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Article

A Monte Carlo Null-Model Test of an Outer-Soddy Completion of the Koide Lepton Triple

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Abstract

The Koide relation $Q = (\sum m_\ell)/(\sum \sqrt{m_\ell})^2 = 2/3$ for the charged leptons has held to one part in 10^5 for over forty years without an accepted derivation and is widely regarded as numerology. This paper takes the relation as a clue rather than an endpoint. Treating lepton mass square roots as Descartes-circle curvatures, the outer root of the Descartes quadratic equals the closed form $\mathcal{F} = e_1 - \sqrt{p_2}$ when Koide holds exactly (Proposition 1); equivalently, $\mathcal{F}^2 = \alpha_K^2\, \mu_\star$ with $\alpha_K^2 = 5/2 - \sqrt{6}$ and $\mu_\star = \sum_\ell m_\ell$ the lepton-sum scale. The three-input symmetric-polynomial identity thus collapses to one dimensionless Koide-determined constant times the lepton-sum scale. Kocik [10] first observed a Descartes-like reading of Koide; our mutually-tangent variant is mathematically distinct but follows the same geometric spirit. The four-curvature completion carries a testable consequence absent from the bare three-mass relation: evaluating the squared fourth curvature numerically, $\mathcal{F}^2 = 95.113$ MeV, and comparing against the strange-quark $\overline{\text{MS}}$ mass at $\mu_\star$ within current lattice precision yields a residual of $+0.04$ MeV against ± 0.69 MeV, about $+0.06\sigma$. The lepton-side quantity is fixed to better than 0.01%; future lattice improvements will sharpen or refute the present numerical agreement. To our knowledge this paper implements the first Monte Carlo null test of the Koide relation under a random-spectrum prior; a Koide-conditioned null-model calibration across four prior shapes pre-registered for the analysis gives hit fractions at the sub-percent level — model-conditional frequencies, not p-values. Scale, input, prior, and filter sensitivities, together with the error budget, are reported; full Monte Carlo protocols, numerical output, and pre-registration are in a companion methods note [15].

Keywords: Koide relation; Descartes circle theorem; Soddy curvature; charged-lepton masses; strange-quark mass; renormalization-group running

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1. Introduction

The Koide relation [1,2]

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} = \frac{2}{3} \tag{1}$$

holds for the observed charged lepton pole masses to one part in 10^5. Using the PDG 2024 values [3] ($m_e = 0.5109989$ MeV, $m_\mu = 105.6584$ MeV, $m_\tau = 1776.93 \pm 0.09$ MeV), one computes $Q = 0.666664$. Although the relation is not part of the standard Standard-Model curriculum, it has been the subject of continuous theoretical investigation since 1981, with derivation attempts, extensions to the neutrino and quark sectors, and geometric reinterpretations appearing across the intervening four decades; we cite the Rivero–Gsponer review [4] as an entry point and [5–9] as representative further reading. We treat (1) as an empirical input.

The Koide relation has accumulated a reputation for numerology over four decades. We do not dispute that characterization as it applied to the bare three-mass relation: no derivation has been accepted, no mechanism has been demonstrated, and the continued absence of either is part of the relation's history. The present work treats Koide as a clue rather than an endpoint, and follows the

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structure the clue points toward through two steps. First, Kocik's 2012 reinterpretation [10] moved the Koide content from a symmetric-polynomial identity among three masses to a circle-geometry condition. Second, completing the three-circle configuration into the four-circle Descartes-Soddy configuration via the ordinary Descartes circle theorem (Section 2) fixes a fourth curvature, and under exact Koide this curvature admits the factorization $\mathcal{F}^2 = \alpha_K^2 \mu_\star$ with $\alpha_K^2 = 5/2 - \sqrt{6}$. The four-curvature completion thus collapses a three-input empirical identity into one dimensionless Koide-determined constant times the lepton-sum scale, and it carries a testable consequence absent from the three-input relation: the squared fourth curvature is a specific mass, and Section 4 reports that this mass coincides with $m_s^{\overline{\mathrm{MS}}}(\mu_\star)$ at $+0.06\sigma$ under current lattice precision. This does not resolve the Koide question and does not constitute a derivation. What it offers is the observation that Koide, approached geometrically and completed to four curvatures, acquires algebraic simplification and a second sub-sigma numerical match without additional free parameters.

Our construction diverges from Kocik's at one technical point. Kocik generalized the Descartes circle formula to circles meeting at a common angle and showed that Koide's $Q = 2/3$ fits within that generalized framework once the fourth curvature is set to zero (a degenerate "line" circle); he does not compute the non-trivial fourth curvature, and he does not connect any geometric quantity to a quark mass. The present work uses the *ordinary* Descartes circle theorem applied to a *mutually tangent* configuration, and does compute the fourth curvature. This is the geometric step that Proposition 1 formalizes: the three-circle condition is closed into a four-circle configuration, and the algebraically simple combination $e_1 - \sqrt{p_2}$ is identified with the fourth curvature $e_1 - 2\sqrt{e_2}$ under exact Koide.

This step does not add mathematical complexity to Koide. The inputs remain the same three lepton masses; the algebraic compression is identical; the only new quantity is the fourth curvature that the Descartes configuration already implies. Proposition 1 is the mathematical contribution, motivated by but not contained in [10].

Cross-sector mass relations of this kind are an established genre in the flavor literature. The golden quark-lepton mass relation [4], which connects charged-lepton and down-type quark masses via $m_\tau m_e m_\mu \approx m_b m_d m_s$, and Rivero's strange-charm-bottom Koide tuple [9] both make contact between lepton and quark sectors algebraically. The present observation arrives by a different route — geometric rather than algebraic, and operating on a single Koide-saturating triple rather than chaining conditions across sectors — but lands in the same numerical neighborhood. We note this convergence in Section 6 without assigning it evidentiary weight beyond what Section 5 supports.

