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Discovery of Truncated M-Fractional Exact Solitons, and Qualitative Analysis to the Generalized Bretherton Model

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Abstract: This paper is concerned about the novel exact solitons to the truncated M-fractional (1+1)-dimensional non-linear generalized Bretherton model with arbitrary constants . This model is used to explain the resonant nonlinear interaction between the waves in different phenomenon, including fluid dynamics, plasma physics, ocean waves, and many others. A series of exact solitons, including bright, dark, periodic, singular, singular-bright, singular-dark, and other solitons are obtained by applying the extended sinh-Gordon equation expansion (EShGEE), and the modified (G'/G^2)-expansion techniques. A novel definition of Fractional derivative provides the solutions distinct from the present solutions. Mathematica software is used to obtain, and verify the solutions. The solutions are shown through 2-D, 3-D, and density plots. The stability process is performed to verify that the solutions are exact and accurate. The modulation instability is used to determine the steady-state stable results to the corresponding equation.

Keywords: generalized Bretherton model; fractional derivatives; stability analysis; modulation instability; analytical methods; exact solitons

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1. Introduction

Naturally taking place natural phenomena are expressed in the form of nonlinear fractional partial differential equations. Many models have been developed in the form of fractional partial differential equations in the different fields of science, and engineering including fractional Phi-4 model [1], fractional Wazwaz-Benjamin-Bona-Mahony model [2], fractional regularized long wave model [3], fractional complex three coupled Maccari's system [4], fractional paraxial nonlinear Schrödinger model [5], and many more.

In this paper, authors used two simple and useful schemes: EShGEE method and modified (G'/G^2)-expansion technique. The concerned schemes have been used for different models. Instantly; EShGEE technique is used for Biswas-Arshed equation [6], hyperbolic and cubic-quintic nonlinear Schrödinger equations [7], generalized non-linear Schrödinger equation [8], Kundu-Eckhaus equation [9], novel liquid crystals model [10], (2+1)-dimensional nonlinear Schrödinger equation with anti-cubic nonlinearity [11], stochastic Phi-4 equation [12], Klein-Gordon-Zakharov equations [13], Nizhnik–Novikov-Veselov system [14], density dependent diffusion-reaction equation [15], Van der Waals equation [16]. Similarly, modified (G'/G^2)-expansion scheme is used for third-order dispersion nonlinear Schrödinger equation [17], Fokas-Lenells equation [18], (1+1)-dimensional classical Boussinesq equation [19], coupled Drinfel'd-Sokolov-Wilson equation [20], Wazwaz Kaur Boussinesq equation [21].

The basic purpose of our work is to discover the new distinct exact solitons to (1+1)-dimensional non-linear generalized Bretherton model along truncated M-fractional derivative. A qualitative analysis of the governing model is also performed.

Motivation of our work is investigate the novel wave solitons to the generalized Bretherton model. The truncated M-fractional derivative fulfills the characteristics of both integer and fractional derivatives. This definition of derivative provides the better solutions than the other definitions. Firstly, both the utilized techniques convert the nonlinear fractional partial differential equations into nonlinear ordinary differential equations (ODEs) then solve the obtained ODEs. The extended sinh-Gordon equation expansion technique provides the dark, bright, dark-bright, singular, singular-bright, and other solitons. The modified (G'/G^2)-expansion scheme gives the periodic wave, kink soliton and other types of soliton solutions.

There are different sections in the paper; the corresponding model and its mathematical treatment are shown in Section 2, the EShGEE approach and exact solitons are mentioned in Section 3, modified (G'/G^2)-expansion technique and its application are shown in Section 4, graphically explanation is mentioned in Section 5, physically description is shown in Section 6, Stability analysis is performed in Section 7, Modulation instability is performed in Section 8, and we concluded our work in Section 9.

Truncated M-fractional derivative (TMFD)

Definition: Consider $v(x):[0,\infty)\rightarrow\Re$, therefore truncated M-fractional derivative of v of order ϵ [22]

$$D^{\epsilon,\varrho}_{M,x}v(x)=\lim_{\epsilon\rightarrow 0}\frac{v\left(x\,E_{\varrho}(\epsilon x^{1-\epsilon})\right)-v(x)}{\epsilon},\quad \epsilon\in(0,1],\ \ \varrho>0,$$

here $E_{\varrho}(.)$ represents a truncated Mittag-Leffler function [23]

$$E_{\varrho}(z)=\sum_{j=0}^{i}\frac{z^j}{\Gamma(\varrho j+1)},\ \ \varrho>0\ \text{ and }\ z\in\mathbf{C}.$$

Properties: Consider a,b $\in\Re$, and g,f are $\epsilon-$ differentiable at a point $x>0$, according to [22]:

$$(a)\ D^{\epsilon,\varrho}_{M,x}(ag(x)+bf(x))=aD^{\epsilon,\varrho}_{M,x}g(x)+bD^{\epsilon,\varrho}_{M,x}f(x)$$

$$(b)\ D^{\epsilon,\varrho}_{M,x}(g(x).f(x))=g(x)D^{\epsilon,\varrho}_{M,x}f(x)+f(x)D^{\epsilon,\varrho}_{M,x}g(x)$$

$$(c)\ D^{\epsilon,\varrho}_{M,x}\left(\frac{g(x)}{f(x)}\right)=\frac{f(x)D^{\epsilon,\varrho}_{M,x}g(x)-g(x)D^{\epsilon,\varrho}_{M,x}f(x)}{(f(x))^2}$$

$$(d)\ D^{\epsilon,\varrho}_{M,x}(B)=0,\ \ \text{where } B \text{ is a constant.}$$

$$(e)\ D^{\epsilon,\varrho}_{M,x}g(x)=\frac{x^{1-\epsilon}}{\Gamma(\varrho+1)}\,\frac{dg(x)}{dx}.$$

2. Model presentation and its mathematical treatment

Bretherton proposed the following partial differential equation [24];

$$v_{tt}+v_{xx}+v_{xxxx}+v-v^2=0. \tag{1}$$

This is a model of a dispersive wave system to explain the resonant nonlinear interaction between three linear models.

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The modified Bretherton equation is given as;

$$v_{tt} + v_{xx} + v_{xxxx} + v - v^3 = 0. \tag{2}$$

Eq.(2) was used by Kudryashov [25]. Different kinds of solitary wave solutions were gained in [26,27].

Our concerning model is a (1+1)-dimensional non-linear generalized Bretherton equation with arbitrary constants given as [28];

$$v_{tt} + av_{xx} + bv_{xxxx} + \mu v + cv^3 = 0. \tag{3}$$

Here $v = v(x,t)$ indicates the wave function, while parameters a,b,c, and μ are arbitrary constants. Eq.(3) is discussed by using different schemes including; improved (G'/G)-expansion scheme [28], extended tanh-function scheme [29].

