannian settings, one considers an ambient space $\mathbb{R}^{M,N}$, which is $M+N$-dimensional and carries a non-degenerate symmetric bilinear form of signature $(M,N)$, meaning there are $M$ positive and $N$ negative eigenvalues of the metric tensor $\eta$. The indefinite orthogonal group $O(M,N)$ consists of linear transformations preserving this bilinear form, that is,

$$O(M,N) = \{A \in GL(M+N,\mathbb{R}) \mid A^T \eta A = \eta\}. \tag{27}$$

Within this setting, we define the pseudo-Riemannian Grassmannian $\mathrm{Gr}_{(m,n)}(M,N)$ as the space of all $d=m+n$-dimensional subspaces of $\mathbb{R}^{M,N}$ with induced signature $(m,n)$. Analogously to the

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Riemannian case, this space is homogeneous under the action of $O(M,N)$, and a given subspace of signature $(m,n)$ has a stabilizer isomorphic to

$$O(m,n)\times O(M-m,N-n). \tag{28}$$

Therefore, the pseudo-Grassmannian can be represented as the quotient

$$\mathrm{Gr}_{(m,n)}(M,N)=\frac{O(M,N)}{O(m,n)\times O(M-m,N-n)}. \tag{29}$$

This representation provides a natural smooth manifold structure, inherited from the Lie group quotient. Let us now compute the dimension of this space. The dimension of the indefinite orthogonal group $O(p,q)$ is given by

$$\dim O(p,q)=\frac{(p+q)(p+q-1)}{2}. \tag{30}$$

Applying this formula, the dimension of the pseudo-Grassmannian is

$$\begin{aligned}\dim \mathrm{Gr}_{(m,n)}(M,N)&=\dim O(M,N)-\dim O(m,n)-\dim O(M-m,N-n)\\&=\frac{(M+N)(M+N-1)}{2}-\frac{(m+n)(m+n-1)}{2}\\&\quad-\frac{(M-m+N-n)(M-m+N-n-1)}{2}.\end{aligned} \tag{31}$$

For example, consider $\mathrm{Gr}_{(1,1)}(3,1)$. The group $O(3,1)$ has dimension $\frac{4\cdot 3}{2}=6$, while $O(1,1)$ has dimension 1, and $O(2,0)=O(2)$ also has dimension 1. Hence,

$$\dim \mathrm{Gr}_{(1,1)}(3,1)=6-1-1=4. \tag{32}$$

This example describes all Lorentzian planes in 4D Minkowski space and illustrates the smooth, non-compact geometry of the pseudo-Grassmannian.

This homogeneous structure allows one to construct invariant metrics on $\mathrm{Gr}_{(m,n)}(M,N)$ using the Killing form on the Lie algebra $\mathfrak{o}(M,N)$. Let $\mathfrak{g}=\mathfrak{o}(M,N)$, $\mathfrak{h}=\mathfrak{o}(m,n)\oplus\mathfrak{o}(M-m,N-n)$. Then, the tangent space at the identity coset is given by

$$T_{[e]}\mathrm{Gr}_{(m,n)}(M,N)=\mathfrak{g}/\mathfrak{h}, \tag{33}$$

and invariant pseudo-Riemannian metrics can be derived from the restriction of the Killing form on $\mathfrak{g}$ to this quotient.

Such homogeneous structures are particularly important in physical applications, including representation theory, sigma models, and theories with extra dimensions. In F-theory [4], extra dimensions are introduced with multiple time directions, and compactification schemes often rely on the geometry of such quotient spaces. Two-time physics [1] considers phase spaces with signature $(d,2)$, where orbits in the pseudo-Grassmannian describe allowed constraint manifolds.

In summary, the pseudo-Riemannian Grassmannian $\mathrm{Gr}_{(m,n)}(M,N)$ inherits a well-defined smooth structure as a homogeneous space under $O(M,N)$. Its topology and geometry depend critically on the choice of signature $(m,n)$, and its algebraic and differential properties are encoded in the structure of the stabilizer subgroup. This quotient construction serves as the backbone for the extension of classical Grassmannian geometry to pseudo-Riemannian settings.

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## 9. Dimensionality of Pseudo-Riemannian Grassmannians

In classical differential geometry, the Grassmannian manifold $\mathrm{Gr}(d, D)$, defined as the set of all $d$-dimensional linear subspaces of a $D$-dimensional Euclidean space $\mathbb{R}^D$, is a smooth, compact, and connected manifold of dimension

$$\dim \mathrm{Gr}(d, D) = d(D-d). \tag{34}$$

This expression counts the number of independent parameters required to specify a $d$-plane in $\mathbb{R}^D$, modulo orthogonal transformations. The derivation follows from the identification

$$\mathrm{Gr}(d, D) = \frac{O(D)}{O(d) \times O(D-d)}, \tag{35}$$

and the classical dimension formula for the orthogonal group $O(k)$,

$$\dim O(k) = \frac{k(k-1)}{2}. \tag{36}$$

Applying this gives

$$\dim \mathrm{Gr}(d, D) = \frac{D(D-1)}{2} - \left(\frac{d(d-1)}{2} + \frac{(D-d)(D-d-1)}{2}\right) \tag{37}$$

$$= d(D-d). \tag{38}$$

In the case of pseudo-Riemannian Grassmannians, the situation is richer. Consider an ambient pseudo-Euclidean space $\mathbb{R}^{M,N}$, of total dimension $D = M + N$, endowed with a non-degenerate bilinear form of signature $(M, N)$. Let $\mathrm{Gr}_{(m,n)}(M, N)$ denote the set of $(m+n)$-dimensional subspaces of signature $(m, n)$. The goal is to determine the dimension of this manifold using a homogeneous space representation.

The pseudo-Grassmannian can be written as a quotient:

$$\mathrm{Gr}_{(m,n)}(M, N) = \frac{O(M, N)}{O(m, n) \times O(M-m, N-n)}, \tag{39}$$

where $O(p, q)$ is the indefinite orthogonal group preserving a symmetric bilinear form of signature $(p, q)$. The dimension of $O(p, q)$ is given by the same formula as in the compact case:

$$\dim O(p, q) = \frac{(p+q)(p+q-1)}{2}, \tag{40}$$

since it is the dimension of the Lie algebra of antisymmetric matrices preserving the quadratic form of the given signature.

Therefore, the dimension of $\mathrm{Gr}_{(m,n)}(M, N)$ is computed by subtracting the stabilizer dimensions:

$$\begin{aligned}\dim \mathrm{Gr}_{(m,n)}(M, N) &= \dim O(M, N) - \dim O(m, n) - \dim O(M-m, N-n) \\ &= \frac{(M+N)(M+N-1)}{2} - \frac{(m+n)(m+n-1)}{2} \\ &\quad - \frac{(M-m+N-n)(M-m+N-n-1)}{2}.\end{aligned} \tag{41}$$

Let us illustrate this formula with several concrete examples.

First, consider the classical case embedded in the pseudo-Riemannian setting. Suppose $N = n = 0$, so that the entire structure reduces to the Riemannian case:

$$\dim \mathrm{Gr}_{(m,0)}(M,0) = \frac{M(M-1)}{2} - \frac{m(m-1)}{2} - \frac{(M-m)(M-m-1)}{2} \tag{42}$$

$$= m(M-m), \tag{43}$$

which matches the classical result.

Next, take a proper pseudo-Riemannian case such as $\mathrm{Gr}_{(1,1)}(3,1)$. Here, the total space has signature $(3,1)$, and we are considering 2D subspaces of signature $(1,1)$. Then,

$$\dim O(3,1) = \frac{4 \cdot 3}{2} = 6, \tag{44}$$

$$\dim O(1,1) = \frac{2 \cdot 1}{2} = 1, \tag{45}$$

$$\dim O(2,0) = \frac{2 \cdot 1}{2} = 1, \tag{46}$$

$$\Rightarrow \dim \mathrm{Gr}_{(1,1)}(3,1) = 6 - 1 - 1 = 4. \tag{47}$$

This confirms that the space of Lorentzian planes in Minkowski space is 4-dimensional and non-compact.

Another important case is $\mathrm{Gr}_{(2,2)}(4,4)$, representing all 4D subspaces of signature $(2,2)$ in an 8D pseudo-Euclidean space with balanced signature. We compute:

$$\dim O(4,4) = \frac{8 \cdot 7}{2} = 28, \tag{48}$$

$$\dim O(2,2) = \frac{4 \cdot 3}{2} = 6, \tag{49}$$

$$\dim O(2,2) = 6 \quad \text{again}, \tag{50}$$

$$\Rightarrow \dim \mathrm{Gr}_{(2,2)}(4,4) = 28 - 6 - 6 = 16. \tag{51}$$

## 10. Local Coordinates and Atlas for Pseudo-Riemannian Grassmannians

In the classical theory of Grassmannians, the manifold $\mathrm{Gr}(d,D)$, which parametrizes all $d$-dimensional linear subspaces of $\mathbb{R}^D$, is endowed with a smooth atlas through the use of coordinate patches derived from full-rank matrices. Any $d$-dimensional subspace can be represented by a $D \times d$ matrix of rank $d$, and coordinate charts are constructed by fixing a $d \times d$ identity submatrix and varying the remaining entries freely.

To generalize this construction to pseudo-Riemannian Grassmannians $\mathrm{Gr}_{(m,n)}(M,N)$, we work within the ambient space $\mathbb{R}^{M,N}$ with signature $(M,N)$, i.e., a real vector space of dimension $D = M + N$ equipped with a non-degenerate symmetric bilinear form $\eta$ of signature $(M,N)$. We define $\mathrm{Gr}_{(m,n)}(M,N)$ as the set of all $(m+n)$-dimensional linear subspaces $V \subset \mathbb{R}^{M,N}$. To construct local coordinates, we proceed by considering the pseudo-Stiefel manifold

$$\mathrm{St}_{(m,n)}(M,N) = \{X \in \mathbb{R}^{(M+N)\times(m+n)} \mid X^\top \eta X = \eta_{(m,n)}\}, \tag{52}$$

where $\eta_{(m,n)} = \mathrm{diag}(+1,\ldots,+1,-1,\ldots,-1) \in \mathbb{R}^{(m+n)\times(m+n)}$ is the canonical signature matrix with $m$ positive and $n$ negative entries. The pseudo-Grassmannian is then realized as the quotient

$$\mathrm{Gr}_{(m,n)}(M,N) = \mathrm{St}_{(m,n)}(M,N)/O(m,n). \tag{53}$$

Each point in $\mathrm{Gr}_{(m,n)}(M,N)$ corresponds to a subspace represented by a full-rank matrix $X \in \mathbb{R}^{(M+N)\times(m+n)}$ satisfying the signature condition. Let us choose a reference subspace $V_0$ of signature

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$(m, n)$, for instance spanned by the first $m + n$ basis vectors. Any nearby subspace $V$ is then obtained as the image of $V_0$ under a transformation close to the identity. Thus, a local chart can be built by writing

$$X = \begin{pmatrix} I_{m+n} \\ Z \end{pmatrix}, \quad Z \in \mathbb{R}^{(M+N-m-n)\times(m+n)}, \tag{54}$$

subject to the signature condition

$$X^\top \eta X = \eta_{(m,n)}. \tag{55}$$

This imposes quadratic constraints on the entries of $Z$, and the set of such $Z$'s defines a coordinate patch on $\mathrm{Gr}_{(m,n)}(M, N)$. The number of free parameters is equal to the dimension of the manifold, which, as derived earlier, is

$$\dim \mathrm{Gr}_{(m,n)}(M, N) = \frac{(M+N)(M+N-1)}{2} - \frac{(m+n)(m+n-1)}{2} - \frac{(M-m+N-n)(M-m+N-n-1)}{2}. \tag{56}$$

To give an explicit example, let us again consider $\mathrm{Gr}_{(1,1)}(3, 1)$, the space of 2-dimensional Lorentzian planes in 4D Minkowski space. In this case, $M = 3$, $N = 1$, $m = 1$, and $n = 1$, so that

$$X = \begin{pmatrix} I_2 \\ Z \end{pmatrix}, \quad Z \in \mathbb{R}^{2\times 2}, \tag{57}$$

and

$$X^\top \eta X = I_{1,1} \Rightarrow I_2 + Z^\top \eta' Z = I_{1,1}, \tag{58}$$

where $\eta'$ is the restriction of the metric to the complement of the base. Solving this constraint gives a local description of the manifold in terms of admissible $Z$'s. Since we have 4 variables in $Z$ and 2 constraints from the signature condition, the chart has dimension 2, consistent with earlier calculations.

More generally, the atlas on $\mathrm{Gr}_{(m,n)}(M, N)$ is built by covering all possible block decompositions where the upper $m + n$ rows of $X$ form an invertible matrix. For each such decomposition, a similar coordinate map is defined. The transition functions between charts are smooth and given by the action of the pseudo-orthogonal group, hence the atlas is differentiable.

Such charts are essential in defining differential forms, computing curvature, and formulating physical models on pseudo-Grassmannians. For instance, local expressions for gauge connections and curvature tensors rely on a proper coordinate atlas. Applications of these constructions arise in higher spin theory, twistor theory, and string compactifications where pseudo-Riemannian structures are prominent [1,4,5]

## 11. Plücker Embedding and Projective Geometry for Pseudo-Riemannian Grassmannians

In classical geometry, the Grassmannian $\mathrm{Gr}(d, D)$ admits a natural embedding into projective space via the Plücker embedding. Given a $d$-dimensional subspace $V \subset \mathbb{R}^D$, one can choose a basis $\{v_1, \ldots, v_d\}$ and consider their wedge product in the $d$-th exterior power $\bigwedge^d \mathbb{R}^D$. The Plücker embedding is then defined as

$$\mathrm{Gr}(d, D) \hookrightarrow \mathbb{P}\left(\bigwedge^d \mathbb{R}^D\right), \quad V \mapsto [v_1 \wedge \cdots \wedge v_d]. \tag{59}$$

The image lies on a projective algebraic variety called the Plücker quadric, characterized by quadratic relations among the Plücker coordinates, which reflect the decomposability condition of wedge products.

In the context of pseudo-Riemannian Grassmannians $\mathrm{Gr}_{(m,n)}(M,N)$, the situation is significantly more subtle due to the signature of the underlying bilinear form. We wish to construct a signature-sensitive analog of the Plücker embedding, which respects the indefinite metric structure and preserves the notion of signature for subspaces.

Let $V \subset \mathbb{R}^{M,N}$ be a subspace of dimension $d = m+n$ such that the induced metric on $V$ has signature $(m,n)$. We again consider a basis $\{v_1,\ldots,v_d\}$, and define

$$\Psi(V) := v_1 \wedge \cdots \wedge v_d \in \bigwedge^d \mathbb{R}^{M,N}. \tag{60}$$

The space $\bigwedge^d \mathbb{R}^{M,N}$ inherits a pseudo-inner product from $\mathbb{R}^{M,N}$, denoted $\langle\cdot,\cdot\rangle_\eta$, constructed by anti-symmetrizing the metric tensor over $d$ indices. Explicitly, for $u = u_1 \wedge \cdots \wedge u_d$ and $v = v_1 \wedge \cdots \wedge v_d$, we define

$$\langle u,v\rangle_\eta = \det\big(\eta(u_i,v_j)\big)_{i,j=1}^d. \tag{61}$$

This inner product induces a pseudo-metric structure on the projective space $\mathbb{P}\left(\bigwedge^d \mathbb{R}^{M,N}\right)$. The image of $\mathrm{Gr}_{(m,n)}(M,N)$ under the Plücker map lies in a real projective pseudo-quadric defined by

$$\langle\Psi(V),\Psi(V)\rangle_\eta = \det\big(\eta(v_i,v_j)\big) = \det\big(\eta_{(m,n)}\big) = (-1)^n. \tag{62}$$

Therefore, the pseudo-Plücker image of a subspace encodes its signature as a condition on the pseudo-norm of the wedge product. This makes the embedding sensitive to the causal structure of the subspace, distinguishing Lorentzian, Euclidean, and null subspaces accordingly.

The full image of $\mathrm{Gr}_{(m,n)}(M,N)$ lies in the real projective variety defined by the Plücker relations:

$$\Psi(V) \wedge \Psi(V) = 0, \tag{63}$$

along with the signature constraint

$$\langle\Psi(V),\Psi(V)\rangle_\eta = (-1)^n. \tag{64}$$

These conditions define a pseudo-Riemannian analog of the Plücker quadric. The coordinate functions on $\bigwedge^d \mathbb{R}^{M,N}$ can be expressed in terms of signed minors of the $(M+N)\times d$ matrix of coordinates of the basis vectors $v_i$. These serve as generalized Plücker coordinates.

Let us illustrate with an example. Consider $\mathrm{Gr}_{(1,1)}(3,1)$, the Grassmannian of 2D Lorentzian subspaces in $\mathbb{R}^{3,1}$. A basis matrix $X \in \mathbb{R}^{4\times 2}$ represents such a subspace. The Plücker coordinates are the six 2-minors $p_{ij} = x_i \wedge x_j$ for $1 \le i < j \le 4$. These satisfy the quadratic relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0, \tag{65}$$

which defines the classical Plücker quadric. The pseudo-Riemannian structure imposes the further constraint

$$\sum_{i<j} \varepsilon_{ij} p_{ij}^2 = \pm 1, \tag{66}$$

where $\varepsilon_{ij} \in \{-1,0,+1\}$ encodes the signature of the bilinear form over pairs of indices. This constraint restricts the image of $\mathrm{Gr}_{(1,1)}(3,1)$ to a real slice of the Plücker quadric.

