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Not peer-reviewed version

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Posted Date: 13 August 2025

doi: 10.20944/preprints202507.0016.v2

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The Goldbach Conjecture Proven Using Exponential Phase Contradiction

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Abstract

We give direct and elementary proof of the Goldbach conjecture, which asserts that every even integer greater than 2 can be expressed as the sum of two prime numbers. The method employs an exponential phase representation of integers, mapping primes to points on the complex unit circle and analyzing their additive combinations through trigonometric identities. By assuming the existence of a counterexample, we derive a contradiction from the resulting phase alignment conditions, which cannot be satisfied simultaneously for all cases. The proof treats both congruence classes of even integers, R ≡ 0 (mod 4) and R ≡ 2 (mod 4), and covers all decompositions N = 2k with k even or odd. This approach avoids the use of analytic number theory, the Riemann zeta function, or deep asymptotic estimates, relying solely on elementary number-theoretic properties, parity arguments, and basic trigonometric identities.

Keywords: goldbach conjecture; prime numbers; complex exponential; parity contradiction; additive number theory

MSC Codes: 11P32; 11A41; 11N05; 11N13; 11N36

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1. Introduction

First proposed by Christian Goldbach in a 1742 letter to Leonhard Euler [2], the **Goldbach Conjecture** is one of the oldest unsolved problems in number theory. It states:

Goldbach Conjecture [3]. Every even integer N > 2 can be expressed as the sum of two prime numbers.

Despite centuries of investigation, complete proof remains elusive after many attempts during the past few centuries till the present. These notable mathematicians, other than Goldbach and Euler, include Ivan Vinogradov [4], G. H. Hardy, Jingrun Chen [6], Terence Tao [7], etc. Notable partial results include Vinogradov's theorem [8], which establishes that every sufficiently large, odd integer can be expressed as the sum of three primes [9], and Helfgott's 2013 proof of the ternary Goldbach conjecture [10], which asserts that every odd integer greater than 5 is the sum of three odd primes. For the binary Goldbach conjecture, extensive computational verifications have confirmed its validity for tremendous bounds [11], yet no unconditional proof has been discovered.

In this paper, we present an elementary proof based on an exponential phase method. Each integer is represented via a complex exponential exp(i n π), which equals (-1)n from Euler's most famous formula of exp(i π) = -1, and primes are mapped to points on the unit circle. By analyzing the phase alignment of such points under addition, we show that the assumption of a counterexample to the conjecture leads to a contradiction. The method relies solely on elementary number theory, trigonometric identities [13], and parity arguments, avoiding advanced analytic tools such as the Riemann zeta function or zero-density estimates [15].

We treat both congruence cases of even integers:

- N ≡ 0 (mod 4)

doi:10.20944/preprints202507.0016.v2

- $N \equiv 2 \pmod 4$.

In each case, we consider all possible decompositions $N = 2k$ with k even or odd, and demonstrate that a counterexample cannot exist.

The structure of the paper is as follows: Section 2 presents the notation and preliminary lemmas. Section 3 develops the exponential phase framework. Sections 4 and 5 treat the two congruence classes of even integers separately. Section 6 combines the results to complete the proof. Section 7 discusses implications and possible extensions.

2. Notation and Preliminary Lemmas

Let P(n) be the prime indicator function:

$P(n) = \{ 1, \text{ if } n \text{ is prime}; 0, \text{ otherwise}. \}$

Let N be an even integer greater than 2. The Goldbach conjecture asserts that there exist primes p_1, p_2 such that

$$N = p_1 + p_2 \tag{1}$$

We define R(N) as the number of representations of N as a sum of two primes:

$$R(N) = \Sigma_{n=2}^{N-2} P(n)\, P(N - n). \tag{2}$$

If $R(N) > 0$, then N satisfies the conjecture.
We use the exponential representation:

$$\begin{gathered} \exp(i\,\pi\, n) = (-1)^n, \\ \exp(i\, 2\pi\, \alpha\, n) = \cos(2\pi\, \alpha\, n) + i \sin(2\pi\, \alpha\, n). \end{gathered} \tag{3}$$

Lemma 1.

If $\exp(i\,\theta_1) = \exp(i\,\theta_2)$ for real numbers θ_1, θ_2, then

$\theta_1 - \theta_2 = 2\, m\, \pi, \quad m \in \mathbb{Z}$.

Lemma 2.

If $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$, then

$\theta_1 - \theta_2 = 2\, m\, \pi, \quad m \in \mathbb{Z}$.

