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Article

Conformable Lagrangian Mechanics of Actuated Pendulum

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Abstract: We construct the conformable actuated pendulum model in the conformable Lagrangian formalism. We solve the equations of motion in the absence of force and in the case of a specific force resulting from torques which generalizes a well known mechanical model. Our study shows that the conformable model captures essential information about the physical system encoded in the parameters which depend on the conformability factor α . This dependence can describe internal variations such as viscous friction, transmission or environmental effects. We solve the equations of motion analytically for $\alpha = 1/2$ and by Frobenius' method for $\alpha \neq 1/2$.

Keywords: conformable calculus; conformable mechanics; fractal mechanics; conformable actuated pendulum

1. Introduction

The conformable derivative was proposed in [1] as an alternative to fractional derivatives that satisfies the Leibniz rule [2]. Soon after, several fundamental mathematical properties were proved in [3–7]. A generalization of the conformable derivative with an exponential scale factor was proposed in [8] and studied in [9], and its extension to an arbitrary time scale was given in [10]. However, it was shown in [11] that the fractional derivatives break the Leibniz rule, particularly for non-integer order. Subsequently, two criteria for fractional derivatives were formulated in [12]. This analysis led to the conclusion that the conformable derivative is not a proper fractional derivative [13,14]. Despite not being a fractional calculus according to these criteria, conformable calculus has recently attracted attention due to its analytical properties. In mathematics, it has been generalized to Hermite polynomials [15], arbitrary time scales [16], formal diffusion equation [17], Banach spaces [18], several variables [19,20], q -deformed derivative [21], soliton solutions for fractional evolution equations [22], mathematical aspects of conformable differential equations with damping terms [23,24], and the effect of the conformable derivative on differential equation [25]. In physics, attempts to interpret the conformable derivative were made in [26–28]. The conformable Newtonian mechanics was given in [29], while the Lagrangian formalism was formulated in [30]. In quantum mechanics, several conformable systems were analyzed in [6,31–41]. Important hints on the nature of the conformable derivative as a fractal were given in [5,42,43]. More recently, it was argued that the conformable derivative is suitable for describing fractal media and fractal distributions of matter in continuum models with fractal density of states in [44]. For a recent review on the applications of conformable calculus in physics, see [45].

Despite the large number of studies on the applications of conformable calculus in physics, there are still unanswered questions about the physical information that can be encoded in a conformable

model. In particular, a crucial problem concerns the generalization of Lagrangian mechanics and the physical effects that can be described by such a system. The variational principle of conformable systems was studied previously in [30,46,47] with emphasis on establishing formal results. In particular, reference [30] gives a detailed construction of the Lagrangian and Hamiltonian formalism for conformable systems that include frictional forces.

In this paper, we address the problem of the physical effects described by conformable calculus in the Lagrangian formulation from the point of view of the relationship between model's physical quantities and parameters and their physical content in the case of a simple mechanical system. This problem allows us to understand if the conformable calculus is applicable to a more abstract or complex system, or there are simple mechanical systems which could be described by conformable mechanical models. We focus our attention on the actuated pendulum, which is a model with many applications in physics and engineering. We construct the conformable actuated pendulum using the Lagrangian formalism, in which the main physical observable defined in terms of the conformable time derivative is the kinetic energy which is quadratic in the conformable derivatives of the position variable. The consistency of the action functional with the conformable calculus requires to define it, in terms of conformable integrals [30]. From the very definition of the conformable Lagrangian functional, the conformable kinetic energy acquires a factor which is a non-integer power of time variable. This immediately implies a redefinition of the mass parameter as well as the interpretation of the conformable model in terms of the non-standard damping effect (damping that is not simply proportional to velocity), which is not present in the standard model. Other time-dependent effects are also captured in the conformable formalism such as decreasing inertia and memory-effect simulation. This suggests that the conformable calculus is suitable for analyzing certain mechanical systems, particularly those with internal friction or complex microstructures (such as viscoelastic materials or materials containing microcracks), where energy dissipation cannot be accurately modeled using standard potential terms or external damping forces; see [48–50]. By mimicking memory effects, the conformable calculus could model materials whose dynamic behavior depends explicitly on coefficients that vary with time, while the introduction of non-standard damping allows the conformable calculus to describe phenomena in which the damping characteristics vary with time¹. By modifying the kinetic energy rather than introducing additional potential terms or non-conservative forces, we can describe complex dissipative behaviors while maintaining the elegance and consistency of the Lagrangian formalism. Moreover, the conformable formalism introduces other time-dependent terms that represent physical effects. This aspect becomes more relevant in the analysis of systems in which damping or inertia vary over time, as these coefficients are related to the physical characteristics and structural properties of the system.

This paper is organized as follows. In Section 2, we briefly review the basic concepts of conformable calculus. We also briefly present the relationship between conformable, fractal, [52,53] and Jackson derivatives [54]. In Section 3, we give the basic relations of the conformable Lagrangian formalism for point particle mechanics and derive the Euler-Lagrange equations from it. We argue here that non-standard damping effects can be modeled by the conformable kinetic energy. In Section 4, we construct the conformable pendulum and derive its equations of motion in the absence of force. Moreover, we derive the general analytic solution of the equation of motion for $\alpha = 1/2$ and the power series solution for $\alpha \neq 1/2$. In Section 5, we construct the conformable actuated pendulum under the influence of a non-conservative generalized force. This model generalizes an arm pendulum connected through a shaft to a mechanical source of torque. We derive the general solutions of the equations of motion for $\alpha = 1/2$ which can be analytically obtained in terms of Bessel functions. Also, we obtain the power series solution for $\alpha \neq 1/2$ using Frobenius' method. In Section 6, we discuss our results and suggest some future research.

¹ Similar descriptions of damping phenomena was given in [51].

2. Fundamentals of Conformable Calculus

In this section, we review the definition of the conformable derivative and integral and their basic properties, whose applications will be discussed throughout this paper. We also revisit an important relationship between the conformable and fractal derivatives which allows us to interpret conformable models. Although it does not meet all the criteria to be classified as a classical fractional derivative, the conformable derivative proves to be very useful and convenient for many practical applications. For more details, we suggest consulting [1,4,5] and the references therein.

2.1. Conformable Derivative and Integral

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a real continuous function in its domain. The *conformable derivative of order* $\alpha \in (0, 1]$ denoted by $D_x^\alpha f(x)$ at $x \in \mathbb{R}$ is defined, in case that the convergence holds, by the following relation

$$D_x^\alpha f(x) := \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}. \quad (1)$$

If the definition can be extended to all $x > 0$, then we say that the function f is α -differentiable on $\mathbb{R}^+$.

The conformable integral of the order α is defined as

$$I_x^\alpha f(x) := \int_0^x \xi^{\alpha-1} f(\xi) d\xi. \quad (2)$$

The definitions of D_x^α and I_x^α establish these two operators as inverse to each other in the following sense

$$D_x^\alpha I_x^\alpha f(x) = f(x), \quad (3)$$

$$I_x^\alpha D_x^\alpha f(x) = f(x) - f(0). \quad (4)$$

The conformable derivative has the following basic properties, for all f and g and all $a, b \in \mathbb{R}$ constants:

$$D_x^\alpha (af(x) + bg(x)) = aD_x^\alpha f(x) + bD_x^\alpha g(x), \quad (5)$$

$$D_x^\alpha (f(x)g(x)) = (D_x^\alpha f(x))g(x) + f(x)D_x^\alpha g(x), \quad (6)$$

$$D_x^\alpha f(g(x)) = (D_{g(x)}^\alpha f(g(x))) (D_x^\alpha g(x)) g(x)^{\alpha-1}. \quad (7)$$

Some important properties involve the conformable fractional derivatives of monomials, which are used to represent functions in power series, of the function quotients and the of the constant functions. Here, we list these properties, which can easily be proven based on their definitions

$$D_x^\alpha (x^p) = px^{p-\alpha}, \quad (8)$$

$$D_x^\alpha \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)D_x^\alpha f(x) - f(x)D_x^\alpha g(x)}{g^2(x)}, \quad (9)$$

$$D_x^\alpha (c) = 0, \quad (10)$$

where p and c are real constants. The above properties are similar to the ones of derivatives in standard calculus.

2.2. Conformable, Fractal and Jackson Derivatives

The algebra of the conformable derivatives is similar to that of standard derivatives, as shown above. In fact, the conformable derivative is related to the fractional and the generalized Jackson derivatives even at the level of their definitions as shown below.

Consider the fractal derivative of a function f in a space with fractal dimension β defined as [52,53]

$$F_{\alpha}^{\beta} f(t_0) = \frac{\partial^{\beta} f(t_0)}{\partial t^{\alpha}} = \lim_{t \rightarrow t_0} \frac{f^{\beta}(t) - f^{\beta}(t_0)}{t^{\alpha} - t_0^{\alpha}}, \quad (11)$$

if the convergence exists, where α is the fractal dimension of $t \in \mathbb{R}^+$ variable. One can easily show that if the variable has a fractal structure, then

$$F_{\alpha} f(t_0) = \frac{\partial f(t_0)}{\partial t^{\alpha}} = \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t^{\alpha} - t_0^{\alpha}} = \frac{t^{1-\alpha}}{\alpha} \frac{df(t_0)}{dt}, \quad (12)$$

for all $\alpha > 0$. On the other hand, from the definition (1), results

$$D_t^{\alpha} f(t) = t^{1-\alpha} \frac{df(t)}{dt}. \quad (13)$$

Then, from equations (12) and (13), we obtain the relation between the conformable and fractal derivatives

$$F_{\alpha} f(t) = \frac{1}{\alpha} D_t^{\alpha} f(t). \quad (14)$$

The equation (14) shows that the fractal and conformable derivatives are formally equal up to a constant. However, they have different interpretations as they belong to different mathematical structures. The concept of fractal derivatives originates from fractal geometry and are associated with systems or functions that exhibit fractal (self-similar) properties, meaning they often describe processes occurring on irregular, non-integer-dimensional spaces [55]. On the other hand, the conformable derivatives generalize classical derivatives and retain many of the fundamental properties of standard derivatives, such as the product rule and chain rule, while extending differentiation to non-integer orders, their interpretation being more straightforward and generally associated with continuous systems.

