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From Chebyshev to Primorials: Establishing the Riemann Hypothesis

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Abstract

The Riemann Hypothesis, one of the most celebrated open problems in mathematics, asserts that all non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = \frac{1}{2}$ and has profound consequences for the distribution of prime numbers. Since Riemann's original 1859 paper, a vast body of work has attempted to settle the question, frequently by examining the asymptotic behavior of arithmetic functions such as Chebyshev's prime-counting function $\theta(x)$. In this work we introduce a new criterion that links the Riemann Hypothesis to the comparative growth of $\theta(x)$ relative to primorial numbers. More precisely, we study the ratio $R(N_k) = \Psi(N_k)/(N_k \log\log N_k)$, where N_k is the k-th primorial and Ψ is the Dedekind function, and show that the Riemann Hypothesis follows from intrinsic monotonicity properties of this ratio. The argument combines Mertens' theorem, the prime number theorem, and an explicit error analysis of the relevant asymptotic expansions to produce a self-contained proof by contradiction. Beyond its implications for the hypothesis itself, the result offers a fresh framework for understanding how the multiplicative structure of primorials governs the analytic behavior of $\zeta(s)$, thereby casting new light on one of mathematics' most enduring mysteries.

Keywords: Riemann Hypothesis; Riemann zeta function; prime numbers; Chebyshev function; primorials; Dedekind Ψ function; Mertens' theorem

MSC: 11M26; 11A25; 11A41; 11N37

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1. Introduction

The Riemann Hypothesis, first proposed by Bernhard Riemann in 1859, asserts that every non-trivial zero of the Riemann zeta function $\zeta(s)$ has real part equal to $\frac{1}{2}$. This conjecture is widely regarded as the foremost unsolved problem in pure mathematics: it constitutes a central part of Hilbert's eighth problem and is one of the seven Clay Mathematics Institute Millennium Prize Problems [1].

The zeta function $\zeta(s)$, defined and analytically continued over the complex plane, has so-called *trivial* zeros at the negative even integers $s = -2, -4, -6, \ldots$ and *non-trivial* zeros in the critical strip $0 < \Re(s) < 1$. Riemann's conjecture is that every non-trivial zero lies on the *critical line* $\Re(s) = \frac{1}{2}$. Far from being a purely theoretical curiosity, the hypothesis has profound implications for the distribution of prime numbers, a subject of fundamental importance in both pure mathematics and computational number theory.

1.1. Overview and Main Result

The proof presented here proceeds entirely by contradiction and rests on two structural lemmas. We summarize the logical architecture before entering the technical details, so that the reader can keep the overall strategy in mind throughout.

The Analytic Engine (Lemma 1: Key Finding)

Lemma 1 provides the *recurring downward step* that the main contradiction will require. For each sufficiently large index n, one can find a later index $n' > n$ such that

$$R(N_{n'}) \; < \; R(N_n),$$

where $R(N_k) = \Psi(N_k)/(N_k \log\log N_k)$ and N_k is the k-th primorial.

The proof of this lemma uses the closed-form identity

$$R(N_k) = \frac{\prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}{\log \theta(p_k)},$$

where θ denotes the Chebyshev function. Under this representation the condition $R(N_{n'}) < R(N_n)$ is equivalent to the logarithmic inequality

$$\frac{\log \theta(p_{n'})}{\log \theta(p_n)} \; > \prod_{p_n < p \leq p_{n'}} \left(1 + \frac{1}{p}\right).$$

Choosing $p_{n'} \approx p_n^{\alpha}$ with $\alpha = \frac{\log p_n}{\log p_n - 1} > 1$, and combining Mertens' theorem, the prime number theorem, and an explicit error analysis of the relevant tail sums, one establishes a strict positive lower bound on the difference of the two sides for all sufficiently large n.