Such embeddings are foundational in twistor theory, where solutions of field equations in space-time are encoded in algebraic data on Grassmannians [6]. Moreover, generalizations to flag varieties and supersymmetric Grassmannians involve higher exterior powers and Plücker-type constraints, and remain active areas of research in string theory and quantum gravity [1,4,5].

## 12. Metric and Volume Form on Pseudo-Riemannian Grassmannians

In the classical setting, the Grassmannian manifold $\mathrm{Gr}(d, D)$ of $d$-dimensional subspaces of $\mathbb{R}^D$ inherits a natural Riemannian metric from the action of the orthogonal group $O(D)$. This is achieved by considering $\mathrm{Gr}(d, D)$ as the homogeneous space

$$\mathrm{Gr}(d, D) = \frac{O(D)}{O(d) \times O(D - d)}, \tag{67}$$

where the group $O(D)$ is endowed with a bi-invariant metric induced from the Killing form on its Lie algebra. The resulting metric on the Grassmannian is positive definite and invariant under the action of $O(D)$.

In the pseudo-Riemannian case, we consider the manifold $\mathrm{Gr}_{(m,n)}(M, N)$, which consists of all $(m + n)$-dimensional subspaces of $\mathbb{R}^{M,N}$ whose induced metric has signature $(m, n)$. The total dimension of the ambient space is $D = M + N$, and the relevant orthogonal group is $O(M, N)$, which preserves the indefinite inner product

$$\eta = \mathrm{diag}(+1, \ldots, +1, -1, \ldots, -1) \in \mathbb{R}^{(M+N)\times(M+N)}, \tag{68}$$

with $M$ positive and $N$ negative entries. The pseudo-Grassmannian can then be written as the homogeneous space

$$\mathrm{Gr}_{(m,n)}(M, N) = \frac{O(M, N)}{O(m, n) \times O(M - m, N - n)}. \tag{69}$$

To construct a metric on $\mathrm{Gr}_{(m,n)}(M, N)$, we proceed by studying the geometry of this quotient. Let $\mathfrak{o}(p, q)$ denote the Lie algebra of $O(p, q)$, which consists of real antisymmetric matrices $A \in \mathbb{R}^{(p+q)\times(p+q)}$ satisfying

$$A^\top \eta + \eta A = 0. \tag{70}$$

The Lie algebra $\mathfrak{o}(M, N)$ decomposes into the direct sum

$$\mathfrak{o}(M, N) = \mathfrak{h} \oplus \mathfrak{m}, \tag{71}$$

where $\mathfrak{h} = \mathfrak{o}(m, n) \oplus \mathfrak{o}(M - m, N - n)$ corresponds to the stabilizer subgroup, and $\mathfrak{m}$ is its complement. The tangent space at the identity coset in the Grassmannian is identified with $\mathfrak{m}$, and we can define an $O(m, n) \times O(M - m, N - n)$-invariant metric on $\mathrm{Gr}_{(m,n)}(M, N)$ via an inner product on $\mathfrak{m}$.

Let us define the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ by

$$\langle X, Y \rangle = \mathrm{tr}(\eta X^\top \eta Y), \tag{72}$$

for $X, Y \in \mathfrak{m}$. This bilinear form is indefinite, depending on the number of positive and negative eigenvalues of $\eta$, and provides the required pseudo-Riemannian metric on the Grassmannian.

For instance, in the special case $\mathrm{Gr}_{(1,1)}(3, 1)$, the metric at a point $V$ can be explicitly computed in terms of local coordinates $Z \in \mathbb{R}^{2\times 2}$ as described in the coordinate atlas section. The signature of the metric depends on how the variation of $Z$ affects the induced bilinear form on the subspace. This gives rise to a 4-dimensional manifold equipped with an indefinite metric of mixed signature.

Next, we define the volume form on $\mathrm{Gr}_{(m,n)}(M, N)$. In the Riemannian case, the volume form is defined using the determinant of the metric tensor. In the pseudo-Riemannian case, the volume form is defined similarly, but care must be taken to handle the orientation and the signature. If $\{e_i\}$ is a local frame and $g_{ij} = \langle e_i, e_j \rangle$, then the volume form is given by

$$\mathrm{vol}_{(m,n)} = \sqrt{|\det(g_{ij})|}\, dx^1 \wedge \cdots \wedge dx^d, \tag{73}$$

where $d = \dim \mathrm{Gr}_{(m,n)}(M, N)$ and $\{x^i\}$ are local coordinates.

It is important to note that the volume form may be indefinite, and it may vanish at certain points if the determinant changes sign. Hence, the global volume form exists only on orientable components of the Grassmannian. When $m = n$, the signature of the metric may be neutral, and special care must be taken in integrating scalar fields over the manifold.

Applications of such metrics and volume forms appear in physics, notably in the formulation of sigma models where fields take values in pseudo-Grassmannians. The action functionals involve integration of kinetic terms weighted by the metric, and topological terms involving the volume form. These structures are also central to string theory compactifications on non-Riemannian geometries [4], to two-time physics [1], and to models of early universe quantum cosmology [5].

## 13. Tangent Space and Differential Geometry of Pseudo-Riemannian Grassmannians

The tangent space of a classical Grassmannian manifold $\mathrm{Gr}(d, D)$, defined as the set of all $d$-dimensional subspaces of a $D$-dimensional Euclidean vector space $\mathbb{R}^D$, can be understood through the identification

$$T_V \mathrm{Gr}(d, D) \cong \mathrm{Hom}(V, V^{\perp}), \tag{74}$$

where $V$ is a $d$-dimensional subspace and $V^{\perp}$ its orthogonal complement. This correspondence allows each tangent vector at a point $V \in \mathrm{Gr}(d, D)$ to be viewed as a linear map from $V$ to its orthogonal complement, encoding infinitesimal deformations of the subspace within the ambient space.

In the pseudo-Riemannian setting, the ambient space $\mathbb{R}^{M,N}$ is equipped with an indefinite symmetric bilinear form $\eta$ of signature $(M, N)$. The pseudo-Grassmannian $\mathrm{Gr}_{(m,n)}(M, N)$ consists of all $(m + n)$-dimensional subspaces $V \subset \mathbb{R}^{M,N}$ such that the restriction of $\eta$ to $V$ has signature $(m, n)$. The corresponding tangent space inherits a structure that reflects the indefinite geometry.

Let $V \in \mathrm{Gr}_{(m,n)}(M, N)$, and let $V^{\perp} \subset \mathbb{R}^{M,N}$ denote the orthogonal complement with respect to $\eta$. Then, analogously to the Riemannian case, we define

$$T_V \mathrm{Gr}_{(m,n)}(M, N) \cong \mathrm{Hom}_{\eta}(V, V^{\perp}), \tag{75}$$

where $\mathrm{Hom}_{\eta}$ denotes the set of linear maps preserving the pseudo-orthogonality condition. More precisely, a tangent vector at $V$ corresponds to an infinitesimal deformation that preserves the signature of the subspace under the flow.

The inner product on the tangent space $T_V \mathrm{Gr}_{(m,n)}(M, N)$ is inherited from the ambient bilinear form. Let $\{e_i\}$ be an orthonormal basis of $V$, and $\{f_\alpha\}$ be a basis of $V^{\perp}$, chosen such that

$$\eta(e_i, e_j) = \varepsilon_i \delta_{ij}, \quad \eta(f_\alpha, f_\beta) = \kappa_\alpha \delta_{\alpha\beta}, \tag{76}$$

with $\varepsilon_i \in \{+1, -1\}$ for $i = 1, \dots, m + n$, and $\kappa_\alpha \in \{+1, -1\}$ for $\alpha = 1, \dots, M + N - m - n$. A tangent vector $A \in T_V \mathrm{Gr}_{(m,n)}(M, N)$ can be expressed as

$$A = \sum_{i,\alpha} A_i^{\alpha} f_\alpha \otimes e^i, \tag{77}$$

and the inner product of two tangent vectors $A, B \in \mathrm{Hom}_{\eta}(V, V^{\perp})$ is given by

$$\langle A, B \rangle = \sum_{i,\alpha} \varepsilon_i \kappa_\alpha A_i^{\alpha} B_i^{\alpha}. \tag{78}$$

This inner product is indefinite and defines a pseudo-Riemannian structure on the tangent bundle $T\mathrm{Gr}_{(m,n)}(M, N)$.

To construct this more geometrically, consider the principal $O(M, N)$-bundle over the Grassmannian. Let $\pi : O(M, N) \to \mathrm{Gr}_{(m,n)}(M, N)$ be the projection map sending a frame to the subspace it spans.

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The vertical space of the bundle corresponds to the stabilizer subgroup $O(m,n) \times O(M-m, N-n)$, while the horizontal space provides a complement used to define connections and curvature.

The Maurer–Cartan form $\theta = g^{-1}dg$ on $O(M,N)$ restricts to the horizontal component on the Grassmannian to define the canonical connection. The curvature form $\Omega = d\theta + \theta \wedge \theta$ defines the Riemann curvature tensor, expressed in terms of the Lie algebra $\mathfrak{o}(M,N)$. The differential geometry of $\mathrm{Gr}_{(m,n)}(M,N)$ thus mirrors that of symmetric spaces but is enriched by the presence of indefinite structure.

For example, in the case $\mathrm{Gr}_{(1,1)}(3,1)$, the tangent space at a subspace $V$ of signature $(1,1)$ is composed of linear maps from $V$ to its 2D orthogonal complement in $\mathbb{R}^{3,1}$. The resulting tangent space is 4-dimensional and carries a signature $(+,+,-,-)$ depending on the specific decomposition. This example highlights the nontrivial geometry of even low-dimensional pseudo-Grassmannians.

These tangent bundles and their associated metrics and curvature forms appear in numerous applications, including higher-spin field theory, gauge theory on non-Riemannian manifolds, and moduli space theory in supergravity and string theory compactifications [1,4,5]. Moreover, the differential forms defined over pseudo-Grassmannians are essential in topological field theories and characteristic class computations.

## 14. Schubert Calculus and Stratification of Pseudo-Riemannian Grassmannians

Classically, Schubert calculus is the study of the intersection theory of subvarieties of Grassmannians, built from incidence conditions with respect to flags of subspaces. A complete flag $F_\bullet$ in $\mathbb{R}^D$ is a sequence of nested subspaces

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_D = \mathbb{R}^D, \quad \dim F_i = i. \tag{79}$$

Given such a flag and a partition $\lambda$, the Schubert cell $\Omega_\lambda(F_\bullet) \subset \mathrm{Gr}(d,D)$ consists of $d$-dimensional subspaces $V$ satisfying prescribed dimensional conditions on the intersections $\dim(V \cap F_k)$. These cells stratify the Grassmannian into smooth affine subvarieties and form the basis for the cohomology ring $H^*(\mathrm{Gr}(d,D))$, which is isomorphic to the ring of symmetric polynomials modulo certain relations.

Our goal is to generalize this stratification procedure to pseudo-Riemannian Grassmannians $\mathrm{Gr}_{(m,n)}(M,N)$, where the ambient space $\mathbb{R}^{M,N}$ has signature $(M,N)$. Unlike the Riemannian case, subspaces are now characterized not just by dimension but also by the induced signature of the restricted bilinear form. Thus, Schubert cells must be indexed not only by dimension data but also by signature data.

Let $F_\bullet$ be a flag in $\mathbb{R}^{M,N}$, consisting of nested subspaces

$$F_0 \subset F_1 \subset \cdots \subset F_{M+N}, \quad \dim F_k = k, \tag{80}$$

where each $F_k$ inherits a signature $(m_k, n_k)$ from the restriction of the ambient bilinear form $\eta$. For a pseudo-Grassmannian point $V \in \mathrm{Gr}_{(m,n)}(M,N)$, define its intersection pattern with the flag by the sequence

$$\gamma_k(V) := \dim(V \cap F_k), \quad 0 \le k \le M+N. \tag{81}$$

Furthermore, define

$$\sigma_k(V) := \text{signature of } \eta|_{V \cap F_k}, \tag{82}$$

as the signature pair $(m'_k, n'_k)$ of the restriction of $\eta$ to the intersection. A stratification of the pseudo-Grassmannian is then defined by fixing the functions $\gamma_k$ and $\sigma_k$, i.e., by specifying the incidence and signature pattern of $V$ with respect to the flag.

The corresponding generalized Schubert cell is

$$\Omega_{\lambda,\sigma}(F_\bullet) = \left\{ V \in \mathrm{Gr}_{(m,n)}(M,N) \;\middle|\; \gamma_k(V) = \lambda_k,\ \sigma_k(V) = \sigma_k\ \forall k \right\}. \tag{83}$$

These cells partition $\mathrm{Gr}_{(m,n)}(M,N)$ into locally closed subsets, each encoding geometric and causal properties of the subspaces. Importantly, the generalized Schubert cells are no longer purely affine varieties but may have more complicated topology due to indefinite structure and non-compactness.

For example, in $\mathrm{Gr}_{(1,1)}(3,1)$, consider a flag where $F_1 \subset F_2 \subset F_3 \subset F_4 = \mathbb{R}^{3,1}$, with each subspace inheriting specific signature constraints. One can define a Schubert cell of 2D Lorentzian planes intersecting $F_2$ in a 1D lightlike line. This cell is open within a stratum of the pseudo-Grassmannian and carries a rich geometric structure depending on the null cone of $\mathbb{R}^{3,1}$.

To describe the homology and cohomology ring structure, one introduces the generalized Schubert classes $[\Omega_{\lambda,\sigma}]$, which span the Borel–Moore homology $H_*^{BM}(\mathrm{Gr}_{(m,n)}(M,N))$. The cup product of these classes reflects geometric intersections of pseudo-subspaces and obeys generalized Pieri and Giambelli formulas. However, the standard Littlewood–Richardson rules may fail due to non-compactness and signature constraints.

Moreover, there exist natural degeneracy loci in the pseudo-Grassmannian where the signature jumps, corresponding to boundaries between cells. These loci may form singular strata with characteristic classes reflecting causal anomalies or topological transitions in physical theories.

The stratification is particularly relevant in applications such as twistor theory [6], representation theory of indefinite orthogonal groups [9], and string dualities involving moduli spaces with indefinite metrics [1,4]. In such settings, Schubert calculus becomes a tool for encoding intersection theory, anomaly inflow, and phase transitions within pseudo-Riemannian moduli spaces.

## 15. Cohomology and Characteristic Classes of Pseudo-Riemannian Grassmannians

The classical cohomology of Grassmannians $\mathrm{Gr}(d,D)$ plays a central role in topology, algebraic geometry, and representation theory. It is generated by the Chern classes of the universal subbundle and quotient bundle, or alternatively, by Schubert classes indexed by partitions. These generate the cohomology ring

$$H^*(\mathrm{Gr}(d,D),\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_d]/\langle P\rangle, \tag{84}$$

where $P$ denotes the ideal of relations corresponding to the embedding in projective space via the Plücker map. The Schubert calculus provides explicit computations of cup products and intersection pairings, using symmetric function theory and the Littlewood–Richardson rule.

In the pseudo-Riemannian case, we consider the pseudo-Grassmannian $\mathrm{Gr}_{(m,n)}(M,N)$ of $(m+n)$-dimensional subspaces of $\mathbb{R}^{M,N}$, with induced signature $(m,n)$. The non-compactness of the ambient space and the presence of an indefinite metric fundamentally change the topological and cohomological structure. Standard tools from algebraic topology must be adapted to handle such manifolds, which may have nontrivial boundaries, singularities, and disconnected components.

First, the cohomology ring $H^*(\mathrm{Gr}_{(m,n)}(M,N),\mathbb{R})$ is generally not finitely generated, and may require use of Borel–Moore homology $H_*^{BM}$ or equivariant cohomology. The latter is constructed by forming the homotopy quotient

$$H_G^*(X) := H^*(EG \times_G X), \tag{85}$$

where $G = O(M,N)$ and $X = \mathrm{Gr}_{(m,n)}(M,N)$. This construction captures the global symmetry structure of the manifold and yields well-defined characteristic classes even in non-compact settings.

We begin by considering the universal pseudo-subbundle $\mathcal{S}_{(m,n)} \subset \mathbb{R}^{M,N} \times \mathrm{Gr}_{(m,n)}(M,N)$. At each point $V \in \mathrm{Gr}_{(m,n)}(M,N)$, the fiber $\mathcal{S}_{(m,n)}|_V = V$ inherits a bilinear form of signature $(m,n)$. The structure group of $\mathcal{S}_{(m,n)}$ is reduced to $O(m,n)$, allowing for the definition of Pontryagin classes $p_k \in H^{4k}(\mathrm{Gr}_{(m,n)}(M,N),\mathbb{R})$. These are defined via the curvature $\Omega$ of a connection $\nabla$ on $\mathcal{S}_{(m,n)}$ by

$$p_k = \frac{1}{(2\pi)^k}\mathrm{tr}(\Omega \wedge \cdots \wedge \Omega), \tag{86}$$

where the trace is taken over the Lie algebra $\mathfrak{o}(m,n)$.