The conformable derivative is also related to the generalized Jackson derivative, and is defined by [54]

$$D_{q,\alpha} f(t) = \lim_{q \rightarrow 1} \left(\frac{d}{dt} \right)_q f(t) = \frac{f(qt) - f(t)}{qt^{\alpha} - t^{\alpha}}, \quad (15)$$

if the convergence holds. This can be seen by making the substitution $q = 1 + \varepsilon t^{-\alpha}$ and taking the limits $\varepsilon \rightarrow 0$ and $q \rightarrow 1$ to obtain [5]

$$D_{\alpha} f(t) = \lim_{q \rightarrow 1} D_{q,\alpha} f(t). \quad (16)$$

The relations among these derivatives show that the conformable derivative can be applied to the same range of problems as the fractal and Jackson derivatives. The relation (13) suggests that the algebra of conformable derivatives is obtained by extending via a tensor product of the standard derivative algebra with fractional powers of the variable. Therefore, this structure exhibits a certain degree of triviality in the functions space over $\mathbb{R}$ as previously analyzed in the literature mentioned in the Introduction.

Classical calculus assumes constant parameters and simple friction or damping laws, which cannot accurately describe complex mechanical systems characterized by internal properties that vary over time. The conformable calculus formalism was chosen because it effectively extends the concept of classical derivatives, offering a more accurate representation of time-dependent properties such as variable inertia, friction, or damping in real pendulum systems.

3. Conformable Lagrangian Mechanics of Point Particle

In this section, we present the conformable generalization of Lagrangian mechanics. The basic principle in generalizing standard mechanical models to fractional or conformable systems is to substitute the standard derivatives for the modified ones. However, a different approach which may or may not lead to the same equations of motion is to construct an action functional by replacing standard derivatives with fractional or conformable derivatives and reinterpreting function spaces. In this approach, physical quantities such as kinetic energy, potential energy, and action are modified from the beginning, incorporating the features encoded within fractional or conformable calculus. Then extending the variational principle to the new functional space provides the equations of motion. Other classical mechanical constructs, such as non-holonomic forces, can be incorporated into the formalism with corresponding interpretations. In the following, we apply these principles to construct the action functional of a conformable classical particle. Our main objective is to examine how conformable calculus contributes to the mathematical and physical insights of these models.

3.1. Conformable Lagrangian Formalism

Consider a conformable particle in $\mathbb{R}^d$ whose instantaneous position is denoted by $q(t) = \{q^i(t)\}$ where $i, j, \dots = 1, 2, \dots, d$. In order to describe the particle's dynamics, we construct a conformable Lagrangian functional for a particle $L^\alpha[q(t), D_t^\alpha q(t)]$ by applying the general principle that requires that standard derivatives be replaced by the conformable ones. If the particle moves under the influence of a general external potential $V[q(t)]$, then the Lagrangian is defined by

$$L^\alpha[q(t), D_t^\alpha q(t)] = \frac{M_\alpha}{2} \sum_{i=1}^d D_t^\alpha q^i(t) D_t^\alpha q^i(t) - V^\alpha[q(t)], \quad (17)$$

where $V^\alpha[q(t)]$ is a potential functional of dimension $ET^{1-\alpha}$. The parameter M_α plays the role of a conformable mass. By dimensional analysis, we can see that $\langle M_\alpha \rangle = ET^{1+\alpha}L^{-2}$ which is natural units $c = 1$ is $\langle M_\alpha \rangle = ET^{\alpha-1}$. In the classical mechanics limit $\alpha \rightarrow 1$, the conformal mass becomes the standard mass as expected

$$\lim_{\alpha \rightarrow 1} M_\alpha = m. \quad (18)$$

As in classical mechanics, we want to interpret the particle's trajectory as the extrema of the action corresponding to $L^\alpha[q(t), D_t^\alpha q(t)]$ in the time interval $[0, T]$. For reasons that will become apparent shortly, we define the action functional using the conformable integral by the relation

$$S^\alpha[q] = \int_0^T L^\alpha[q(t), D_t^\alpha q(t)] d\mu_t^\alpha, \quad (19)$$

where $d\mu_t^\alpha = t^{\alpha-1}dt$ is the integration measure of conformable integral.

To derive the Euler-Lagrange equations, we consider variations $\delta q^i(t)$ of the functions $q^i(t)$ that vanish at the boundary of the time interval $[0, T]$. The variation of the action functional is given by

$$\delta S^\alpha = \int_0^T \left[\frac{\partial L^\alpha}{\partial q^i} \delta q^i + \frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} \delta (D_t^\alpha q^i) \right] t^{\alpha-1} dt. \quad (20)$$

We recall that the variation δ commutes with the multiplication by $t^{1-\alpha}$ because $t^{1-\alpha}$ is a function of the independent variable t and does not depend on the varied quantity $q(t)$. As a result, the conformable derivative has the following property

$$\delta (D_t^\alpha q^i) = D_t^\alpha (\delta q^i). \quad (21)$$

By using the relation (21) and integrating by parts the second term, we obtain

$$\int_0^T \frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} D_t^\alpha (\delta q^i) t^{\alpha-1} dt = \frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} \delta q^i t^{\alpha-1} \Big|_0^T - \int_0^T D_t^\alpha \left(\frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} t^{\alpha-1} \right) \delta q^i dt, \quad (22)$$

Since δq^i vanishes at the endpoints, the boundary term is zero, and we are left with

$$\int_0^T \frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} D_t^\alpha (\delta q^i) t^{\alpha-1} dt = - \int_0^T D_t^\alpha \left(\frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} t^{\alpha-1} \right) \delta q^i dt. \quad (23)$$

Substituting equation (23) back into (20), the variation of the action becomes

$$\delta S^\alpha = \int_0^T \left[\frac{\partial L^\alpha}{\partial q^i} - D_t^\alpha \left(\frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} t^{\alpha-1} \right) \right] \delta q^i t^{\alpha-1} dt. \quad (24)$$

For the action to be stationary ($\delta S^\alpha = 0$) for arbitrary variations δq^i , the integrand must vanish. This gives the conformable Euler-Lagrange equations

$$\frac{\partial L^\alpha}{\partial q^i} - D_t^\alpha \left(\frac{\partial L^\alpha}{\partial (D_t^\alpha q^i)} t^{\alpha-1} \right) = 0. \quad (25)$$

For the Lagrangian functional from (17), we obtain the following equations of motion

$$M_\alpha (D_t^\alpha)^2 q^i(t) - \frac{\partial V^\alpha[q(t)]}{\partial q^i(t)} = 0, \quad (26)$$

where we adopt the notation

$$(D_t^\alpha)^2 q^i(t) = D_t^\alpha [D_t^\alpha q^i(t)]. \quad (27)$$

Using the property (13), the equation (26) can be expressed as a differential equation in standard derivatives

$$M_\alpha t^{2-2\alpha} \frac{d^2 q^i(t)}{dt^2} + (1-\alpha) M_\alpha t^{1-2\alpha} \frac{dq^i(t)}{dt} - \frac{\partial V^\alpha}{\partial q^i(t)} = 0. \quad (28)$$

This shows a time-dependent damping term proportional to $\frac{dq^i(t)}{dt}$ with coefficient $(1-\alpha)M_\alpha t^{1-2\alpha}$, and a time-dependent "mass" term $M_\alpha(t) = M_\alpha t^{2(1-\alpha)}$ multiplying the standard acceleration.

By analogy with classical mechanics, we define the conformable linear momentum by

$$p_\alpha^i(t) := \frac{\partial L}{\partial D_t^\alpha q^i(t)}, \quad p_\alpha^i(t) = M_\alpha D_t^\alpha q^i(t). \quad (29)$$

Then the conformable force can be defined as

$$F_\alpha^i(t) := D_t^\alpha p_\alpha^i(t), \quad (30)$$

which allows us to interpret the equations of motion (26) as the conformable version of Newton's second law

$$M_\alpha (D_t^\alpha)^2 q^i(t) = F_\alpha^i(t). \quad (31)$$

The conformable derivative, due to its mathematical properties, allows us to construct a conformable classical mechanics similar to the standard one. For example, we define the Hamiltonian functional by analogy with classical mechanics

$$H^\alpha[q(t), p_\alpha(t)] := \sum_{i=1}^n p_\alpha^i(t) D_t^\alpha q^i(t) - L^\alpha[q(t), D_t^\alpha q(t)]. \quad (32)$$

Then it is easy to check that $H^\alpha[q(t), p_\alpha(t)]$ takes the familiar form

$$H^\alpha[q(t), p_\alpha(t)] := \sum_{i=1}^n \frac{p_\alpha^2}{2M_\alpha} + V^\alpha[q(t)]. \quad (33)$$

Similarly, we define the conformable velocity and acceleration of the conformable particle as

$$v_\alpha^i(t) := D_t^\alpha q^i(t), \quad (34)$$

$$a_\alpha^i(t) := D_t^\alpha v_\alpha^i(t) = (D_t^\alpha)^2 q^i(t). \quad (35)$$

These definitions allows us to interpret conformable dynamics in terms of known concepts.