The lepton-side quantity $\mathcal{F}^2$ is fixed to better than 0.01% precision. Each future FLAG compilation of $m_s^{\overline{\mathrm{MS}}}$ is therefore a direct test of the numerical match reported here; Section 4 discloses the order in which the lepton geometry and the quark comparison were computed. The contribution of this paper is the computation of $\mathcal{F}$ and the disclosure of the resulting numerical match, together with a sensitivity analysis characterizing how sharply the match depends on the input lepton triple, the evaluation scale, the sampling prior, and the perturbative cutoff.

One further observation bears on the timing of this report. The lepton-derived quantity $\mathcal{F}^2 = 95.113$ MeV has been computable since the lepton masses were measured with their current precision, which is decades ago. What has changed is the quark side: the 1σ uncertainty on $m_s^{\overline{\mathrm{MS}}}(2\text{ GeV})$ from lattice QCD was of order 5 MeV in the early 2000s and has narrowed to ± 0.68 MeV in FLAG 2024. The acceptance window used in the null-model tests of Section 5 scales with this uncertainty; at earlier precision levels, $\mathcal{F}^2$ sitting $+0.04$ MeV from the central value would have been submerged within a much wider band. The narrowing of the lattice acceptance window over this period illustrates this directly (see Section 5.2, Figure 2): the shaded $\pm\sigma_{m_s}(2\text{ GeV})$ band narrows progressively over the plotted period, and the lepton-derived target line enters the 1σ window only in the most recent determinations. Section 5.6 tabulates the corresponding trajectory of the residual at $\mu_\star$ as an exploratory consistency check. The observation is not that something obvious was overlooked; it is that the discriminatory power to notice it has only recently arrived.

A methodological point reinforces this timing. The charged leptons have long been measured to the precision needed to support a discriminating Monte Carlo null test of any mass-relation coincidence within the lepton sector. The same methodology applied to quark-involving relations has been much less informative historically: lattice uncertainties on $m_s^{\overline{\mathrm{MS}}}$ of order several MeV produced acceptance windows wide enough that random spectra cleared them at tens of percent, leaving such a test nearly content-free. FLAG 2024 narrows that window by roughly an order of magnitude. Separately, the Soddy completion converts the bare three-mass Koide identity into a four-mass constraint: the outer Soddy curvature must match a specific hadronic observable in addition to the three leptons satisfying Koide. A Monte Carlo test against this four-mass structure is correspondingly more fragile than a test against the bare three-mass relation. The combination of tighter lattice input and the four-mass matching requirement is what first makes a random-spectrum MC test substantive here; the results of Section 5 exploit that window.

2. The Koide Relation as a Descartes Condition

We summarize the ordinary-Descartes construction used in this paper for completeness and to fix notation. As discussed in Section 1, this construction is distinct from the generalized-angle reading in [10].

The Descartes circle theorem [11] states that four mutually tangent circles in the plane, with curvatures k_1, k_2, k_3, k_4, satisfy

$$(k_1 + k_2 + k_3 + k_4)^2 \;=\; 2\,(k_1^2 + k_2^2 + k_3^2 + k_4^2). \tag{2}$$

Given three curvatures k_1, k_2, k_3, equation (2) is a quadratic in k_4 with two real solutions

$$k_4^{\pm} \;=\; (k_1 + k_2 + k_3) \;\pm\; 2\sqrt{k_1k_2 + k_2k_3 + k_1k_3}, \tag{3}$$

corresponding respectively to the inner and outer Soddy circles tangent to the original three. By convention, the larger-curvature solution k_4^+ corresponds to the smaller circle (the inner Soddy circle, nestled in the gap between the original three), and the smaller-curvature solution k_4^- is conventionally called the "outer" Soddy circle. When k_4^- is negative, it corresponds to a large circle enclosing the original three; when k_4^- is positive, as is the case for the observed lepton triple ($k_4^- \approx 9.75\ \mathrm{MeV}^{1/2}$), it corresponds instead to a fourth circle externally tangent to the three but smaller than k_e^{-1} in radius. The "outer" label refers to the smaller of the two algebraic roots of the Descartes quadratic, not to literal enclosure. The relevant quantity for our purposes is this curvature $k_4^- = e_1 - 2\sqrt{e_2}$.

Proposition 1 (Koide and the outer Soddy curvature)**.** *Identify $k_i \equiv \sqrt{m_i}$ for $i \in \{e, \mu, \tau\}$. Let $e_1 = \sum_i \sqrt{m_i}$, $e_2 = \sum_{i<j} \sqrt{m_i m_j}$, and $p_2 = \sum_i m_i$. The outer Soddy curvature of the Descartes configuration (k_e, k_μ, k_τ) is*

$$k_4^- \;=\; e_1 - 2\sqrt{e_2}. \tag{4}$$

The algebraically simpler expression

$$\mathcal{F} \;\equiv\; e_1 - \sqrt{p_2} \;=\; \left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right) - \sqrt{m_e + m_\mu + m_\tau} \tag{5}$$

coincides with k_4^- if and only if the Koide relation $Q \equiv p_2/e_1^2 = 2/3$ holds exactly.

Proof. The Soddy form follows from solving the quadratic (2) for k_4 given k_1, k_2, k_3, using the identity $e_1^2 = p_2 + 2e_2$ to simplify the discriminant; the smaller-curvature root is $k_4^- = e_1 - 2\sqrt{e_2}$. Setting $e_1 - 2\sqrt{e_2} = e_1 - \sqrt{p_2}$ yields $4e_2 = p_2$. Combined with $e_1^2 = p_2 + 2e_2$, this gives $e_1^2 = p_2 + p_2/2 = (3/2)p_2$, equivalently $p_2/e_1^2 = 2/3$, which is the Koide condition. Conversely, if $Q = 2/3$ then $e_2 = e_1^2/6 = p_2/4$, so $2\sqrt{e_2} = \sqrt{p_2}$ and the two expressions agree. □

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Proposition 1 states the precise algebraic content of the standard-Descartes reading: under exact Koide, the dimensionally simple combination $e_1 - \sqrt{p_2}$, formed without reference to circles, is in fact the smaller Descartes-Soddy root of the configuration whose three input curvatures are the lepton mass square roots. The observed leptons satisfy $Q = 0.666664$, so the equality of (4) and (5) holds to one part in 10^5. In what follows we adopt $\mathcal{F}$ to denote the exact outer Soddy curvature k_4^- for all numerical computations.