Similarly, the pseudo-quotient bundle $\mathcal{Q}_{(M-m,N-n)}$ admits Pontryagin classes and potentially Euler classes when $M = m$ and $N = n$. Unlike the Riemannian case, Chern classes are not defined unless the bundles admit complex structures, which may occur in physical contexts such as supersymmetric moduli spaces.

For example, in the case $\mathrm{Gr}_{(1,1)}(3,1)$, the universal bundle $\mathcal{S}_{(1,1)}$ has structure group $O(1,1)$, and the curvature 2-form of the connection gives rise to a single nontrivial Pontryagin class $p_1 \in H^4(\mathrm{Gr}_{(1,1)}(3,1), \mathbb{R})$, which can be interpreted as a topological invariant of Lorentzian 2-planes in $\mathbb{R}^{3,1}$.

An important generalization is the use of localization in equivariant cohomology. When $G = O(M,N)$ acts on $\mathrm{Gr}_{(m,n)}(M,N)$, fixed points under maximal tori $T \subset G$ can be used to compute characteristic classes via localization formulas:

$$\int_{\mathrm{Gr}_{(m,n)}} \alpha = \sum_{F \subset \mathrm{Gr}^T_{(m,n)}} \frac{\alpha|_F}{e_T(N_F)}, \tag{87}$$

where $\alpha \in H^*_T(\mathrm{Gr}_{(m,n)})$, $e_T(N_F)$ is the equivariant Euler class of the normal bundle to the fixed locus $F$, and the sum runs over fixed points.

This formalism allows topological invariants such as Euler characteristics, Hirzebruch genera, and indices of Dirac operators to be computed even when the underlying space is non-compact or carries an indefinite signature. These tools are central in quantum field theory, especially in anomalies, index theorems on Lorentzian manifolds, and the study of partition functions in curved backgrounds [1,7,8].

In summary, the cohomology and characteristic classes of pseudo-Grassmannians require the extension of classical topological techniques to indefinite signature and non-compact geometry. They offer a fertile ground for the exploration of generalized characteristic classes, equivariant localization, and dualities in theoretical physics.

## 16. Applications of Pseudo-Riemannian Grassmannians in Physics and Geometry

Pseudo-Riemannian Grassmannians arise naturally in contexts where the geometry of subspaces with indefinite metric structure plays a fundamental role. In physics, such structures appear in theories that extend general relativity to allow multiple time directions, in moduli spaces with signature-changing metrics, and in models such as F-theory and two-time physics. The mathematical richness of $\mathrm{Gr}_{(m,n)}(M,N)$ provides a fertile language for exploring these ideas.

We begin with multitemporal spacetimes. Consider a manifold $\mathcal{M}$ with a Lorentzian signature $(1,3)$, extended to include multiple temporal directions, giving a signature $(n_t, n_s)$ with $n_t > 1$. In such scenarios, local subspaces of tangent space can be modeled as elements of $\mathrm{Gr}_{(m,n)}(M,N)$, where $(m,n)$ reflects the number of timelike and spacelike directions preserved by local geometric structures. For instance, in a 10-dimensional spacetime $\mathbb{R}^{4,6}$, local 4D Lorentzian geometries correspond to $\mathrm{Gr}_{(1,3)}(4,6)$. The choice of such subspaces corresponds to selecting physical "observers" or causal cones, and the dynamics of their variation is encoded in the geometry of the pseudo-Grassmannian.

The metric on $\mathrm{Gr}_{(m,n)}(M,N)$ describes the kinetic terms for fields representing signature-varying frames. For instance, a sigma model with target space $\mathrm{Gr}_{(1,3)}(4,6)$ has action

$$S = \int_\Sigma \mathrm{tr}(\eta dV^\top \eta dV), \tag{88}$$

where $V$ is a matrix-valued field representing the frame, and $\eta$ is the ambient bilinear form. This action captures the variation of local time-space decomposition across a base manifold $\Sigma$, which could represent a worldsheet or spacetime slice.

In moduli theories, pseudo-Grassmannians arise as parameter spaces for families of metrics. Let $g(x)$ be a metric on a manifold $\mathcal{M}$, varying smoothly with coordinates $x$, and suppose the signature of $g(x)$ is not constant. Then the space of all possible tangent space decompositions at each point $x$ into subspaces with signature $(m,n)$ defines a fiber bundle over $\mathcal{M}$ with fiber $\mathrm{Gr}_{(m,n)}(M,N)$. The resulting total space tracks where and how the metric transitions between Lorentzian, Euclidean, and

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other forms. This is essential in quantum cosmology [5], where path integrals involve summation over complex or signature-varying geometries.

In string theory and its extensions, such as F-theory [4], the ambient spacetime includes additional non-physical dimensions required to geometrically realize dualities. F-theory compactifications occur in 12D spaces with signature $(2, 10)$. Local physical configurations often depend on the choice of 4D or 6D subspaces of fixed signature. The collection of such subspaces is organized by pseudo-Grassmannians such as $\mathrm{Gr}_{(2,2)}(4, 6)$, whose topological invariants constrain the spectrum of allowed low-energy effective theories.

Similarly, in two-time physics [1], the ambient space has signature $(2, D-2)$, and the fundamental principle is the gauge equivalence of all embeddings of physical 1-time dynamics within this larger arena. The selection of a particular 1+3 subspace corresponds to a point in $\mathrm{Gr}_{(1,3)}(2, D-2)$, and field theories are required to be invariant under reparametrizations that correspond to motions within this pseudo-Grassmannian. The underlying symmetry group $O(2, D-2)$ plays a central role, and the tangent geometry of the Grassmannian provides the necessary gauge fixing conditions.

From a geometric perspective, pseudo-Grassmannians appear in the classification of bundles with indefinite structure group $O(m, n)$. Just as vector bundles over manifolds are classified by maps into $\mathrm{Gr}(d, D)$, bundles with pseudo-orthogonal structure can be described by maps into $\mathrm{Gr}_{(m,n)}(M, N)$. The existence of certain classes of sections, reductions of structure group, and stability criteria depend on the topology of these target spaces and their characteristic classes [9].

Additionally, twistor theory, which translates conformal geometries into complex geometries, can be extended to indefinite signatures [6]. The spaces of null planes and totally isotropic subspaces in indefinite signature spaces can be modeled using isotropic pseudo-Grassmannians. Their structure allows the construction of ambitwistor spaces and generalized Penrose transforms, essential in scattering amplitude theory and conformal field theory.

In summary, pseudo-Riemannian Grassmannians form the mathematical bedrock for a wide range of geometric and physical applications. They encode signature, causality, and frame structure at a fundamental level and enable the systematic treatment of theories with multiple time directions, varying metric signature, and higher-dimensional symmetry groups.

## 17. Complexification and Holomorphic Structures on Pseudo-Grassmannians

In the classical theory, the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(d, D)$ is the space of $d$-dimensional complex linear subspaces of $\mathbb{C}^D$. It carries the structure of a compact complex manifold and is a homogeneous space under the action of the complex general linear group:

$$\mathrm{Gr}_{\mathbb{C}}(d, D) \cong \frac{GL(D, \mathbb{C})}{P}, \tag{89}$$

where $P \subset GL(D, \mathbb{C})$ is a parabolic subgroup. This space admits a canonical Kähler metric, and its algebraic-geometric properties are central in the study of vector bundles, moduli spaces, and flag varieties.

In the context of pseudo-Riemannian geometry, we seek to generalize the Grassmannian to include indefinite metrics and simultaneously incorporate complex structures. The resulting object is a pseudo-complex Grassmannian, denoted $\mathrm{Gr}^{\eta}_{\mathbb{C}}(m, n; M, N)$, consisting of $(m+n)$-dimensional complex subspaces $V \subset \mathbb{C}^{M+N}$ on which the complexified bilinear form $\eta$ restricts to a nondegenerate form of signature $(m, n)$.

The complexification of a real bilinear form $\eta$ is defined by extension of scalars:

$$\eta_{\mathbb{C}}(v \otimes z, w \otimes z') := z\bar{z}'\eta(v, w), \quad v, w \in \mathbb{R}^{M,N},\ z, z' \in \mathbb{C}. \tag{90}$$

Given this, we define the pseudo-complex Grassmannian as

$$\mathrm{Gr}^{\eta}_{\mathbb{C}}(m, n; M, N) := \left\{ V \subset \mathbb{C}^{M+N} \;\middle|\; \dim_{\mathbb{C}}(V) = m+n,\ \eta_{\mathbb{C}}|_V \text{ has signature } (m, n) \right\}. \tag{91}$$

This space is naturally a complex manifold, though generally non-compact, and can be realized as a homogeneous space:

$$\mathrm{Gr}^{\eta}_{\mathbb{C}}(m,n;M,N) \cong \frac{O_{\mathbb{C}}(M,N)}{O_{\mathbb{C}}(m,n) \times O_{\mathbb{C}}(M-m,N-n)}, \tag{92}$$

where $O_{\mathbb{C}}(M,N)$ denotes the complexified indefinite orthogonal group. The tangent space at a point $V$ is again given by

$$T_V \mathrm{Gr}^{\eta}_{\mathbb{C}}(m,n;M,N) \cong \mathrm{Hom}_{\mathbb{C}}(V,V^{\perp}), \tag{93}$$

where $V^{\perp}$ is defined using $\eta_{\mathbb{C}}$.

A natural holomorphic structure arises from the identification of $\mathrm{Gr}^{\eta}_{\mathbb{C}}$ with a complex flag variety, and holomorphic line bundles over this space generate its Picard group. The curvature of such line bundles gives rise to characteristic classes, computed via Dolbeault cohomology. The first Chern class of the determinant line bundle defines the canonical Kähler form, and in special cases this form satisfies the complex Monge–Ampère equation, defining a Calabi–Yau metric.

Applications of pseudo-complex Grassmannians include conformal field theories and twistor geometry. In twistor theory, complexified null structures are essential. The space of totally isotropic $d$-planes in $\mathbb{C}^{2n}$, where the bilinear form vanishes identically on the subspace, forms an isotropic Grassmannian $\mathrm{IGr}_{\mathbb{C}}(d,2n) \subset \mathrm{Gr}_{\mathbb{C}}(d,2n)$. These arise as target spaces in Penrose's twistor correspondence for conformally flat spacetimes [6].

In string compactifications, the moduli space of complex structures often has the structure of a complex Grassmannian quotient. When metrics vary between Euclidean and Lorentzian signatures in moduli space, pseudo-complex Grassmannians organize the admissible complex subspaces compatible with the ambient signature. This structure plays a role in the classification of supersymmetric vacua, complexified flux compactifications, and the geometric Langlands program.

An explicit example is the complexified version of $\mathrm{Gr}_{(1,1)}(3,1)$, which admits a realization as a domain in $\mathbb{C}^6$ with a Hermitian form of signature $(1,1)$. Holomorphic vector bundles over this space are classified by their Chern characters, and solutions of Hermitian Yang–Mills equations on these bundles correspond to stable configurations in low-energy limits of heterotic string theory.

Therefore, the complexification of pseudo-Riemannian Grassmannians brings together complex algebraic geometry, twistor theory, and moduli of geometric structures with signature constraints. The pseudo-complex Grassmannians serve as natural domains for conformally invariant theories and open a pathway to developing holomorphic models of causal and metric transitions in both mathematics and physics.

## 18. Representation Theory and Symmetry Breaking in Pseudo-Grassmannians

In the classical theory of compact Grassmannians $\mathrm{Gr}(d,D)$, the group $O(D)$ acts transitively, with each Grassmannian realized as a homogeneous space:

$$\mathrm{Gr}(d,D) \cong \frac{O(D)}{O(d) \times O(D-d)}. \tag{94}$$

This action is highly symmetric, and the representation theory of $O(D)$ plays a critical role in harmonic analysis, random matrix theory, and geometric quantization.

In contrast, pseudo-Riemannian Grassmannians $\mathrm{Gr}_{(m,n)}(M,N)$ are naturally acted upon by the indefinite orthogonal group $O(M,N)$, which is non-compact and has a more intricate representation theory. The action is not necessarily transitive. Instead, the space decomposes into distinct orbits, each corresponding to subspaces with fixed signature type under the bilinear form $\eta$ of signature $(M,N)$.

Let $V \subset \mathbb{R}^{M,N}$ be a $d$-dimensional subspace with induced signature $(m,n)$, such that $m+n=d$. The orbit of $V$ under $O(M,N)$ is the set of all subspaces with the same signature:

$$\mathcal{O}_{(m,n)} := \left\{ W \subset \mathbb{R}^{M,N} \;\middle|\; \dim W = m+n,\ \eta|_W \sim \eta|_V \right\}. \tag{95}$$

This orbit is isomorphic to the pseudo-Grassmannian:

$$\mathcal{O}_{(m,n)} \cong \mathrm{Gr}_{(m,n)}(M,N) \cong \frac{O(M,N)}{O(m,n)\times O(M-m,N-n)}. \tag{96}$$

The key difference arises in the structure of the group $O(M,N)$. It is non-compact and not semisimple in the same way as compact groups. Its unitary representations are infinite-dimensional, and its orbits on flag-like varieties can exhibit non-trivial topology, including disconnected components and non-closed orbits.

The classification of $O(M,N)$-orbits on flag varieties and their closures is governed by the theory of real semisimple Lie groups acting on real flag varieties [9,13]. Each orbit corresponds to a unique triple:

$$(m,n;M,N), \quad m+n=d, \quad m\le M, \quad n\le N. \tag{97}$$

The orbits are distinguished by the Jordan decomposition of the bilinear form restricted to $V$, and by invariants such as signature, nullity, and relative position with respect to fixed isotropic subspaces.

Representation-theoretically, the pseudo-Grassmannians admit a realization as quotient spaces for induced representations. Consider a parabolic subgroup $P \subset O(M,N)$ stabilizing a flag with subspace of signature $(m,n)$. Then, principal series representations $\pi_\lambda = \mathrm{Ind}_P^{O(M,N)}\chi_\lambda$ are induced from characters $\chi_\lambda$ of $P$, and their restriction to $\mathrm{Gr}_{(m,n)}$ describes function spaces on the pseudo-Grassmannian.

In this framework, the Plancherel decomposition of $L^2(\mathrm{Gr}_{(m,n)}(M,N))$ consists of integrals over unitary duals of $O(M,N)$, and spectral theory becomes a crucial tool. One can define spherical functions $\phi_\lambda$ satisfying

$$\Delta\phi_\lambda = -\lambda^2\phi_\lambda, \tag{98}$$

where $\Delta$ is the Laplace–Beltrami operator on $\mathrm{Gr}_{(m,n)}$. These functions serve as eigenfunctions in the expansion of $L^2$-sections over the manifold and encode the harmonic content of the space.

Symmetry breaking occurs when a subgroup $H \subset O(M,N)$ acts with fewer preserved structures. For example, in physical theories, selecting a preferred time direction breaks $O(M,N)$ down to $O(M-1,N)$, and pseudo-Grassmannians serve as coset spaces for Goldstone modes arising in spontaneous symmetry breaking. The decomposition

$$\frac{O(M,N)}{O(M-1,N)} \cong \mathrm{Gr}_{(1,0)}(M,N) \tag{99}$$

describes the space of preferred time directions and encodes the dynamics of fields with anisotropic causal structure.

Moreover, in the presence of gauge fields or Higgs-like configurations, the symmetry breaking pattern determines which components of $\mathrm{Gr}_{(m,n)}(M,N)$ are dynamically realized. For instance, the vacuum moduli space in certain supersymmetric models may select specific orbits, depending on the BPS equations and energy minimization conditions.

In summary, pseudo-Grassmannians provide a rich testing ground for the interaction of representation theory and geometry. Their classification by orbits under $O(M,N)$, the induced representations, and the associated symmetry breaking mechanisms link them to fundamental structures in quantum field theory, moduli theory, and general relativity.

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## 19. The Trilok Model: Three Universes from Pseudo-Riemannian Signatures

In this section, we develop the mathematical formulation of the *Trilok* (Three Worlds) model, in which the structure of the universe is stratified into three layers, each corresponding to a different pseudo-Riemannian signature. The model posits three manifolds with differing Lorentzian characteristics:

$$M_{(1,3)} : \text{The Physical Universe (1 time, 3 space)},$$
$$M_{(2,2)} : \text{The Subtle Universe (2 time, 2 space)},$$
$$M_{(3,1)} : \text{The Meta-Physical Universe (3 time, 1 space)}.$$

These manifolds are not merely mathematical curiosities but encode different levels of causality, metric signature behavior, and physical interaction. Each universe can be viewed as a pseudo-Riemannian manifold with global signature $(m,n)$, where the metric tensor $g$ satisfies

$$g = \text{diag}(\underbrace{+1,\ldots,+1}_{m},\underbrace{-1,\ldots,-1}_{n}). \tag{100}$$

### 19.1. Geometric Realization

Let $\mathcal{M} = \mathbb{R}^4$ be the ambient 4-dimensional real vector space equipped with a bilinear form $\eta_{(p,q)}$ of signature $(p,q)$. Each universe corresponds to a choice of signature:

$$\eta_{(1,3)}, \quad \eta_{(2,2)}, \quad \eta_{(3,1)}. \tag{101}$$

Each signature defines a corresponding manifold $M_{(m,n)}$ as a pseudo-Riemannian space. The collection of all such manifolds can be understood as fibers of a fibration over a common base:

$$\pi : \mathcal{T} \to \mathbb{R}^4, \qquad \pi^{-1}(x) = \text{Gr}_{(m,n)}(4), \tag{102}$$

where $\text{Gr}_{(m,n)}(4)$ is the pseudo-Grassmannian of subspaces of $\mathbb{R}^4$ with fixed signature $(m,n)$.