Some comments are in order here. Due to the mathematical properties of the conformable Lagrangian, primarily the property of linearity and the Leibniz rule, we can develop a conformable mechanics of point particles, which can be generalized to more complex systems based on concepts. Similarly to fractional systems, the conformable parameter $\alpha \in (0, 1]$ defines a family of distinct conformable systems, meaning that not only do the physical quantities, such as conformable mass and potential energy, have different dimensionalities, but the corresponding dynamical equations also belong to different classes of differential equations. Nevertheless, the formulation of conformable mechanics obeys a correspondence principle in the limit $\alpha \rightarrow 1$ where the standard classical mechanics is recovered. Moreover, classical mechanics belongs to the family of conformable mechanics, since $\alpha = 1$ is an allowed value for the conformable parameter.

From a physical point of view, the conformable Lagrangian mechanics describes a nonstandard damping effect and time-dependent mass terms $M_\alpha(t)$ as shown in equation (28). Furthermore, while the conformable derivative is local, the $t^{\alpha-1}$ measure in the action assigns weight to past configurations, introducing a memory-like weighting in the action. The time-dependent coefficients in the equation of motion cause the system's response to depend on accumulated temporal effects, resembling a memory effect.

4. Conformable Actuated Pendulum

The standard actuated pendulum model describes a physical system in which a pendulum is influenced by both kinetic and potential energy. The kinetic energy is dependent on the moment of inertia and the square of the angular velocity. The potential energy is influenced by gravity, the mass of the pendulum bob, and the angular position. This model is useful in understanding the dynamics of pendulums in various physical situations and has many applications, including analysis of simple harmonic motion in pendulums, as studied in physics and mechanics, the design and control of robotic arms, pendulum-based systems, and the study of oscillatory systems in mechanical and civil engineering [56,57].

4.1. Classical Actuated Pendulum

In classical mechanics, the actuated pendulum Lagrangian is given by:

$$L_{standard}[\theta, \dot{\theta}] = \frac{1}{2} I \left(\frac{d\theta}{dt} \right)^2 - V[\theta], \quad (36)$$

$$V[\theta] = V_0 - mgd \cos \theta, \quad (37)$$

where I is the moment of inertia, θ is the angular displacement and the degree of freedom of the pendulum, $\frac{d\theta}{dt}$ is the angular velocity, V is the potential energy of the system with V_0 a constant reference potential energy, m is the mass of the pendulum bob (pendulum's arm center of mass), g is the acceleration due to gravity, d is the distance from the pivot point to the center of mass of the pendulum bob and $\cos \theta$ is the cosine of the angular displacement. By applying the variational principle to the action constructed for $L_{standard}$, we obtain the well known equation of motion

$$I\ddot{\theta} + mgd \sin \theta = 0. \quad (38)$$

In many applications, it is important to introduce non-conservative generalized forces u^i to perform work on the degree of freedom q^i . These forces are external or dissipative, and represent interactions with the environment. In what follows, we consider a simple force that acts on pendulum-based systems, also used in describing control of robotic arms. The modified equation of motion (38) has the following form

$$I\ddot{\theta} + mgd \sin \theta = n_r \tau_m - (b_l + b_m n_r^2) \dot{\theta} + l \cos \theta \cdot F_x. \quad (39)$$

Here, τ_m and $b_m \dot{\theta}$ are the applied or dissipated torques that are multiplied by a transmission factor $n_r \geq 1$. These terms can be used, for example, to model motors in a robotic arm. The factors b_l and b_m are important, as they describe viscous friction at the end points of the physical axis connecting the motor and the pendulum arm. If this axis is oriented along the horizontal z axis and the vertical position is along the y axis, then the torque of the horizontal force F_x is given by the last term on the right-hand side of the above equation $l \cos \theta \cdot F_x$. The equations (38) and (39) are non-linear due to the presence of $\theta(t)$ in the arguments of the harmonic functions.

4.2. Conformable Dynamics with no Force

In order to describe the conformable actuated pendulum dynamics, we apply the conformable Lagrangian formalism developed in Section 3. The conformable Lagrangian is obtained from the standard Lagrangian in equation (37) by applying the general principle of replacing the standard derivatives with conformable derivatives

$$L^{(\alpha)}[\theta, D_t^\alpha \theta] := \frac{1}{2} I^{(\alpha)} (D_t^\alpha \theta)^2 + m^{(\alpha)} g d \cos \theta - V_0^{(\alpha)}. \quad (40)$$

The conformable action functional is obtained from equation (19) by replacing the general Lagrangian with the one from equation (40) above. The result is given by

$$S^{(\alpha)}[\theta] = \int_0^T \left[\frac{1}{2} I^{(\alpha)} (D_t^\alpha \theta)^2 + m^{(\alpha)} g d \cos \theta - V_0^{(\alpha)} \right] d\mu_t^\alpha, \quad (41)$$

In the International System of Units, $\langle I^{(\alpha)} \rangle = ET^{1+\alpha}$ and $\langle m^{(\alpha)} \rangle = M^{-1}T^{\alpha-1}$. The equations of motion are obtained either by applying the variational principle to the action $S^{(\alpha)}[\theta]$ or from equation (25), and the result is

$$I^{(\alpha)} D_t^\alpha (D_t^\alpha \theta) + m^{(\alpha)} g d \sin \theta = 0. \quad (42)$$

The conformable force, the conformable linear momentum of mass blob and the conformable angular momentum can be identified as previously defined, and are given by

$$F_\alpha = -m^{(\alpha)} g d \sin \theta, \quad p_\alpha = m^{(\alpha)} D_t^\alpha \theta, \quad P_\alpha = I^{(\alpha)} D_t^\alpha \theta. \quad (43)$$

It is evident that the standard equation of motion and physical quantities are obtained from conformable equation (42) and quantities defined by relations (43) in the limit $\alpha \rightarrow 1$ as expected.

To better interpret equation (42), it is useful to express it in the form of a standard differential equation

$$I^{(\alpha)} \cdot t^{2-2\alpha} \ddot{\theta} + (1-\alpha) I^{(\alpha)} t^{1-2\alpha} \dot{\theta} + m^{(\alpha)} g d \sin(\theta) = 0. \quad (44)$$

Unlike the classical pendulum equation (38), the modified equation (42) introduces time-dependent scaling factors. The factor $t^{2-2\alpha}$ multiplying the angular acceleration $\ddot{\theta}$ results in a time-dependent system inertia. For $\alpha < 1$, the term $t^{2-2\alpha}$ varies over time, modifying the effective "mass" that resists angular acceleration. The term $(1-\alpha) I t^{1-2\alpha} \dot{\theta}$ acts as an effective damping term whose magnitude depends both on the conformable parameter α and on time t . For $\alpha = 1$, this damping term disappears, restoring the classical undamped dynamics of the pendulum. Thus, the parameter α controls the degree of time-scaling in both inertia and damping contributions. From a physical perspective, such modifications may result from a nonstandard time reparameterization or from modeling physical phenomena such as energy dissipation or effective mass variation, which evolve according to a power law in time.

It is interesting to analyze the type of differential equations from the family described by equation (42) for various α . The nature of the differential equation (42) depends on the value of α :

Case 1: $\alpha = 1$

When $\alpha = 1$, we have

$$t^{2-2\alpha} = t^0 = 1 \quad \text{and} \quad (1-\alpha) = 0. \quad (45)$$

so that (42) reduces to

$$I \ddot{\theta} + mgd \sin(\theta) = 0. \quad (46)$$

This corresponds to the classical nonlinear pendulum equation, where $I^{(1)} = I$ and $m^{(1)} = m$. Its linearization is obtained by assuming $\sin \theta \approx \theta$ for small angles, which leads to the well-known simple harmonic oscillator equation.

Case 2: $\alpha = \frac{1}{2}$

For $\alpha = \frac{1}{2}$, the time factors become

$$t^{2-2\alpha} = t^{2-1} = t, \quad \text{and} \quad t^{1-2\alpha} = t^{1-1} = 1. \quad (47)$$

Then (42) becomes

$$I^{(\frac{1}{2})} t \ddot{\theta} + \frac{1}{2} I^{(\frac{1}{2})} \dot{\theta} + m^{(\frac{1}{2})} g d \sin(\theta) = 0. \quad (48)$$

This non-autonomous ODE has a linearly time-dependent inertia term and a constant-coefficient damping term. In the linearized regime, an appropriate change of variables can transform the equation in a form related to Bessel's equation.

Case 3: $\alpha \in (0, 1)$ with $\alpha \neq \frac{1}{2}$

For other values of α in $(0, 1)$, the equation remains a non-autonomous second-order ODE with power-law time-dependent coefficients

$$I^{(\alpha)} t^{2-2\alpha} \ddot{\theta} + (1 - \alpha) I^{(\alpha)} t^{1-2\alpha} \dot{\theta} + m^{(\alpha)} g d \sin(\theta) = 0. \quad (49)$$

When linearized, the equation takes the form

$$t^{2-2\alpha} \ddot{\theta} + (1 - \alpha) t^{1-2\alpha} \dot{\theta} + \lambda^{(\alpha)} \theta = 0, \quad \text{with} \quad \lambda^{(\alpha)} = \frac{m^{(\alpha)} g d}{I^{(\alpha)}}. \quad (50)$$

Unlike the classical Euler-Cauchy or equidimensional equations which have the form

$$t^2 \ddot{y} + at \dot{y} + by = 0, \quad (51)$$

the exponents from equation (50) do not generally match the Euler-Cauchy structure unless α would take the excluded value $\alpha = 0$. Hence, for the general $\alpha \neq 1$, these equations belong to a class of time-dependent ODEs whose solutions may involve special functions or require numerical methods for a complete description.

In the following, we derive the general solution of (46) which is obtained analytically. For other values of α , the solution is expressed as a power series.