Koide factorization of $\mathcal{F}$.

A useful corollary of Proposition 1 is the factorization

$$\mathcal{F} \;=\; \alpha_K \sqrt{\mu_\star}, \qquad \alpha_K \;\equiv\; \sqrt{\tfrac{3}{2}} - 1 \;\approx\; 0.22474, \tag{6}$$

where $\mu_\star \equiv p_2 = \sum_i m_i$ is the lepton-sum scale. Equivalently,

$$\mathcal{F}^2 \;=\; \alpha_K^2\, \mu_\star, \qquad \alpha_K^2 \;=\; \tfrac{5}{2} - \sqrt{6} \;\approx\; 0.05051. \tag{7}$$

This follows by substituting the Koide identity $e_1 = \sqrt{3/2}\sqrt{p_2}$ into the algebraic surrogate (5). The factorization makes explicit the role of $\mu_\star$ as the lepton-side mass scale carried by $\mathcal{F}^2$, with α_K a dimensionless constant fixed entirely by $Q = 2/3$. It also realizes the structural compression described in the introduction: the three-input symmetric-polynomial identity of the bare Koide relation becomes a single dimensionless constant multiplying a single mass scale. Figure 1 shows the geometric configuration schematically.

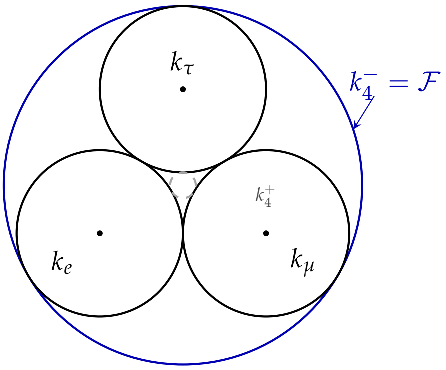

Figure 1. *Sign-convention schematic for the two roots of the Descartes quadratic; the drawing depicts the enclosing-circle orientation that corresponds to negative k_4^-, not the observed-lepton case.* The blue k_4^- circle is shown in the enclosing ($k_4^- < 0$) configuration solely to illustrate the conventional "outer" nomenclature; it is not the observed lepton geometry. The three input circles (solid black) carry curvatures $k_i = \sqrt{m_i}$ for $i \in \{e, \mu, \tau\}$. The two Soddy circles tangent to all three are shown: the inner Soddy circle k_4^+ (dashed gray, in the central gap) and the smaller-curvature root k_4^- (blue), conventionally called the "outer" Soddy circle. For the observed lepton triple, $k_4^- \approx 9.75\ \mathrm{MeV}^{1/2}$ is positive, so the actual blue circle would be a small fourth circle externally tangent to the three input circles, not an enclosing one. When the Koide relation $Q = 2/3$ holds exactly, the outer Soddy curvature k_4^- equals the surrogate $\mathcal{F} = e_1 - \sqrt{p_2}$. *The figure is a topological schematic, not drawn to scale.*

Although Kocik [10] gives the geometric identification of Koide as a Descartes condition, to our knowledge the outer Soddy curvature $\mathcal{F}$ has not previously been computed and compared with hadronic observables. We discuss the relation of the present work to existing Koide–quark literature, in particular Rivero's algebraic "waterfall" [9], in Section 6.

3. The Outer Soddy Curvature of the Lepton Triple

We have defined the exact comparison quantity as the outer Soddy curvature $\mathcal{F} \equiv k_4^- = e_1 - 2\sqrt{e_2}$ in Equation (4). Two further remarks before computing $\mathcal{F}$.

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The function $\mathcal{F}$ in the Descartes framework.

$\mathcal{F}$ is not chosen from a catalog of symmetric polynomials. Once the Koide relation is read as a Descartes condition (Proposition 1), the Descartes-determined quantities of dimension $[\text{mass}]^{1/2}$ are the two Soddy curvatures $k_4^{\pm} = e_1 \pm 2\sqrt{e_2}$. The choice between them is binary (inner versus outer), not a continuous selection. We adopt the outer (smaller-curvature) root k_4^-. The inner root squared at the observed lepton triple evaluates to $(k_4^+)^2 \approx 9.32$ GeV, in the gap between m_b and m_t and matching no quark mass at any reasonable scale (see also row 3 of the comparison family in Section 5.6); the inner/outer distinction is therefore empirically one-sided at the observed lepton triple rather than a live two-way selection. The integer power n in any subsequent comparison is fixed at $n = 2$ by dimensional consistency: $\mathcal{F}$ has units of $\text{MeV}^{1/2}$, so only $\mathcal{F}^2$ is dimensionally commensurate with a quark mass. The remaining selection freedom — the choice of comparison quark and the choice of evaluation scale — contributes to the look-elsewhere effect [12] that we quantify explicitly in Section 5.

Numerical value.

At the PDG 2024 central charged-lepton pole masses,

$$\mathcal{F} \;=\; 9.7526\ \text{MeV}^{1/2}, \qquad \mathcal{F}^2 \;=\; 95.1134\ \text{MeV}. \tag{8}$$

The propagated uncertainty from PDG lepton inputs is $\sigma_{\mathcal{F}^2} = 0.010$ MeV (Section 5), dominated by the tau mass uncertainty. The value reported in (8) is computed as the exact outer Soddy curvature $k_4^- = e_1 - 2\sqrt{e_2} = 9.752607\ \text{MeV}^{1/2}$. The algebraic surrogate (5), $e_1 - \sqrt{p_2} = 9.752823\ \text{MeV}^{1/2}$, agrees with k_4^- to one part in 10^5; the corresponding difference in the squared quantity is 0.004 MeV, smaller than the lepton-side measurement uncertainty and negligible compared to the dominant lattice uncertainty of ≈ 0.69 MeV. In what follows, $\mathcal{F}$ denotes the exact outer Soddy curvature k_4^- throughout. As a consistency check, the factorization (7) gives $\mathcal{F}^2 = \alpha_K^2 \mu_\star = 0.05051 \times 1883.10 = 95.116$ MeV, consistent with the algebraic surrogate value (95.118 MeV) to within the observed Koide residual.