The (1+1)-dimensional nonlinear generalized Bretherton model with arbitrary constant in the concept of TMFD is given as;

$$D^{2\epsilon,\varrho}_{M,t} v + aD^{2\epsilon,\varrho}_{M,x} v + bD^{4\epsilon,\varrho}_{M,x} v + \mu v + cv^3 = 0. \tag{4}$$

Consider a wave transformation;

$$v(x,t) = V(\mho), \qquad \mho = \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \lambda t^\epsilon). \tag{5}$$

where λ is a soliton velocity.

Using Eq.(5) into Eq.(4), results into;

$$\left(a + \lambda^2\right) V'' + bV^{(4)} + cV^3 + \mu V = 0 \tag{6}$$

Natural number m is calculated by applying the Homogeneous Balance technique into Eq.(6), and balancing the terms $V^{(4)}$ and V^3, we get $m = 2$.

3. Explanation and application of EShGEE method

3.1. Description

Here we will mention some of the stages of the scheme.

Stage 1:

Supposing a non-linear fractional PDE:

$$F(g, D^{\epsilon,\varrho}_{M,t} g^2, g^2 g_x, g_x, ...) = 0. \tag{7}$$

Where, $g = g(x,t)$ is a wave profile.

Considering a wave transformation:

$$g(x,t) = G(\mho), \ \ \mho = \frac{(1+\varrho)}{\epsilon}(x^\epsilon + \kappa t^\theta). \tag{8}$$

Inserting Eq. (8) in Eq. (7), yields:

$$H(G, G^2 G', G'', ...) = 0. \tag{9}$$

Stage 2:

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Suppose the roots of Eq. (9) given below:

$$G(f) = \alpha_0 + \sum_{j=1}^{m} (\beta_j \sinh(f) + \alpha_j \cosh(f))^j. \tag{10}$$

Here α_0, α_j, β_j $(j = 1, 2, 3, ..., m)$ are to be found. Suppose a new profile f of $\mho$ that fulfill:

$$\frac{df}{d\mho} = \sinh(f). \tag{11}$$

Natural number m is calculate by applying the Homogeneous Balance scheme. Eq. (11) is achieved from the given equation:

$$q_{xt} = \kappa \sinh(v). \tag{12}$$

From [30], we obtain the results for Eq. (12) shown as:

$$\sinh f(\mho) = \pm \operatorname{csch}(\mho) \quad or \quad \cosh f(\mho) = \pm \coth(\mho). \tag{13}$$

And

$$\sinh f(\mho) = \pm \iota\, sech(\mho) \quad or \quad \cosh f(\mho) = \pm \tanh(\mho). \tag{14}$$

$\iota^2 = -1$.

Stage 3:

Inserting Eq. (10) and Eq. (11) in the Eq. (9), results a system including $f'^k(\mho) \sinh^l f(\mho) \cosh^m f(\mho)$ $(k = 0, 1;\; l = 0, 1;\; m = 0, 1, 2, ...)$. Putting each co-efficient of $f'^k(\mho) \sinh^l f(\mho) \cosh^m f(\mho)$ equal to 0, to achieve a set of consisting α_j and $\beta_j (j = 1, 2, 3...m)$ and others.

Stage 4:

By solving the gained set, yields results for unknowns. From obtained solutions, Eqs. (13) and (14), yields the solutions for Eq. (9) given as:

$$G(\mho) = \alpha_0 + \sum_{j=1}^{m} (\pm\beta_j \operatorname{csch}(\mho) \pm \alpha_j \coth(\mho))^j. \tag{15}$$

And

$$G(\mho) = \alpha_0 + \sum_{j=1}^{m} (\pm\iota\beta_j sech(\mho) \pm \alpha_j \tanh(\mho))^j. \tag{16}$$

From this method, one may gain the sech, csch, tanh and coth consisting results.

3.2. Application to the EShGEE scheme

Eq.(10) changes to the given form for $m = 2$:

$$V(\mho) = \alpha_0 + \alpha_1 \cosh(f(\mho)) + \beta_1 \sinh(f(\mho)) + (\alpha_2 \cosh(f(\mho)) + \beta_2 \sinh(f(\mho)))^2. \tag{17}$$

Using Eq.(17) into Eq.(6) along Eq.(11), we gain a system containing α_0, α_1,α_2, β_1, β_2, λ, and other parameters. By manumitting, results into the given sets :

Set 1:

$$\{\alpha_0 = \frac{\sqrt{2b} - \sqrt{-30b}}{\sqrt{c}}, \alpha_1 = 0, \beta_1 = 0, \alpha_2 = \pm\frac{\sqrt[4]{-15}\, 2^{3/4} \sqrt[4]{b}}{\sqrt[4]{c}}, \beta_2 = 0,$$
$$\lambda = -\sqrt{-a - 2\sqrt{-15}\, b + 10b}, \mu = 4(7b - \sqrt{-15}\, b)\}. \tag{18}$$

$$v_1(x,t) = \frac{\sqrt{2b}}{\sqrt{c}} \left(1 \pm \sqrt{-15} \coth(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{-a - 2\sqrt{-15}\, b + 10b\, t^\epsilon}))\right). \tag{19}$$

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$$v_2(x,t) = \frac{\sqrt{2b}}{\sqrt{c}}\left(1 \pm \sqrt{-15}\tanh(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{-a-2\sqrt{-15}\,b+10b}\,t^\epsilon))\right). \tag{20}$$

Set 2;

$$\{\alpha_0 = \frac{\sqrt{2b}-\sqrt{-30b}}{\sqrt{c}}, \alpha_1 = 0, \beta_1 = 0, \alpha_2 = \pm\frac{\sqrt[4]{-15}\,2^{3/4}\sqrt[4]{b}}{\sqrt[4]{c}}, \beta_2 = 0,$$
$$\lambda = \sqrt{-a-2\sqrt{-15}\,b+10b}, \mu = 4(7b-\sqrt{-15}\,b)\}. \tag{21}$$

$$v_1(x,t) = \frac{\sqrt{2b}}{\sqrt{c}}\left(1 \pm \sqrt{-15}\coth(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{-a-2\sqrt{-15}\,b+10b}\,t^\epsilon))\right). \tag{22}$$

$$v_2(x,t) = \frac{\sqrt{2b}}{\sqrt{c}}\left(1 \pm \sqrt{-15}\tanh(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{-a-2\sqrt{-15}\,b+10b}\,t^\epsilon))\right). \tag{23}$$

Set 3;

$$\{\alpha_0 = -\frac{\sqrt{-30b}+\sqrt{2b}}{\sqrt{c}}, \alpha_1 = 0, \beta_1 = 0, \alpha_2 = \pm\frac{\sqrt[4]{-15}2^{3/4}\sqrt[4]{b}}{\sqrt[4]{c}}, \beta_2 = 0,$$
$$\lambda = -\sqrt{-a+10b+2\sqrt{-15}b}, \mu = 4(\sqrt{-15}b+7b)\}. \tag{24}$$

$$v_1(x,t) = \frac{\sqrt{2b}}{\sqrt{c}}\left(-1 \pm \sqrt{-15}\coth(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{-a+10b+2\sqrt{-15}b}\,t^\epsilon))\right). \tag{25}$$

$$v_2(x,t) = \frac{\sqrt{2b}}{\sqrt{c}}\left(-1 \pm \sqrt{-15}\tanh(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{-a+10b+2\sqrt{-15}b}\,t^\epsilon))\right). \tag{26}$$