The Logical Backbone (Lemma 2: Main Insight)

Lemma 2 is the engine of the contradiction. Suppose, for the sake of argument, that the Riemann Hypothesis is false. By Proposition 2, this assumption forces the set

$$\left\{ n \in \mathbb{N} \mid R(N_n) < \tfrac{e^{\gamma}}{\zeta(2)} \right\}$$

to be infinite. Lemma 2 shows that if, beyond some threshold index n_0, every occurrence of $R(N_n) < e^{\gamma}/\zeta(2)$ is followed by a later index $n' > n$ with $R(N_{n'}) < R(N_n)$, then one can build an infinite strictly decreasing sequence

$$R(N_{n_1}) \; > \; R(N_{n_2}) \; > \; R(N_{n_3}) \; > \; \cdots$$

bounded below by zero, hence convergent to some limit $L \geq 0$. Since $(R(N_{n_j}))_{j \geq 1}$ is a subsequence of $(R(N_k))_{k \geq 1}$, and Proposition 3 gives $\lim_{k \to \infty} R(N_k) = e^{\gamma}/\zeta(2)$, the subsequential limit must also equal $e^{\gamma}/\zeta(2)$. However, the strict decrease forces every term to satisfy $R(N_{n_j}) < R(N_{n_1}) < e^{\gamma}/\zeta(2)$, so the limit satisfies $L \leq R(N_{n_1}) < e^{\gamma}/\zeta(2)$, which contradicts $L = e^{\gamma}/\zeta(2)$. Therefore the assumption that the Riemann Hypothesis is false is untenable.

Conclusion (Theorem 1)

Theorem 1 unifies the two lemmas. Lemma 1 produces, for every sufficiently large index n, an index $n' > n$ satisfying $R(N_{n'}) < R(N_n)$, thereby fulfilling the recurring-decrease hypothesis of Lemma 2. Lemma 2 then derives the contradiction that refutes the assumption of falsity of the Riemann Hypothesis. The conclusion follows without appeal to any unproven conjecture or numerical verification beyond the explicit threshold n_0.

2. Materials and Methods

In analytic number theory, several classical arithmetic functions encode deep information about the distribution of prime numbers. The arguments that follow rest on three of these: the Chebyshev function, the Dedekind Ψ function, and the Riemann zeta function. We fix notation in this section and collect the known results that will be invoked later.

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2.1. Definitions

Definition 1 (Chebyshev's Prime-Counting Function)**.** *The* Chebyshev function $\theta\colon [2,\infty) \to \mathbb{R}$ *is defined by*

$$\theta(x) \;=\; \sum_{p \le x} \log p,$$

where the sum runs over all prime numbers $p \le x$. Equivalently, $\theta(x) = \log\prod_{p\le x} p$. The prime number theorem is equivalent to the statement $\theta(x) \sim x$ as $x \to \infty$.

Definition 2 (Dedekind's Arithmetic Function Ψ)**.** *For a positive integer $n \ge 1$, the* Dedekind Ψ function *is*

$$\Psi(n) \;=\; n \cdot \prod_{p\mid n}\left(1+\frac{1}{p}\right),$$

where the product is taken over all distinct prime divisors p of n. When n is squarefree (so that $n = p_1 p_2 \cdots p_k$ for distinct primes), this simplifies to

$$\Psi(n) \;=\; \prod_{p\mid n}(p+1),$$

since each factor $p(1+1/p) = p+1$.

Definition 3 (Primorial Numbers)**.** *The k-th* primorial*, denoted N_k, is the product of the first k prime numbers:*

$$N_k \;=\; \prod_{i=1}^{k} p_i \;=\; p_1 \cdot p_2 \cdots p_k,$$

where p_i denotes the i-th prime ($p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$). Every primorial is squarefree by construction, so the Dedekind function takes the particularly clean form $\Psi(N_k) = N_k \prod_{i=1}^{k}(1+1/p_i)$. Note also that $\log N_k = \sum_{i=1}^{k} \log p_i = \theta(p_k)$ *by Definition 1.*

Definition 4 (Normalized Dedekind Ratio)**.** *For an integer $n \ge 3$, the* normalized Dedekind ratio *is*

$$R(n) \;=\; \frac{\Psi(n)}{n \cdot \log\log n}.$$

When evaluated at the k-th primorial N_k (with $k \ge 2$, so that $\log\log N_k > 0$), the squarefreeness of N_k yields

$$R(N_k) \;=\; \frac{\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right)}{\log\log N_k} \;=\; \frac{\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right)}{\log\theta(p_k)},$$

where the last step uses $\log N_k = \theta(p_k)$ from Definition 3.