4. The Observation

Evaluate the strange-quark $\overline{\text{MS}}$ mass at the natural lepton-sum scale,

$$\mu_\star \;\equiv\; m_e + m_\mu + m_\tau \;=\; 1883.1\ \text{MeV}, \tag{9}$$

by running the FLAG 2024 $N_f = 2{+}1{+}1$ estimate [13] $m_s^{\overline{\text{MS}}}(2\ \text{GeV}) = 93.44 \pm 0.68$ MeV via four-loop QCD [14] with $\alpha_s(M_Z) = 0.1180$. We treat the quoted $m_s(2\ \text{GeV})$ as an $n_f = 4$ $\overline{\text{MS}}$ mass and evolve it to $\mu_\star$ at fixed $n_f = 4$ using four-loop renormalization-group running, since both 2 GeV and $\mu_\star \approx 1.88$ GeV lie above m_c. This gives

$$m_s^{\overline{\text{MS}}}(\mu_\star) \;=\; 95.07 \pm 0.69\ \text{MeV}, \tag{10}$$

where the uncertainty includes the lattice input, the propagation of α_s uncertainty, and four-loop truncation, with the lattice contribution dominating; we decompose this in Section 5. The two sides of the comparison are in different renormalization schemes: $\mathcal{F}$ is constructed from charged-lepton pole masses, while $m_s^{\overline{\text{MS}}}(\mu_\star)$ is the FLAG lattice $\overline{\text{MS}}$ average run via four-loop QCD. We make no claim of scheme-equivalence between the two sides; the comparison is reported as a numerical relation between objects defined in different schemes, not as a derived equality. Comparing,

$$\mathcal{F}^2 - m_s^{\overline{\text{MS}}}(\mu_\star) \;=\; +0.038\ \text{MeV}, \tag{11}$$

or about +0.055 standard deviations of the lattice uncertainty. For context, the PDG 2024 quark-mass review [3] quotes $m_s^{\overline{\text{MS}}}(2\ \text{GeV}) = 92.93 \pm 0.63$ MeV from the latest $N_f = 2{+}1{+}1$ calculations; using this value in place of FLAG shifts the comparison residual by approximately +0.5 MeV (residual $\approx +0.9\sigma$), remaining well within 1σ. The numerical match is robust to this choice of low-scale input at the level of the conclusions we draw.

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Methodological note on order of operations.

We disclose the order of operations leading to this observation, because it bears on the look-elsewhere assessment in Section 5. The author was investigating Kocik's geometric reading [10] of the Koide relation and computed the fourth (outer Soddy) curvature of the Koide-saturating lepton triple as a geometric question, with the original motivation being whether the missing fourth curvature might admit a particle interpretation. The numerical agreement $\mathcal{F}^2 \approx m_s^{\overline{\mathrm{MS}}}(\mu_\star)$ was noticed only after the geometric computation was complete, on comparison with a previously consulted reference value for the strange-quark mass. The function $\mathcal{F}$ was therefore not selected from a class of candidate symmetric polynomials of lepton masses; once one takes the ordinary-Descartes reading, it is determined up to the binary inner/outer choice. The choice of evaluation scale deserves separate scrutiny, which we address empirically in Section 5.3.

5. Robustness

We report robustness checks across six axes: (i) propagation of input uncertainties and decomposition of the error budget, (ii) continuous scale sensitivity around $\mu_\star$, (iii) discrete alternative scale prescriptions, (iv) a null-model Monte Carlo across four prior shapes pre-registered for the Monte Carlo analysis and drawn from the flavor-model literature, (v) lower-cutoff filter sensitivity, and (vi) three supplementary checks (measurement-noise bootstrap, a comparison family of lepton functions, and an exploratory temporal trajectory), each described briefly here and in full in the companion methods note [15]. Central values, lattice-input propagation, and all Monte Carlo output are reproducible from the version-pinned GitHub repository [16].

All robustness scans in this section use $\mathcal{F}^2 \equiv (k_4^-)^2$ as the comparison quantity, where k_4^- is the exact outer Soddy curvature defined in (4).

5.1. Input Sensitivity and Error Budget

Propagating PDG 2024 lepton uncertainties through the exact curvature (4) yields $\sigma_{\mathcal{F}^2}^{\mathrm{lepton}} = 0.010$ MeV, dominated by $\sigma_{m_\tau} = 0.09$ MeV. We decompose the running uncertainty on $m_s^{\overline{\mathrm{MS}}}(\mu_\star)$ by varying each input independently:

Source	σ contribution (MeV)
FLAG lattice $m_s(2\,\mathrm{GeV}) = 93.44 \pm 0.68$	0.692
$\alpha_s(M_Z) = 0.1180 \pm 0.0009$	0.051
Charm threshold $m_c(m_c) = 1.273 \pm 0.005$ GeV (not used for 2 GeV $\to \mu_\star$ evolution)	0.000
Four-loop truncation (4L vs 3L)	0.039
Quadrature sum	0.695

The lattice input dominates by an order of magnitude over all other sources. Decomposing the total variance: the FLAG $m_s(2\,\mathrm{GeV})$ input accounts for approximately 99.1%, α_s uncertainty for 0.54%, four-loop truncation for 0.31%, and the lepton-side propagation for 0.02%. The lepton side of (11) is essentially exact at present precision.

One systematic absent from the above decomposition is the QED scheme conversion on the lepton side. Converting the lepton pole masses to $\overline{\mathrm{MS}}$ at leading QED order shifts $\mathcal{F}^2$ by approximately $-\alpha\,\mathcal{F}^2/\pi \approx -0.22$ MeV, moving the comparison residual from $+0.038$ MeV to approximately -0.18 MeV. The scheme-insensitive band for the residual is approximately $[-0.18, +0.04]$ MeV, entirely within 0.3σ of the dominant lattice uncertainty.