Set 4;

$$\{\alpha_0 = -\frac{\sqrt{-30b}+\sqrt{2b}}{\sqrt{c}}, \alpha_1 = 0, \beta_1 = 0, \alpha_2 = \pm\frac{\sqrt[4]{-15}2^{3/4}\sqrt[4]{b}}{\sqrt[4]{c}}, \beta_2 = 0,$$
$$\lambda = \sqrt{-a+10b+2\sqrt{-15}b}, \mu = 4(\sqrt{-15}b+7b)\}. \tag{27}$$

$$v_1(x,t) = \frac{\sqrt{2b}}{\sqrt{c}}\left(-1 \pm \sqrt{-15}\coth(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{-a+10b+2\sqrt{-15}b}\,t^\epsilon))\right). \tag{28}$$

$$v_2(x,t) = \frac{\sqrt{2b}}{\sqrt{c}}\left(-1 \pm \sqrt{-15}\tanh(2\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{-a+10b+2\sqrt{-15}b}\,t^\epsilon))\right). \tag{29}$$

Set 5;

$$\{\alpha_0 = -\frac{2\sqrt{-30b}}{\sqrt{c}}, \alpha_1 = 0, \beta_1 = 0, \alpha_2 = \pm\frac{\sqrt[4]{-15}2^{3/4}\sqrt[4]{b}}{\sqrt[4]{c}}, \beta_2 = 0, \lambda = -\sqrt{-a-20b}, \mu = 64b\}. \tag{30}$$

$$v_1(x,t) = \frac{\left(2\sqrt{-30b}\right)}{\sqrt{c}}\mathrm{csch}^2(\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{-a-20b}\,t^\epsilon)). \tag{31}$$

$$v_2(x,t) = -\frac{\left(2\sqrt{-30b}\right)}{\sqrt{c}}\mathrm{sech}^2(\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{-a-20b}\,t^\epsilon)). \tag{32}$$

Set 6;

$$\{\alpha_0 = -\frac{2\sqrt{-30b}}{\sqrt{c}}, \alpha_1 = 0, \beta_1 = 0, \alpha_2 = \pm\frac{\sqrt[4]{-15}2^{3/4}\sqrt[4]{b}}{\sqrt[4]{c}}, \beta_2 = 0, \lambda = \sqrt{-a-20b}, \mu = 64b\}. \tag{33}$$

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$$v_1(x,t) = \frac{\left(2\sqrt{-30b}\right)}{\sqrt{c}}\text{csch}^2\left(\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{-a-20b}\, t^\epsilon)\right). \tag{34}$$

$$v_2(x,t) = -\frac{\left(2\sqrt{-30b}\right)}{\sqrt{c}}\text{sech}^2\left(\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{-a-20b}\, t^\epsilon)\right). \tag{35}$$

4. Explanation of modified $(G'/G^2)-$ expansion scheme

In this section, we will explain the main steps of this scheme [17].

Step 1:

Suppose the Eqns. (7), (8) and (9).

Step 2:

Suppose the result for Eq.(9) given as;

$$Q(\mho) = \sum_{j=0}^{m} \alpha_j \left(\frac{G'}{G^2}\right)^j, \tag{36}$$

where $\alpha_j(j = 0,1,2,3,...,m)$ are unknowns while $\alpha_j \neq 0$. The function G=$G(\mho)$ fulfills the given ODE,

$$\left(\frac{G'}{G^2}\right)' = \lambda_0 + \lambda_1\left(\frac{G'}{G^2}\right)^2, \tag{37}$$

here λ_0 and λ_1 are the constants. One can obtain the following cases to Eq. (37) depends on the conditions of λ_0:

Case 1:if $\lambda_0\lambda_1 < 0$, then

$$\left(\frac{G'}{G^2}\right) = -\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1} + \frac{\sqrt{|\lambda_0\lambda_1|}}{2}\left[\frac{C_1\sinh(\sqrt{\lambda_0\lambda_1}\,\mho) + C_2\cosh(\sqrt{\lambda_0\lambda_1}\,\mho)}{C_1\cosh(\sqrt{\lambda_0\lambda_1}\,\mho) + C_2\sinh(\sqrt{\lambda_0\lambda_1}\,\mho)}\right], \tag{38}$$

Case 2: if $\lambda_0\lambda_1 > 0$, then

$$\left(\frac{G'}{G^2}\right) = \sqrt{\frac{\lambda_0}{\lambda_1}}\left[\frac{C_1\cos(\sqrt{\lambda_0\lambda_1}\,\mho) + C_2\sin(\sqrt{\lambda_0\lambda_1}\,\mho)}{C_1\sin(\sqrt{\lambda_0\lambda_1}\,\mho) - C_2\sin(\sqrt{\lambda_0\lambda_1}\,\mho)}\right], \tag{39}$$

Case 3: if $\lambda_0 = 0$ and $\lambda_1 \neq 0$, then

$$\left(\frac{G'}{G^2}\right) = -\frac{C_1}{\lambda_1(C_1\mho + C_2)}. \tag{40}$$

Here C_1 and C_2 are constants.

Step 3:

Putting Eq. (36) in the Eq. (9) along Eq. (37), and collecting coefficients of every power of $(\frac{G'}{G^2})^j$ to 0, then solving that algebraic equations obtained including $\alpha_j, \lambda_0, \lambda_1, \nu$ and others.

Step 4:

Eq. (36) of which α_j, ν and other parameters that are obtained in the step 3 into the Eq. (9), one can gain the results of Eq. (7).

4.1. Application

For $m = 2$, Eq.(36) reduces into:

$$Q(\mho) = \alpha_0 + \alpha_1\frac{G'(\mho)}{G^2(\mho)} + \alpha_2\left(\frac{G'(\mho)}{G^2(\mho)}\right)^2 \tag{41}$$

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Here α_0, α_1 and α_2 are unknowns. By putting Eq. (41) with Eq. (37) into the Eq. (6), and by solving by Maple software, results into the given sets:

Set 1:

$$\left\{\alpha_0 = \pm\frac{2\lambda_0\lambda_1\sqrt{-30\,cb}}{c}, \alpha_1 = 0, \alpha_2 = \pm\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}, \lambda = -\sqrt{20\,b\lambda_0\lambda_1 - a}, \mu = 64\,\lambda_0{}^2\lambda_1{}^2 b\right\}. \tag{42}$$

If $\lambda_0\lambda_1 < 0$, we have

$$v(x,t) = \pm\frac{2\lambda_1\sqrt{-30\,cb}}{c}(\lambda_0 + \lambda_1(-\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1} + \frac{\sqrt{|\lambda_0\lambda_1|}}{2}\ ((C_1\sinh(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon} (x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) + C_2\cosh(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)))/(C_1\cosh(\sqrt{\lambda_0\lambda_1} \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) + C_2\sinh(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon))))))^2). \tag{43}$$