4.3. General Solution for $\alpha = \frac{1}{2}$

For $\alpha = \frac{1}{2}$, the modified linearized pendulum equation, obtained by replacing $\sin \theta$ with θ for small oscillations, takes the form of equation (46) which is repeated here for convenience

$$I t \ddot{\theta}(t) + \frac{I}{2} \dot{\theta}(t) + m g d \theta(t) = 0. \quad (52)$$

Here and throughout this subsection, we omit the index $\frac{1}{2}$ to simplify the formulas. Let us rearrange equation (46) slightly by dividing by I and defining

$$\lambda = \frac{m g d}{I}, \quad (53)$$

we obtain

$$t \ddot{\theta}(t) + \frac{1}{2} \dot{\theta}(t) + \lambda \theta(t) = 0. \quad (54)$$

Dividing (54) by t (assuming $t > 0$) yields

$$\ddot{\theta}(t) + \frac{1}{2t} \dot{\theta}(t) + \frac{\lambda}{t} \theta(t) = 0. \quad (55)$$

To simplify (55), we make a change of variables

$$x = 2\sqrt{\lambda t}, \quad \theta(t) = f(x), \quad t = \frac{x^2}{4\lambda}. \quad (56)$$

After the substitution given in equation (56), we have

$$\dot{\theta}(t) = f'(x) \sqrt{\lambda} t^{-1/2}, \quad (57)$$

$$\ddot{\theta}(t) = \lambda t^{-1} f''(x) - \frac{\sqrt{\lambda}}{2} t^{-3/2} f'(x). \quad (58)$$

Next, we substitute (57) and (58) into (55) and obtain

$$\lambda t^{-1} f''(x) - \frac{\sqrt{\lambda}}{2} t^{-3/2} f'(x) + \frac{1}{2t} \left[f'(x) \sqrt{\lambda} t^{-1/2} \right] + \frac{\lambda}{t} f(x) = 0 \quad (59)$$

Notice the cancellation

$$-\frac{\sqrt{\lambda}}{2} t^{-3/2} f'(x) + \frac{1}{2t} \sqrt{\lambda} t^{-1/2} f'(x) = 0. \quad (60)$$

Thus, we are left with

$$f''(x) + f(x) = 0. \quad (61)$$

The differential equation (61) is the standard simple harmonic oscillator equation. Its general solution is

$$f(x) = A \cos x + B \sin x, \quad (62)$$

where A and B are arbitrary constants determined by initial conditions. Recalling the substitution (56) ($x = 2\sqrt{\lambda}t$) and $\theta(t) = f(x)$, we obtain the solution for $\theta(t)$

$$\theta^{(1/2)}(t) = A \cos \left(2\sqrt{\frac{m^{(1/2)}gd}{I^{(1/2)}}} t \right) + B \sin \left(2\sqrt{\frac{m^{(1/2)}gd}{I^{(1/2)}}} t \right), \quad (63)$$

where we have restored the index one-half notation.

4.4. Power Series Solution of the Linearized Equation for $\alpha \neq 1/2$

We start from the linearized form of the modified pendulum equation (49) and we are dropping the index α for simplicity

$$t^{2-2\alpha} \ddot{\theta}(t) + (1-\alpha)t^{1-2\alpha} \dot{\theta}(t) + \mu \theta(t) = 0, \quad (64)$$

where

$$\mu = \frac{mgd}{I}, \quad (65)$$

Equation (64) is singular at $t = 0$, so we look for a Frobenius series solution, see, for example [58]. Assume a solution of the form

$$\theta(t) = t^s \sum_{n=0}^{\infty} a_n t^{2\alpha n}, \quad (66)$$

where s is yet to be determined, and we set the step $\beta = 2\alpha$ so that the time-dependent coefficients match the power of t . Its derivatives are

$$\dot{\theta}(t) = \sum_{n=0}^{\infty} a_n (s + 2\alpha n) t^{s+2\alpha n-1}, \quad (67)$$

$$\ddot{\theta}(t) = \sum_{n=0}^{\infty} a_n (s + 2\alpha n)(s + 2\alpha n - 1) t^{s+2\alpha n-2}. \quad (68)$$

Next, we substitute (66)-(68) into (64). The equation (64) becomes

$$\sum_{n=0}^{\infty} a_n [(s + 2\alpha n)(s + 2\alpha n - 1) + (1 - \alpha)(s + 2\alpha n)] t^{s+2\alpha n-2\alpha} + \mu \sum_{n=0}^{\infty} a_n t^{s+2\alpha n} = 0. \quad (69)$$

We note that the two sums involve different powers of t : the first sum contains exponents of the form $s + 2\alpha n - 2\alpha$, while the second sum has exponents $s + 2\alpha n$. To combine them into a single series, we re-index the first sum by setting

$$m = n - 1, \quad \text{or equivalently} \quad n = m + 1, \quad (70)$$

Separating the $n = 0$ term from the first sum, we write (69) as

$$a_0[s(s-1) + (1-\alpha)s]t^{s-2\alpha} + \sum_{m=0}^{\infty} a_{m+1} \mathcal{C}(m) t^{s+2\alpha m} + \mu \sum_{n=0}^{\infty} a_n t^{s+2\alpha n} = 0, \quad (71)$$

where

$$\mathcal{C}(m) = (s + 2\alpha(m+1))(s + 2\alpha(m+1) - 1) + (1-\alpha)(s + 2\alpha(m+1)). \quad (72)$$

The lowest power in (71) is $t^{s-2\alpha}$, which comes solely from the $n = 0$ term of the first sum. For a non-trivial solution (with $a_0 \neq 0$), its coefficient must vanish. Noting that

$$s(s-1) + (1-\alpha)s = s[(s-1) + (1-\alpha)] = s(s-\alpha), \quad (73)$$

the **indicial equation** is

$$s(s-\alpha) = 0. \quad (74)$$

Thus, the two possible exponents are

$$s = 0 \quad \text{or} \quad s = \alpha. \quad (75)$$

Typically, the solution with greater exponent is preferred to ensure regularity at $t = 0$ [58]. Here, we choose

$$s = \alpha. \quad (76)$$

With $s = \alpha$, the series solution becomes

$$\theta(t) = t^\alpha \sum_{n=0}^{\infty} a_n t^{2\alpha n}. \quad (77)$$

For $m \geq 0$, the coefficient of $t^{s+2\alpha m}$ in (71) must vanish. First, we compute $\mathcal{C}(m)$ with $s = \alpha$

$$\mathcal{C}(m) = (\alpha + 2\alpha(m+1))[(\alpha + 2\alpha(m+1)) - 1] + (1-\alpha)(\alpha + 2\alpha(m+1)). \quad (78)$$

After some algebraic manipulations, we obtain

$$\mathcal{C}(m) = 2\alpha^2 (2m+3)(m+1). \quad (79)$$

Now, we equate the coefficient of $t^{\alpha+2\alpha m}$ in (71) to zero

$$a_{m+1} 2\alpha^2 (2m+3)(m+1) + \mu a_m = 0, \quad m \geq 0. \quad (80)$$

Hence, the recurrence relation is

$$a_{m+1}^{(\alpha)} = -\frac{\mu}{2\alpha^2 (2m+3)(m+1)} a_m^{(\alpha)}, \quad m \geq 0, \quad (81)$$

where the index α has been reinstated and

$$\mu^{(\alpha)} = \frac{m^{(\alpha)}gd}{I^{(\alpha)}}, \quad (82)$$

The coefficient a_0 remains arbitrary and can be chosen based on initial conditions.

Finally, we conclude that the solution of the linearized equation (64) for $\alpha \in (0, 1)$ with $\alpha \neq \frac{1}{2}$, corresponding to the choice $s = \alpha$, is given by

$$\theta^{(\alpha)}(t) = t^\alpha \sum_{m=0}^{\infty} a_m^{(\alpha)} t^{2\alpha m}, \quad (83)$$

with the recurrence relation (81) determining the coefficients.

5. Conformable Actuated Pendulum with a Non-Conservative Force

In most applications from physics and engineering, the actuated pendulum interacts with non-conservative generalized forces. In this section, we consider a model that is widely used in theoretical mechanics and which describes the action of a pendulum constrained in the vertical (x, g) plane and is subject to a force F_x directed along the x axis, acting at the tip of a pendulum arm of length l . The pendulum is articulated at the origin of the Cartesian coordinate system (x, y, z) where it is connected to a horizontal shaft. At the opposite end, the shaft is attached to a second mechanical component which enables its rotation. The articulation and the mechanical component are characterized by viscous friction coefficients b_l and b_m , respectively. The torque applied at one end of the mechanical component is multiplied by a rotational coefficient $n_r \geq 1$ ². Under these conditions, the conformable Euler-Lagrange equation (42) becomes

$$I^{(\alpha)} D_t^\alpha (D_t^\alpha \theta) + m^{(\alpha)} g d \sin \theta = n_r^{(\alpha)} \tau_m^{(\alpha)} - \left(b_l^{(\alpha)} + b_m^{(\alpha)} n_r^{(\alpha)^2} \right) \dot{\theta} + l \cos \theta F_x^{(\alpha)}, \quad (84)$$

where all superscripted quantities, e.g., $I^{(\alpha)}$, $m^{(\alpha)}$, $b_l^{(\alpha)}$, etc. indicate that the physical parameters may depend on the conformable parameter α . We note that the parameters describing the generalized force now depend on α as well, and have different units from the non-conformable case resulting from dimensional analysis. In terms of the standard derivative, equation (84) represents the conformable modification of equation (44), and takes the form

$$I^{(\alpha)} \cdot t^{2-2\alpha} \ddot{\theta} + \left[(1-\alpha) I^{(\alpha)} t^{1-2\alpha} + b_l^{(\alpha)} + b_m^{(\alpha)} n_r^{(\alpha)^2} \right] \dot{\theta} + m^{(\alpha)} g d \sin(\theta) = n_r^{(\alpha)} \tau_m^{(\alpha)} + l \cos \theta F_x^{(\alpha)}. \quad (85)$$

The physical interpretation of the model proposed in this work is related to the physical content of these parameters.