5.2. Continuous scale sensitivity

We compute $m_s^{\overline{\mathrm{MS}}}(\mu)$ for $\mu/\mu_\star \in [0.5, 4.0]$ and compare against the constant $\mathcal{F}^2$. The intervals over which $|\mathcal{F}^2 - m_s^{\overline{\mathrm{MS}}}(\mu)|$ remains within one and two standard deviations of $\sigma_{m_s}(\mu)$ are

$$1\sigma \text{ window:} \quad \mu \in [1845,\, 1921]\ \mathrm{MeV} \quad \approx \quad \mu_\star(1 \pm 2.0\%), \tag{12}$$

$$2\sigma \text{ window:} \quad \mu \in [1808,\, 1996]\ \mathrm{MeV} \quad \approx \quad \mu_\star(1 \pm 5.0\%). \tag{13}$$

The 1σ window is narrow (about $\pm 2.0\%$), confirming that the relation is picking out a specific scale. Running $m_s^{\overline{\text{MS}}}$ from 2 GeV to $\mu_\star \approx 1883$ MeV shifts the mass by $+1.63$ MeV ($+1.7\%$), approximately 39 times larger than the comparison residual; the scale prescription is not cosmetic. Figure 2 shows historical PDG and FLAG determinations of $m_s^{\overline{\text{MS}}}(2\ \text{GeV})$ for context.

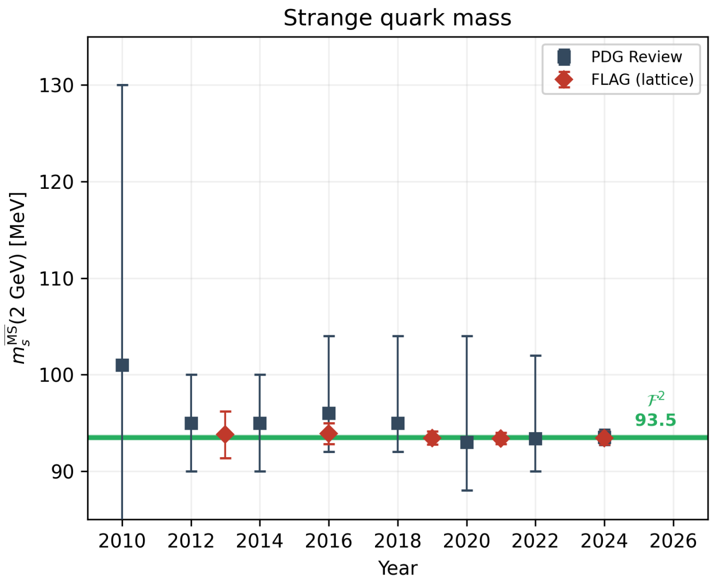

Figure 2. Historical FLAG lattice determinations (diamonds) of $m_s^{\overline{\text{MS}}}(2\ \text{GeV})$ from 2013 to 2024, alongside PDG Review values (squares) for context, with shaded $\pm\sigma_{m_s}$ bands at each FLAG determination. The horizontal green line marks the lepton-derived target $\mathcal{F}^2$ evolved back to 2 GeV (≈ 93.48 MeV), sitting $+0.04$ MeV above the FLAG 2024 central value of 93.44 MeV. The shaded bands illustrate the central point of the closing paragraph of Section 1: the 1σ acceptance window has narrowed from $\sim\pm 5$ MeV in the early 2000s to ± 0.68 MeV in FLAG 2024, and the lepton-derived target enters the 1σ window only with the most recent lattice generation. The observation is not that something obvious was missed; it is that the discriminatory power to notice it has only recently arrived. *This plot is not an independent validation of the relation.* Future updates to PDG and FLAG averages will continue to test the value directly, since the lepton-side quantity is essentially stationary at present precision. Historical data tabulated in `results/ms_history.csv` in the accompanying repository [16].

5.3. Alternative Scale Prescriptions

We test eight scale prescriptions and compute $m_s^{\overline{\text{MS}}}(\mu)$ at each, comparing against $\mathcal{F}^2 = 95.1134$ MeV:

Prescription	μ (MeV)	Hit (within 1σ)?
$\mu_\star = m_e + m_\mu + m_\tau$ (baseline)	1883.10	yes ($+0.06\sigma$)
$m_\mu + m_\tau$	1882.59	yes (trivially equivalent)
m_τ	1776.93	no (-2.32σ)
2 GeV (conventional reference, not lepton-derived)	2000.00	no ($+2.46\sigma$)
$(m_e + m_\mu + m_\tau)/3$	627.70	non-perturbative
$\sqrt{m_\mu m_\tau}$	433.30	non-perturbative
$(m_e m_\mu m_\tau)^{1/3}$	45.78	non-perturbative
$3/(1/m_e + 1/m_\mu + 1/m_\tau)$ (harmonic)	1.53	non-perturbative

Of the four prescriptions in the perturbative regime, only $\sum m_\ell$ and the trivially equivalent $m_\mu + m_\tau$ produce hits. The next-best lepton-derived alternative (m_τ alone) misses by -2.32σ, and the conventional reference scale of 2 GeV misses by $+2.46\sigma$.

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5.4. Random-Spectrum Null-Model Test

The Koide relation has attracted a substantial literature over four decades — of order fifty published works, with approximately thirty in peer-reviewed venues, spanning extensions to the quark and neutrino sectors, geometric reinterpretations, and dynamical mechanisms; a curated survey is maintained in the companion methods note [15] and in the project repository [16]. To our knowledge, no prior work implements a Monte Carlo null test of the Koide relation under a random-spectrum prior; the analysis here extends the methodology of the anarchy [17,18] and landscape [19,20] programs to this observable.

We characterize the rarity of the observation under a family of Koide-conditioned toy null models that draw random lepton-like triples from physically motivated priors. The scope of the pre-registration is worth making explicit: the numerical observation in Section 4 predates the Monte Carlo design. The pre-registration, committed to the companion methods note [15] before any simulation was executed, applies only to the prior shapes, sample sizes, hit criteria, cutoffs, and seeds of the analysis that follows, not to the discovery of the observation itself. This distinction is important for interpreting the hit fractions below.