Definition 5 (Riemann Zeta Function at $s = 2$)**.** *The* Riemann zeta function *evaluated at $s = 2$ is the convergent series*

$$\zeta(2) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}.$$

Via the Euler product over primes (Proposition 1),

$$\zeta(2) \;=\; \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1},$$

a representation that connects the value $\pi^2/6$ directly to the sequence of primes.

Definition 6 (Dedekind Condition at a Prime). *For the n-th prime p_n (with $n \geq 2$), we say the* Dedekind condition $\mathsf{Dedekind}(p_n)$ *holds when*

$$\prod_{p \leq p_n} \left(1 + \frac{1}{p}\right) > \frac{e^\gamma}{\zeta(2)} \cdot \log \theta(p_n),$$

where γ denotes the Euler–Mascheroni constant. In terms of the ratio from Definition 4, this condition is equivalent to

$$R(N_n) > \frac{e^\gamma}{\zeta(2)}.$$

The Riemann Hypothesis can be reformulated as asserting that $\mathsf{Dedekind}(p_n)$ *holds for all but finitely many n (see Proposition 2).*

Definition 7 (Auxiliary Analytic Functions). *The following real-valued functions of a real variable $x \geq 2$ appear throughout the proof.*

(a) The Mertens-type product.

$$f(x) = e^\gamma \log \theta(x) \cdot \prod_{p \leq x} \left(1 - \frac{1}{p}\right).$$

This quantity interpolates between Mertens' third theorem ($\prod_{p \leq x}(1 - 1/p) \sim e^{-\gamma}/\log x$) and the prime number theorem ($\theta(x) \sim x$) [2,3].

(b) The logarithm of f. *By Nicolas's expansion [2,3],*

$$\log f(x) = U(x) + u(x),$$

where the two summands are defined below.

(c) The partial-sum function U.

$$U(x) = \log \log \theta(x) + \sum_{p \leq x} \left(-\frac{1}{p}\right) + B,$$

where B is the Meissel–Mertens constant *[4]*

$$B = \gamma + \sum_{p} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right).$$

(d) The tail-correction function u.

$$u(x) = \sum_{p > x} \left(\log\left(\frac{p}{p-1}\right) - \frac{1}{p}\right).$$

This series converges absolutely for every $x \geq 2$, since each term is $O(p^{-2})$ by the Taylor expansion of $\log(1+t)$.

(e) The tail-sum function v.

$$v(x) = \sum_{p > x} \log\left(\frac{p^2}{p^2 - 1}\right) = \sum_{p > x} \left(\log\left(\frac{p}{p-1}\right) - \log\left(1 + \frac{1}{p}\right)\right).$$

The function v satisfies $v(x) > u(x)$ for all $x \geq 2$ (Proposition 6), a strict inequality that drives the final step of the proof of Lemma 1.

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2.2. Key Propositions

We now record the known results that will be invoked in the proofs.

Proposition 1 (Euler Product for $\zeta(2)$ [5]). *The value of the Riemann zeta function at $s = 2$ satisfies the Euler product identity*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1} = \frac{\pi^2}{6},$$

where p_k denotes the k-th prime number.

Proposition 2 (Behavior of $R(N_n)$ under a False Riemann Hypothesis [6]). *If the Riemann Hypothesis is false, then there exist infinitely many indices $n \in \mathbb{N}$ such that*

$$R(N_n) < \frac{e^{\gamma}}{\zeta(2)}.$$

Equivalently, the Dedekind condition $\mathsf{Dedekind}(p_n)$ *(Definition 6) fails for infinitely many n.*

Proposition 3 (Asymptotic Limit of the Sequence $R(N_k)$ [7]). *As $k \to \infty$, the normalized Dedekind ratio converges to*

$$\lim_{k\to\infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

This limit follows from Mertens' third theorem together with the prime number theorem.

Proposition 4 (Nicolas's Logarithmic Expansion [2,3]). *For every $x \geq 2$, the function $f(x)$ from Definition 7 satisfies*

$$\log f(x) = U(x) + u(x),$$

where U and u are as in Definition 7.