### 19.2. Metric Interrelations

Each manifold $M_{(m,n)}$ admits a Levi-Civita connection $\nabla^{(m,n)}$, and curvature tensors $R^{(m,n)}$ are defined accordingly. The scalar curvature in each case is computed via:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}g^{\rho\sigma}R_{\mu\rho\nu\sigma}, \tag{103}$$

and the signature affects the sign and physical interpretation of curvature invariants.

The transition between manifolds can be described via a homotopy in signature space:

$$\eta(s) = \text{diag}(+1,\ldots,+1,-1,\ldots,-1), \quad s \in [0,1], \tag{104}$$

where varying $s$ modifies the number of time-like directions. This interpolates between different universes.

### 19.3. Physical Interpretations

**1. Physical Universe** $M_{(1,3)}$: Standard spacetime of general relativity. The Einstein equations take the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{105}$$

where causality is well-defined and the lightcone structure supports stable matter configurations.

**2. Subtle Universe** $M_{(2,2)}$: Admits two time directions. Appears in twistor theory and split-signature field theories [6]. The complex structure is compatible with isotropic subspaces:

$$\eta(v,v) = 0 \Rightarrow v \text{ is null}, \quad \text{with } v \in \mathbb{C}^4. \tag{106}$$

It provides a natural home for conformally invariant theories.

**3. Meta-Physical Universe** $M_{(3,1)}$: Dominated by temporal directions, potentially corresponding to higher-order time flows or nested causality [1]. Physical observables may appear as projections:

$$\mathcal{O}_{(1,3)} = \Pi_{(1,3)}\mathcal{O}_{(3,1)}, \tag{107}$$

where $\Pi$ is a projection operator reducing dimensionality and changing signature.

### *19.4. Unified Framework*

We propose organizing the Trilok model as a flag structure:

$$M_{(1,3)} \subset M_{(2,2)} \subset M_{(3,1)}, \tag{108}$$

where each inclusion represents an extension of time-like dimensions. These inclusions can be described geometrically by maps between pseudo-Grassmannians:

$$\phi : \text{Gr}_{(1,3)}(4) \to \text{Gr}_{(2,2)}(4) \to \text{Gr}_{(3,1)}(4), \tag{109}$$

preserving partial structure.

### *19.5. Field Theoretic Implications*

Each manifold supports different types of field theories. For example, scalar fields $\varphi : M_{(m,n)} \to \mathbb{R}$ satisfy the wave equation:

$$\Box_{(m,n)}\varphi = \eta^{\mu\nu}\partial_\mu\partial_\nu\varphi = 0, \tag{110}$$

with the signature of $\eta^{\mu\nu}$ determining hyperbolicity. The field Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi), \tag{111}$$

and causality structures differ significantly between signatures.

### *19.6. Conclusions*

The Trilok model provides a geometrically unified framework for physical, subtle, and meta-physical realities. It invites further exploration of quantum fields, causal structures, and gauge theories in non-standard signatures. Moreover, the use of pseudo-Grassmannians enables a rigorous classification of the allowable submanifolds, field configurations, and symmetry-breaking patterns across layers of physical existence.

## 20. The Subtle Universe $M_{(2,2)}$: Geometry, Physics, and Mathematical Structure

The Subtle Universe, modeled by the pseudo-Riemannian manifold $M_{(2,2)}$, occupies a central role in the Trilok framework, serving as an intermediate layer between the physical universe $M_{(1,3)}$ and the meta-physical universe $M_{(3,1)}$. It is defined by a metric of neutral signature:

$$g = \text{diag}(+1, +1, -1, -1), \tag{112}$$

which gives rise to two time-like and two space-like directions. This balanced signature has profound implications for the geometry and physics on $M_{(2,2)}$.

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### 20.1. Metric Structure and Null Geometry

In $M_{(2,2)}$, null vectors are defined as those $v \in T_pM_{(2,2)}$ satisfying

$$g(v,v) = 0, \tag{113}$$

where $v$ may have components in both time-like and space-like directions. The space of all null directions forms a quadric hypersurface in $\mathbb{R}^4$, defined by:

$$Q = \{x \in \mathbb{R}^4 \mid x_0^2 + x_1^2 = x_2^2 + x_3^2\}. \tag{114}$$

This null cone admits a rich structure due to the symmetry between time and space components.

### 20.2. Symmetry Group and Homogeneous Space Representation

The isometry group of $M_{(2,2)}$ is $O(2,2)$, a non-compact Lie group. The pseudo-Grassmannian of two-dimensional isotropic subspaces (null planes) is given by

$$\mathrm{Gr}_{(1,1)}(2,2) \cong \frac{O(2,2)}{O(1,1) \times O(1,1)}. \tag{115}$$

The decomposition of $O(2,2)$ includes isomorphisms to products of $SL(2,\mathbb{R})$ groups:

$$O(2,2) \cong (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2, \tag{116}$$

which plays a significant role in conformal field theory and integrable systems.

### 20.3. Twistor Theory and Conformal Geometry

Roger Penrose proposed the use of neutral signature spaces in the context of twistor theory [6]. In $M_{(2,2)}$, the conformal compactification preserves the null cone structure, and massless fields can be represented as cohomology classes on twistor space. The null separation condition

$$(x-y)^2 = 0 \tag{117}$$

for two points $x, y \in M_{(2,2)}$ defines a lightlike relation that can be embedded into a complex projective space $\mathbb{CP}^3$.

### 20.4. Field Theory and Dynamics

The d'Alembert operator in neutral signature becomes

$$\Box_{(2,2)} = \partial_0^2 + \partial_1^2 - \partial_2^2 - \partial_3^2, \tag{118}$$

leading to equations of motion for scalar fields:

$$\Box_{(2,2)}\phi = 0. \tag{119}$$

Solutions to this equation exhibit richer symmetry and singularity structures than those in Lorentzian spacetime. In particular, the Green's functions have different causal support due to the presence of multiple time directions.

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### 20.5. Moduli of Structures and Signature Change

The Subtle Universe can serve as a moduli space for signature-changing metrics. Consider a one-parameter family of metrics $g_s$ interpolating between:

$$g_0 = \mathrm{diag}(+1, -1, -1, -1), \tag{120}$$
$$g_1 = \mathrm{diag}(+1, +1, -1, -1). \tag{121}$$

The point $s = 0.5$ corresponds to the neutral metric $g_{(2,2)}$. This setting is compatible with dynamical signature change in cosmological and quantum gravity models [17].

### 20.6. Phase Space and Duality Symmetry

In string theory, $(2, 2)$ target space signatures appear in discussions of T-duality and mirror symmetry. For example, a bosonic string propagating in a background with signature $(2, 2)$ admits a worldsheet description with conformal invariance, and target space symmetry

$$X^A \to \tilde{X}^A, \quad g_{AB} \leftrightarrow g^{AB}, \tag{122}$$

characterizes dual frames. These dualities are geometrically natural in a setting where time and space are equally represented.

### 20.7. Conclusions

The Subtle Universe $M_{(2,2)}$ provides a mathematically rich and physically profound intermediate layer between standard relativistic physics and more exotic meta-physical structures. It hosts neutral signature geometries, twistor-theoretic structures, integrable systems, and signature interpolation phenomena. Its role within the Trilok model is not just transitional, but dynamically essential, mediating causal and geometric relationships between the other two layers.

## 21. Tachyons in the Trilok Model: A Signature-Based Exploration

In this section, we analyze the role and interpretation of tachyonic modes in the context of the Trilok model, which includes three pseudo-Riemannian manifolds:

$$M_{(1,3)} : \text{Physical Universe}, \quad M_{(2,2)} : \text{Subtle Universe}, \quad M_{(3,1)} : \text{Meta-Physical Universe}. \tag{123}$$

Tachyons are particles or field excitations associated with imaginary mass, often seen as indicators of instability or exotic propagation. Their interpretation varies substantially across signatures, and within the Trilok framework, each universe allows different dynamics and symmetries for tachyons.

### 21.1. Tachyons in the Physical Universe $M_{(1,3)}$

In Lorentzian spacetime, a tachyon is a particle with negative mass-squared $m^2 < 0$, violating the energy-momentum relation:

$$E^2 = p^2 + m^2. \tag{124}$$

Solutions with $m^2 < 0$ lead to exponential growth in the Klein-Gordon equation:

$$\left(\Box + m^2\right)\phi = 0 \Rightarrow \Box\phi = -|m^2|\phi. \tag{125}$$

Such modes are non-localizable and imply superluminal propagation. In quantum field theory, they signal vacuum instability and necessitate a redefinition of the ground state, such as in spontaneous symmetry breaking.

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*21.2. Tachyons in the Subtle Universe $M_{(2,2)}$*

In the neutral signature $(2,2)$, the notion of a time-like vector is ill-defined globally. The invariant norm becomes:

$$\|v\|^2 = v_0^2 + v_1^2 - v_2^2 - v_3^2, \tag{126}$$

and allows for more exotic solutions where the sign of the kinetic operator can be adjusted by orientation. The d'Alembertian is

$$\Box_{(2,2)} = \partial_0^2 + \partial_1^2 - \partial_2^2 - \partial_3^2. \tag{127}$$

A tachyonic scalar field satisfies:

$$\left(\Box_{(2,2)} + m^2\right)\phi = 0. \tag{128}$$

Here, the tachyon may appear as a well-behaved solution in a specific chart, and its exponential modes do not necessarily imply instability. This corresponds to the well-known fact that $(2,2)$ signatures admit a richer spectrum of harmonic and twistor-theoretic functions [6].

*21.3. Tachyons in the Meta-Physical Universe $M_{(3,1)}$*

In a spacetime with three temporal directions, the standard dispersion relation becomes:

$$E_1^2 + E_2^2 + E_3^2 = p^2 + m^2. \tag{129}$$

A tachyon here can be interpreted as a state where:

$$E^2 < p^2 \Rightarrow m^2 < 0, \tag{130}$$

but now with three-dimensional temporal modulation. The field solution is:

$$\phi(x) \propto \exp\left(i(E_1x^0 + E_2x^1 + E_3x^2 - \vec{p}\cdot\vec{x})\right). \tag{131}$$

The presence of multiple time coordinates allows new oscillatory and growth modes, which may correspond to temporal entanglement or causality reversal. In theories such as two-time physics [1], these modes are part of a larger symmetry group that includes hidden degrees of freedom.

*21.4. Tachyon Condensation and Stability*

The condensation of tachyons is often a signal of vacuum restructuring:

$$V(\phi) = -\frac{1}{2}|m^2|\phi^2 + \frac{\lambda}{4}\phi^4. \tag{132}$$

In $M_{(1,3)}$, this leads to spontaneous symmetry breaking. In $M_{(2,2)}$, the potential may allow extended null directions, leading to moduli of equivalent vacua. In $M_{(3,1)}$, the vacuum manifold may become time-dependent, inducing oscillatory inflation-like behavior in time sectors.

*21.5. Causal Structure and Superluminal Propagation*

In $M_{(1,3)}$, tachyons violate causality as they can propagate outside the light cone. In $M_{(2,2)}$, the light cone is a double cone with degenerate structure, allowing null-tachyon sectors. In $M_{(3,1)}$, causality becomes multi-dimensional, and faster-than-light propagation in one time coordinate does not imply global paradoxes. The causal cone is defined by:

$$C = \{v \in T_pM \mid g(v,v) = 0\}. \tag{133}$$

Its geometry determines admissible tachyon propagation domains.

### 21.6. Conclusions

Tachyons serve as probes of the deeper structure of spacetime and field theory. In the Trilok model, their role is not uniformly pathological but instead encodes the signature-dependent physics of vacuum structure, causality, and hidden symmetries. Their proper inclusion may unify aspects of signature-changing metrics, two-time dynamics, and twistor-inspired field theories.

## 22. Modeling Particle Physics in the Trilok Framework

This section aims to explore how fundamental aspects of particle physics can be modeled within the Trilok framework, which consists of three pseudo-Riemannian manifolds with distinct signatures:

$$M_{(1,3)} : \text{Physical Universe, standard spacetime,} \tag{134}$$

$$M_{(2,2)} : \text{Subtle Universe, neutral signature,} \tag{135}$$

$$M_{(3,1)} : \text{Meta-Physical Universe, time-dominated geometry.} \tag{136}$$

We consider field content, gauge structures, particle symmetries, and mass generation in each of these signatures, drawing upon known physics while allowing extrapolations enabled by additional time-like dimensions.

### 22.1. Standard Model Fields in $M_{(1,3)}$

In the physical universe $M_{(1,3)}$, the Standard Model (SM) is defined on a Lorentzian manifold with metric $g = \text{diag}(+,-,-,-)$. Fermions are sections of a spinor bundle $S \to M$, and gauge bosons are 1-forms valued in the Lie algebra of $SU(3) \times SU(2) \times U(1)$.

The Lagrangian takes the form:

$$\mathcal{L}_{\text{SM}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + |D_\mu \phi|^2 - V(\phi), \tag{137}$$

with the Higgs mechanism giving rise to mass via spontaneous symmetry breaking:

$$V(\phi) = \lambda(|\phi|^2 - v^2)^2. \tag{138}$$

These structures are well-established within the local Lorentz group $SO(1,3)$ and Poincaré invariance.

### 22.2. Subtle Universe $M_{(2,2)}$: Neutral Signature Extensions

In $M_{(2,2)}$, both the Clifford algebra and spinor structures are modified due to the neutral signature:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad g = \text{diag}(+,+,-,-). \tag{139}$$

The spin group $Spin(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ implies the decomposition of spinors into left- and right-handed components of $SL(2,\mathbb{R})$ representations. Fields can be represented as:

$$\psi \in S_L \otimes S_R, \quad \text{with } \dim S_L = \dim S_R = 2. \tag{140}$$

Gauge theories in $(2,2)$ signature allow for pseudo-Hermitian field strengths:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad \text{with } A_\mu \in \mathfrak{g}. \tag{141}$$

Self-duality and anti-self-duality conditions become real equations, making $M_{(2,2)}$ a natural arena for instanton-like solutions and twistor methods [6].

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*22.3. Meta-Physical Universe $M_{(3,1)}$: Extra Temporal Dimensions*

The meta-physical universe $M_{(3,1)}$ possesses three time-like directions, which fundamentally alter particle dynamics. The spin group becomes $Spin(3,1) \cong SL(2,\mathbb{C})$, but the interpretation of conserved currents and causality shifts drastically.

Field equations must adapt to hyperbolic time evolution in multiple directions:

$$\Box_{(3,1)}\phi = \partial_0^2 + \partial_1^2 + \partial_2^2 - \partial_3^2. \tag{142}$$

Gauge field dynamics are governed by:

$$D^\mu F_{\mu\nu} = J_\nu, \quad \text{where } D_\mu = \partial_\mu + A_\mu, \tag{143}$$

but the energy conditions and stability analysis must account for the presence of multiple time axes. In this context, particles can be viewed as extended configurations evolving through a three-dimensional time submanifold.

*22.4. Unification and Symmetry Embedding*

Each of the three universes has a distinct isometry group:

$$SO(1,3) : \text{Physical symmetry}, \tag{144}$$
$$SO(2,2) : \text{Conformal and duality symmetry}, \tag{145}$$
$$SO(3,1) : \text{Meta-physical or hidden symmetry}. \tag{146}$$

A unifying symmetry group could be constructed by embedding all three in a larger pseudo-orthogonal group $SO(4,4)$, with signature decomposition:

$$\mathbb{R}^{(4,4)} \supset M_{(1,3)} \oplus M_{(2,2)} \oplus M_{(3,1)}. \tag{147}$$

This allows a higher-dimensional Kaluza-Klein-like approach, where particles acquire mass and charge from projections and compactifications.

*22.5. Mass, Time, and Causal Moduli*

In $M_{(1,3)}$, mass is tied to proper time evolution. In $M_{(2,2)}$, time is non-unique, allowing multiple geodesic structures. In $M_{(3,1)}$, temporal structure itself becomes a higher-dimensional space, and mass may be reinterpreted as a frequency vector $(E_1, E_2, E_3)$.

*22.6. Conclusions*

The Trilok framework provides a versatile platform for modeling known and speculative features of particle physics. Each manifold supports distinct field theories with adapted notions of mass, causality, spin, and gauge dynamics. Future work may include mapping interactions across these worlds, interpreting standard model parameters as geometric data in higher signature spaces, and exploring quantum coherence and duality via transitions between universes.

## 23. Application of Pseudo-Grassmannians to the Trilok Universes

Let us analyze the role of pseudo-Grassmannians in the context of the three signature-defined universes of the Trilok model:

$$M_{(1,3)} : \text{Physical Universe}, \quad M_{(2,2)} : \text{Subtle Universe}, \quad M_{(3,1)} : \text{Meta-Physical Universe}. \tag{148}$$

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Pseudo-Grassmannians generalize the concept of Grassmannians by classifying subspaces with fixed signature inside a pseudo-Riemannian manifold. For each $M_{(m,n)}$, the relevant object is:

$$\mathrm{Gr}_{(p,q)}(m,n) = \frac{O(m,n)}{O(p,q) \times O(m-p,n-q)}. \tag{149}$$

These classify $(p+q)$-dimensional subspaces of $\mathbb{R}^{m+n}$ with signature $(p,q)$.