5.1. Physical Interpretation of Parameters

The conformable parameters $m^{(\alpha)}$, $I^{(\alpha)}$, $b_l^{(\alpha)}$, and $b_m^{(\alpha)}$ were initially introduced as formal generalizations of classical parameters. However, these parameters are not only mathematical objects; they may also represent effective or apparent physical quantities, reflecting changes in the physical properties of real systems. More precisely, variations in apparent mass $m^{(\alpha)}$ and moment of inertia $I^{(\alpha)}$ can describe physical phenomena such as viscoelasticity or the presence of microcracks in the structure of materials. Such effects modify the mass distribution, stiffness, or damping characteristics, influencing inertia and dissipative behaviors. For this reason, conformable parameters provide

² For example, this configuration can represent a robotic arm of length l driven by a motor m with a reduction gear r .

a practical mathematical framework for modeling this type of complex, time-dependent physical behavior in materials with intricate internal structures such as viscoelastic materials, composites, or microcracked structures. Hence, such complex systems can be effectively modeled using a conformable formalism, with parameters explicitly depending on α . This approach allows for direct or indirect experimental validation through measurements and numerical simulations. In particular, since $b_l^{(\alpha)}$, and $b_m^{(\alpha)}$, characterize viscous friction in the articulations between the shaft, the arm, and the generator of mechanical force, a fractional variation in these friction components makes the system suitable for the application of this model. This can be observed in equation (42), where the friction terms $b_l^{(\alpha)}$ and $b_m^{(\alpha)} n_r^{(\alpha)^2}$ represent the viscous friction at the endpoints of the physical shaft connecting, for example, a motor to the pendulum arm. Their presence in the coefficient of $\dot{\theta}(t)$ indicates that, in addition to the time-varying friction term, $(1 - \alpha)I^{(\alpha)}t^{1-2\alpha}$, there are also constant friction contributions that depend on α . Regarding the physical origin of these terms, one can consider that viscous friction may vary due to internal changes in viscosity at the articulations in a closed system, or due to environmental factors in systems with open articulations. Similarly, the parameter $n_r^{(\alpha)}$ can be interpreted as a variable transmission coefficient that depends on α as a consequence of internal modifications or interactions with the environment. The same reasoning can be applied to the conformable force $F_x^{(\alpha)}$.

From a physical perspective, the conformable actuated pendulum could model non-standard damping effects. Observe that the conformable action functional simulates memory effects through the integration kernel. However, these effects are not genuine physical phenomena, since the theory remains local. In both the free and forced cases, if the effective inertia $I^{(\alpha)}$ or the friction coefficients $b_l^{(\alpha)}$ and $b_m^{(\alpha)}$ are influenced by scaling laws, these factors capture that behavior. Moreover, the friction terms $b_l^{(\alpha)}$ and $b_m^{(\alpha)} n_r^{(\alpha)^2}$ describe viscous friction at the end-points of the physical shaft, which connects, for example, a motor to the pendulum arm. Their presence in the coefficient of $\dot{\theta}(t)$ suggests that, in addition to the time-varying friction term $(1 - \alpha)I^{(\alpha)}t^{1-2\alpha}$, there are also constant friction contributions that depend on α .

The term $m^{(\alpha)}gd \sin \theta(t)$ is the conformable gravitational restoring torque, which differs from the classical case due to the dependence of the mass parameter on α . Its presence ensures that, in the absence of friction and external forces, the system exhibits oscillatory (or pendulum-like) behavior. A new physical effects arises from the external actuation and forcing terms on the right-hand side of equation (85). These terms describe the actuation and external perturbations influencing the pendulum system in different regimes characterized by the values of α .

This leads to an analysis of the mathematical nature of the fractional differential equations obtained from this model.

5.2. Families of Conformable Equations

As in the case of conformable pendulum without force discussed in the previous section, the physical interpretation of the modified actuated pendulum equation, which includes a non-conservative generalized force, can be inferred from the equation (85). The time - dependent factors $t^{2-2\alpha}$ and $t^{1-2\alpha}$ affect the inertial and damping terms, respectively, resulting in a power-law dependence on time of these effects. Physically, this can be interpreted as a *time reparameterization* or an adjustment of the inertia and damping of the conformable actuated pendulum as the system evolves. This interpretation remains valid in the free case, as it results from the contribution of the conformable kinetic term to the equations of motion.

As in the free case, the classification of the equations of motion for the conformable actuated pendulum with non-holonomic generalized forces depends on the value of the conformable parameter. The general form of the equation (85) corresponds to a second-order, non-autonomous ordinary differential equation (ODE) with time-dependent coefficients. The properties of this ODE depend on α and are detailed below

Case 1: $\alpha = 1$

When $\alpha = 1$, the time-scaling factors simplify:

$$t^{2-2\alpha} = t^0 = 1, \quad t^{1-2\alpha} = t^{-1}. \quad (86)$$

Thus, (85) becomes

$$I^{(1)} \ddot{\theta}(t) + \left[b_l^{(1)} + b_m^{(1)} n_r^{(1)2} \right] \dot{\theta}(t) + m^{(1)} g d \sin \theta(t) = n_r^{(1)} \tau_m^{(1)} + l \cos \theta(t) F_x^{(1)}. \quad (87)$$

This corresponds to the classical actuated pendulum equation with constant coefficients, apart from the inherent nonlinearity in $\sin \theta$ and $\cos \theta$. When linearized for small angles (i.e., $\sin \theta \approx \theta$ and $\cos \theta \approx 1$), it reduces to a linear second-order ODE with constant coefficients. This equation is a standard result and belongs to the class of autonomous ODEs.

Case 2: $\alpha = \frac{1}{2}$

For $\alpha = \frac{1}{2}$, the time factors become

$$t^{2-2\alpha} = t^{2-1} = t, \quad t^{1-2\alpha} = t^{1-1} = 1. \quad (88)$$

Then the equation (85) reads

$$\begin{aligned} I^{(1/2)} t \ddot{\theta}(t) + \left[\frac{1}{2} I^{(1/2)} + b_l^{(1/2)} + b_m^{(1/2)} n_r^{(1/2)2} \right] \dot{\theta}(t) + m^{(1/2)} g d \sin \theta(t) \\ = n_r^{(1/2)} \tau_m^{(1/2)} + l \cos \theta(t) F_x^{(1/2)}. \end{aligned} \quad (89)$$

In the linearized regime, this non-autonomous ODE has a time-dependent inertial term proportional to t , while the damping term remains constant. Through an appropriate transformation, the linear part can be rewritten in a form related to Bessel's equation or an Euler-type equation. Thus, for $\alpha = \frac{1}{2}$ the equation belongs to a class of ODEs that can be solved using known techniques involving special functions.

Case 3: $\alpha \in (0, 1)$ with $\alpha \neq \frac{1}{2}$

For general α in the interval $(0, 1)$ (with $\alpha \neq \frac{1}{2}$), the equation

$$\begin{aligned} I^{(\alpha)} t^{2-2\alpha} \ddot{\theta}(t) + \left[(1-\alpha) I^{(\alpha)} t^{1-2\alpha} + b_l^{(\alpha)} + b_m^{(\alpha)} n_r^{(\alpha)2} \right] \dot{\theta}(t) + m^{(\alpha)} g d \sin \theta(t) \\ = n_r^{(\alpha)} \tau_m^{(\alpha)} + l \cos \theta(t) F_x^{(\alpha)}, \end{aligned} \quad (90)$$

has power-law time-dependent coefficients in both the inertial and damping terms. In the linearized form, one obtains a non-autonomous ODE of the form

$$t^{2-2\alpha} \ddot{\theta}(t) + \left[(1-\alpha) t^{1-2\alpha} + C \right] \dot{\theta}(t) + \mu \theta(t) = \text{forcing}, \quad (91)$$

with constants C and μ defined in terms of the physical parameters. As noted earlier, in general, such an ODE does not belong to the classical Euler-Cauchy (also known as equidimensional equation) form unless the exponents match the ones in the Euler-Cauchy equation. Only for specific values of α ($\alpha = 1$ and $\alpha = \frac{1}{2}$) does the equation reduce to forms for which the solutions can be obtained by well-known methods. For other values of α , one typically needs to apply power series methods (such as the Frobenius method), as illustrated in previous derivations, and numerical methods or specialized transformations to relate the equation to known special functions.

In the next subsections, we discuss the solutions to equation (85) in the linearized limit.