The sampler draws m_3 from a prior on the heaviest mass and $r = m_1/m_3$ from a conditional prior on the ratio; m_2 is then determined analytically from the Koide condition $Q(m_1, m_2, m_3) = 2/3$. Requiring the "minus" branch of the Koide quadratic to yield the physical ordering $m_1 < m_2 < m_3$ is algebraically equivalent to $m_3/m_1 > (4 + \sqrt{18})^2 \approx 67.9$; triples failing this constraint are rejected mathematically (they do not correspond to points on the physical Koide sheet), not statistically. For each retained triple we compute $\mathcal{F}^{(i)}$ and $\mu_\star^{(i)}$, run $m_s^{\overline{\rm MS}}(\mu_\star^{(i)})$ from the FLAG 2 GeV input, apply the cutoff $\mu_\star^{(i)} > 1$ GeV, and record a hit if $|\mathcal{F}^{(i)\,2} - m_s^{\overline{\rm MS}}(\mu_\star^{(i)})| < \sigma_{m_s}(\mu_\star^{(i)})$. Sample size is $N = 10{,}000$ Koide-valid triples per prior.

Four prior shapes are pre-registered, each attached to an established position in the flavor-model literature:

- **A1 (Donoghue log-uniform), primary.** m_3 log-uniform on [1 MeV, 200 GeV]; r log-uniform on $[10^{-3}, 10^{-1}]$. Motivated by the observation [19] that the observed charged-fermion spectrum is approximately log-uniform over six decades.
- **A2 (Hall–Salem–Watari log-normal).** Log-normal m_3 centered on the geometric mean of the Standard-Model charged-fermion masses (median 0.1 GeV) with $\log\sigma = 2.5$; log-normal r with median 0.01 and $\log\sigma = 1.5$. Motivated by the log-normal-like mass distributions that arise in Gaussian-landscape models [20].
- **A3 (Yukawa-anarchy singular-value ratio).** m_3 log-uniform; $r = \sigma_{\rm min}/\sigma_{\rm max}$ of an i.i.d. $\mathcal{N}(0,1)$ 3×3 matrix, the canonical anarchy ratio distribution [17,18].
- **A4 (linear-uniform stress test).** m_3 and r linear-uniform on the A1 supports. Included only to bound the sensitivity of the hit fraction to a worst-reasonable-case prior; not a physics proposal.

Table 1. Random-spectrum hit fractions across four prior shapes pre-registered for the Monte Carlo analysis, $N = 10{,}000$ Koide-valid triples per prior, $\mu_\star > 1$ GeV cutoff. Seeds 2026–2029 (pre-registered). CIs are exact Clopper–Pearson [21] intervals from sampling noise only.

Prior shape	Hit fraction	95% CI
A1: Donoghue log-uniform (primary)	0.340%	[0.236, 0.475]
A2: Hall–Salem–Watari log-normal	0.590%	[0.449, 0.760]
A3: Yukawa-anarchy singular-value ratio	0.250%	[0.162, 0.369]
A4: linear-uniform stress test	0.020%	[0.002, 0.072]

Across the three physics-motivated priors (A1, A2, A3), the hit fraction varies by a factor of 2.4, with A1 primary value $p_{\rm A1} = 0.340\%$ [0.236, 0.475]. Including the linear stress test widens the range to a factor of ≈ 29, driven by A4's concentration of m_3 above 10 GeV — a region in which typical residuals are orders of magnitude from the hit band.

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A remark on the anarchy prior.

Singular-value ratios of i.i.d. Gaussian matrices have median ≈ 0.14, well above the Koide threshold $r < 1/67.9 \approx 0.0147$ derived above. Consequently only $\approx 2.5\%$ of A3 draws produce triples that lie on the physical Koide sheet in the first place; the remaining 97.5% are rejected as mathematically inadmissible, not as statistically unlikely. This is a property of the Koide manifold under anarchic priors, independent of the outer-Soddy observation; we record it for completeness and do not assign evidentiary weight to it in the comparison of (11).

5.5. Filter Sensitivity

Varying the lower cutoff $\mu_{\min}$ on $\mu_\star$ for the primary A1 prior:

$\mu_{\min}$	A1 hit fraction
0.7 GeV (extrapolated)	0.33%
1.0 GeV (baseline)	0.34%
1.5 GeV	0.37%
2.0 GeV (excludes the observed lepton triple)	0.00%

Across the three nontrivial cutoffs the hit fraction varies by a factor of 1.1. The result is stable under reasonable variations of the lower cutoff.

5.6. Supplementary Checks

Three supplementary tests are summarized here; full protocols, numerical output, and pre-registration appear in the companion methods note [15].

Measurement-noise bootstrap.

A parametric bootstrap [22] of the PDG 2024 charged-lepton masses (four variants: uncorrelated 1σ baseline, correlated $\rho = 0.5$, tightened 0.5σ, loosened 2σ, $N = 2{,}000$ per variant) places the observed residual $+0.038$ MeV at the 5.2% percentile of the baseline noise distribution. The fraction of bootstrap draws producing a 1σ hit is 65.7% (B1 baseline), consistent with the Gaussian expectation $\mathrm{erf}(1/\sqrt{2}) \approx 68.3\%$; this is a consistency check confirming that the engine correctly propagates lepton input uncertainties through the Soddy pipeline and the $\overline{\mathrm{MS}}$ running. Null B does not on its own argue against a noise-level explanation.

Comparison family of lepton functions.

We pre-specified 24 functions of (m_e, m_μ, m_τ) of algebraic complexity comparable to $\mathcal{F}^2$, frozen at the pre-registration commit of [15]. Table 2 gives the full list with values on observed leptons. Two of the 24 fall within 1σ of $m_s^{\overline{\mathrm{MS}}}(\mu_\star)$: $\mathcal{F}^2$ itself and the algebraic surrogate $(e_1 - \sqrt{p_2})^2$ of (5), which is Proposition 1's identity and differs from the first by 0.004 MeV at present lepton precision. The next-closest miss (row 22) is at $+6.73\sigma$; the remaining 21 variants miss by $\geq 16\sigma$. The same 24 variants give per-variant random-spectrum hit rates in the narrow range 0–0.45% at $N = 2{,}000$ under the A1 prior, indicating that the observed selectivity on (m_e, m_μ, m_τ) is not driven by a structural bias of the enumerated family toward small residuals.