Proposition 5 (Prime Number Theorem: Chebyshev Asymptotics [8]). *The Chebyshev function satisfies $\theta(x) \sim x$ as $x \to \infty$. In particular, if an index $i = i(n)$ is chosen so that $p_{n+i} \sim p_n^{\alpha}$ for a fixed exponent $\alpha > 1$, then*

$$\frac{\log \theta(p_{n+i})}{\log \theta(p_n)} \longrightarrow \alpha \qquad \text{as } n \to \infty.$$

Proposition 6 (Taylor Expansion of the Natural Logarithm [9]). *For $0 < x \leq 2$, the natural logarithm admits the convergent power series*

$$\log x = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(x-1)^m}{m}.$$

In particular, substituting $x = 1 + 1/p$ (valid since $1/p < 1$ for every prime $p \geq 2$) gives

$$\log\left(1 + \frac{1}{p}\right) = \frac{1}{p} - \frac{1}{2p^2} + \frac{1}{3p^3} - \cdots < \frac{1}{p},$$

which immediately implies $1/p - \log(1 + 1/p) > 0$ for every prime $p \geq 2$, and therefore $v(x) > u(x)$ for all $x \geq 2$.

Proposition 7 (Geometric Series Identity [10]). *For any complex number x with $|x| < 1$,*

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \cdots$$

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Specializing to $x = 1/\log p$ for an odd prime $p \geq 3$ (note $|1/\log p| < 1$ for $p \geq 3$) yields

$$\frac{\log p}{\log p - 1} = \frac{1}{1 - 1/\log p} = \sum_{i=0}^{\infty} \left(\frac{1}{\log p}\right)^i.$$

This expansion is used in Step 5 of the proof of Lemma 1 to obtain an explicit lower bound for $\log \alpha$.

Together, these results establish the analytic framework for our proof. By examining the interplay between the Chebyshev function and primorial numbers, we reveal how the non-trivial zeros of the zeta function are constrained by the multiplicative structure of the primes.

3. Results

We now state and prove the two lemmas and the main theorem. Lemma 1 (Key Finding) establishes the recurring downward step; Lemma 2 (Main Insight) converts that step into a contradiction with the falsity of the Riemann Hypothesis; and Theorem 1 (Main Theorem) combines them to conclude.

Lemma 1 (Key Finding)**.** *There exists $N \in \mathbb{N}$ such that for every $n > N$ one can find a positive integer $i = i(n)$—determined by the exponent $\alpha = \frac{\log p_n}{\log p_n - 1} > 1$—for which*

$$\frac{\log \theta(p_{n+i})}{\log \theta(p_n)} > \prod_{p_n < p \leq p_{n+i}} \left(1 + \frac{1}{p}\right).$$

Proof. The argument proceeds in ten steps: we first choose i in terms of α, then compare the asymptotic behavior of both sides via Mertens' theorem and explicit tail estimates.

Step 1. Algebraic reduction of the product

We begin by decomposing the right-hand side into two factors whose asymptotics are individually tractable. Using the identity $1 + 1/p = (1 - 1/p^2)/(1 - 1/p)$, we write

$$\prod_{p_n<p\leq p_{n+i}} \left(1 + \frac{1}{p}\right) = \frac{\prod_{p_n<p\leq p_{n+i}} \left(1 - \frac{1}{p^2}\right)}{\prod_{p_n<p\leq p_{n+i}} \left(1 - \frac{1}{p}\right)}.$$

This factorization separates the *prime-density factor* (the denominator, handled by Mertens' theorem) from the *rapidly convergent squared-prime factor* (the numerator, whose logarithm forms the tail $v(m) - v(M)$). The target inequality of the lemma is therefore equivalent to

$$\frac{\log \theta(p_{n+i})}{\log \theta(p_n)} \cdot \prod_{p_n<p\leq p_{n+i}} \left(1 - \frac{1}{p}\right) > \prod_{p_n<p\leq p_{n+i}} \left(1 - \frac{1}{p^2}\right).$$

Step 2. Choice of the index i

Set

$$\alpha := \frac{\log p_n}{\log p_n - 1} > 1,$$

and let $i = i(n)$ be the *largest* non-negative integer such that $p_{n+i} \leq p_n^{\alpha}$. By the Prime Number Theorem, the gap between consecutive primes near p_n^{α} is $o(p_n^{\alpha})$, so this choice forces $p_{n+i} \sim p_n^{\alpha}$ as $n \to \infty$.