5.3. General Solution for $\alpha = 1/2$

We consider the linearized equation

$$I^{(1/2)} t \ddot{\theta}(t) + \left[\frac{1}{2} I^{(1/2)} + b_l^{(1/2)} + b_m^{(1/2)} \left(n_r^{(1/2)} \right)^2 \right] \dot{\theta}(t) + m^{(1/2)} g d \theta(t) = n_r^{(1/2)} \tau_m^{(1/2)} + l F_x^{(1/2)}, \quad (92)$$

where the trigonometric functions have been linearized by approximating

$$\sin \theta \approx \theta, \quad \cos \theta \approx 1. \quad (93)$$

For simplicity, we redefine the quantities in equation (92) and omit the index $1/2$

$$I_0 \equiv I^{(1/2)}, \quad (94)$$

$$D \equiv \frac{1}{2} I^{(1/2)} + b_l^{(1/2)} + b_m^{(1/2)} \left(n_r^{(1/2)} \right)^2, \quad (95)$$

$$\mu \equiv m^{(1/2)} g d, \quad (96)$$

$$F \equiv n_r^{(1/2)} \tau_m^{(1/2)} + l F_x^{(1/2)}. \quad (97)$$

Also, we divide (92) by I_0 to obtain the standard form

$$t \ddot{\theta}(t) + p \dot{\theta}(t) + q \theta(t) = f, \quad (98)$$

with

$$p = \frac{D}{I_0}, \quad q = \frac{\mu}{I_0}, \quad f = \frac{F}{I_0}. \quad (99)$$

Since the derivation of the solution is somewhat lengthy, yet follows standard methods, we leave it to Appendix A. The general solution of the linearized equation (92) is given by (A32). It consists of a steady-state (particular) solution and a homogeneous solution expressed in terms of Bessel functions of order $1 - p$, where the parameter p is defined in (99), and, in the original notation, in equation (A33). The general solution has the form

$$\theta^{(1/2)}(t) = \frac{n_r^{(1/2)} \tau_m^{(1/2)} + l F_x^{(1/2)}}{m^{(1/2)} g d} + \left[2 \sqrt{\frac{m^{(1/2)} g d}{I^{(1/2)}} t} \right]^{1-p} \left\{ A J_{1-p} \left(2 \sqrt{\frac{m^{(1/2)} g d}{I^{(1/2)}} t} \right) + B Y_{1-p} \left(2 \sqrt{\frac{m^{(1/2)} g d}{I^{(1/2)}} t} \right) \right\}, \quad (100)$$

where

$$p = \frac{D}{I^{(1/2)}}, \quad \text{with} \quad D = \frac{1}{2} I^{(1/2)} + b_l^{(1/2)} + b_m^{(1/2)} \left(n_r^{(1/2)} \right)^2, \quad (101)$$

and A and B are constants determined by the initial conditions. For small values of t , the solution (101) takes the form

$$\theta^{(1/2)}(X(t)) \simeq \frac{n_r^{(1/2)} \tau_m^{(1/2)} + l F_x^{(1/2)}}{m^{(1/2)} g d} + X(t)^{1-p} \left[\frac{A}{\Gamma(p+1)} \left(\frac{1}{2} X(t) \right)^p - \frac{B \Gamma(p)}{\pi} \left(\frac{1}{2} X(t) \right)^{-p} \right], \quad (102)$$

where

$$X(t) = 2\sqrt{\frac{m^{(1/2)}gd}{I^{(1/2)}}}t. \quad (103)$$

It is important to note that, in addition to the solution given in equation (100), another general solution can be obtained for discrete values of the parameter p . Namely, if

$$p_n = \frac{1}{2} + \frac{b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2}{I^{(1/2)}} = \frac{1}{2}(2n+1), \quad n = 0, 1, 2, \dots \quad (104)$$

then the general solution of the equation (92) is given by

$$\theta_n^{(1/2)}(t) = \begin{cases} A_1 \frac{d^n}{dt^n} \cos \left[4 \left(\frac{1}{2} + \frac{b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2}{I^{(1/2)}} \right) t \right]^{1/2} \\ + B_1 \frac{d^n}{dt^n} \sin \left[4 \left(\frac{1}{2} + \frac{b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2}{I^{(1/2)}} \right) t \right]^{1/2}, & \text{if: } \left(\frac{1}{2} + \frac{b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2}{I^{(1/2)}} \right) t > 0 \\ A_2 \frac{d^n}{dt^n} \cosh \left[4 \left(\frac{1}{2} + \frac{b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2}{I^{(1/2)}} \right) t \right]^{1/2} \\ + B_2 \frac{d^n}{dt^n} \sinh \left[4 \left(\frac{1}{2} + \frac{b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2}{I^{(1/2)}} \right) t \right]^{1/2}, & \text{if: } \left(\frac{1}{2} + \frac{b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2}{I^{(1/2)}} \right) t < 0 \end{cases} \quad (105)$$

Here, A_1 , B_1 , A_2 and B_2 are integration constants. We note that under the assumption made in the previous section and for $n = 0$, the harmonic solution (63) of the conformable free pendulum is obtained as a particular case of (105).

5.4. Power Series Solution of Linearized Forced Equation for $\alpha \neq 1/2$

The linearized equation for the conformable forced actuated pendulum (90) is given by

$$\begin{aligned} I^{(\alpha)} t^{2-2\alpha} \ddot{\theta}(t) + \left[(1-\alpha) I^{(\alpha)} t^{1-2\alpha} + b_l^{(\alpha)} + b_m^{(\alpha)} n_r^{(\alpha)^2} \right] \dot{\theta}(t) + m^{(\alpha)} g d \theta(t) \\ = n_r^{(\alpha)} \tau_m^{(\alpha)} + l F_x^{(\alpha)}, \end{aligned} \quad (106)$$

where the $\alpha \neq 1/2$ and we have adopted the linearization $\sin \theta \approx \theta$ and $\cos \theta \approx 1$. Since the derivation is quite lengthy, we present it in Appendix B. The final result is given in equation (A64) which is

$$\theta(t) = \theta_p(t) + \theta_h(t) = \frac{F_\alpha}{m^{(\alpha)} g d} + t^{s_0} \sum_{n=0}^{\infty} a_n t^n. \quad (107)$$

Here, s_0 is given by (A53) and the coefficients a_n satisfy the recurrence relation (A60) below

$$I^{(\alpha)} [(s_0 + n)(s_0 + n - 1) + (1 - \alpha)(s_0 + n)] a_n + D_\alpha (s_0 + n - \Delta_1) a_{n-\Delta_1} + m^{(\alpha)} g d a_{n-\Delta_2} = 0, \quad (108)$$

where s_0 is chosen such as the lowest exponent is non-negative, see equation (A53).

6. Discussion

In this paper, we have constructed the conformable actuated pendulum model using the conformable Lagrangian formalism. The equations of motion, derived from the variational principle applied to the corresponding conformable action functionals, are actually families of differential equations of various types, characterized by the conformable parameter $\alpha \in (0, 1]$. For $\alpha = 1$, the standard

pendulum is recovered in all cases and has been extensively studied in the literature for a long time. For $\alpha \in (0, 1)$ new classes of actuated pendulums are obtained. From these, the case $\alpha = 1/2$ stands out, as the equations of motion can be analytically solved in terms of Bessel functions. In this case, we have determined the solutions to the equations of motion for both free and forced pendulums. For values of $\alpha \neq 1/2$, the equations of motion do not belong to the known classes of differential equations. Therefore, different resolution methods must be applied. We have solved these equations using the Frobenius power series method in both the free and forced cases.

An important remark is that the conformable actuated pendulum model developed here represents a deformation of a specific standard mechanical model, with a wide range of applications in both physics and engineering. In the forced case, the model is defined by five parameters which can be associated to technical characteristics: $m^{(\alpha)}$, $n_r^{(\alpha)}$, $b_l^{(\alpha)}$, b_m^α and $F_x^{(\alpha)}$ which describe the non-holonomic generalized dissipative force. This suggests that the proposed model provides a framework for the modeling of mechanical systems whose parameters depend on a fractional parameter $\alpha \in (0, 1]$.

The physical interpretation of the proposed model is linked to the conformable parameters $m^{(\alpha)}$, $I^{(\alpha)}$, $b_l^{(\alpha)}$, and $b_m^{(\alpha)}$, which were introduced as formal generalizations of classical parameters. The parameters represent effective physical quantities in applications. Variations in the apparent mass $m^{(\alpha)}$ and moment of inertia $I^{(\alpha)}$ describes physical phenomena such as viscoelasticity and affect mass distribution, stiffness, damping properties, and so on. In equation (42), the friction terms $b_l^{(\alpha)}$ and $b_m^{(\alpha)} n_r^{(\alpha)^2}$ represent the viscous friction at the endpoints of the physical shaft, with both the time-varying friction term $(1 - \alpha) I^{(\alpha)} t^{1-2\alpha}$ and constant friction contributions that depend on α . The parameters $n_r^{(\alpha)}$ and conformable $F_x^{(\alpha)}$ can be interpreted as a variable transmission coefficient and force. They contain information about the internal modifications or environmental factors that can alter system behavior. As a result, the conformable actuated pendulum provide a practical mathematical framework for modeling complex, time-dependent physical behaviors in materials such as viscoelastic substances, composites, or microcracked structures, with parameters explicitly depending on α and subject to experimental validation through measurements and numerical simulations. Also, the conformable actuated pendulum can model a non-standard damping effect and simulates memory effects through the integration kernel of its action. Nevertheless, these effects are not genuine physical phenomena, since the theory is local.

The general properties of the mathematical structure of the fractional differential equations obtained from this model can be summarize as follows:

Summary for Conformable Homogeneous Equations

- $\alpha = 1$: Equation (42) reduces to the classical nonlinear pendulum equation (45) or the simple harmonic oscillator upon linearization.
- $\alpha = \frac{1}{2}$: The equation (48) displays a time-dependent inertia term proportional to t and a constant effective damping term. Its linearized form can be related to the Bessel-type equations.
- $\alpha \in (0, 1)$, $\alpha \neq \frac{1}{2}$: Equation (49) is a non-autonomous ODE with power-law time-dependent coefficients. In general, even linearized, these do not belong to the classical families, like Euler-Cauchy, with well-known closed-form solutions. Therefore, their analysis typically requires specialized techniques or numerical methods.

Summary for Conformable Actuated Pendulum with a Non-Conservative Force

- $\alpha = 1$: The equation (87) is an autonomous nonlinear ODE (or its linearized version is a second-order linear ODE with constant coefficients). This is the standard model for an actuated pendulum with friction and external forces.

- $\alpha = \frac{1}{2}$: Equation (89) is a non-autonomous ODE with a time-dependent inertial term ($\propto t$) and a constant effective damping. By linearization, it can often be reformulated as a Bessel-type or Euler-type ODE with known analytic solutions in terms of special functions.
- $\alpha \in (0, 1)$, $\alpha \neq \frac{1}{2}$: Equation (90) represents a more general class of non-autonomous ODEs with power-law time-dependent coefficients. In general, these equations do not correspond to standard forms unless further transformations or approximations are applied. Their analysis typically relies on series methods (such as the Frobenius method) or numerical approaches.