If $\lambda_0\lambda_1 > 0$, we have

$$v(x,t) = \pm\frac{2\lambda_1\sqrt{-30\,cb}}{c}(\lambda_0 + \lambda_1(\sqrt{\frac{\lambda_0}{\lambda_1}}\ ((C_1\cos(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) + C_2\sin(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)))/(C_1\sin(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon} (x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) - C_2\sin(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon + \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon))))))^2). \tag{44}$$

Set 2:

$$\left\{\alpha_0 = \pm\frac{2\lambda_0\lambda_1\sqrt{-30\,cb}}{c}, \alpha_1 = 0, \alpha_2 = \pm\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}, \lambda = \sqrt{20\,b\lambda_0\lambda_1 - a}, \mu = 64\,\lambda_0{}^2\lambda_1{}^2 b\right\}. \tag{45}$$

If $\lambda_0\lambda_1 < 0$, we have

$$v(x,t) = \pm\frac{2\lambda_1\sqrt{-30\,cb}}{c}(\lambda_0 + \lambda_1(-\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1} + \frac{\sqrt{|\lambda_0\lambda_1|}}{2}\ ((C_1\sinh(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon} (x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) + C_2\cosh(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)))/(C_1\cosh(\sqrt{\lambda_0\lambda_1} \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) + C_2\sinh(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon))))))^2). \tag{46}$$

If $\lambda_0\lambda_1 > 0$, we have

$$v(x,t) = \pm\frac{2\lambda_1\sqrt{-30\,cb}}{c}(\lambda_0 + \lambda_1(\sqrt{\frac{\lambda_0}{\lambda_1}}\ ((C_1\cos(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) + C_2\sin(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)))/(C_1\sin(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon} (x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon)) - C_2\sin(\sqrt{\lambda_0\lambda_1}\ \frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon - \sqrt{20\,b\lambda_0\lambda_1 - a}\ t^\epsilon))))))^2). \tag{47}$$

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Set 3:

$$\{\alpha_0 = \mp\frac{(-\sqrt{-30\,cb}+\sqrt{2}\sqrt{cb})\lambda_1\lambda_0}{c}, \alpha_1 = 0, \alpha_2 = \pm\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c},$$
$$\lambda = -\frac{\sqrt{-\sqrt{-30\,cb}(30\,\sqrt{2}\sqrt{cb}\lambda_0\lambda_1 b + 10\,\lambda_0\lambda_1 b\sqrt{-30\,cb} + a\sqrt{-30\,cb})}}{\sqrt{-30\,cb}},$$
$$\mu = \frac{-4\lambda_0{}^2\lambda_1{}^2 b(\sqrt{2}\sqrt{cb}+9\,\sqrt{-30\,cb})}{2\,\sqrt{2}\sqrt{cb}-\sqrt{-30\,cb}}\} \quad (48)$$

Case 1:if $\lambda_0\lambda_1 < 0$, then

$$v(x,t) = \mp\frac{(-\sqrt{-30\,cb}+\sqrt{2}\sqrt{cb})\lambda_1\lambda_0}{c} \pm \frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}(-\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1}+\frac{\sqrt{|\lambda_0\lambda_1|}}{2}$$
$$(\frac{C_1\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) + C_2\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}{C_1\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) + C_2\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}))^2. \quad (49)$$

Case 2: if $\lambda_0\lambda_1 > 0$, then

$$v(x,t) = \mp\frac{(-\sqrt{-30\,cb}+\sqrt{2}\sqrt{cb})\lambda_1\lambda_0}{c} \pm \frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}(\sqrt{\frac{\lambda_0}{\lambda_1}}$$
$$(\frac{C_1\cos(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) + C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}{C_1\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) - C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}))^2. \quad (50)$$

where λ is given in Eq.(48).
Set 4:

$$\{\alpha_0 = \mp\frac{(-\sqrt{-30\,cb}+\sqrt{2}\sqrt{cb})\lambda_1\lambda_0}{c}, \alpha_1 = 0, \alpha_2 = \pm\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c},$$
$$\lambda = \frac{\sqrt{-\sqrt{-30\,cb}(30\,\sqrt{2}\sqrt{cb}\lambda_0\lambda_1 b + 10\,\lambda_0\lambda_1 b\sqrt{-30\,cb} + a\sqrt{-30\,cb})}}{\sqrt{-30\,cb}},$$
$$\mu = \frac{-4\lambda_0{}^2\lambda_1{}^2 b(\sqrt{2}\sqrt{cb}+9\,\sqrt{-30\,cb})}{2\,\sqrt{2}\sqrt{cb}-\sqrt{-30\,cb}}\} \quad (51)$$

Case 1:if $\lambda_0\lambda_1 < 0$, then

$$v(x,t) = \mp\frac{(-\sqrt{-30\,cb}+\sqrt{2}\sqrt{cb})\lambda_1\lambda_0}{c} \pm \frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}(-\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1}+\frac{\sqrt{|\lambda_0\lambda_1|}}{2}$$
$$(\frac{C_1\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) + C_2\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}{C_1\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) + C_2\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}))^2. \quad (52)$$

Case 2: if $\lambda_0\lambda_1 > 0$, then

$$v(x,t) = \mp\frac{(-\sqrt{-30\,cb}+\sqrt{2}\sqrt{cb})\lambda_1\lambda_0}{c} \pm \frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}(\sqrt{\frac{\lambda_0}{\lambda_1}}$$
$$(\frac{C_1\cos(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) + C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}{C_1\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon)) - C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}))^2. \quad (53)$$

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where λ is given in Eq.(51).

Set 5:

$$\{\alpha_0=\mp\frac{(\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb})\lambda_1\lambda_0}{c},\alpha_1=0,\alpha_2=\mp\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c},$$
$$\lambda=-\frac{\sqrt{\sqrt{-30\,cb}(30\,\sqrt{2}\sqrt{cb}\lambda_0\lambda_1b-10\,\lambda_0\lambda_1b\sqrt{-30\,cb}-a\sqrt{-30\,cb})}}{\sqrt{-30\,cb}},$$
$$\mu=-4\,\frac{\lambda_0{}^2\lambda_1{}^2b(\sqrt{2}\sqrt{cb}-9\,\sqrt{-30\,cb})}{2\,\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb}}\}. \tag{54}$$

Case 1:if $\lambda_0\lambda_1<0$, then

$$v(x,t)=\mp\frac{(\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb})\lambda_1\lambda_0}{c}\mp\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}(-\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1}+\frac{\sqrt{|\lambda_0\lambda_1|}}{2}$$
$$(\frac{C_1\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))+C_2\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}{C_1\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))+C_2\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}))^2. \tag{55}$$

Case 2: if $\lambda_0\lambda_1>0$, then

$$v(x,t)=\mp\frac{(\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb})\lambda_1\lambda_0}{c}\mp\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}(\sqrt{\frac{\lambda_0}{\lambda_1}}$$
$$(\frac{C_1\cos(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))+C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}{C_1\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))-C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}))^2. \tag{56}$$

where λ is given in Eq.(54).