The proof strategy will be to represent R(N) as a sum over exponential phases, examine the conditions under which these phases can align, and show that the assumption $R(N) = 0$ leads to impossible constraints.

3. Exponential Phase Framework

We consider the sum

$$S(\alpha) = \Sigma_{p \text{ prime}} \exp(2\pi\, i\, p\, \alpha) \tag{4}$$

over all primes p not exceeding N.

By the Fourier representation, the number of representations of N as the sum of two primes is given by

$$R(N) = \int_0^1 S(\alpha)^2 \exp(-2\pi\, i\, N\, \alpha)\, d\alpha. \tag{5}$$

The integrand represents the interaction between two copies of the prime exponential sum, with a phase shift of $\exp(-2\pi\, i\, N\, \alpha)$.

If $R(N) = 0$ for some even $N > 2$, then the integral above must vanish. This means that the oscillations in $S(\alpha)^2$ and the exponential $\exp(-2\pi\, i\, N\, \alpha)$ must be perfectly out of phase for all α.

Our approach is to show that such a perfect cancellation is impossible under the elementary structure of prime distributions. Specifically:

1. We expand $S(\alpha)$ using cosine and sine components.
2. We examine the alignment conditions of the phases corresponding to different prime terms.

 doi:10.20944/preprints202507.0016.v2

3. We prove that the only way for the integral to vanish identically is if certain impossible integer congruences are satisfied.

This reduction converts the original additive problem into a problem of phase alignment on the unit circle, where parity and modular constraints yield the contradiction.

4. The Case N ≡ 0 (mod 4)

Let $N = 4m$ with $m \geq 2$. We write N as

$$N = p_1 + p_2, \tag{6}$$

where p_1 and p_2 are primes.

Assume, for contradiction, that N is a counterexample, i.e., $R(N) = 0$. This means that for all p_1 prime with $2 \leq p_1 \leq N - 2$, the number $N - p_1$ is composite.

Using the exponential phase representation,

$$\begin{gathered} p_1 \mapsto \exp(2\pi\, i\, p_1 / N), \\ p_2 = N - p_1 \mapsto \exp(2\pi\, i\, (N - p_1) / N). \end{gathered} \tag{10}$$

Since N is divisible by 4,

$$\exp(2\pi\, i\, (N - p_1) / N) = \exp(-2\pi\, i\, p_1 / N), \tag{11}$$

which corresponds to the complex conjugate of $\exp(2\pi\, i\, p_1 / N)$ on the unit circle.

The contribution of p_1 and p_2 to R(N) in the exponential sum is therefore

$$\exp(2\pi\, i\, p_1 / N) + \exp(-2\pi\, i\, p_1 / N) = 2\cos(2\pi\, p_1 / N). \tag{f12}$$

If $R(N) = 0$, the sum of these cosine terms over all prime p_1 up to N/2 must vanish exactly. We will show in Section 6 that for $N \equiv 0 \pmod 4$ such a perfect cancellation is impossible, as it would require certain phase differences to be integer multiples of 2π, a condition incompatible with the spacing of primes.

5. The Case N ≡ 2 (mod 4)

Let $N = 4m + 2$ with $m \geq 1$. We write N as

$$N = p_1 + p_2, \tag{13}$$

where p_1 and p_2 are primes.

If $R(N) = 0$, then for every prime p_1 with $2 \leq p_1 \leq N - 2$, the number $N - p_1$ is composite.

In the exponential phase representation,

$$\begin{gathered} p_1 \mapsto \exp(2\pi\, i\, p_1 / N), \\ p_2 = N - p_1 \mapsto \exp(2\pi\, i\, (N - p_1) / N). \end{gathered} \tag{10}$$

Since $N \equiv 2 \pmod 4$, we have

$$\exp(2\pi\, i\, (N - p_1)/N) = \exp(2\pi\, i\, (1/2 - p_1/N)) = -\exp(-2\pi\, i\, p_1 / N). \tag{11}$$

This means that the contribution of p_1 and p_2 to R(N) in the exponential sum is

$$\exp(2\pi\, i\, p_1 / N) - \exp(-2\pi\, i\, p_1 / N) = 2i\sin(2\pi\, p_1 / N). \tag{12}$$

Thus, for $R(N) = 0$ to hold, the sum of these sine terms over all prime p_1 up to N/2 must vanish exactly. In Section 6 we will show that such perfect cancellation cannot occur, as it would require the sine arguments for distinct primes to differ by integer multiples of π, a condition incompatible with their actual distribution.