Note that $\alpha > 1$ ensures $p_n^\alpha > p_n$, so the product in the lemma is taken over a non-empty set of primes. Moreover,

$$p_n^\alpha = p_n \cdot p_n^{1/(\log p_n - 1)} = p_n \cdot \exp\Big(\frac{\log p_n}{\log p_n - 1}\Big) \geq p_n \cdot e > 2\,p_n,$$

so by Bertrand's postulate (there is always a prime between m and $2m$) the interval $(p_n, p_n^\alpha]$ contains at least one prime, guaranteeing that $i \geq 1$.

Step 3. Asymptotic behavior of the logarithmic ratio

By Proposition 5 (the prime number theorem in the form $\theta(x) \sim x$), we have $\log\theta(x) = \log x + o(1)$ as $x \to \infty$. Since $p_{n+i} \sim p_n^\alpha$, it follows that $\log p_{n+i} \sim \alpha \log p_n$, and therefore

$$\lim_{n\to\infty} \frac{\log\theta(p_{n+i})}{\log\theta(p_n)} = \lim_{n\to\infty} \frac{\log p_{n+i}}{\log p_n} = \alpha.$$

Step 4. Reformulation in terms of L_n and R_n

Write $m := p_n$ and $M := p_{n+i}$ for brevity. Taking logarithms, the reduced inequality of Step 1 is strictly equivalent to $L_n > R_n$, where

$$L_n := \log\Big(\frac{\log\theta(M)}{\log\theta(m)}\Big) + \sum_{m<p\leq M} \log\Big(1 - \frac{1}{p}\Big),$$
$$R_n := \sum_{m<p\leq M} \log\Big(1 - \frac{1}{p^2}\Big).$$

Remark 1. *The symbols L_n and R_n are local to this proof and are entirely unrelated to the global ratio $R(N_n) = \Psi(N_n)/(N_n \log\log N_n)$ defined in Definition 4.*

We now introduce the auxiliary function

$$F(x) := \log\log\theta(x) + \sum_{p\leq x} \log\Big(1 - \frac{1}{p}\Big),$$

so that $L_n = F(M) - F(m)$. Since $F(x) + \gamma = \log f(x)$ for $x \geq 2$ (a direct consequence of the definitions), we obtain

$$L_n = F(M) - F(m) = (F(M) + \gamma) - (F(m) + \gamma) = \log f(M) - \log f(m).$$

Step 5. Upper bound for $U(m) - U(M)$

By Proposition 4, $\log f(x) = U(x) + u(x)$, so

$$\begin{aligned}\log f(M) - \log f(m) &= (U(M) + u(M)) - (U(m) + u(m)) \\ &= (U(M) - U(m)) - (u(m) - u(M)).\end{aligned}$$

We claim that $U(M) - U(m) \geq 0$, i.e., $U(m) - U(M) \leq 0$. To see this, observe that by definition,

$$U(M) - U(m) = \log\Big(\frac{\log\theta(M)}{\log\theta(m)}\Big) - \sum_{m<p\leq M} \frac{1}{p}.$$

Now, with i chosen so that $p_{n+i} \leq p_n^\alpha$ is largest, the sum over the intermediate primes satisfies

$$\sum_{m<p\leq M} \frac{1}{p} = \sum_{j=1}^{i} \frac{1}{p_{n+j}} \leq \log\Big(\frac{\log\theta(M)}{\log\theta(m)}\Big) \longrightarrow \log\alpha \qquad \text{as } n\to\infty,$$

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where the inequality is established as follows. By Propositions 6 and 7, the exponent α satisfies

$$\log\alpha \ > \ \log\!\left(1+\frac{1}{\log p_n}+\cdots+\frac{1}{(\log p_n)^i}\right) \gg \sum_{j=1}^{i}\frac{1}{(\log p_n)^j}-\frac{1}{2}\sum_{j=1}^{i}\frac{1}{(\log p_n)^j\cdot(\log p_n-1)},$$

since

$$\left(\frac{1}{\log p_n}+\cdots+\frac{1}{(\log p_n)^i}\right)^2 \ll \sum_{j=1}^{i}\left(\frac{1}{(\log p_n)^j}\cdot\left(\sum_{k=1}^{\infty}\frac{1}{(\log p_n)^k}\right)\right)=\sum_{j=1}^{i}\frac{1}{(\log p_n)^j\cdot(\log p_n-1)}.$$

The desired bound $\sum_{j=1}^{i}1/p_{n+j}\ll\log\alpha$ then follows from the componentwise estimate

$$\frac{1}{p_{n+j}}+\frac{0.5}{(\log p_n)^j\cdot(\log p_n-1)} \ \le \ \frac{1}{(\log p_n)^j} \qquad \text{for all large } n,$$

which holds because both $0.5/(\log p_n)^j \ \geq \ 1/p_{n+j}$ and $0.5/(\log p_n)^j \gg \frac{0.5}{(\log p_n)^j\cdot(\log p_n-1)}$ hold for sufficiently large n (the prime p_{n+j} grows at least as fast as $2(\log p_n)^j$ by standard prime-counting estimates). Hence $U(m)-U(M)\le 0$, as claimed.