Set 6:

$$\{\alpha_0=\mp\frac{(\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb})\lambda_1\lambda_0}{c},\alpha_1=0,\alpha_2=\mp\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c},$$
$$\lambda=\frac{\sqrt{\sqrt{-30\,cb}(30\,\sqrt{2}\sqrt{cb}\lambda_0\lambda_1b-10\,\lambda_0\lambda_1b\sqrt{-30\,cb}-a\sqrt{-30\,cb})}}{\sqrt{-30\,cb}},$$
$$\mu=-4\,\frac{\lambda_0{}^2\lambda_1{}^2b(\sqrt{2}\sqrt{cb}-9\,\sqrt{-30\,cb})}{2\,\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb}}\}. \tag{57}$$

Case 1:if $\lambda_0\lambda_1<0$, then

$$v(x,t)=\mp\frac{(\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb})\lambda_1\lambda_0}{c}\mp\frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}(-\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1}+\frac{\sqrt{|\lambda_0\lambda_1|}}{2}$$
$$(\frac{C_1\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))+C_2\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}{C_1\cosh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))+C_2\sinh(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^\epsilon-\lambda t^\epsilon))}))^2. \tag{58}$$

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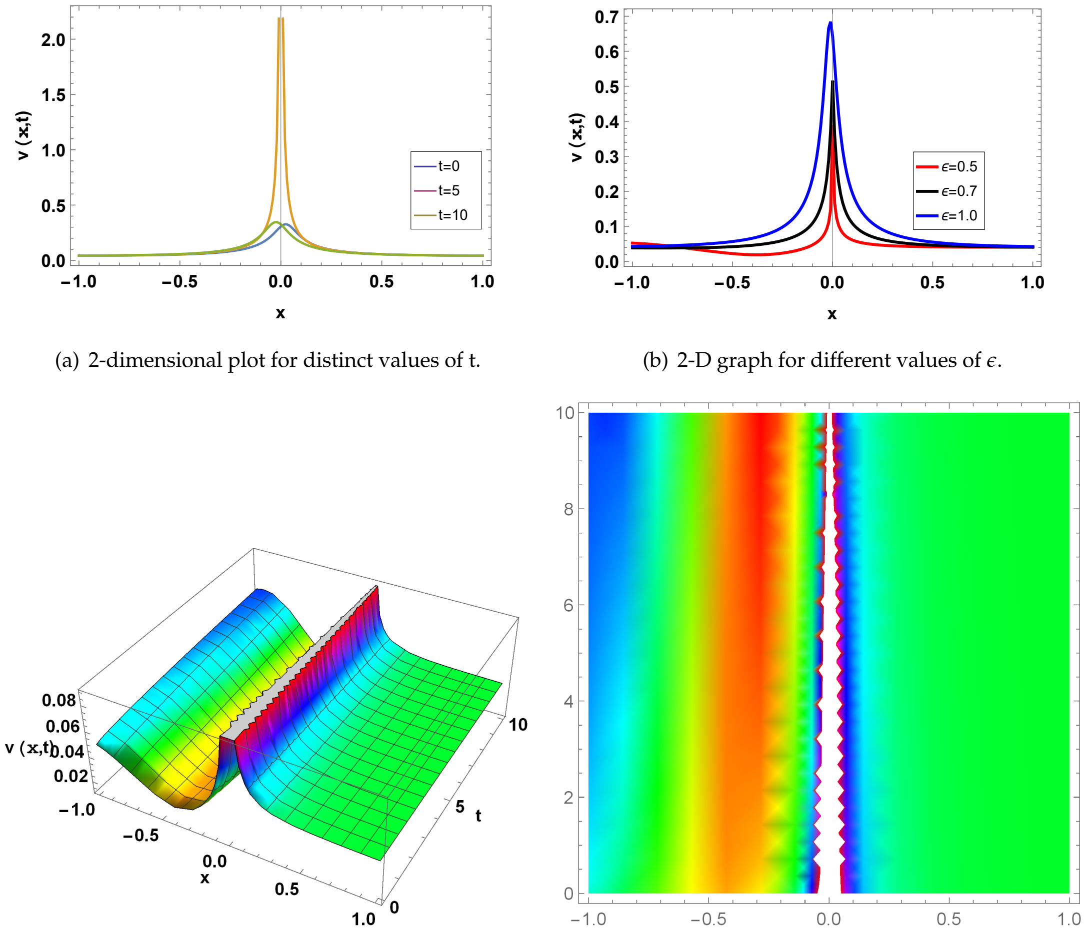
Case 2: if $\lambda_0\lambda_1 > 0$, then

$$v(x,t) = \mp\frac{(\sqrt{2}\sqrt{cb}+\sqrt{-30\,cb})\lambda_1\lambda_0}{c} \mp \frac{2\sqrt{-30\,cb}\lambda_1{}^2}{c}\Big(\sqrt{\frac{\lambda_0}{\lambda_1}}\ \Big((C_1\cos(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^{\epsilon}-\lambda t^{\epsilon}))+C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^{\epsilon}-\lambda t^{\epsilon})))/(C_1\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^{\epsilon}-\lambda t^{\epsilon}))-C_2\sin(\sqrt{\lambda_0\lambda_1}\,\frac{\Gamma(1+\varrho)}{\epsilon}(x^{\epsilon}-\lambda t^{\epsilon}))\Big)\Big)\Big)^2. \quad (59)$$

where λ is given in Eq.(57).

5. Graphically explanation

Here, we shown the gained solutions by 2-D, 3-D and contour graphs. The 2-D plots also drawn for different values of ϵ.



(a) 2-dimensional plot for distinct values of t.

(b) 2-D graph for different values of ϵ.

(c) 3-D graph for $\epsilon = 0.6$

(d) Density graph for $\epsilon = 0.6$

Figure 1. Plot for $v(x,t)$ is shown in Eq.(19).

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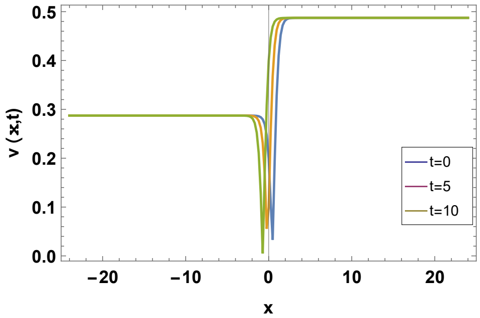
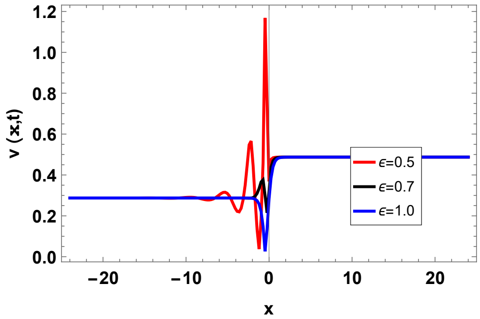
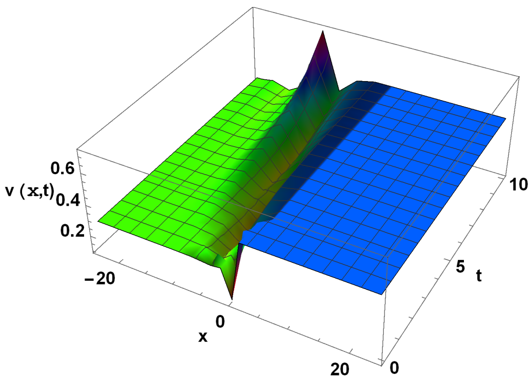