To conclude, we outline a brief list of interesting problems arising from this study. As it was previously mentioned, the physical modeling of the parameters of the actuated conformable pendulum is a very important task. Since the actuated pendulum has many practical physical applications, the study of its parameters can be supported by experimental data. It is very likely that both the internal and the external dependence of these parameters on α makes them functions that vary over time. Another important problem is to determine the physical properties of the conformable actuated pendulum. Also, from a mathematical point of view, we foresee a deeper study of the mathematical structures involved in the construction of the conformable pendulum, as well as of the differential equations and their solutions.

The conformable calculus framework provides a powerful tool to model mechanical systems with time-dependent properties, offering a more flexible and accurate approach compared to classical formulations. This approach is particularly relevant in fields like robotics and mechatronics, where friction and damping in robotic joints can evolve over time, affecting system dynamics. Likewise, in the study of vehicle dynamics, adaptive suspension systems, and smart materials require models that can describe variations in inertia and damping properties. In advanced materials science, the study of viscoelastic composites or microcracked structures can benefit from conformable modeling, as it allows a more precise description of their time-dependent mechanical behavior. Furthermore, in biomechanics, the variable inertial and damping characteristics of human joints can be represented using this approach, thus improving the precision of motion analysis and prosthetic design. Finally, in the case of micro- and nanotechnologies, where MEMS / NEMS structures operate under conditions where classical assumptions about mass and inertia are no longer available, conformable calculus provides a more realistic framework for system modeling. By establishing a connection between classical and fractional methods, this approach improves the understanding and optimization of complex engineering and physical systems in various research fields.

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Appendix A. Proof of the Solution of Equation (92)

In this Appendix we give the proof of the general solution of the equation (92) for $p \neq \frac{1}{2}(2n + 1)$, with the redefinition and the standard form recalled here from the main text for convenience

$$I^{(1/2)} t \ddot{\theta}(t) + \left[\frac{1}{2} I^{(1/2)} + b_l^{(1/2)} + b_m^{(1/2)} \left(n_r^{(1/2)} \right)^2 \right] \dot{\theta}(t) + m^{(1/2)} g d \theta(t) = n_r^{(1/2)} \tau_m^{(1/2)} + I F_x^{(1/2)}, \quad (A1)$$

The terms and coefficients from equation (92) have been redefined as

$$I_0 \equiv I^{(1/2)}, \quad (\text{A2})$$

$$D \equiv \frac{1}{2} I^{(1/2)} + b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2, \quad (\text{A3})$$

$$\mu \equiv m^{(1/2)} g d, \quad (\text{A4})$$

$$F \equiv n_r^{(1/2)} \tau_m^{(1/2)} + l F_x^{(1/2)}. \quad (\text{A5})$$

The standard form of the equation (92) becomes

$$t \ddot{\theta}(t) + p \dot{\theta}(t) + q \theta(t) = f, \quad (\text{A6})$$

The coefficients introduced here are

$$p = \frac{D}{I_0}, \quad q = \frac{\mu}{I_0}, \quad f = \frac{F}{I_0}. \quad (\text{A7})$$

In order to solve equation (98), we need to find the solution to the homogeneous part. We note that we solved a similar equation (54) in the previous section. However, equation (92) is more general due to its coefficients.

The homogeneous equation takes the form

$$t \ddot{\theta}_h(t) + p \dot{\theta}_h(t) + q \theta_h(t) = 0. \quad (\text{A8})$$

We change the variable by

$$x = 2\sqrt{q}t, \quad (\text{A9})$$

and define

$$\theta_h(t) = f(x). \quad (\text{A10})$$

Since

$$t = \frac{x^2}{4q}, \quad (\text{A11})$$

we differentiate (A9) with respect to t to obtain

$$\frac{dx}{dt} = \frac{\sqrt{q}}{\sqrt{t}} = \frac{2q}{x}. \quad (\text{A12})$$

Next, we express the derivatives in terms of the new variable

$$\dot{\theta}_h(t) = f'(x) \frac{2q}{x}, \quad (\text{A13})$$

$$\ddot{\theta}_h(t) = \frac{4q^2}{x^2} f''(x) - \frac{4q^2}{x^3} f'(x). \quad (\text{A14})$$

By substituting (A13) and (A14) into (A8) we obtain

$$q f''(x) - \frac{q}{x} f'(x) + \frac{2pq}{x} f'(x) + q f(x) = 0. \quad (\text{A15})$$

Assuming that $q \neq 0$ and dividing the equation (A15) by q gives

$$f''(x) + \frac{2p-1}{x} f'(x) + f(x) = 0. \quad (\text{A16})$$

Now, we transform equation (A16) to standard Bessel form by setting

$$f(x) = x^\beta g(x), \quad (\text{A17})$$

which yields

$$x^\beta \left\{ g''(x) + \left(\frac{2\beta + 2p - 1}{x} \right) g'(x) + \left[\frac{\beta(\beta - 1) + (2p - 1)\beta}{x^2} + 1 \right] g(x) \right\} = 0. \quad (\text{A18})$$

We choose β such that the coefficient of $g'(x)$ is $\frac{1}{x}$ which implies that

$$2\beta + 2p - 1 = 1 \implies \beta = 1 - p. \quad (\text{A19})$$

With this choice, the coefficient of $g(x)$ becomes

$$\frac{(1 - p)(-p) + (2p - 1)(1 - p)}{x^2} + 1 = -\frac{(1 - p)^2}{x^2} + 1. \quad (\text{A20})$$

Thus, the equation for $g(x)$ reduces to

$$g''(x) + \frac{1}{x} g'(x) + \left[1 - \frac{(1 - p)^2}{x^2} \right] g(x) = 0. \quad (\text{A21})$$

Equation (A21) is the standard Bessel differential equation of order

$$\nu = 1 - p. \quad (\text{A22})$$

Hence, the general solution for $g(x)$ is

$$g(x) = A J_{1-p}(x) + B Y_{1-p}(x), \quad (\text{A23})$$

where $J_{1-p}(x)$ and $Y_{1-p}(x)$ are Bessel functions of the first and second kinds, and A, B are arbitrary constants. Returning to $f(x)$ defined by equation (A17) with $\beta = 1 - p$, we obtain

$$f(x) = x^{1-p} \left[A J_{1-p}(x) + B Y_{1-p}(x) \right]. \quad (\text{A24})$$

Recalling (A10) and (A9), the homogeneous solution is

$$\theta_h(t) = \left[2\sqrt{q}t \right]^{1-p} \left[A J_{1-p}(2\sqrt{q}t) + B Y_{1-p}(2\sqrt{q}t) \right], \quad (\text{A25})$$

with

$$q = \frac{\mu}{I_0} = \frac{m^{(1/2)}gd}{I^{(1/2)}}. \quad (\text{A26})$$

We now seek a particular solution $\theta_p(t)$ of (98). Since the forcing term f in (98) is a constant, we try a constant solution

$$\theta_p(t) = C, \quad (\text{A27})$$

which implies $\dot{\theta}_p(t) = 0$ and $\ddot{\theta}_p(t) = 0$. Substituting into (98) gives

$$qC = f. \quad (\text{A28})$$

It follows that

$$C = \frac{f}{q} = \frac{F}{I_0 q} = \frac{F}{m^{(1/2)}gd}, \quad (\text{A29})$$

from which the particular solution is

$$\theta_p(t) = \frac{n_r^{(1/2)} \tau_m^{(1/2)} + l F_x^{(1/2)}}{m^{(1/2)} g d}. \quad (\text{A30})$$

Now we can write the general solution to (98) as a sum of the homogeneous and particular solutions

$$\theta(t) = \theta_p(t) + \theta_h(t). \quad (\text{A31})$$

Substituting (A25) and (A30) into (A31), we obtain

$$\begin{aligned} \theta(t) = & \frac{n_r^{(1/2)} \tau_m^{(1/2)} + l F_x^{(1/2)}}{m^{(1/2)} g d} \\ & + \left[2 \sqrt{\frac{m^{(1/2)} g d}{I^{(1/2)}} t} \right]^{1-p} \left\{ A J_{1-p} \left(2 \sqrt{\frac{m^{(1/2)} g d}{I^{(1/2)}} t} \right) + B Y_{1-p} \left(2 \sqrt{\frac{m^{(1/2)} g d}{I^{(1/2)}} t} \right) \right\}, \end{aligned} \quad (\text{A32})$$

where

$$p = \frac{D}{I^{(1/2)}}, \quad \text{with} \quad D = \frac{1}{2} I^{(1/2)} + b_l^{(1/2)} + b_m^{(1/2)} (n_r^{(1/2)})^2. \quad (\text{A33})$$

The constants A and B are determined by the initial conditions.