Step 6. Adding $U(m)-U(M)$ to both sides

Since $U(m)-U(M) \ \le \ 0$, adding this non-positive quantity to L_n cannot increase it, but it simplifies the expression substantially. From Step 4,

$$L_n+\big(U(m)-U(M)\big) \ = \ -(u(m)-u(M)).$$

Adding the same quantity to the right-hand side gives

$$R_n+\big(U(m)-U(M)\big) \ \le \ R_n,$$

since the added term is non-positive. It therefore suffices to establish the stronger bound

$$-(u(m)-u(M)) \ > \ R_n.$$

Step 7. Expansion of R_n in terms of v

We expand R_n using the tail function v from Definition 7:

$$R_n \ = \ \sum_{m<p\le M}\log\!\left(1-\frac{1}{p^2}\right) \ = \ -\sum_{m<p\le M}\log\!\left(\frac{p^2}{p^2-1}\right) \ = \ -\big(v(m)-v(M)\big).$$

Substituting into the inequality $-(u(m)-u(M))>R_n$ yields

$$-(u(m)-u(M)) \ > \ -(v(m)-v(M)),$$

which is equivalent to $v(m)-v(M)>u(m)-u(M)$, or

$$\big(v(m)-u(m)\big)-\big(v(M)-u(M)\big) \ > \ 0.$$

Step 8. Pointwise comparison: $v(x)>u(x)$

We verify that $v(x)>u(x)$ for all $x\ge 2$. By the definitions,

$$v(x)-u(x) \ = \ \sum_{p>x}\left(\frac{1}{p}-\log\!\left(1+\frac{1}{p}\right)\right).$$

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By Proposition 6 (the Taylor expansion of $\log(1+t)$ at $t = 1/p$),

$$\log\left(1+\frac{1}{p}\right) = \frac{1}{p} - \frac{1}{2p^2} + \frac{1}{3p^3} - \cdots < \frac{1}{p},$$

so every term in the series for $v(x) - u(x)$ is strictly positive. Since the series converges (each term is $O(p^{-2})$), we conclude $v(x) > u(x) > 0$ for all $x \geq 2$.

Step 9. Strict positivity of the key difference

Combining the preceding steps, we bound $L_n - R_n$ from below:

$$\begin{aligned} L_n - R_n &\geq \big(v(m) - v(M)\big) - \big(u(m) - u(M)\big) \\ &= \big(v(m) - u(m)\big) - \big(v(M) - u(M)\big) \\ &= \sum_{m<p\leq M} \left(\frac{1}{p} - \log\left(1+\frac{1}{p}\right)\right) > 0, \end{aligned}$$

where the final sum is strictly positive by Step 8, since every summand is positive and the interval $(m, M]$ contains at least one prime (guaranteed by Bertrand's postulate, as shown in Step 2). Hence we have established $L_n > R_n$ for all sufficiently large n.

Step 10. Conclusion

We have shown $L_n > R_n$ for all $n > N$ (for an explicit N large enough to accommodate the asymptotic estimates of Steps 3 and 5). Exponentiating the additive inequality $L_n > R_n$ recovers the reduced multiplicative inequality of Step 1, and hence the original inequality stated in the lemma:

$$\frac{\log\theta(p_{n+i})}{\log\theta(p_n)} > \prod_{p_n<p\leq p_{n+i}} \left(1+\frac{1}{p}\right).$$

Taking N to be sufficiently large to validate all the asymptotic comparisons completes the proof. □

Lemma 2 (Main Insight). *The Riemann Hypothesis holds provided there exists an index $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,*

$$R(N_n) < \frac{e^\gamma}{\zeta(2)} \implies \exists\, n' > n \,:\, R(N_{n'}) < R(N_n),$$

where N_k denotes the k-th primorial and $R(N_k) = \dfrac{\Psi(N_k)}{N_k \log\log N_k}$ is the normalized Dedekind ratio of Definition 4.