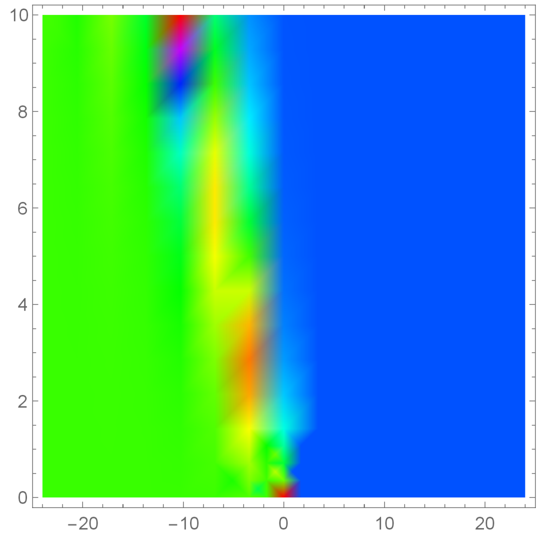
(a) 2-dimensional plot for distinct values of t.



(b) 2-D graph for different values of ϵ.



(c) 3-D graph for $\epsilon = 0.6$



(d) Density graph for $\epsilon = 0.6$

Figure 2. Plot for $v(x,t)$ is shown in Eq.(20).

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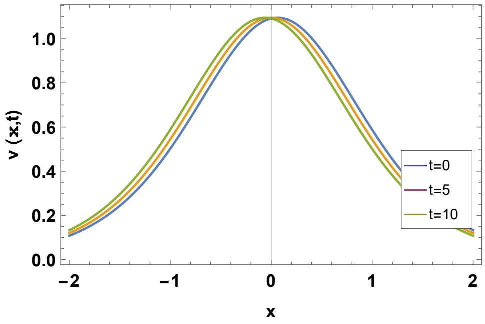
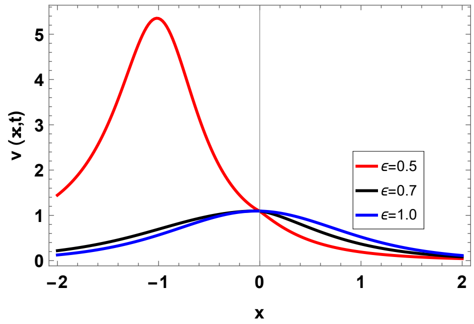
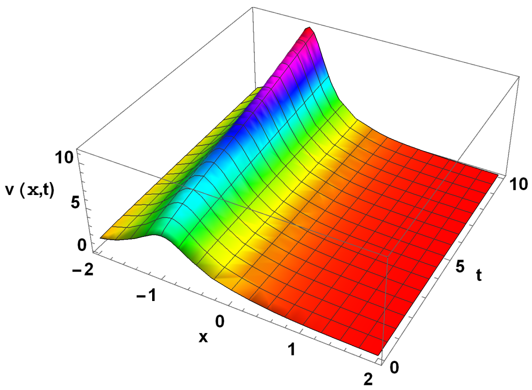


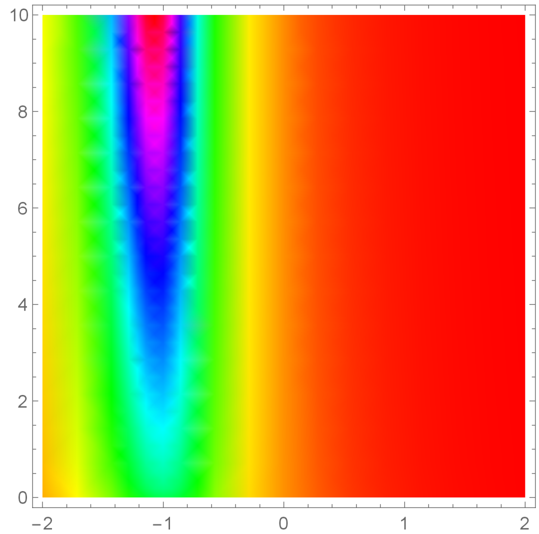
(a) 2-D graph for different values of t.



(b) 2-D graph for different values of ϵ.



(c) 3-D graph for $\epsilon = 0.6$



(d) Density graph for $\epsilon = 0.6$

Figure 3. Plot for $v(x,t)$ is shown in Eq.(32).

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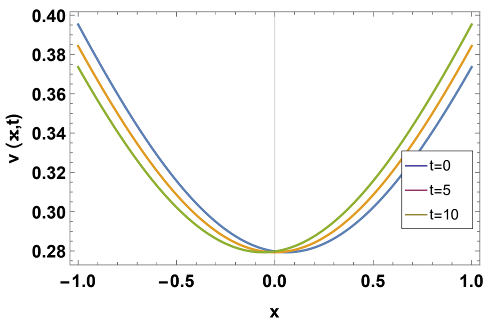
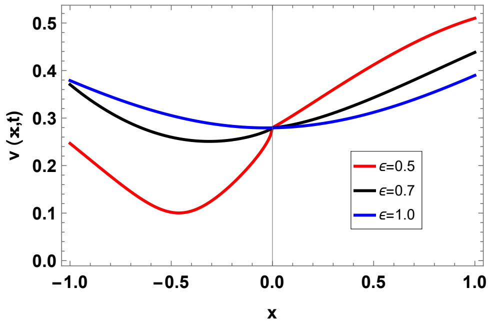
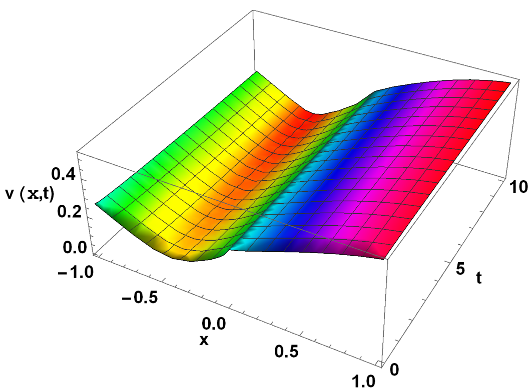


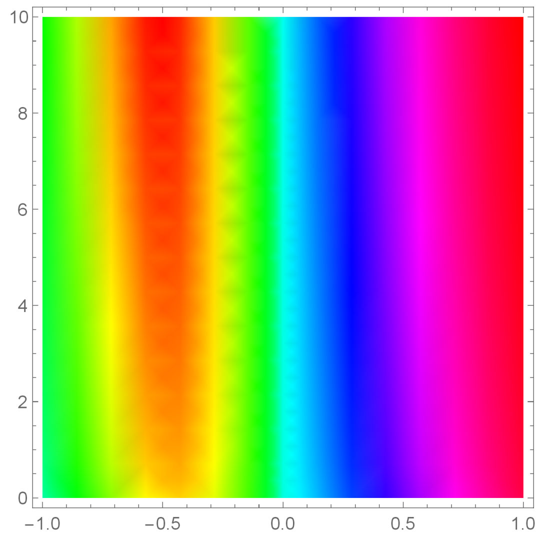
(a) 2-dimensional plot for distinct values of t.