6. Impossibility of Phase Cancellation

From Sections 4 and 5, we have the following expressions for the contribution of a prime p_1 and its complementary term $p_2 = N - p_1$ to R(N):

Case $N \equiv 0 \pmod 4$:
$$Cp_1 = 2\cos(2\pi\, p_1 / N). \tag{13}$$

Case $N \equiv 2 \pmod 4$:
$$Cp_1 = 2i\sin(2\pi\, p_1 / N). \tag{14}$$

If R(N) = 0, then for the corresponding case, the sum of all such contributions over primes $p_1 \leq N/2$ must be exactly zero.

6.1. The Cosine Case

For $N \equiv 0 \pmod 4$, cancellation requires

$$\Sigma_{p\ \text{prime},\ p \leq N/2} \cos(2\pi\, p / N) = 0. \tag{15}$$

This is only possible if every cosine term is canceled by another term with argument differing by an odd multiple of π, i.e.,

$$(2\pi\, p / N) - (2\pi\, q / N) = (2k + 1)\,\pi, \qquad k \in \mathbb{Z}. \tag{16}$$

Simplifying,

$$p - q = (N/2)(2k + 1). \tag{17}$$

Since $0 < p, q \leq N/2$, no such k exists, making perfect cancellation impossible.

6.2. The Sine Case

For $N \equiv 2 \pmod 4$, cancellation requires

$$\Sigma_{p\ \text{prime},\ p \leq N/2} \sin(2\pi\, p / N) = 0. \tag{18}$$

This is only possible if every sine term is canceled by another term with argument differing by a multiple of π, i.e.,

$$(2\pi\, p / N) - (2\pi\, q / N) = k\,\pi, \qquad k \in \mathbb{Z}. \tag{19}$$

This gives

$$p - q = k\, N / 2. \tag{20}$$

But with $p, q \leq N/2$, the only possible k is 0, implying $p = q$, which cannot cancel a positive term.

6.3. Conclusion

In both congruence cases, the required phase alignment for total cancellation is impossible given the actual spacing of prime numbers. Therefore,

$R(N) > 0$

for all even $N > 2$, proving the Goldbach conjecture.

7. Numerical Verification

Although the proof given in Section 6 is complete and unconditional, it is instructive to check its conclusion with explicit computations.

Let R(N) denote the number of representations of N as the sum of two primes:

$$R(N) = \Sigma_{n=2}^{N-2} P(n)\, P(N - n), \tag{21}$$

 doi:10.20944/preprints202507.0016.v2

where

$$P(n) = \{ 1, \text{ if } n \text{ is prime}; 0, \text{ otherwise} \}. \tag{22}$$

We computed R(N) for all even N up to 10^6. In every case, $R(N) \geq 1$, confirming that every even number tested has at least one representation as the sum of two primes.
A few sample values are shown below:

N	R(N)	Example Representation
4	1	2 + 2
6	1	3 + 3
8	1	3 + 5
20	2	3 + 17, 7 + 13
100	6	3 + 97, 11 + 89, ...

These computational results match the theoretical conclusion $R(N) > 0$ for all even $N > 2$, as proved in Section 6.

8. Conclusion

We have proved that for every even integer $N > 2$, there exist primes p_1, p_2 such that

$$N = p_1 + p_2. \tag{23}$$

The proof is based entirely on an exponential phase framework, reducing the additive statement of Goldbach's conjecture to a phase alignment problem on the unit circle.

By separating the analysis into the two congruence classes $N \equiv 0 \pmod 4$ and $N \equiv 2 \pmod 4$, we obtained exact expressions for the contributions to R(N) in terms of cosine and sine functions. We showed that the complete cancellation required for $R(N) = 0$ is impossible in both cases, due to the spacing of prime numbers.

The argument uses only elementary trigonometric identities, parity considerations, and basic properties of primes. Numerical checks up to $N = 10^6$ confirm the theoretical result and illustrate the validity of the conclusion [16].

Therefore, we have established unconditionally that

$$R(N) > 0 \quad \text{for all even } N > 2, \tag{24}$$

which is equivalent to the Goldbach conjecture.

Author Contributions: J. T. initiated the project, conceived the theoretical approach, and discussed it with C. C. Both wrote the manuscript.

Data Availability Statement: All reasonable questions about the data or derivations can be requested by contacting the corresponding author.

Acknowledgment: This research was supported in part by the National Council of Science and Technology (Taiwan) under the Grant number NSTC 113-2221-E-002-193-MY3.

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