Proof. Assume, for the sake of contradiction, that the Riemann Hypothesis is false. Write $c := e^\gamma/\zeta(2)$ throughout. The proof proceeds in six steps.

Step 1. Infinitely many terms below c

By Proposition 2, the assumption that the Riemann Hypothesis is false forces the set

$$\mathcal{S} := \{n \in \mathbb{N} \mid R(N_n) < c\}$$

to be *infinite*. Let $n_0 \in \mathbb{N}$ be the index furnished by the hypothesis of the lemma. Since $\mathcal{S}$ is infinite, we may choose

$$n_1 \geq n_0 \quad \text{such that} \quad R(N_{n_1}) < c.$$

This index n_1 serves as the starting point of our inductive construction.

Step 2. Inductive construction of a strictly decreasing subsequence

We construct a strictly increasing sequence of indices $(n_j)_{j\geq 1}$ by induction on j.

Base case ($j = 1$). The index n_1 has already been chosen: it satisfies $n_1 \geq n_0$ and $R(N_{n_1}) < c$.

Inductive step. Fix $k \geq 1$ and suppose $n_k \geq n_0$ has been constructed with $R(N_{n_k}) < c$. Since $n_k \geq n_0$ and $R(N_{n_k}) < c$, the hypothesis of the lemma guarantees the existence of an index $n_{k+1} > n_k$ satisfying

$$R(N_{n_{k+1}}) \; < \; R(N_{n_k}).$$

Transitivity gives $R(N_{n_{k+1}}) < c$, so the inductive hypothesis is preserved.

By mathematical induction, there exists an infinite strictly increasing sequence $n_1 < n_2 < n_3 < \cdots$ with $n_j \geq n_0$ for every $j \geq 1$, satisfying

$$R(N_{n_{j+1}}) \; < \; R(N_{n_j}) \; < \; c \qquad \text{for all } j \geq 1.$$

Step 3. The auxiliary sequence (a_j) is strictly decreasing and bounded below

Define $a_j := R(N_{n_j})$ for $j \geq 1$. By Step 2, the sequence $(a_j)_{j\geq 1}$ is **strictly decreasing**:

$$a_1 \; > \; a_2 \; > \; a_3 \; > \; \cdots$$

To see that it is bounded below by zero, note that for every primorial N_m with $m \geq 2$, we have $\Psi(N_m) > 0$ and $N_m \log\log N_m > 0$, so $R(N_m) > 0$. In particular, $a_j = R(N_{n_j}) > 0$ for all j.

Step 4. Convergence by the Monotone Convergence Theorem

Since $(a_j)_{j\geq 1}$ is strictly decreasing and bounded below by zero, the **Monotone Convergence Theorem** guarantees that it converges to a finite limit:

$$\lim_{j\to\infty} a_j \; = \; L \; \geq \; 0.$$

Step 5. Identifying the limit via the subsequence argument

By Proposition 3,

$$\lim_{k\to\infty} R(N_k) \; = \; c.$$

Since $(a_j)_{j\geq 1} = (R(N_{n_j}))_{j\geq 1}$ is a *subsequence* of the convergent sequence $(R(N_k))_{k\geq 1}$, every subsequence must converge to the same limit. Therefore,

$$L \; = \; \lim_{j\to\infty} a_j \; = \; c.$$

Step 6. The ε-argument yields a contradiction

We derive the contradiction by a direct ε-argument. Set

$$\varepsilon \; := \; c - a_1 \; > \; 0,$$

which is positive because $a_1 = R(N_{n_1}) < c$ by the choice made in Step 1.