(b) 2-D graph for different values of ϵ.



(c) 3-D graph for $\epsilon = 0.6$



(d) Density graph for $\epsilon = 0.6$

Figure 4. Plot for $v(x,t)$ is shown in Eq.(43).

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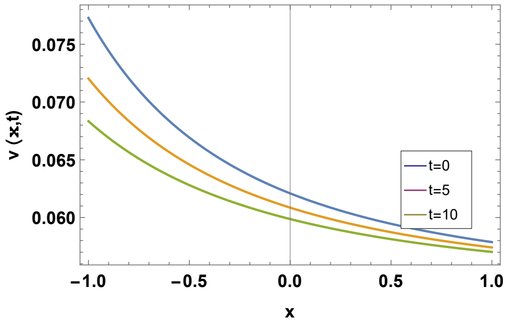
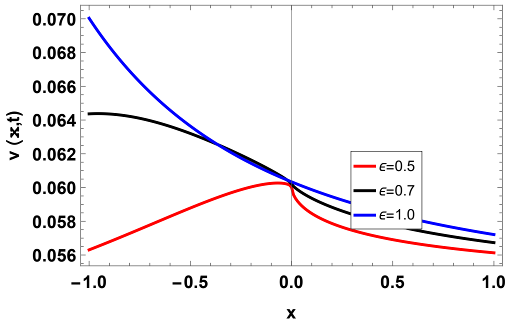
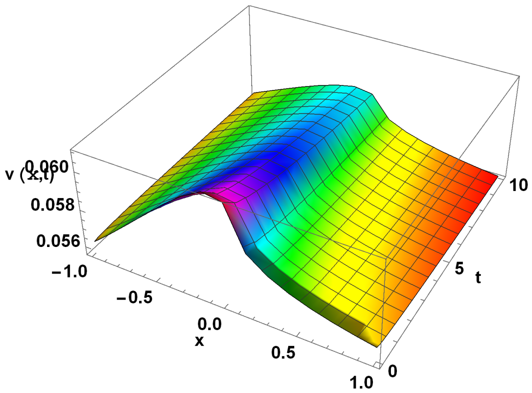


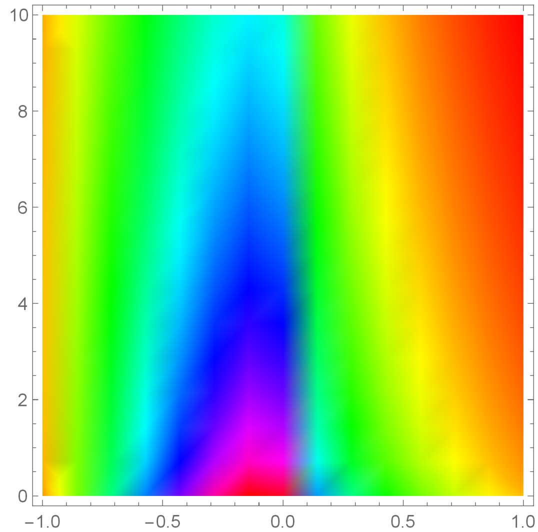
(a) 2-dimensional plot for distinct values of t.



(b) 2-D graph for different values of ϵ.



(c) 3-D graph for $\epsilon = 0.6$



(d) Density graph for $\epsilon = 0.6$

Figure 5. Plot for $v(x,t)$ is shown in Eq.(44).

6. Physically Interpretation

Here, we will describe the dynamical behaviour of the solutions of the truncated M-fractional generalized Bretherton model.

Figure 1: represents a singular soliton at the values of; $a = -0.0001, b = -0.0001, c = -2$, and $\varrho = 0.5$. Fig(a) represents a 2-D graph for $-1 < x < 1$ at $\epsilon = 1$, where the Blue line for $t = 0$, Orange line at $t = 5$, and Green line at $t = 10$. Fig(b) represents a two-dimensional graph for $-1 < x < 1$ at $t \in (0,10)$, while Red line at $\epsilon = 0.5$, Black-line at $\epsilon = 0.7$, and Blue-line at $\epsilon = 1.0$. Fig(c) represents a 3-dimensional graph at $\epsilon = 0.6$ for $t \in (0,10)$. Fig(d) shows a density plot for $\epsilon = 0.6$ at $t \in (0,10)$.

Figure 2; represents a dark soliton at the values of; $a = 0.005, b = 0.005, c = 1$, and $\varrho = 0.5$. Fig(a) represents a 2-D graph for $-24 < x < 24$ at $\epsilon = 1$, where the Blue line for $t = 0$, Orange line at $t = 5$, and Green line at $t = 10$. Fig(b) represents a two-dimensional graph for $-24 < x < 24$ at $t \in (0,10)$, while Red line at $\epsilon = 0.5$, Black-line at $\epsilon = 0.7$, and Blue-line at $\epsilon = 1.0$. Fig(c) represents a 3-dimensional graph at $\epsilon = 0.6$ for $t \in (0,10)$. Fig(d) shows a density plot for $\epsilon = 0.6$ at $t \in (0,10)$.

Figure 3; represents a bright soliton at the values of; $a = 0.001, b = -0.0001, c = -0.01$, and $\varrho = 0.5$. Fig(a) represents a 2-D graph for $-2 < x < 2$ at $\epsilon = 1$, where the Blue line for $t = 0$, Orange line at $t = 5$, and Green line at $t = 10$. Fig(b) represents a two-dimensional graph for $-2 < x < 2$ at $t \in (0,10)$, while Red line at $\epsilon = 0.5$, Black-line at $\epsilon = 0.7$, and Blue-line at $\epsilon = 1.0$. Fig(c) represents a 3-dimensional graph at $\epsilon = 0.6$ for $t \in (0,10)$. Fig(d) shows a density plot for $\epsilon = 0.6$ at $t \in (0,10)$.