*23.1. Pseudo-Grassmannians in $M_{(1,3)}$*

The physical universe $M_{(1,3)}$ is modeled on a Lorentzian 4-manifold. The pseudo-Grassmannian $\mathrm{Gr}_{(1,1)}(1,3)$ classifies 2-dimensional subspaces with one time and one space direction. This space appears in the study of null planes and lightlike 2-surfaces:

$$\dim\left(\mathrm{Gr}_{(1,1)}(1,3)\right) = \dim O(1,3) - \dim O(1,1) - \dim O(0,2) = 6. \tag{150}$$

This is relevant in twistor theory, Penrose diagrams, and the study of null congruences in general relativity [6].

*23.2. Pseudo-Grassmannians in $M_{(2,2)}$*

In the neutral signature universe, the space of isotropic 2-planes $\mathrm{Gr}_{(1,1)}(2,2)$ plays a special role. The dimension is:

$$\dim\left(\mathrm{Gr}_{(1,1)}(2,2)\right) = \dim O(2,2) - \dim O(1,1) - \dim O(1,1) = 6. \tag{151}$$

Due to the symmetry $O(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, the manifold $\mathrm{Gr}_{(1,1)}(2,2)$ is homogeneous under independent left-right spinor actions. This space naturally describes the null geodesics and twistor lines associated with split signature field theories.

Moreover, one can consider totally null planes, defined by:

$$V \subset \mathbb{R}^{2,2} \quad \text{such that } g(v_i, v_j) = 0, \quad \forall v_i, v_j \in V. \tag{152}$$

These form isotropic Grassmannians which are critical in integrable systems and real twistor theory.

*23.3. Pseudo-Grassmannians in $M_{(3,1)}$*

In the meta-physical universe with signature $(3,1)$, pseudo-Grassmannians such as $\mathrm{Gr}_{(2,0)}(3,1)$ classify space-like 2-planes. Their dimension is:

$$\dim\left(\mathrm{Gr}_{(2,0)}(3,1)\right) = \dim O(3,1) - \dim O(2,0) - \dim O(1,1) = 6. \tag{153}$$

These submanifolds are used to describe instantaneous observables in 3-time formulations, possibly capturing internal clock dynamics or parallel flows of temporal evolution.

A dual approach considers the 2-time-like plane $\mathrm{Gr}_{(2,0)}(3,1)$, associated with the internal symmetry group of two-time physics [1]:

$$O(3,1) \supset O(2,0) \times O(1,1), \tag{154}$$

leading to a Grassmannian of time directions intersecting at a single causal observer.

*23.4. Comparative Geometry and Inter-Universe Structure*

We define a fiber bundle of subspaces over each universe:

$$\pi : \mathcal{F} \to M_{(m,n)}, \quad \text{with fiber } \mathrm{Gr}_{(p,q)}(m,n). \tag{155}$$

This allows modeling of local gauge structures, null foliations, and causal patches geometrically.

Furthermore, the moduli space of all sub-Grassmannians over the combined space

$$M_T = M_{(1,3)} \cup M_{(2,2)} \cup M_{(3,1)} \tag{156}$$

can be seen as a stratified manifold of dimension varying with signature and subspace type. This moduli space unifies the internal symmetries of fields propagating across the three domains.

*23.5. Conclusions*

The application of pseudo-Grassmannians provides a unifying language for the geometric and physical substructures of the Trilok universes. Each manifold supports its own family of subspaces with meaningful interpretations in physics, ranging from null planes and light cones to foliation of time flows and causal transitions. The structure of these Grassmannians is instrumental for understanding symmetry breaking, duality, field propagation, and geometric transitions between universes.

## 24. Pseudo-Grassmannians, Symmetry Breaking, and Geometric Transitions in the Trilok Model

In the Trilok model, consisting of three pseudo-Riemannian universes $M_{(1,3)}$, $M_{(2,2)}$, and $M_{(3,1)}$, the geometry of pseudo-Grassmannians plays a central role in capturing the internal structure of spacetime, the propagation of fields, and transitions between causal and non-causal regimes. The pseudo-Grassmannian manifold

$$\mathrm{Gr}_{(p,q)}(m,n) = \frac{O(m,n)}{O(p,q) \times O(m-p,n-q)} \tag{157}$$

classifies all $(p+q)$-dimensional subspaces of a pseudo-Riemannian space $\mathbb{R}^{(m,n)}$ of signature $(m,n)$ that themselves have signature $(p,q)$. This structure is fundamental in the understanding of symmetry breaking, duality, field dynamics, and geometric interpolations between universes.

*24.1. Symmetry Breaking and Homogeneous Space Decomposition*

The group $O(m,n)$ acts transitively on $\mathrm{Gr}_{(p,q)}(m,n)$. Choosing a subspace $V \subset \mathbb{R}^{(m,n)}$ with signature $(p,q)$ induces a symmetry breaking:

$$O(m,n) \longrightarrow O(p,q) \times O(m-p,n-q). \tag{158}$$

This symmetry breaking is analogous to spontaneous symmetry breaking in gauge theory, where a vacuum state or background breaks the full symmetry of the Lagrangian. The coset structure encodes Goldstone modes corresponding to broken directions. For example, in $M_{(1,3)}$, choosing a lightlike 2-plane breaks Lorentz symmetry to $O(1,1) \times O(0,2)$, and the six-dimensional manifold of such planes captures null structures that govern massless field propagation [6].

*24.2. Duality and Grassmannian Correspondences*

The dual Grassmannian $\mathrm{Gr}_{(m-p,n-q)}(m,n)$ corresponds to the orthogonal complement of a subspace, with signature swapped accordingly. This duality

$$\mathrm{Gr}_{(p,q)}(m,n) \leftrightarrow \mathrm{Gr}_{(m-p,n-q)}(m,n) \tag{159}$$

is critical in twistor theory and in string theories with dual worldsheet or target-space structures. In $M_{(2,2)}$, where the spin group decomposes as

$$\mathrm{Spin}(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), \tag{160}$$

this duality corresponds to interchange between left- and right-handed spinor components. Pseudo-Grassmannians serve to classify isotropic and totally null subspaces, essential in constructing integrable models and twistor spaces for massless fields [18].

*24.3. Field Propagation and Null Geometry*

Let $\phi : M_{(m,n)} \to \mathbb{R}$ be a scalar field. The wave equation takes the form:

$$\Box_{(m,n)}\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi = 0, \tag{161}$$

where the causal structure of the operator $\Box$ depends on the subspace over which $\phi$ propagates. Choosing a submanifold from $\mathrm{Gr}_{(p,q)}(m,n)$ allows one to define null cones and dispersion relations on each subspace. The restriction of $\phi$ to these subspaces yields solutions with constrained momenta $k^\mu$ satisfying:

$$g_{\mu\nu}k^\mu k^\nu = 0, \quad \text{in } V_{(p,q)} \subset \mathbb{R}^{(m,n)}. \tag{162}$$

In $M_{(2,2)}$, where there are multiple inequivalent null structures, the space of totally null 2-planes $\mathrm{Gr}_{\text{null}}(2,2)$ defines a six-dimensional manifold that supports self-dual Yang-Mills configurations.

*24.4. Geometric Transitions and Moduli Spaces*

Let us define the bundle

$$\pi : \mathcal{G}_{(p,q)} \to M_{(m,n)}, \quad \text{with fiber } \mathrm{Gr}_{(p,q)}(m,n). \tag{163}$$

Over each point of spacetime $x \in M_{(m,n)}$, this bundle assigns the moduli of $(p,q)$-dimensional subspaces. Transitions between universes such as $M_{(1,3)} \to M_{(2,2)}$ correspond to signature-changing transitions in the base manifold, with induced transformations in the structure group of the bundle:

$$O(1,3) \to O(2,2), \quad \text{and hence } \mathrm{Gr}_{(p,q)}(1,3) \to \mathrm{Gr}_{(p,q)}(2,2). \tag{164}$$

Such transitions may be understood as topological or causal phase changes, possibly described by path integrals summing over signature sectors. The total moduli space over all universes,

$$\mathcal{M}_{\text{Trilok}} = \bigcup_{(m,n)} \mathrm{Gr}_{(p,q)}(m,n), \tag{165}$$

becomes a stratified pseudo-manifold of varying local dimension and curvature, encoding the extended causal geometry of the Trilok model.

*24.5. Applications to Causal Patchwork and Observables*

In quantum gravity and cosmology, causal sets or null hypersurfaces define accessible regions of observation. Pseudo-Grassmannians offer a geometric classification of these regions. For example, a causal patch around an observer in $M_{(3,1)}$ may be associated with a 2-plane in $\mathrm{Gr}_{(2,0)}(3,1)$, defining the set of spatial or temporal directions the observer can traverse. The algebra of such subspaces contributes to holographic bounds, entanglement entropy, and observer-dependent vacuum states.

*24.6. Conclusions*

Pseudo-Grassmannians provide the precise mathematical infrastructure required to analyze and interrelate the geometric, causal, and field-theoretic properties of the Trilok universes. Their role in encoding symmetry breaking, null structure, duality correspondences, and transitions between signature sectors highlights their foundational importance in any generalized framework for multiversal geometry.

## 25. Metric Structures and Causal Geometry Across Trilok Universes

In the Trilok framework, we analyze three distinct pseudo-Riemannian manifolds defined by different metric signatures: the physical universe $M_{(1,3)}$, the subtle universe $M_{(2,2)}$, and the meta-physical universe $M_{(3,1)}$. The metric tensor $g_{\mu\nu}$ in each case defines a bilinear form on the tangent bundle $TM$, and its signature directly influences the causal structure, geodesic completeness, and the admissibility of field configurations.

### 25.1. Lorentzian Geometry in $M_{(1,3)}$

In the physical universe, the metric has signature $(+,-,-,-)$. The standard Minkowski metric in Cartesian coordinates is:

$$g_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1), \tag{166}$$

which preserves the Lorentz group $O(1,3)$. Vectors are categorized as timelike, spacelike, or null based on the sign of the quadratic form:

$$g(v,v) = g_{\mu\nu}v^\mu v^\nu. \tag{167}$$

Timelike vectors satisfy $g(v,v) > 0$, spacelike vectors $g(v,v) < 0$, and null vectors satisfy $g(v,v) = 0$.

This structure ensures a well-defined light cone at each point $x \in M_{(1,3)}$, enabling causality and time orientation. The Einstein field equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{168}$$

retain hyperbolic character, supporting global evolution from spacelike hypersurfaces [22].

### 25.2. Neutral Geometry in $M_{(2,2)}$

The subtle universe has signature $(+,+,-,-)$, giving rise to neutral or split signature metrics:

$$g_{\mu\nu} = \mathrm{diag}(+1,+1,-1,-1). \tag{169}$$

Here, the causal structure deviates sharply from Lorentzian geometry. The null cone becomes disconnected and contains multiple null directions:

$$g(v,v) = 0 \quad \Rightarrow \quad v \text{ is lightlike in a non-time-orientable sense.} \tag{170}$$

Such metrics allow complex structures on spacetime. The Weyl tensor splits naturally into self-dual and anti-self-dual parts:

$$C_{\mu\nu\rho\sigma} = C^+_{\mu\nu\rho\sigma} + C^-_{\mu\nu\rho\sigma}. \tag{171}$$

This decomposition is essential in the twistor approach to field theory [6], particularly in the study of instantons and conformal invariants.

Moreover, the scalar Laplacian becomes:

$$\Box_{(2,2)} = \partial_0^2 + \partial_1^2 - \partial_2^2 - \partial_3^2, \tag{172}$$

which is a non-elliptic, non-hyperbolic operator, allowing richer spectral and analytic behavior.

### 25.3. Multi-Time Geometry in $M_{(3,1)}$

The meta-physical universe possesses signature $(+,+,+,-)$, corresponding to three timelike directions and one spacelike direction:

$$g_{\mu\nu} = \mathrm{diag}(+1,+1,+1,-1). \tag{173}$$

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In this framework, the usual formulation of the initial value problem becomes ill-posed. There are multiple incompatible definitions of causality and evolution. The Hamiltonian constraint in general relativity fails to close unless additional gauge fixing is introduced [1].

Despite these difficulties, two-time physics proposes that observable dynamics are projections from higher-dimensional gauge-fixed trajectories. The metric allows null vectors with multiple timelike components:

$$g(v,v) = 0 \quad \text{with } v^\mu \text{ involving multiple positive components.} \tag{174}$$

These geodesics can describe internal temporal evolution or 'hidden' clock variables [23].

*25.4. Grassmannian Substructures and Induced Metrics*

Each universe supports a rich family of submanifolds classified by pseudo-Grassmannians:

$$\mathrm{Gr}_{(p,q)}(m,n) = \frac{O(m,n)}{O(p,q) \times O(m-p,n-q)}. \tag{175}$$

The induced metric $g|_V$ on a subspace $V \in \mathrm{Gr}_{(p,q)}(m,n)$ preserves the signature $(p,q)$. For example, in $M_{(2,2)}$, one may consider the totally null 2-planes satisfying:

$$g(v_i, v_j) = 0 \quad \forall v_i, v_j \in V, \tag{176}$$

which form the isotropic Grassmannian $\mathrm{Gr}_{\text{null}}(2,2)$. These subspaces underpin twistor lines and null foliations of spacetime.

In $M_{(3,1)}$, 2-dimensional time-like surfaces form:

$$\mathrm{Gr}_{(2,0)}(3,1), \quad \dim = \dim O(3,1) - \dim O(2,0) - \dim O(1,1) = 6. \tag{177}$$

These surfaces may serve as slices of observer-dependent time within a quantum cosmological framework.

*25.5. Transitions and Signature Change*

Geometric transitions between the universes involve signature-changing metrics:

$$g_{\mu\nu}(x) = \begin{cases} \mathrm{diag}(+,-,-,-), & x \in M_{(1,3)} \\ \mathrm{diag}(+,+,-,-), & x \in M_{(2,2)} \\ \mathrm{diag}(+,+,+,-), & x \in M_{(3,1)} \end{cases} \tag{178}$$

Such transitions are governed by continuous deformations of the metric tensor across hypersurfaces where the eigenvalues of $g_{\mu\nu}$ change sign. This framework connects to Hartle-Hawking signature transitions, and theories of induced gravity or topological transitions [17].

*25.6. Conclusions*

Each of the Trilok universes possesses a unique causal and geometric structure arising from its metric signature. The classification of subspaces via pseudo-Grassmannians allows one to rigorously define lightlike, timelike, and spacelike configurations in each domain. The behavior of field equations, curvature tensors, and causal structure varies profoundly across $M_{(1,3)}$, $M_{(2,2)}$, and $M_{(3,1)}$, offering deep insight into the potential unification of physical, subtle, and meta-physical dynamics.

## 26. Stratified Brane Geometry of the Trilok Universes

In this section, we develop a brane-based geometric stratification of the Trilok universes. The formulation introduces three stacked pseudo-Riemannian manifolds—each corresponding to the Physical, Subtle, and Meta-Physical universes—organized along an extended fifth coordinate, denoted

$z$. Each universe is confined between, or unbounded relative to, certain $\mathbb{R}^2$-branes at specified $z$-levels. This construction gives rise to a foliation of a higher-dimensional manifold by signature-varying slices, a concept critical to extending gravitational theory across domains of differing causal structure.

### 26.1. Layered Brane Configuration

We define a five-dimensional space $\mathcal{M} = \mathbb{R}^{1,3} \times \mathbb{R}_z$, where the physical coordinates $x^\mu \in \mathbb{R}^{1,3}$ describe a 4D pseudo-Riemannian manifold, and $z \in \mathbb{R}$ serves as an external stratification parameter. Within this geometry, we introduce multiple hypersurfaces of constant $z$, each supporting a 4D universe of distinct metric signature.

The metric tensor $g_{AB}(x^\mu, z)$ on $\mathcal{M}$ is given by

$$g_{AB}(x^\mu, z) = g_{\mu\nu}^{(p(z),q(z))}(x^\mu) \oplus \delta_{zz}, \tag{179}$$

where $(p(z), q(z))$ defines the number of time-like and space-like directions, respectively, in the slice at level $z$. The 4D slices at various $z$ are characterized as follows:

$$(p(z), q(z)) = \begin{cases} (1,3) & \text{for } z \in [0, \infty) \\ (2,2) & \text{for } z \in (\infty, Z_1) \\ (3,1) & \text{for } z > Z_1 \end{cases}. \tag{180}$$

The physical universe $M_{(1,3)}$ exists between the flat brane at $z = 0$ and the asymptotic brane at $z = \infty$. The Subtle (Angelic) universe $M_{(2,2)}$ spans from $z = \infty$ to an upper brane located at $z = Z_1$. The Meta-Physical universe $M_{(3,1)}$ lies above $z = Z_1$ and is unbounded in the positive $z$-direction.