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Table 2. The 24 comparison functions of $(m_1, m_2, m_3) \equiv (m_e, m_\mu, m_\tau)$ pre-specified in `engine/algorithm_variants.py` before Monte Carlo execution, evaluated on PDG 2024 observed lepton masses. e_k denote elementary symmetric polynomials in $\sqrt{m_i}$; $p_2 = \sum_i m_i$. Only two variants (boldface) fall within 1σ of $m_s^{\overline{\rm MS}}(\mu_\star) = 95.07 \pm 0.69$ MeV; these two are algebraically identical under exact Koide. Three features of the frozen catalog are disclosed rather than silently edited: rows 10, 11, and 12 collide at observed-lepton precision (e_2 and $e_1^2/6$ coincide under exact Koide); rows 17 and 18 are literal duplicates; row 24 has mass dimension 3/2 rather than 1, and its value reflects the code's evaluation in GeV-based units. All three are retained per pre-registration. Code names, per-variant pseudo-universe hit rates, and full per-row references appear in the companion methods note [15].

#	Expression	Value (MeV)
1	$(e_1 - 2\sqrt{e_2})^2 = \mathcal{F}^2$ outer Soddy	**95.113**
2	$(e_1 - \sqrt{p_2})^2$ Prop. 1 surrogate	**95.118**
3	$(e_1 + 2\sqrt{e_2})^2$ inner Soddy	9320.44
4	$\left[2(\sqrt{m_1} + \sqrt{m_3}) + \sqrt{3}\sqrt{m_1 + 4\sqrt{m_1 m_3} + m_3}\right]^2$ Koide "+" branch squared	25983.85
5	e_1^2	2824.66
6	$e_1^2/3$	941.55
7	$e_1^2 - p_2 \ (= 2e_2)$	941.56
8	$2e_2 - p_2$	−941.54
9	$p_2 - e_2$	1412.32
10	$e_2 = \sum_{i<j}\sqrt{m_i m_j}$	470.78
11	e_2 (duplicate enumeration)	470.78
12	$e_1^2/6$ ($= e_2$ under exact Koide)	470.78
13	$p_2 = \sum_i m_i \quad (= \mu_\star)$	1883.10
14	$p_2/3$ arithmetic mean	627.70
15	$(m_1 m_2 m_3)^{1/3}$ geometric mean	45.78
16	$3/\sum_i m_i^{-1}$ harmonic mean	1.53
17	$\sqrt{m_1 m_3}$	30.13
18	$\sqrt{m_3}\cdot\sqrt{m_1}$ (= row 17)	30.13
19	$\sqrt{m_2 m_3}$	433.30
20	$m_1 + m_2$	106.17
21	$m_3 - m_2$	1671.27
22	$m_2 m_3/(m_2 + m_3)$ reduced mass of (m_μ, m_τ)	99.73
23	$m_2^2/(m_1 + m_3)$	6.28
24	$\left[(m_1 m_2 m_3)^{1/4}\right]^2 = (m_1 m_2 m_3)^{1/2}$ (dim. 3/2; see caption)	9.80

Temporal trajectory.

As an exploratory consistency check — not a hypothesis test — we tabulate the residual $\mathcal{F}^2 - m_s^{\overline{\rm MS}}(\mu_\star)$ across five FLAG compilations of $m_s^{\overline{\rm MS}}(2\,\text{GeV})$ spanning 2013 to 2024, over which the reported lattice uncertainty narrowed from ± 2.4 MeV to ± 0.68 MeV:

Epoch	$m_s^{\overline{\rm MS}}(\mu_\star)$ (MeV)	Residual (MeV)	Within 1σ?
FLAG 2013	95.44 ± 2.44	-0.33	yes
FLAG 2016	95.54 ± 1.12	-0.43	yes
FLAG 2019	95.08 ± 0.69	$+0.038$	yes
FLAG 2021	95.08 ± 0.69	$+0.038$	yes
FLAG 2024	95.08 ± 0.69	$+0.038$	yes

$\mathcal{F}^2$ is consistent with the 1σ band at every epoch, including the earliest two at which the lattice uncertainty was approximately $3\times$ the present value. A coincidence-null Monte Carlo (companion methods note [15], §8) drawing $m_s^{\rm true} \sim U(60, 130)$ MeV and sampling each epoch's central value with its published uncertainty yields an all-five-epochs hit fraction of 0.26%. This figure is a conditional frequency under the stated construction, not a p-value, and does not account for correlations among FLAG 2019, 2021, and 2024, which share most of their underlying lattice inputs. Included as descriptive context for the trajectory only.

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5.7. Interpretation

The hit fractions and tail frequencies reported in Sections 5.4–5.6 are conditional frequencies under specific Monte Carlo constructions over explicitly chosen null models. They are not p-values [23,24], do not constitute hypothesis tests against any physical null, and should not be interpreted as evidence against chance in any objective sense. They characterize the observation's numerical behavior within the constructions we built; we draw no further inferential conclusion from them. In particular, the 0.26% figure reported for the temporal trajectory and the 0.34% primary random-spectrum hit fraction are not combinable into a single compound "significance"; they probe different axes of the same observation under separate and partly overlapping constructions, and their independence is not established.

6. Caveats and Non-Claims

We have not derived $\mathcal{F}^2 = m_s^{\overline{\text{MS}}}(\mu_\star)$ from any Lagrangian, symmetry, or theoretical framework, and we propose no mechanism. The Monte Carlo characterization of Section 5 reports the matchs behavior under a Koide-conditioned toy null. We hand the question of mechanism to model builders.

Not the only algebraic route to a comparable number.