Figure 4; represents a kink soliton at the values of; $a = 0.0001, b = -0.0001, c = -0.01, \lambda_0 = -0.5, \lambda_1 = 1, C_1 = 0.5, C_2 = 0.3$, and $\varrho = 0.5$. Fig(a) represents a 2-D graph for $-1 < x < 1$ at $\epsilon = 1$, where the Blue line for $t = 0$, Orange line at $t = 5$, and Green line at $t = 10$. Fig(b) represents a two-dimensional

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graph for $-1 < x < 1$ at $t \in (0, 10)$, while Red line at $\epsilon = 0.5$, Black-line at $\epsilon = 0.7$, and Blue-line at $\epsilon = 1.0$. Fig(c) represents a 3-dimensional graph at $\epsilon = 0.6$ for $t \in (0, 10)$. Fig(d) shows a density plot for $\epsilon = 0.6$ at $t \in (0, 10)$.
Figure 5; represents a periodic wave solution at the values of; $a = -0.01, b = 0.0001, c = -0.01, \lambda_0 = -0.5, \lambda_1 = -0.1, C_1 = 0.5, C_2 = -0.3$, and $\varrho = 0.5$. Fig(a) represents a 2-D graph for $-1 < x < 1$ at $\epsilon = 1$, where the Blue line for $t = 0$, Orange line at $t = 5$, and Green line at $t = 10$. Fig(b) represents a two-dimensional graph for $-1 < x < 1$ at $t \in (0, 10)$, while Red line at $\epsilon = 0.5$, Black-line at $\epsilon = 0.7$, and Blue-line at $\epsilon = 1.0$. Fig(c) represents a 3-dimensional graph at $\epsilon = 0.6$ for $t \in (0, 10)$. Fig(d) shows a density plot for $\epsilon = 0.6$ at $t \in (0, 10)$.

7. Stability Analysis

Here, the stability analysis of the concerning model is discussed. The stability analysis is used for many equations likely, [31,32]. For the Eq.(3) stability analysis, one takes the Hamiltonian transformation given as,

$$\mathcal{S} = \frac{1}{2}\int_{-\infty}^{\infty} v^2\, dx, \tag{60}$$

Here, $\mathcal{S}$ denotes a momentum factor, while $h(x,t)$ denotes the power of possibility. The necessary condition for the stable solutions is given as;

$$\frac{\partial \mathcal{S}}{\partial \lambda} > 0, \tag{61}$$

here λ indicates a wave speed, inserting Eq.(32) into Eq.(60) results;

$$\mathcal{S} = \frac{1}{2}\int_{-6}^{6} (-\frac{(2\sqrt{-30b})}{\sqrt{c}}\text{sech}^2((x + \sqrt{-a-20b}\; t)))^2 dx, \tag{62}$$

by using the criterion given in Eq.(61), we get

$$\frac{1920e^{24}(1-e^{24})bt(4e^{12}\sinh(2t\sqrt{-a-20b}) + (1+e^{24})\sinh(4t\sqrt{-a-20b}))}{c(2e^{12}\cosh(2t\sqrt{-a-20b}) + e^{24}+1)^4} > 0. \tag{63}$$

Hence, Eq.(3) denotes a stable non-linear fractional equation because the condition is satisfied.

8. Modulation instability (MI)

We take the following transformation for the steady-state result of generalized Bretherton model [33]:

$$v(x,t) = \big(V(x,t) + \sqrt{\tau}\big)e^{\imath\tau t}. \tag{64}$$

Here τ shows the optical power of normalized. Inserting Eq.(64) into Eq.(3). By linearizing, one gets

$$aV_{xx} + bV_{xxxx} + \mu\sqrt{\tau} - \tau^{5/2} + 2\imath\tau V_t + V_{tt} + \mu V - \tau^2 V = 0. \tag{65}$$

Consider the solution of Eq.(65) in the form;

$$V(x,t) = A_1 e^{\imath(px-\rho t)} + A_2 e^{-\imath(px-\rho t)}. \tag{66}$$

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here ρ and p represent the frequency and normalized wave number of perturbation respectively. Putting the Eq.(66) into Eq.(65). By summing up the co-efficients of $e^{\iota(px-\rho t)}$ and $e^{\iota(px-\rho t)}$, one gets the dispersion solution by solving the determinant of the coefficient matrix.

$$a^2p^4 - 2abp^6 - 2a\mu p^2 + 2ap^2\rho^2 + 2ap^2\tau^2 + b^2p^8 + 2b\mu p^4 - 2bp^4\rho^2 - 2bp^4\tau^2 + \mu^2 - 2\mu\rho^2 - 2\mu\tau^2 + \rho^4 - 2\rho^2\tau^2 + \tau^4 = 0. \quad (67)$$

Determining the dispersion solution from Eq.(67) for μ, results

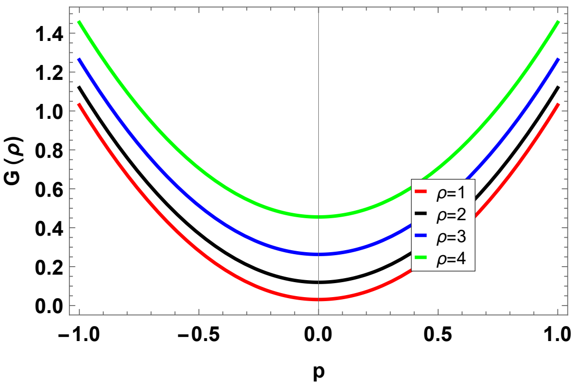
$$\mu = \pm 2\rho\tau \pm \sqrt{a^2p^4 - 2abp^6 + 2ap^2\rho^2 + 2ap^2\tau^2 + b^2p^8 - 2bp^4\rho^2 - 2bp^4\tau^2 + \rho^4 + 2\rho^2\tau^2 + \tau^4}. \quad (68)$$

The obtained dispersion form represents the stability of steady state. If a wave number μ is complex then the steady state solution will not be stable because the perturbation grows gradually. If a wave number μ is real then steady state change into the stable against small perturbations. A steady state result is not stable when;

$$a^2p^4 - 2abp^6 + 2ap^2\rho^2 + 2ap^2\tau^2 + b^2p^8 - 2bp^4\rho^2 - 2bp^4\tau^2 + \rho^4 + 2\rho^2\tau^2 + \tau^4 < 0. \quad (69)$$

MI gain spectrum $G(\rho)$ is achieved as;

$$\begin{aligned} G(\rho) &= 2Im(\mu) \\ &= \pm\sqrt{a^2p^4 - 2abp^6 + 2ap^2\rho^2 + 2ap^2\tau^2 + b^2p^8 - 2bp^4\rho^2 - 2bp^4\tau^2 + \rho^4 + 2\rho^2\tau^2 + \tau^4}. \quad (70) \end{aligned}$$



(a)

Figure 6. Gain spectrum of MI at different values of ρ.

9. Conclusion

We successfully obtained a new kinds of exact solitons of (1+1)-dimensional non-nonlinear generalized Bretherton model. A series of exact soliton solutions, including bright, dark, periodic, singular, singular-bright, singular-dark, and other solitons are obtained by applying the extended sinh-Gordon equation expansion (EShGEE), and the modified (G'/G^2)-expansion techniques. A novel definition of Fractional derivative provides the solutions distinct from the present solutions. Mathematica software is used to obtain, and verify the solutions. The solutions are shown by 2D, 3D, and density graphs. The results are valuable in various areas of applied sciences and engineering. At the end, it is conclude that the used techniques are easy to use and provide the useful results.

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