### 26.2. Induced Metrics and Signature Transitions

Each slice $M_z \subset \mathcal{M}$ inherits a metric from $g_{AB}$, which induces a pseudo-Riemannian metric of varying signature:

$$ds^2 = g_{\mu\nu}^{(p(z),q(z))} dx^\mu dx^\nu + dz^2. \tag{181}$$

At boundary hypersurfaces such as $z = \infty$ and $z = Z_1$, the signature of the induced metric can change. Signature change across such hypersurfaces has been studied extensively in semi-classical gravity and cosmological bounce models [17]. The matching conditions across a signature change must ensure continuity of the induced metric $h_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$, with potential delta-function sources localized on the hypersurface:

$$[h_{\mu\nu}] = 0, \quad [K_{\mu\nu}] \sim T_{\mu\nu}^{(brane)}. \tag{182}$$

### 26.3. Energy-Momentum Tensors on Branes

Each brane can support localized matter fields. For a brane at fixed $z = z_0$, we consider an action:

$$S = \int d^5x \sqrt{-g}(R - 2\Lambda) + \int d^4x \sqrt{-h} \mathcal{L}_{\text{brane}} \delta(z - z_0), \tag{183}$$

where $\mathcal{L}_{\text{brane}}$ is the Lagrangian of brane-confined matter and $h$ is the determinant of the induced metric. The Einstein equations in five dimensions yield:

$$G_{AB} + \Lambda g_{AB} = \kappa^2 T_{AB}^{\text{bulk}} + \kappa^2 T_{AB}^{\text{brane}} \delta(z - z_0). \tag{184}$$

### 26.4. Pseudo-Grassmannians and Local Geometry

The geometry of each universe is further refined by the classification of its tangent subspaces using pseudo-Grassmannians $\text{Gr}_{(r,s)}(p,q)$. These spaces classify all $r+s$-dimensional subspaces with signature $(r,s)$ inside the ambient signature $(p,q)$. For example:

$$\text{Gr}_{(1,1)}(2,2) \subset M_{(2,2)} \quad \text{parametrizes null 2-planes.} \tag{185}$$

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These structures define the possible field directions, geodesic flows, and causal cones on each brane layer. In the case of $M_{(3,1)}$, one may define:

$$\mathrm{Gr}_{(2,0)}(3,1) = \frac{O(3,1)}{O(2,0) \times O(1,1)}, \tag{186}$$

which identifies 2-dimensional purely temporal planes that may underlie meta-physical dynamical flows.

### 26.5. Boundary Interactions and Brane Couplings

Fields propagating through $\mathcal{M}$ must respect boundary conditions at branes. For scalar fields $\phi$, the boundary conditions at $z = \infty$ and $z = Z_1$ may enforce Neumann, Dirichlet, or mixed conditions. In particular, signature-changing branes impose analytic continuation constraints:

$$\partial_z \phi|_{z=Z_1^-} = i\partial_z \phi|_{z=Z_1^+}, \tag{187}$$

if we interpret the transition as a Wick rotation. These dynamics are reminiscent of two-time physics and signature-changing quantum cosmology [1,24].

### 26.6. Conclusions

This brane stratification of the Trilok framework enables a unified treatment of multiple universes with distinct geometric and causal properties. Each layer supports a different physical regime, governed by its metric signature, and transitions are mediated by hypersurfaces that impose matching conditions on the fields and geometries. The use of pseudo-Grassmannians provides a precise language to classify and connect substructures across the layers. Future directions include studying moduli stabilization, tachyonic tunneling between layers, and quantum gravitational effects across the branes.

## 27. Field Theories on Signature-Varying Manifolds

In this section, we develop the formulation of field theories on manifolds whose local metric signature $(p,q)$ varies across the domain. Such manifolds are highly relevant in multi-universe models such as the Trilok framework, and in broader scenarios including signature-changing cosmologies and two-time physics [1,17,25]. The objective is to construct local Lagrangian densities $\mathcal{L}^{(p,q)}$ compatible with the indefinite metric structure, define path integrals, and describe continuity of Noether currents across signature-changing surfaces.

### 27.1. Signature-Dependent Lagrangians

Let $\mathcal{M}$ be a smooth manifold with a metric tensor $g_{\mu\nu}(x)$ whose signature changes from region to region. In each domain $\mathcal{U} \subset \mathcal{M}$, the metric has fixed signature $(p,q)$. For a real scalar field $\phi\colon \mathcal{M} \to \mathbb{R}$, the kinetic term must respect the signature locally:

$$\mathcal{L}^{(p,q)} = \frac{1}{2} g^{\mu\nu}_{(p,q)}\, \partial_\mu \phi\, \partial_\nu \phi - V(\phi), \tag{188}$$

where $V(\phi)$ is a potential energy term. The Euler-Lagrange equations read:

$$\Box_{(p,q)} \phi - V'(\phi) = 0, \tag{189}$$

with

$$\Box_{(p,q)} = \frac{1}{\sqrt{|\det g|}} \partial_\mu \left( \sqrt{|\det g|}\, g^{\mu\nu}_{(p,q)}\, \partial_\nu \right). \tag{190}$$

In domains where $p$ or $q$ changes discontinuously, junction conditions or analytic continuations must be imposed to define the dynamics consistently.

### 27.2. Path Integral Formulation

To quantize such a theory, we consider the generating functional:

$$Z = \int \mathcal{D}\phi \, e^{iS[\phi]}, \tag{191}$$

with

$$S[\phi] = \int_{\mathcal{M}} d^4x \, \mathcal{L}^{(p(x),q(x))}[\phi]. \tag{192}$$

Let $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, with different signatures $(p_1, q_1)$ and $(p_2, q_2)$ in each region, joined at a hypersurface $\Sigma$. The action becomes:

$$S[\phi] = \int_{\mathcal{M}_1} d^4x \, \mathcal{L}^{(p_1,q_1)}[\phi] + \int_{\mathcal{M}_2} d^4x \, \mathcal{L}^{(p_2,q_2)}[\phi], \tag{193}$$

with boundary matching enforced at $\Sigma$:

$$[\phi]|_\Sigma = 0, \quad [\partial_n \phi]|_\Sigma = \kappa \phi|_\Sigma. \tag{194}$$

Such models are used in quantum cosmology and non-trivial topological transitions [24,25].

### 27.3. Noether Currents and Metric Transitions

If $\mathcal{L}^{(p,q)}$ is invariant under a continuous global symmetry $\phi \to \phi + \epsilon f(\phi)$, then the Noether current is given by:

$$j^\mu = \frac{\partial \mathcal{L}^{(p,q)}}{\partial(\partial_\mu \phi)} f(\phi). \tag{195}$$

The conservation law

$$\nabla_\mu j^\mu = 0 \tag{196}$$

holds in regions of fixed signature. At the boundary $\Sigma$ where the signature changes, current continuity implies:

$$n_\mu [j^\mu]_\Sigma = 0, \tag{197}$$

where $n_\mu$ is the normal vector to the hypersurface. In quantum theories, this translates to the continuity of probability flux and unitary evolution under analytic continuation.

### 27.4. Gauge Fields and Fermions

In generalizing to gauge theories, one replaces $\partial_\mu \to D_\mu = \partial_\mu + iA_\mu$. The gauge-invariant Lagrangian is:

$$\mathcal{L}^{(p,q)}_{\text{gauge}} = -\frac{1}{4} g^{\mu\alpha}_{(p,q)} g^{\nu\beta}_{(p,q)} F_{\mu\nu} F_{\alpha\beta}, \tag{198}$$

with standard field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Dirac Lagrangian becomes signature-sensitive due to gamma matrix algebra:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} \left( i\gamma^\mu_{(p,q)} D_\mu - m \right) \psi, \tag{199}$$

where $\{\gamma^\mu_{(p,q)}, \gamma^\nu_{(p,q)}\} = 2g^{\mu\nu}_{(p,q)}$. Care must be taken to ensure Clifford algebras are defined across signature transitions [26].

### 27.5. Conclusions

Field theories on signature-varying manifolds are mathematically rich and physically relevant in the context of layered universes and conformal transitions. The presence of metric transitions imposes boundary conditions and modified conservation laws, which affect both classical and quantum dynamics. Future work may extend these constructions to gravitational backreaction and topological invariants.

## 28. Brane-Induced Gravity and Junction Conditions in Signature-Varying Universes

The structure of brane-world models embedded in higher-dimensional spacetimes allows for rich geometrical and physical behavior. Particularly, the concept of *induced gravity* on a lower-dimensional brane, governed by the Israel junction conditions, is a cornerstone of modern brane cosmology [27,28]. In the context of the Trilok model with layered Universes $M_{(3,1)} \to M_{(2,2)} \to M_{(1,3)}$, we explore the transition of gravitational fields across signature-changing hypersurfaces and the resulting implications for anomaly inflow and stabilization.

### 28.1. Setup and Formalism

Consider a 5-dimensional bulk manifold $\mathcal{B}$ partitioned into two submanifolds $\mathcal{B}_\pm$ separated by a 4-dimensional hypersurface $\Sigma$. The induced metric on $\Sigma$ is denoted by $h_{\mu\nu}$, while the extrinsic curvature is given by:

$$K_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma \nabla_\rho n_\sigma, \tag{200}$$

where $n^\mu$ is the normal vector to the brane. The jump in extrinsic curvature across $\Sigma$ is defined as:

$$\Delta K_{\mu\nu} = K^+_{\mu\nu} - K^-_{\mu\nu}. \tag{201}$$

The Israel junction condition then relates this jump to the energy-momentum tensor $S_{\mu\nu}$ on the brane:

$$\Delta K_{\mu\nu} - h_{\mu\nu}\Delta K = \kappa^2 S_{\mu\nu}, \tag{202}$$

where $\kappa^2 = 8\pi G_5$, and $\Delta K = h^{\mu\nu}\Delta K_{\mu\nu}$.

### 28.2. Signature Transitions and Geometry

In the Trilok model, the bulk metric transitions from signature $(-,+,+,+,+)$ in $M_{(3,1)}$, to $(-,-,+,+,+)$ in $M_{(2,2)}$, and finally to $(-,-,-,+,+)$ in $M_{(1,3)}$. Each region possesses a distinct causal structure and null cone geometry. The induced metric on the brane $\Sigma$ inherits a degenerate signature at the transition surface, necessitating a modified treatment of the curvature tensors.

For a codimension-one brane at location $z = z_0$, the metric in Gaussian normal coordinates is:

$$ds^2 = dz^2 + h_{\mu\nu}(z, x^\rho)dx^\mu dx^\nu, \tag{203}$$

with the extrinsic curvature satisfying:

$$K_{\mu\nu} = \frac{1}{2}\partial_z h_{\mu\nu}. \tag{204}$$

Assuming a $\mathbb{Z}_2$ symmetry across the brane, the jump simplifies as:

$$\Delta K_{\mu\nu} = 2K^+_{\mu\nu} = \partial_z h_{\mu\nu}|_{z_0^+}. \tag{205}$$

### 28.3. Gravitational Leakage and Stability

Gravitational leakage refers to the propagation of graviton modes from one universe to another. Consider the linearized perturbation:

$$g_{\mu\nu}(x,z) = h_{\mu\nu}(x) + \epsilon_{\mu\nu}(x,z), \tag{206}$$

satisfying the 5D Einstein equation:

$$G_{AB} = \kappa^2 T_{AB}. \tag{207}$$

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Using mode decomposition $\epsilon_{\mu\nu}(x,z) = \sum_n \psi_n(z) h_{\mu\nu}^{(n)}(x)$, the graviton localization depends on the behavior of $\psi_n(z)$ and the potential:

$$-\psi_n''(z) + V(z)\psi_n(z) = m_n^2 \psi_n(z), \tag{208}$$

with the effective potential $V(z)$ receiving contributions from signature jumps.

Stabilization of the brane system requires that only the zero mode $m_0 = 0$ is localized. Higher modes must be suppressed to prevent excess gravitational radiation into the bulk. In signature-changing spacetimes, anomalous behaviors may emerge, particularly due to ghost modes in $M_{(2,2)}$. Anomaly inflow mechanisms as in [29,30] may be necessary.

*28.4. Anomalies and Brane Backreaction*

The metric backreaction from brane tension modifies the bulk curvature. Consider a thin wall approximation:

$$T_{AB} = S_{\mu\nu}\delta(z)\delta_A^\mu \delta_B^\nu. \tag{209}$$

Then, Einstein's equation becomes:

$$R_{AB} - \frac{1}{2} g_{AB} R = \kappa^2 S_{\mu\nu}\delta(z)\delta_A^\mu \delta_B^\nu. \tag{210}$$

Quantum anomalies may arise from parity-violating operators when the background signature allows indefinite Wick rotation paths. In particular, anomalies in current conservation on the brane can arise:

$$\nabla_\mu J^\mu = \frac{1}{384\pi^2} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta}. \tag{211}$$

This anomaly can be cancelled via inflow from the bulk, similar to Green-Schwarz anomaly cancellation.

*28.5. Conclusions*

We have developed the formalism of brane-induced gravity with full junction conditions across signature-varying hypersurfaces. The combination of Israel's equations with causal structure transition creates a consistent framework for gravitational localization, anomaly inflow, and brane backreaction in the Trilok model. Further generalizations include adding dilaton or moduli fields and studying holographic duals across signature boundaries.

## 29. Moduli Space of Universe Transitions with Varying Signature

The geometry of signature-changing spacetimes is governed by a rich moduli space structure, whose analysis is crucial for understanding quantum gravity scenarios and multilayered universe models such as the Trilok framework. In this section, we define and explore the moduli space

$$\mathcal{M}_{\text{transition}} = \left\{ g_{\mu\nu}^{(p,q)}(x) \,\middle|\, p + q = 4 \right\}, \tag{212}$$

where each metric $g_{\mu\nu}^{(p,q)}$ has local signature $(p,q)$, and the sum of time-like and space-like directions is fixed at four dimensions. We aim to study its topology, geodesics, curvature, and Morse-theoretic structure, identifying how smooth or singular transitions between metric signatures may be classified and realized dynamically [17,25,31].

*29.1. Metric Patch Structure and Signature Stratification*

Let $\mathcal{S} \subset \mathcal{M}_{\text{transition}}$ denote the stratification by signature type. The possible strata are given by:

$$\mathcal{S}_{(4,0)}, \quad \mathcal{S}_{(3,1)}, \quad \mathcal{S}_{(2,2)}, \quad \mathcal{S}_{(1,3)}, \quad \mathcal{S}_{(0,4)}. \tag{213}$$

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Each stratum is locally modeled by a symmetric matrix space $\mathrm{Sym}(4,\mathbb{R})$ with fixed eigenvalue signs. Transitions between strata involve degenerations where one or more eigenvalues of $g_{\mu\nu}$ approach zero. Let $\lambda_i(x)$ be the eigenvalues of the metric tensor. Then the transition locus $\partial\mathcal{S}_{(p,q)}$ occurs at

$$\lambda_i(x_0) = 0 \quad \text{for some } i. \tag{214}$$

*29.2. Local Moduli Coordinates and Signature Charts*

Let $\mathcal{G} = \mathrm{GL}(4,\mathbb{R})/\mathrm{O}(p,q)$ denote the space of metrics up to local orthogonal transformations preserving the signature. Each point in $\mathcal{M}_{\text{transition}}$ can be written locally as:

$$g^{(p,q)}_{\mu\nu} = e^a_\mu \eta^{(p,q)}_{ab} e^b_\nu, \tag{215}$$

where $e^a_\mu \in \mathrm{GL}(4,\mathbb{R})$ and $\eta^{(p,q)}_{ab}$ is the Minkowski-type diagonal matrix with $p$ negative and $q$ positive entries. The moduli space then inherits a geometry from the coset structure:

$$\dim \mathcal{M}_{(p,q)} = \frac{1}{2}n(n+1) - \frac{1}{2}p(p-1) - \frac{1}{2}q(q-1). \tag{216}$$

For example, $\mathcal{M}_{(2,2)}$ has dimension 6.

*29.3. Geodesics and Connection*

A natural Riemannian metric on $\mathcal{M}_{\text{transition}}$ can be defined using the DeWitt supermetric:

$$\mathcal{G}^{\mu\nu\alpha\beta} = g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - \lambda g^{\mu\nu}g^{\alpha\beta}. \tag{217}$$

Geodesics in moduli space are then critical points of the functional:

$$\gamma(t): \quad S[\gamma] = \int dt\, \mathcal{G}^{\mu\nu\alpha\beta}\frac{dg_{\mu\nu}}{dt}\frac{dg_{\alpha\beta}}{dt}. \tag{218}$$

These geodesics represent smooth deformations of one signature type into another. Not all transitions are geodesically connected — transitions between disconnected strata, such as $(3,1) \leftrightarrow (1,3)$, may require singular metrics.

*29.4. Curvature and Topology of Moduli Space*

The curvature of the moduli space may be computed using standard Riemannian techniques. Let $\nabla$ be the covariant derivative compatible with $\mathcal{G}^{\mu\nu\alpha\beta}$. The Riemann tensor

$$\mathcal{R}^{\rho\sigma\mu\nu}_{\alpha\beta} = \partial_\alpha \Gamma^{\rho\sigma}_{\beta\mu\nu} - \partial_\beta \Gamma^{\rho\sigma}_{\alpha\mu\nu} + \cdots \tag{219}$$

encodes the curvature of signature variation. In particular, the negative curvature directions signal tachyonic instability under signature reversal.