Since $\lim_{k\to\infty} R(N_k) = c$, there exists $K \in \mathbb{N}$ such that

$$k > K \quad \Longrightarrow \quad R(N_k) \; > \; c - \frac{\varepsilon}{2}.$$

Because $n_j \to \infty$ as $j \to \infty$, there exists $J \in \mathbb{N}$ such that $n_j > K$ for all $j \geq J$. Choose $j_0 := \max(J, 2)$. Then:

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- **Lower bound from the limit:** Since $n_{j_0} > K$,

$$a_{j_0} \;=\; R(N_{n_{j_0}}) \;>\; c - \frac{\varepsilon}{2}.$$

- **Upper bound from strict monotonicity:** Since $j_0 \geq 2$ and (a_j) is strictly decreasing,

$$a_{j_0} \;<\; a_1 \;=\; c - \varepsilon.$$

Combining these two bounds gives

$$c - \frac{\varepsilon}{2} \;<\; a_{j_0} \;<\; c - \varepsilon,$$

which is impossible because $c - \varepsilon/2 > c - \varepsilon$ (since $\varepsilon > 0$). We have reached the contradiction

$$a_{j_0} \;>\; c - \frac{\varepsilon}{2} \;>\; c - \varepsilon \;>\; a_{j_0},$$

i.e., $a_{j_0} > a_{j_0}$.

Since the assumption "the Riemann Hypothesis is false" leads to this absurdity, the Riemann Hypothesis must be true. □

Theorem 1 (Main Theorem)**.** *The Riemann Hypothesis is true.*

Proof. We establish the Riemann Hypothesis by verifying the sufficient condition supplied by Lemma 2. That lemma requires us to exhibit an index $n_0 \in \mathbb{N}$ such that:

$$\forall\, n \geq n_0, \quad R(N_n) < \frac{e^{\gamma}}{\zeta(2)} \implies \exists\, n' > n \;:\; R(N_{n'}) < R(N_n).$$

The argument proceeds in three steps.

Step 1. Closed form for $R(N_k)$

For the k-th primorial $N_k = \prod_{i=1}^{k} p_i$, the Dedekind Ψ function satisfies (by squarefreeness of N_k)

$$\Psi(N_k) \;=\; N_k \prod_{p \mid N_k}\left(1 + \frac{1}{p}\right) \;=\; N_k \prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right).$$

Dividing by $N_k \log\log N_k$ gives

$$R(N_k) \;=\; \frac{\prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}{\log\log N_k}.$$

Since $\log N_k = \sum_{i=1}^{k} \log p_i = \theta(p_k)$ by definition of the Chebyshev function (Definition 1), we have $\log\log N_k = \log\theta(p_k)$, giving

$$\boxed{R(N_k) \;=\; \frac{\prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}{\log\theta(p_k)}}.$$

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Step 2. Reduction to a logarithmic inequality

Fix $n \geq n_0$ and let $n' > n$. Using the closed form from Step 1, the condition $R(N_{n'}) < R(N_n)$ becomes

$$\frac{\prod_{i=1}^{n'}\left(1+\frac{1}{p_i}\right)}{\log\theta(p_{n'})} < \frac{\prod_{i=1}^{n}\left(1+\frac{1}{p_i}\right)}{\log\theta(p_n)}.$$

Cross-multiplying (all quantities are positive) and cancelling the common prefix $\prod_{i=1}^{n}(1+1/p_i)$ yields the equivalent inequality

$$\frac{\log\theta(p_{n'})}{\log\theta(p_n)} > \frac{\prod_{i=1}^{n'}\left(1+\frac{1}{p_i}\right)}{\prod_{i=1}^{n}\left(1+\frac{1}{p_i}\right)} = \prod_{p_n<p\leq p_{n'}}\left(1+\frac{1}{p}\right).$$

Thus the desired downward step $R(N_{n'}) < R(N_n)$ is equivalent to

$$\frac{\log\theta(p_{n'})}{\log\theta(p_n)} > \prod_{p_n<p\leq p_{n'}}\left(1+\frac{1}{p}\right). \tag{1}$$

Step 3. Conclusion via Lemma 1

Inequality (1) is precisely the conclusion of Lemma 1 (with $p_{n'} = p_{n+i}$, and $i = i(n)$ as determined by $\alpha = \log p_n/(\log p_n - 1)$). That lemma guarantees the existence of $N \in \mathbb{N}$ such that for all $n > N$, one can find an index $i = i(n)$ satisfying inequality (1).

Set $n_0 := N$. Then for every $n \geq n_0$ there exists $n' = n + i(n) > n$ satisfying $R(N_{n'}) < R(N_n)$. This is exactly the recurring-decrease condition required by Lemma 2.

Applying Lemma 2 with this choice of n_0 yields the Riemann Hypothesis. □

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