Rivero [9] showed independently in 2011 that the quark triple (s, c, b) with $-\sqrt{m_s}$ satisfies a Koide-like relation $Q \approx 2/3$, and that chained Koide conditions (the "Koide waterfall") connect lepton and quark sectors algebraically; this construction recovers m_s in the same numerical neighborhood as the Soddy curvature reported here. The two routes are conceptually distinct: Rivero's construction is purely algebraic and chains Koide conditions across triples, whereas the present construction is geometric and operates on a single Koide-saturating triple via the Descartes circle theorem. Independent algebraic and geometric routes landing in the same numerical neighborhood could indicate shared underlying structure; it could equally indicate that lepton-mass combinations near 1 GeV are not scarce near the strange-quark mass, a consideration that bears on the evidentiary weight assigned to either route individually.

Not a charm or bottom relation.

$\mathcal{F}$ has units of MeV$^{1/2}$, so only $\mathcal{F}^2$ carries mass dimension one; comparisons $\mathcal{F}^n$ for $n \neq 2$ are not unit-invariant. Restricting to the dimensionally consistent comparison, $\mathcal{F}^2 \approx 95$ MeV is well outside both the charm range ($m_c \sim 1.3$ GeV) and the bottom range ($m_b \sim 4.2$ GeV) at any reasonable scale. We restrict the present note to the strange sector.

7. Discussion and Conclusions

If future work were to establish that the observation is not coincidental, possible interpretations fall into three broad classes. First, the Koide relation may have a geometric origin that has not been recognized, and the Soddy connection may be its geometric content; in that case the present observation extends the same geometric structure into the strange sector via the natural mass scale of the lepton triple. Second, the lepton-sum scale $\mu_\star = \sum m_\ell$ may be the natural matching scale of some flavor mechanism that produces m_s from leptonic input, in which case the observation is a datum that any such mechanism should seek to reproduce. Third, the observation may be a coincidence of the kind that Koide's relation itself might be: an unexplained numerical regularity that nonetheless remains stable as measurements improve. We do not argue for any of the three.

Author's tentative interpretation.

The Koide relation has held to one part in 10^5 for over four decades without an accepted derivation, and $\mathcal{F}^2$ is consistent with $m_s^{\overline{\text{MS}}}$ at the natural lepton-sum scale within current lattice precision. In the author's view, regularities of this kind are best understood as shadows of underlying algebraic structure: the Descartes–Soddy construction is a visualization of an algebraic identity, not an explanation for it. The three possibilities enumerated above are offered as a neutral map of interpretations; the

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author does not argue for any particular one. We report the observation in the possibility that, if such an underlying algebraic structure exists and is eventually identified, the present numerical match may serve as an additional constraint to narrow the search for it.

Outlook.

The present work reports a single numerical observation and deliberately stops there. One item lies outside the scope of this paper but may be of interest as a cross-reference: the factorization $\mathcal{F}^2 = \alpha_K^2 \mu_\star$ with $\alpha_K^2 = 5/2 - \sqrt{6}$ makes the content of the numerical match explicit. What the present observation picks out is the quantity $\alpha_K^2 \mu_\star$ at the lepton-sum scale, with α_K a dimensionless Koide-determined coupling fixed by $Q = 2/3$. It does not contribute to the evidentiary base of the present observation.

Conclusions.

We have introduced an ordinary-Descartes reading of the Koide lepton triple, computed the resulting outer Soddy curvature $\mathcal{F}$, and reported the numerical relation $\mathcal{F}^2 = 95.113$ MeV, consistent with $m_s^{\overline{\text{MS}}}(\mu_\star)$ (residual $+0.038$ MeV against ± 0.69 MeV). The observation is stable across four prior shapes pre-registered for the Monte Carlo analysis, eight scale prescriptions, and the non-excluding lower-cutoff choices tested, and it is consistent with the 1σ band at every FLAG compilation between 2013 and 2024. The lepton-side quantity is essentially stationary at present precision. Both FLAG (compilations every 2–3 years; most recent FLAG 2024) and PDG (biennial Review of Particle Physics; most recent PDG 2024) are ongoing, and each successor edition will test the numerical match reported here without further analysis on the lepton side. Figure 2 shows the narrowing of lattice uncertainties over the past decade; continued narrowing in the next cycle will sharpen or refute the agreement directly. The bare Koide relation compresses under four-curvature completion to the one-constant form $\mathcal{F}^2 = \alpha_K^2 \mu_\star$ with $\alpha_K^2 = 5/2 - \sqrt{6}$, and that compression carries a second sub-sigma numerical match as its first non-trivial output. The result is presented as a precise numerical regularity whose theoretical status, if any, remains open.

8. Computational Note

Every Monte Carlo result reported in this paper is backed by a committed JSON artifact in `results/` of the version-pinned GitHub repository [16] and is reproducible byte-for-byte from the pre-committed seeds. The §5.1 error-budget decomposition is an analytic propagation from the stated input uncertainties, verifiable by inspection rather than a Monte Carlo output. Full protocols, the pre-registration document, and all Monte Carlo output (per-draw data, hit fractions with confidence intervals, the full 24-function comparison family, and the temporal trajectory) are in the companion methods note [15]. Four-loop $\overline{\text{MS}}$ running is performed via CRunDec [14] with $\alpha_s(M_Z) = 0.1180$ and flavor thresholds at $m_c(m_c) = 1.273$ GeV and $m_b(m_b) = 4.183$ GeV. Lepton inputs are PDG 2024 charged-lepton pole masses. Quark inputs are the FLAG 2024 $N_f = 2{+}1{+}1$ Table 11 of [13] estimate for m_s and PDG 2024 summary-table values for m_c and m_b. The core four-test sequence (A/B/C/D) completes in under five minutes on a laptop; the full robustness suite including the filter-sensitivity and alternative-scales drivers completes in under ten minutes.

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bottom Koide tuple, which converges with the present geometric route. The author also thanks the lattice QCD community, whose successive improvements to the strange-quark mass determination made the numerical observation reported here testable. Computational infrastructure and institutional support were provided by Applied Dynamics Research, Sapporo.

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