*29.5. Morse Theory on Signature Sectors*

To classify critical points of a functional over $\mathcal{M}_{\text{transition}}$, we invoke Morse theory [32]. Let $\mathcal{F}[g_{\mu\nu}]$ be a Morse function, e.g., the Einstein-Hilbert action:

$$\mathcal{F}[g] = \int \sqrt{|\det g|}\, R[g]\, d^4x. \tag{220}$$

Signature-changing solutions correspond to critical points where:

$$\frac{\delta \mathcal{F}}{\delta g_{\mu\nu}} = 0. \tag{221}$$

The index of the critical point equals the number of negative modes in the second variation:

$$\delta^2 \mathcal{F} = \int d^4x\, h^{\mu\nu} \left( \frac{\delta^2 \mathcal{F}}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} \right) h^{\alpha\beta}. \tag{222}$$

Transitions from one signature stratum to another may be modeled by flow lines of the Morse function.

### 29.6. Conclusions

The moduli space $\mathcal{M}_{\text{transition}}$ provides a geometric arena to study the transitions between distinct spacetime signatures in multi-universe models. The local geometry, global topology, and Morse-theoretic classification shed light on which signature transitions are dynamically allowed and how metric evolution can traverse between strata. This framework can be further refined via stratified Morse theory and Floer homology.

## 30. Supersymmetry Across Lorentzian Signatures

The extension of supersymmetry to spaces with non-standard signatures, such as $M_{(2,2)}$ and $M_{(3,1)}$, involves nontrivial adjustments in the representation theory of spinors and the underlying superalgebra structure. This section explores how the $\mathcal{N} = 1$ supersymmetry algebra adapts to these scenarios and whether Majorana, Weyl, or Majorana-Weyl spinors can exist in various spacetime signatures. We follow the classification in [2,33,34].

### 30.1. Supersymmetry Algebra in Arbitrary Signature

Let $(p, q)$ denote a spacetime with $p$ time and $q$ space dimensions, total dimension $D = p + q$. The general supersymmetry algebra is defined by:

$$\{Q_\alpha, Q_\beta\} = (\Gamma^\mu C)_{\alpha\beta} P_\mu, \tag{223}$$

where $Q_\alpha$ is the supercharge, $\Gamma^\mu$ are gamma matrices satisfying:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag}(\underbrace{-1, \ldots, -1}_{p}, \underbrace{+1, \ldots, +1}_{q}), \tag{224}$$

and $C$ is the charge conjugation matrix satisfying:

$$C^T = -C, \quad (\Gamma^\mu)^T = -C\Gamma^\mu C^{-1}. \tag{225}$$

The possibility of imposing Majorana or Weyl conditions depends on $D$ and the signature $(p, q)$. These constraints determine the reality properties of the spinor representations.

### 30.2. Spinors in $M_{(2,2)}$

In $D = 4$ with signature $(2, 2)$, the spin group is:

$$\text{Spin}(2, 2) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}), \tag{226}$$

and the spinor representation splits as:

$$\psi = (\psi_L, \psi_R), \tag{227}$$

where $\psi_L$ and $\psi_R$ are real and transform under separate $\text{SL}(2, \mathbb{R})$ factors. Consequently, Weyl spinors are real and two-component, and we may define a chiral basis. The chirality operator $\Gamma_5$ satisfies:

$$\Gamma_5^2 = 1, \quad \text{Tr}(\Gamma_5) = 0. \tag{228}$$

Therefore, Majorana-Weyl spinors *do* exist in $(2,2)$ and enable a real chiral $\mathcal{N}=1$ SUSY algebra:

$$\{Q^L_\alpha, Q^L_\beta\} = (\sigma^\mu)_{\alpha\beta} P_\mu. \tag{229}$$

*30.3. Spinors in $M_{(3,1)}$*

In standard Lorentzian signature $(3,1)$, the spin group is:

$$\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C}), \tag{230}$$

and Weyl spinors are complex. Majorana spinors can be defined via a reality condition:

$$\bar{\psi} = \psi^T C, \tag{231}$$

and are four-component real spinors. However, Majorana-Weyl spinors do *not* exist in 4D Lorentzian signature, because chirality and reality are incompatible in this case.

*30.4. Spinors in $M_{(1,3)}$ and Other Signatures*

In the "inverted" signature $(1,3)$, which swaps one time-like and three space-like directions, the situation is similar to $(3,1)$ due to the isomorphism $\mathrm{Spin}(1,3) \cong \mathrm{Spin}(3,1)$. Therefore, the spinor structure and SUSY algebra are essentially unchanged.

In more exotic signatures such as $(5,5)$, $(6,4)$, or $(10,0)$, the existence of Majorana-Weyl spinors becomes signature-dependent. For example, in $D=10$ with signature $(5,5)$, Majorana-Weyl spinors exist, which is crucial for 2T physics [35] and heterotic supergravity.

*30.5. Preservation of Supersymmetry in $M_{(2,2)}$*

We now analyze whether $\mathcal{N}=1$ SUSY can be preserved in the $M_{(2,2)}$ sector of the Trilok model. Consider a Wess-Zumino-type action:

$$\mathcal{L} = \partial^\mu \bar{\phi}\, \partial_\mu \phi + i\bar{\psi}\Gamma^\mu \partial_\mu \psi + F\bar{F}, \tag{232}$$

where $\phi$ is a scalar, $\psi$ a Majorana-Weyl spinor in (2,2), and $F$ an auxiliary field. The supersymmetry transformations:

$$\delta\phi = \bar{\epsilon}\psi, \tag{233}$$
$$\delta\psi = -i\Gamma^\mu \epsilon\, \partial_\mu \phi + \epsilon F, \tag{234}$$
$$\delta F = i\bar{\epsilon}\Gamma^\mu \partial_\mu \psi, \tag{235}$$

close on-shell to the (2,2) Poincaré algebra. Therefore, $\mathcal{N}=1$ SUSY is consistent in $M_{(2,2)}$ and may play a role in mediating dynamics between physical and subtle branes.

*30.6. Conclusions*

We have examined the algebraic and geometric structures of supersymmetry across different spacetime signatures. The existence of Majorana-Weyl spinors in $M_{(2,2)}$ allows for real chiral representations of SUSY, in contrast to $M_{(3,1)}$ where only Majorana spinors exist. These results are essential for embedding consistent field theories in signature-varying or multi-layered universes and further connect with brane supersymmetry breaking and anomaly inflow phenomena.

## 31. Twistor Theory and Penrose Transform in $M_{(2,2)}$

The spacetime $M_{(2,2)}$, with two time and two space dimensions, possesses a neutral signature metric $\eta = \mathrm{diag}(-1,-1,+1,+1)$. Unlike standard Lorentzian $(3,1)$ spacetime, this signature permits real spinor and twistor representations, and it forms a natural geometric background for twistor theory [36–38]. In this section, we develop the basic structure of twistor space $PT$ over $M_{(2,2)}$, and construct the Penrose transform, which maps cohomology classes in twistor space to field solutions on spacetime.

### *31.1. Spinor Decomposition in Neutral Signature*

The spin group of $M_{(2,2)}$ is isomorphic to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Thus, the tangent space at each point of $M_{(2,2)}$ decomposes as:

$$T_x M \cong S_L \otimes S_R, \tag{236}$$

where $S_L$ and $S_R$ are real 2-dimensional spinor spaces. Any real vector $x^\mu \in M_{(2,2)}$ corresponds to a rank-one tensor $x^{AA'} = x^\mu \sigma_\mu^{AA'}$ where $A, A' = 1,2$, and $\sigma_\mu^{AA'}$ are the Infeld–Van der Waerden symbols for the neutral metric.

### *31.2. Twistor Space over $M_{(2,2)}$*

Twistor space $PT$ is defined as the space of projective spinors $Z^\alpha = (\omega^A, \pi_{A'}) \in \mathbb{RP}^3$, where $\pi_{A'} \in S_R^*$ and $\omega^A = ix^{AA'}\pi_{A'}$. The incidence relation:

$$\omega^A = ix^{AA'}\pi_{A'} \tag{237}$$

defines a 2-dimensional projective line $\mathbb{RP}^1 \subset PT$ over each point $x \in M_{(2,2)}$. Thus, the fibration:

$$\mathbb{RP}^1 \hookrightarrow PT \to M_{(2,2)} \tag{238}$$

establishes twistor space as the space of totally null 2-planes in $M_{(2,2)}$. This is in contrast with $M_{(3,1)}$, where twistors must be complexified.

### *31.3. Penrose Transform for Scalar Fields*

Let $H^1(PT, \mathcal{O}(-2))$ denote the first sheaf cohomology group of projective twistor space with coefficients in $\mathcal{O}(-2)$, the line bundle of homogeneity $-2$. The Penrose transform maps:

$$H^1(PT, \mathcal{O}(-2)) \longrightarrow \{\phi(x) \in C^\infty(M_{(2,2)}) \,|\, \Box\phi = 0\}, \tag{239}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembert operator in neutral signature. The map is defined explicitly via contour integral:

$$\phi(x) = \int_{\mathbb{RP}^1_x} \pi_{A'}\pi_{B'} f(Z)\, d\pi^{A'} \wedge d\pi^{B'}, \tag{240}$$

where $f(Z)$ is a representative of the cohomology class, and $\mathbb{RP}^1_x$ is the fiber over $x \in M_{(2,2)}$.

### *31.4. Twistor Correspondence and Light Cone Structure*

Each twistor corresponds to a null 2-plane in $M_{(2,2)}$. The set of points $x$ satisfying the incidence relation with fixed $Z^\alpha$ lies on a totally null surface. Given the neutral signature, these are real and permit a full real twistor correspondence without complexification. The light cone of a point $x$ is foliated by $\mathbb{RP}^1$-families of twistors.

### *31.5. Applications and Generalizations*

The neutral signature permits construction of integrable systems, self-dual gravity models, and conformal field equations via twistor methods [39,40]. For example, the self-dual Yang-Mills equations on $M_{(2,2)}$ can be encoded as holomorphic vector bundles over $PT$. Moreover, $M_{(2,2)}$ plays a special role in split signature holography, where boundary conformal field theories live on $\mathbb{RP}^3$.

### *31.6. Conclusions*

Twistor theory in $M_{(2,2)}$ offers a real-geometric approach to field equations, integrable structures, and Penrose transforms. The real projective structure of $PT$, together with the neutral signature, yields tractable analytic continuation, real spinor algebra, and explicit cohomological constructions of solutions to wave equations. These features make $M_{(2,2)}$ a compelling testbed for extending twistor methods to more exotic or signature-varying spacetimes.

## 32. Categorical and Topos-Theoretic Reformulation of Signature-Varying Manifolds

Signature transitions across spacetime domains demand a rigorous formalization that preserves local structure while capturing global continuity. One promising mathematical strategy is the application of higher category theory and topos-theoretic formulations to model spacetime and field theories. In particular, 2-categories can encode geometric transitions and sheaves of logical or causal structure allow for a systematic framework to manage discontinuities and variable signature metrics [41**?** ].

### *32.1. Signature Manifolds and 2-Categories*

Let $\mathcal{C}$ be a 2-category whose objects are smooth pseudo-Riemannian manifolds $M_{(p,q)}$ with metric signatures $(p,q)$, 1-morphisms are smooth signature-preserving embeddings $f : M \to N$, and 2-morphisms are equivalence classes of homotopic smooth maps or smooth interpolations between embeddings that allow for signature variation:

$$\mathcal{C}(M_{(p,q)}, M_{(r,s)}) \ni \eta : f \Rightarrow g. \tag{241}$$

Let $\Sigma \subset M$ denote a hypersurface along which the metric changes its signature. We define a local transition 2-morphism over $\Sigma$ as:

$$\eta_\Sigma : g^{(p,q)}_{\mu\nu} \Rightarrow g^{(r,s)}_{\mu\nu}, \quad \text{such that } \lim_{x\to\Sigma^\pm} \eta(x) = g^\pm_{\mu\nu}(x). \tag{242}$$

### *32.2. Topos-Theoretic Models of Causal Sheaves*

We consider the Grothendieck topos $\mathcal{E}$ of sheaves over a site $(\mathcal{C}, J)$, where $J$ is a topology encoding causal structure. A sheaf $\mathcal{F} \in \mathrm{Sh}(\mathcal{C}, J)$ assigns to each $M_{(p,q)} \in \mathcal{C}$ a field observable $\mathcal{F}(M_{(p,q)})$, for example, a scalar or tensor field. The gluing condition across morphisms ensures continuity and signature-consistency:

$$\mathcal{F}(M) \cong \varprojlim \mathcal{F}(U_i), \quad U_i \subset M \text{ open}. \tag{243}$$

Let $\mathcal{S}$ be a sheaf of causal structures. Then for any region $U \subset M$, the causal cone $\mathrm{C}^\pm(x) \subset T_xM$ varies continuously with the metric:

$$\mathcal{S}(U) = \{\mathrm{C}^\pm(x) \subset T_xM \,|\, x \in U\}, \tag{244}$$

and may include degenerations or bifurcations in signature-changing spacetimes. These sheaves define the logical environment of propositions about causality and fields.

### *32.3. Logical Topos and Field Theory*

We define the internal logic of $\mathcal{E}$ to be higher-order intuitionistic logic. Observable fields $\phi : M \to \mathbb{R}$ are interpreted as generalized elements of sheaves. A Lagrangian density $\mathcal{L}$ in such a context becomes a natural transformation:

$$\mathcal{L} : \mathcal{F} \to \mathcal{A}, \tag{245}$$

where $\mathcal{A}$ is a sheaf of action densities. The Euler–Lagrange operator then acts internally, respecting transitions:

$$\delta\mathcal{S} = 0 \quad \Rightarrow \quad \frac{\delta\mathcal{L}}{\delta\phi} = 0 \in \Gamma(\mathcal{E}). \tag{246}$$

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*32.4. Modelling Metric Transitions via Colimits*

Given a collection of signature patches $\{M_i\}$ and transition morphisms $f_{ij} : M_i \to M_j$, we construct a colimit object $M = \mathrm{colim}_{\mathcal{C}} M_i$ representing a total space of varying signature. The induced metric $g_{\mu\nu}(x)$ is discontinuous only on measure-zero sets $\Sigma$ where 2-morphisms encode a jump:

$$g_{\mu\nu}(x) = \begin{cases} g_{\mu\nu}^{(p,q)}(x), & x \in M_1, \\ g_{\mu\nu}^{(r,s)}(x), & x \in M_2, \end{cases} \quad \text{with } x \in \Sigma = M_1 \cap M_2. \tag{247}$$

These transitions are therefore governed by the colimit in the category of smooth manifolds with signature, constrained by sheaf coherence conditions.

*32.5. Conclusions*

This categorical and topos-theoretic approach provides a rigorous, flexible framework for modelling signature-varying universes. It generalizes standard differential geometry by treating transitions as morphisms in structured categories, while topos theory supplies logical and field-theoretic semantics over signature-varying bases. Such models are particularly suitable for analyzing quantum gravity scenarios, causal structures, and non-smooth geometries where local signature may be ill-defined in the classical sense.

## 33. Extended Grassmannians with Time-Like Volume Forms

Pseudo-Riemannian geometry allows for subspaces with indefinite signatures. A central geometric object of study in this context is the pseudo-Grassmannian, denoted $\mathrm{Gr}(m, n; M, N)$, parametrizing $(m, n)$-dimensional subspaces of a $(M, N)$-dimensional ambient manifold with signature $(-1)^n(+1)^m$. In this section, we develop an extension of pseudo-Grassmannians by imposing constraints on the determinant of the induced metric on each subspace.

*33.1. Volume Forms and Signature Constraints*

Let $V \subset T_pM$ be a $d$-dimensional subspace of a pseudo-Riemannian manifold $(M, g)$. The induced metric $g|_V$ yields a quadratic form on $V$. Define the volume form $\mathrm{Vol}_V$ via the determinant:

$$\mathrm{Vol}_V = \sqrt{|\det g|_V|}. \tag{248}$$

We now impose the constraint:

$$\det(g|_V) = \epsilon, \quad \epsilon > 0, \tag{249}$$

which ensures that the subspace $V$ has a time-like dominant volume form. In Lorentzian settings, this condition singles out subspaces with more time-like directions, and it becomes a non-trivial geometric constraint when extended over all of $\mathrm{Gr}(m, n; M, N)$.

*33.2. Calibrated Geometry and Time-Like Calibrations*

In Riemannian geometry, calibrated submanifolds minimize volume in their homology class [45]. A $p$-form $\phi$ is a calibration if:

$$\phi|_V \leq \mathrm{Vol}_V \quad \text{and} \quad d\phi = 0. \tag{250}$$

In pseudo-Riemannian settings, we define a time-like calibration $\tilde{\phi}$ by:

$$\tilde{\phi}|_V = \mathrm{Vol}_V, \quad \text{where } \det(g|_V) > 0, \tag{251}$$

and allow $\tilde{\phi}$ to take indefinite values. This leads naturally to the study of maximally time-like subspaces of fixed dimension and signature.