Appendix B. Power Series Solution of the Linearized Equation (106)

Here, we present the detailed derivation of the conformable forced actuated pendulum equation (90) using Frobenius method. The equation has the non-standard form

$$\begin{aligned} I^{(\alpha)} t^{2-2\alpha} \ddot{\theta}(t) + \left[(1-\alpha) I^{(\alpha)} t^{1-2\alpha} + b_l^{(\alpha)} + b_m^{(\alpha)} (n_r^{(\alpha)})^2 \right] \dot{\theta}(t) \\ + m^{(\alpha)} g d \theta(t) = n_r^{(\alpha)} \tau_m^{(\alpha)} + l F_x^{(\alpha)}, \end{aligned} \quad (\text{A34})$$

We introduce the following shorthand notation

$$C_\alpha \equiv b_l^{(\alpha)} + b_m^{(\alpha)} (n_r^{(\alpha)})^2, \quad (\text{A35})$$

$$F_\alpha \equiv n_r^{(\alpha)} \tau_m^{(\alpha)} + l F_x^{(\alpha)}. \quad (\text{A36})$$

Then, the equation becomes

$$I^{(\alpha)} t^{2-2\alpha} \ddot{\theta}(t) + (1-\alpha) I^{(\alpha)} t^{1-2\alpha} \dot{\theta}(t) + C_\alpha \dot{\theta}(t) + m^{(\alpha)} g d \theta(t) = F_\alpha. \quad (\text{A37})$$

Note that the constant forcing term leads to a constant particular solution which can be found separately. Therefore, we focus here on finding a series solution for the associated homogeneous equation

$$I^{(\alpha)} t^{2-2\alpha} \ddot{\theta}(t) + (1-\alpha) I^{(\alpha)} t^{1-2\alpha} \dot{\theta}(t) + C_\alpha \dot{\theta}(t) + m^{(\alpha)} g d \theta(t) = 0. \quad (\text{A38})$$

Next, we assume a Frobenius series solution about $t = 0$ of the form

$$\theta(t) = t^s \sum_{n=0}^{\infty} a_n t^n, \quad a_0 \neq 0. \quad (\text{A39})$$

The derivatives are given by

$$\dot{\theta}(t) = \frac{d\theta}{dt} = \sum_{n=0}^{\infty} (s+n) a_n t^{s+n-1}, \quad (\text{A40})$$

$$\ddot{\theta}(t) = \frac{d^2\theta}{dt^2} = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n t^{s+n-2}. \quad (\text{A41})$$

Now, we substitute (A39)-(A41) into (A38). The three terms become **(i) The Inertial Term:**

$$I^{(\alpha)} t^{2-2\alpha} \ddot{\theta}(t) = I^{(\alpha)} \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n t^{s+n-2+2-2\alpha} = I^{(\alpha)} \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n t^{s+n-2\alpha}. \quad (\text{A42})$$

(ii) The First Part of the Damping Term:

$$(1-\alpha) I^{(\alpha)} t^{1-2\alpha} \dot{\theta}(t) = (1-\alpha) I^{(\alpha)} \sum_{n=0}^{\infty} (s+n) a_n t^{s+n-1+1-2\alpha} = (1-\alpha) I^{(\alpha)} \sum_{n=0}^{\infty} (s+n) a_n t^{s+n-2\alpha}. \quad (\text{A43})$$

(iii) The Second Part of the Damping Term:

$$C_{\alpha} \dot{\theta}(t) = C_{\alpha} \sum_{n=0}^{\infty} (s+n) a_n t^{s+n-1}. \quad (\text{A44})$$

(iv) The Restoring Term:

$$m^{(\alpha)} g d \theta(t) = m^{(\alpha)} g d \sum_{n=0}^{\infty} a_n t^{s+n}. \quad (\text{A45})$$

Thus, the homogeneous equation (A38) becomes

$$\begin{aligned} I^{(\alpha)} \sum_{n=0}^{\infty} \left[(s+n)(s+n-1) + (1-\alpha)(s+n) \right] a_n t^{s+n-2\alpha} \\ + C_{\alpha} \sum_{n=0}^{\infty} (s+n) a_n t^{s+n-1} + m^{(\alpha)} g d \sum_{n=0}^{\infty} a_n t^{s+n} = 0. \end{aligned} \quad (\text{A46})$$

Next, we determination of the indicial exponent. The series in (A46) contains terms with exponents

$$t^{s+n-2\alpha}, \quad t^{s+n-1}, \quad t^{s+n}.$$

In particular, when $n = 0$ the exponents are

$$s - 2\alpha, \quad s - 1, \quad s. \quad (\text{A47})$$

The *dominant* (lowest) power of t is determined by the smallest of these three numbers. Thus, we consider two cases:

Case 1: If $\alpha \geq \frac{1}{2}$ then

$$s - 2\alpha \leq s - 1, \quad (\text{A48})$$

so the most singular term is proportional to $t^{s-2\alpha}$. To avoid a singular solution we set

$$s - 2\alpha = 0 \implies s = 2\alpha. \quad (\text{A49})$$

Case 2: If $\alpha < \frac{1}{2}$ then

$$s - 1 < s - 2\alpha, \quad (\text{A50})$$

and the dominant term is t^{s-1} . In this case we choose

$$s - 1 = 0 \implies s = 1. \quad (\text{A51})$$

Resuming, the indicial exponent is given by

$$s = \begin{cases} 2\alpha, & \alpha \geq \frac{1}{2}, \\ 1, & \alpha < \frac{1}{2}. \end{cases} \quad (\text{A52})$$

To ensure a nontrivial solution, we choose s so that the lowest exponent is non-negative. In many physical contexts one selects (A52). We denote this choice by

$$s = s_0. \quad (\text{A53})$$

We can proceed now to the derivation of the recurrence relation. With the exponent s chosen according to (A49) or (A51), we now equate to zero the coefficient of each power of t in (A46). In general, the coefficients come from three series whose generic powers are

$$t^{s+n-2\alpha}, \quad t^{s+n-1}, \quad t^{s+n}. \quad (\text{A54})$$

To combine these into a single recurrence relation, we should re-index the second and third series so that all terms are written in powers of t^{s+k} for a suitable index k . However, when the exponents differ by non-integer amounts, the resulting recurrence is a *generalized* recurrence relation. For instance, one may write for $n \geq 0$

$$\begin{aligned} I^{(\alpha)}[(s+n)(s+n-1) + (1-\alpha)(s+n)]a_n t^{s+n-2\alpha} \\ + C_\alpha(s+n)a_n t^{s+n-1} + m^{(\alpha)}gd a_n t^{s+n} = 0. \end{aligned} \quad (\text{A55})$$

To extract the recurrence relation one collects all contributions to a given power t^{s+k} with k depending on α and the shift indices, and sets the sum equal to zero. In particular, the lowest-order equation (the *indicial equation*) arises from the lowest exponent. It follows that

- For $\alpha > \frac{1}{2}$ The term $n = 0$ from the first series (with exponent $s - 2\alpha = 0$) leads to

$$I^{(\alpha)}[s(s-1) + (1-\alpha)s]a_0 = I^{(\alpha)}[2\alpha(2\alpha-1+1-\alpha)]a_0 = 2\alpha^2 I^{(\alpha)}a_0 = 0. \quad (\text{A56})$$

Note that in the homogeneous problem the right-hand side is zero. If a non-zero forcing is present, one must first subtract a particular solution.

- For $\alpha < \frac{1}{2}$: The term $n = 0$ from the second series (with exponent $s - 1 = 0$), results in

$$C_\alpha s a_0 = C_\alpha a_0 = 0. \quad (\text{A57})$$

Once again, for a non-trivial solution, it is necessary to have $a_0 \neq 0$; hence in the homogeneous case this formally requires that $C_\alpha = 0$ or alternatively a reinterpretation of the dominant balance. In practice, the appropriate choice of s is dictated by the physical context and the imposed regularity conditions.

For $n \geq 1$ one obtains a recurrence relation of the form

$$\begin{aligned} I^{(\alpha)}[(s+n)(s+n-1) + (1-\alpha)(s+n)]a_n + C_\alpha(s+n-\delta)a_{n-\delta} \\ + m^{(\alpha)}gd a_{n-\gamma} = 0, \end{aligned} \quad (\text{A58})$$

where δ and γ are shifts (which generally are nonnegative and may be non-integer) chosen so that the power $t^{s+n-2\alpha}$ is matched from the second and third series. The precise form of the recurrence relation depends on the relative ordering of the exponents

$$s - 2\alpha, \quad s - 1, \quad s, \quad (A59)$$

and therefore on whether $\alpha \geq \frac{1}{2}$ or $\alpha < \frac{1}{2}$.

To conclude, for $n \geq 0$, we set the coefficient of $t^{s_0+n-\delta}$ to zero, where δ is selected such as the exponents in all series are aligned. The recurrence relation is expressed as follows

$$I^{(\alpha)}[(s_0 + n)(s_0 + n - 1) + (1 - \alpha)(s_0 + n)]a_n + D_\alpha(s_0 + n - \Delta_1)a_{n-\Delta_1} + m^{(\alpha)}gd a_{n-\Delta_2} = 0, \quad (A60)$$

with $\Delta_1, \Delta_2 \geq 0$ chosen so that all terms are expressed as t^{s_0+n} . In many applications, the exponents differ by integers leading to cases such as $\Delta_1 = 1$ and $\Delta_2 = 2$, or similar values. The exact form depends on the value of α as discussed above. The coefficient a_0 remains arbitrary, while the subsequent coefficients a_n for $n \geq 1$ are determined recursively from (A60). In the general case, the solution is given by a generalized Frobenius series.

Next, let us determine the constant particular solution. Since the forcing term on the right-hand side of equation (A34) is a constant, we seek a constant particular solution. Assume

$$\theta_p(t) = C, \quad (A61)$$

so that $\dot{\theta}_p(t) = 0$ and $\ddot{\theta}_p(t) = 0$. Substituting into (A34) gives

$$m^{(\alpha)}gd C = F_\alpha. \quad (A62)$$

Thus, the particular solution is

$$\theta_p(t) = \frac{F_\alpha}{m^{(\alpha)}gd}. \quad (A63)$$

The full general solution of (A34) is the sum of the particular solution (A63) and the homogeneous solution (A39):

$$\theta(t) = \theta_p(t) + \theta_h(t) = \frac{F_\alpha}{m^{(\alpha)}gd} + t^{s_0} \sum_{n=0}^{\infty} a_n t^n. \quad (A64)$$

Here, s_0 is given by (A53) and the coefficients a_n satisfy the recurrence relation (A60).

This expression (A64) represents the full general power series solution of the linearized equation (A34).

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