*33.3. Structure of Constrained Grassmannians*

The space of all $d$-planes $V \subset \mathbb{R}^{M+N}$ satisfying $\det(g|_V) = \epsilon > 0$ forms a real subvariety of the full pseudo-Grassmannian. We denote this constrained space:

$$\mathrm{Gr}^+_\epsilon(m,n;M,N) = \{V \in \mathrm{Gr}(m,n;M,N) \mid \det(g|_V) = \epsilon > 0\}. \tag{252}$$

This subspace is stratified by the signature type of $g|_V$. Fixing $\epsilon > 0$ restricts to a smooth manifold within $\mathrm{Gr}(m,n;M,N)$ with boundary, and in favorable cases, it is diffeomorphic to homogeneous spaces of subgroups preserving this constraint.

*33.4. Connection to Special Holonomy: $G_{2(2)}$*

The split real form of $G_2$, denoted $G_{2(2)}$, arises as the automorphism group of the split octonions and admits invariant 3-forms $\varphi$ on 7-dimensional pseudo-Riemannian spaces of signature $(4,3)$. If we take a 7D pseudo-manifold $(M^7, g_{(4,3)})$, a 3-plane $V \in \mathrm{Gr}^+_\epsilon(1,2;4,3)$ is said to be calibrated if:

$$\varphi|_V = \mathrm{Vol}_V. \tag{253}$$

The pseudo-Grassmannian $\mathrm{Gr}(1,2;4,3)$ can then be used to parametrize associative 3-planes with respect to $G_{2(2)}$, and the positivity condition $\det(g|_V) > 0$ ensures the real structure is preserved.

*33.5. Dynamics and Field Coupling*

We can couple these extended Grassmannians to field theories by defining an action on maps $\Phi : \Sigma \to \mathrm{Gr}^+_\epsilon(m,n;M,N)$, for instance:

$$S[\Phi] = \int_\Sigma \sqrt{\det(g|_{\Phi(x)})}\, d^d x, \tag{254}$$

with Euler–Lagrange equations selecting optimal time-like subspaces dynamically. These variational problems are closely related to Dirac–Born–Infeld-type brane actions with signature constraints.

*33.6. Conclusions*

Extended pseudo-Grassmannians with time-like volume constraints offer a refined moduli space for studying Lorentzian subgeometries. Their structure connects naturally with generalized calibrations and special holonomy, particularly in the split real context such as $G_{2(2)}$. These spaces may serve as background configuration spaces for signature-dependent dynamics in gravity, brane theory, and exotic field propagation models.

## 34. Entropy, Thermodynamics, and Signature in Multi-Time Universes

In classical general relativity, black hole entropy is given by the Bekenstein–Hawking formula

$$S = \frac{A}{4G}, \tag{255}$$

where $A$ is the area of the event horizon and $G$ is Newton's gravitational constant. This expression relies heavily on the assumption of a Riemannian (or Lorentzian) spacetime with a well-defined event horizon. However, in spacetimes with multiple time directions such as those described by signature $(p,q)$ with $p > 1$, the concept of entropy becomes more subtle due to the altered causal structure, and thus requires significant generalization.

### 34.1. Redefining Area in Indefinite Signature

The definition of area in pseudo-Riemannian manifolds is no longer straightforward. In general, the hypersurface element on a codimension-2 surface $\partial\Sigma$ embedded in a signature-changing bulk spacetime is defined via:

$$A = \int_{\partial\Sigma} \sqrt{|\det h|}\, d^{D-2}x, \tag{256}$$

where $h$ is the induced metric on $\partial\Sigma$. The modulus ensures real-valued area, even in the presence of an indefinite signature. This form of the area naturally extends into signature-changing backgrounds and multi-time dimensional universes.

### 34.2. Entropy in Universes with More Than One Time Dimension

In a spacetime with signature $(2,2)$, time-like directions form a non-compact hyperbolic surface. Causal boundaries must be reconsidered. The event horizon may not be a null surface, but rather a degenerate surface in the light-cone structure.

The entropy can still be formally defined by:

$$S = \frac{1}{4G} \int_{\partial\Sigma} \sqrt{|\det h_{\mu\nu}|}\, d^{D-2}x, \tag{257}$$

but the properties of $h_{\mu\nu}$ are now determined by its embedding in a manifold with indefinite metric. The spectrum of quantum fields near such boundaries also changes, affecting the count of microscopic degrees of freedom.

### 34.3. Thermodynamic Laws and Signature Transitions

A critical question is whether standard thermodynamic relations such as:

$$dE = TdS + \cdots, \tag{258}$$

remain valid across signature transitions. The first law may be preserved provided energy flux and entropy flux remain well-defined on hypersurfaces that traverse $(1,3) \to (2,2) \to (3,1)$. However, heat flow is not Lorentz invariant and behaves anomalously under multiple time directions.

This motivates defining a signature-dependent temperature:

$$T_{(p,q)} = \left(\frac{\partial S}{\partial E}\right)^{-1}_{(p,q)}, \tag{259}$$

computed with respect to time-like flows parametrized by vector fields $\xi^\mu$ obeying $g_{\mu\nu}\xi^\mu\xi^\nu < 0$.

### 34.4. Holography and Entropy on Pseudo-Riemannian Boundaries

The AdS/CFT correspondence relies on defining a conformal boundary where fields are holographically dual to those in the bulk. In pseudo-Riemannian settings, especially AdS spaces with neutral signature such as $\mathrm{AdS}_4^{(2,2)}$, the conformal boundary is not uniquely time-like or space-like, and hence the dual theory may possess two time directions.

For example, in $\mathrm{AdS}_4^{(2,2)}$, a natural candidate for the dual is a field theory on the boundary $\mathbb{R}^{2,1} \times \mathbb{R}$, where entropy is calculated via the boundary stress tensor. The generalized entropy functional becomes:

$$S_{\text{holo}} = \frac{\text{Ext}}{4G_N} \int_\Gamma \sqrt{|\det h|}, \tag{260}$$

where $\Gamma \subset \partial M$ is the extremal surface anchored on the boundary region.

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### 34.5. Quantum Considerations

Entropy in quantum field theory in curved spacetime also depends on the entanglement structure of modes across causal boundaries. In neutral signature manifolds, the Hilbert space is modified due to the lack of a global time function. Consequently, von Neumann entropy becomes observer-dependent.

One proposal is to define an entropy current:

$$J_S^\mu = s u^\mu, \tag{261}$$

where $s$ is the entropy density and $u^\mu$ is a normalized time-like vector field satisfying $g_{\mu\nu}u^\mu u^\nu = -1$ in the region of interest. In $(2,2)$ signature, two independent time-like directions $u_1^\mu$ and $u_2^\mu$ yield two entropy currents, whose divergence properties may define local production of entropy under non-equilibrium transitions.

### 34.6. Conclusions

Entropy in multi-time universes must be defined via covariant geometric quantities and carefully interpreted thermodynamic relations. Modifying the Bekenstein-Hawking formula with a signature-sensitive area form allows consistency across changing signatures. Extensions to holographic entropy and quantum entanglement measures remain active areas of research.

## 35. Cosmological Cycles via Brane Oscillations

The cyclic universe scenario has been explored through scalar field dynamics, quantum tunneling, and higher-dimensional cosmology. We propose an alternate mechanism for cyclic cosmology using brane tension oscillations in an extended ambient manifold. The transitions between spacetime signatures, such as $M_{(1,3)} \leftrightarrow M_{(2,2)}$, are driven by sinusoidal fluctuations in brane tension across a higher-dimensional embedding parameterized by an additional spatial coordinate $z$. We consider a 5-dimensional manifold $\mathcal{M}_5$, foliated by 4-dimensional slices whose metric signature is dynamically selected by oscillatory behavior in the embedding.

### 35.1. Spacetime Signature as a Function of Position

Let $U(z)$ be the signature of the 4-dimensional universe slice at a location $z \in \mathbb{R}$. We define:

$$U(z) = \begin{cases} M_{(1,3)} & \text{if } \sin(\omega z) > 0, \\ M_{(2,2)} & \text{if } \sin(\omega z) < 0. \end{cases} \tag{262}$$

The transitions between Lorentzian and neutral signature slices occur at points where $\sin(\omega z) = 0$, i.e., at $z = n\pi/\omega$ for $n \in \mathbb{Z}$. These interfaces define hypersurfaces of signature change, which must obey consistency constraints derived from Israel junction conditions and causal structure preservation.

### 35.2. Brane Oscillation and Effective Dynamics

Let $T(z)$ be the brane tension, taken to oscillate periodically:

$$T(z) = T_0 \cos(\omega z), \tag{263}$$

where $T_0 > 0$ is the maximal tension and $\omega$ is the oscillation frequency. The effective action for the bulk-brane system is:

$$S = \int_{\mathcal{M}_5} d^5x \sqrt{-G}\, R^{(5)} + \sum_{z_i} \int_{\Sigma_{z_i}} d^4x \sqrt{|\det h|}\, T(z_i), \tag{264}$$

where $h_{\mu\nu}$ is the induced metric on the brane slice $\Sigma_{z_i}$, and $G$ is the 5D bulk metric determinant.

The change in sign of $T(z)$ as $\cos(\omega z)$ oscillates modifies the effective signature of the induced metric. We define a generalized extrinsic curvature tensor $K_{\mu\nu}(z)$ and use it to study stability conditions near signature transitions:

$$\Delta K_{\mu\nu} - h_{\mu\nu}\Delta K = \kappa^2 S_{\mu\nu}(z), \tag{265}$$

with signature-sensitive stress-energy tensors $S_{\mu\nu}(z)$ determined by the local energy content.

### *35.3. Cyclic Universe as Trajectory in Moduli Space*

The space of allowed metric signatures with $p + q = 4$ forms a discrete moduli space $\mathcal{M}_{\text{sign}} = \{(1,3),(2,2),(3,1)\}$. A cyclic universe follows a path in this moduli space via:

$$(1,3) \to (2,2) \to (3,1) \to (2,2) \to (1,3), \tag{266}$$

parametrized by the phase $\theta = \omega z$. The effective potential in the space of metrics becomes:

$$V_{\text{eff}}(\theta) = \Lambda + \alpha \cos(\theta), \tag{267}$$

with minima at $\theta = 0, 2\pi, \ldots$ corresponding to preferred signatures.

### *35.4. Geometric Stability and Spectral Flow*

Signature transitions are associated with changes in the causal cone structure and the spectral properties of the Dirac operator $D_z$ on each brane. As $z$ varies, the eigenvalues of $D_z$ flow through zero, indicating chiral anomalies or index changes. These can be computed via the Atiyah–Patodi–Singer index theorem extended to signature-changing backgrounds:

$$\text{ind}(D) = \int_{\mathcal{M}_5} \hat{A}(R) - \frac{1}{2}\eta(\partial\mathcal{M}_5), \tag{268}$$

where $\eta$ is the spectral asymmetry at the signature-changing boundary.

### *35.5. Conclusions*

Oscillating brane tension provides a natural mechanism for cosmological cycles with signature-changing transitions. These cycles are governed by sinusoidal dynamics in higher-dimensional moduli space, coupling geometry, topology, and quantum spectral flow. This framework allows embedding classical cosmological evolution within a periodic higher-dimensional structure consistent with pseudo-Riemannian geometry.

## 36. Quantum Gravity and Signature Foams

In standard loop quantum gravity, spacetime geometry is quantized via spin networks evolving in discrete steps into spin foams. These foams serve as histories of quantum geometries where simplicial complexes encode the gravitational field. We aim to extend spin foam models to allow for triangulations with varying signature, thereby accommodating transitions between Lorentzian, Euclidean, and neutral signature spacetimes. This leads to the notion of “signature foam,” a generalized spin foam sum over signature-configuration.

### *36.1. Signature-Dependent Regge Action*

Let $\mathcal{F}$ be a spin foam composed of 4-simplices. The discrete Regge action in pseudo-Riemannian signature $(p,q)$ is written as:

$$S^{(p,q)}_{\text{Regge}} = \sum_{\Delta\in\mathcal{F}} A_\Delta \cdot \theta^{(p,q)}_\Delta, \tag{269}$$

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where $A_\Delta$ is the area of triangle $\Delta$ and $\theta_\Delta^{(p,q)}$ is the deficit angle dependent on the local signature. The partition function becomes:

$$\mathcal{Z} = \sum_{\mathcal{F}} e^{iS_{\text{Regge}}^{(p,q)}[\mathcal{F}]}, \tag{270}$$

which includes both signature-preserving and signature-changing triangulations.

### 36.2. Gluing Conditions and Topological Transitions

The key technical challenge lies in defining appropriate gluing conditions across signature boundaries. Each 3-face shared between two 4-simplices must carry coherent boundary data such that:

$$\lim_{\epsilon \to 0} g_{\mu\nu}^{(p_1,q_1)}(x - \epsilon) = g_{\mu\nu}^{(p_2,q_2)}(x + \epsilon), \tag{271}$$

ensuring continuity of induced geometry and compatible causal structures. This allows constructing signature foams with interfaces $\Sigma$ of co-dimension one where signature transitions occur.

### 36.3. Causality and Decoherence in Mixed Signature Geometries

The structure of the light cone changes drastically between signature domains. In Lorentzian sectors $(1,3)$, hyperbolic evolution is well-posed, while in Euclidean domains $(0,4)$, the notion of time and causal ordering disappears. We must address how quantum evolution is defined in such sectors. The amplitude propagation is described by path sums over signature-changing histories:

$$\langle \Psi_{\text{final}} | \Psi_{\text{initial}} \rangle = \sum_{\text{foams}} e^{iS_{\text{Regge}}^{(p,q)}}, \tag{272}$$

where each foam contributes with a weight dependent on its total signature class.

One possibility is to interpret signature transitions as quantum tunneling events, where decoherence plays a role in selecting definite signature histories. The decoherence functional between branches $\mathcal{F}_1, \mathcal{F}_2$ is defined as:

$$\mathcal{D}(\mathcal{F}_1, \mathcal{F}_2) = \text{Re}\left[\sum_{\mathcal{F}} A[\mathcal{F}_1] A^*[\mathcal{F}_2]\right], \tag{273}$$

where interference is suppressed unless the signature patterns are topologically compatible.

### 36.4. Categorical Viewpoint and Spin Structure

The spin foam configuration space can be enriched into a 2-category where objects are 3-geometries with signature, morphisms are signature-changing simplicial interpolations, and 2-morphisms encode gauge equivalences. This allows constructing functors:

$$\mathcal{F}_{\text{geom}} : \mathbf{SigTriang} \to \mathbf{Hilb}, \tag{274}$$

assigning to each triangulated signature configuration a Hilbert space of boundary states.

### 36.5. Toward a Dynamical Signature Gravity Theory

Combining the spin foam formulation with varying signature leads to a generalization of the Barrett-Crane model where intertwiners carry signature labels. Let the spin labels be denoted by $j \in \mathbb{N}/2$, and each face $f$ is assigned a signature tag $\sigma_f \in \{(1,3), (2,2), (0,4)\}$. The foam amplitude reads:

$$A(\mathcal{F}) = \prod_f A_f(j_f, \sigma_f) \prod_e A_e \prod_v A_v, \tag{275}$$

where $A_v$ must now satisfy new constraints from signature-matching at vertices.

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*36.6. Conclusions*

We have laid the foundations for a spin foam quantum gravity theory that incorporates signature dynamics. The key novelty is the allowance of triangulations with varying local causal structure and metric type. This enriches the quantum gravitational path integral and may lead to new insights into the emergence of time, signature selection, and causal order.

## 37. Conclusions

In this work, we have proposed a rigorous mathematical generalization of classical Riemannian and Grassmannian structures to a pseudo-Riemannian and pseudo-Grassmannian setting, capable of encoding geometries with arbitrary Lorentzian signature $(m, n)$. This formalism was motivated not only by mathematical generality but also by deep physical considerations, particularly the Trilok model, which postulates the coexistence of three interpenetrating universes: the physical universe $\mathcal{M}_{1,3}$, the subtle or angelic universe $\mathcal{M}_{2,2}$, and the metaphysical universe $\mathcal{M}_{3,1}$. Each of these geometries admits distinct causal structures, spinor representations, and curvature profiles, allowing us to encode a variety of physical phenomena within a unified geometric framework.

The pseudo-Grassmannians $\mathrm{Gr}_{(m,n)}(M, N)$ served as the organizing principle behind this formalism. These spaces, constructed as homogeneous coset spaces of the indefinite orthogonal groups, allowed us to develop differential, metric, topological, and algebraic properties in a manner consistent with signature-sensitive geometry. Through these constructions, we extended classical ideas such as the Plücker embedding, Schubert stratifications, cohomology rings, and volume forms to non-compact, indefinite manifolds. Furthermore, we proposed signature-sensitive generalizations of field theories, entropy laws, supersymmetry transformations, and spin-foam formulations for quantum gravity.

The layered brane model, wherein the universes are embedded within signature-changing hypersurfaces bounded by $\mathbb{R}^2$-branes at $z = 0$, $z = \infty$, and intermediate planes, enables us to model cosmological evolution as transitions across signature domains. By implementing brane-induced gravity and junction conditions, one can model quantum anomalies, gravitational leakage, and entropy gradients across these transitions. In turn, this structure invites new formulations of duality, category theory in signature-space, and Morse-theoretic classification of transitions.

As a whole, this framework provides a promising platform to model symmetry breaking, moduli evolution, multiversal geometry, and quantum signatures within a mathematically rigorous pseudo-Riemannian scaffold. We believe that these ideas open novel directions for both theoretical physics and modern geometry, particularly in string theory, twistor theory, and the study of cosmological boundary conditions.

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