A formalization of the concept of a numeral system

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- **Abstract.** We consider finite and unconditionally convergent infinite expansions of elements of a given topological monoid G in some base $B \subset G$ as words of the alphabet B, identify insignificantly different words and define a multiplication and a topology on the set of classes of these words. Classical numeral systems are particular cases of this construction. Then we study algebraic and topological properties of the obtained monoid and, for some cases, find conditions under which it is canonically topologically isomorphic to the initial one.

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- **Keywords:** topological monoid, numeral system.

Introduction

Different numeral systems are used for calculations with real and complex numbers and begin being used for other topological algebraic structures. In particular, a wide survey of positional numeral systems in R can be found in [10]. The numeral system in C with the radix $2i$ and the digits 0, 1, 2, 3 (s.c. quarter-imaginary one) is also proposed there. Other numeral systems in C are considered in [5] and [8]. Some numeral systems in $\mathrm{R}[x]/(x^n + 1)$ with $n \geq 2$ are studied in [9]. Calculations in these structures are used in physics.

A. D. Markovskii, an author of many algorithms of fast calculations (see [11], [12]), widely uses non-standard representations of reals. For example, those are expansions in the bases $\mathcal{B}_1 = \{\ln(1 + 2^{-n})\}_{n \in \mathrm{N}_0}$ in $(\mathrm{R}_0^+, +)$ and $\mathcal{B}_2 = \{1 + 2^{-n}\}_{n \in \mathrm{N}_0}$ in $([1; \infty[, \cdot)$ for approximate computations of values of the real exponential function. For other purposes, some of his algorithms operate with expansions in appropriate algebraic extensions of R, in particular, in $\mathrm{R}[x]/(x^2 - 1)$ and in $\mathrm{R}[x]/(x^2)$.

The base $\mathcal{B} = \{x^n/n!\}_{n \in \mathrm{N}_0}$ in $\mathcal{G} = \mathrm{R}[x]/(L)$ with $L = a_0 x^k + \ldots + a_k$ can be used for the approximate solving of the linear differential equation $a_0 x^{(k)} + \ldots + a_k x = 0$ since co-ordinates of the element $\exp(xt) = \sum_0^\infty t^n x^n/n!$ in the base $1, \ldots, x^{k-1}$ of $\mathcal{G}$ depending on the real parameter t form a fundamental system of its solutions with the unit Cauchy matrix at the point $t = 0$.

Generally, a numeral system in a topological monoid $\mathcal{G}$ assigns to its elements their expansions in some base $\mathcal{B} \subset \mathcal{G}$, i.e. represents them as words of the alphabet $\mathcal{B}$. Endowing the set of these words with algebraic and topological structures simulating the corresponding structures of $\mathcal{G}$, we can reduce approximate calculations with these elements to transformations of words. Moreover, as in the case of numbers, such representations can sometimes be useful for proofs of properties of this monoid.

The main purpose of this paper is to find constructions of these structures for arbitrary $\mathcal{B}$ and $\mathcal{G}$. In paper [1], it was done for abelian topological groups. This paper also contains a lot of examples, but there are no theorems proved in this paper there. In papers [2] and [3], the results of [1] were partially transferred onto n-semigroups. However, it became clear now that topological monoids are the natural domain of application of these ideas.

First, we introduce a topology on the free monoid $\widehat{\mathcal{W}}$ generated by the set of expansions in the base $\mathcal{B}$ and prove some its properties, in particular, a criterion of compactness of its subsets. Then we identify expansions which can be obtained from each other by means of simple transformations. This equivalence is said to be algebraic. It is a congruence, and we obtain the quotient monoid $\mathcal{W}'$.

In order to reproduce the main topological properties of $\mathcal{G}$ on $\mathcal{W}'$, we define its topology as the quotient one in the category where $\mathcal{G}$ is studied. Three cases occur in this paper which are not the only possible or mutually exclusive ones. These are the case T) for the category of topological monoids, the case CR) where the underlying space of $\mathcal{G}$ is assumed to be $T_{3\frac{1}{2}}$, and the case U) for uniform monoids.

This topology on $\mathcal{W}'$ can be non-T_0. Therefore, we identify its topologically not distinguishable elements and obtain the quotient monoid $\mathcal{W}$. For

example, for the decimal numeral system in $(\mathrm{R}_0^+, +)$, the expansions $1, 00\ldots$ and $0, 99\ldots$ are topologically equivalent.

Any identity which is true for one of the monoids $\mathcal{G}$, $\mathcal{W}'$ and $\mathcal{W}$, is also true for the others.

Then we define the concept of a numeral system which remained informal until now: *a $\mathcal{B}$-numeral system is a section of the canonical surmorphism $\pi\colon \mathcal{W} \to \mathcal{G}$ assigning to each element of the linear hull $\langle\mathcal{B}\rangle$ its class of finite expansions.* We prove our main theorem which was announced in [2]:

In the cases CR) and U), the following statements are equivalent:

i) There exists a continuous (in the case CR)) or uniformly continuous (in the case U)) $\mathcal{B}$-numeral system;

ii) The map π is an isomorphism of topological (respectively, uniform) monoids.

The third statement of this theorem links the existence of a continuous $\mathcal{B}$-numeral system with the possibility of an extension onto $\mathcal{G}$ of a continuous map given on $\langle\mathcal{B}\rangle$.

In the last section, we show that the standard numeral systems in $(\mathrm{R}_0^+, +)$ are particular cases of the above construction.

1   The construction and properties of the monoid of expansions

A) For a given Hausdorff topological monoid $\mathcal{G}$, denote its multiplication by $\diamond$ (we write $a \diamond b$ instead of $\diamond(a, b)$), its topology by τ and its identity by 1.

Let $\mathcal{B}$ be its fixed subset.

Definition 1.1. A *permissible simple (or 1-tuple) word* of the alphabet $\mathcal{B}$ is any sequence $B = \{b_i\}_{i\in\mathrm{N}}$ of elements of $\mathcal{B}$ satisfying the following condition:

For each neighborhood O of 1, there exists a natural n such that the values of the product $\prod_{i\in I} b_i$ belongs to O for any finite or infinite strictly increasing sequence of natural numbers $I = \{i_1, i_2, \ldots\}$ with $i_1 > n$.

For a series in a linear normed space, to meet this requirement means to be unconditionally convergent.

The *meaning* $\hat{\pi}(B)$ of a given permissible simple word B is the product of all its letters: $\hat{\pi}(B) = \prod_{i\in\mathrm{N}} b_i$. This product converges by the previous definition.

Definition 1.2. The subset $\mathcal{B}$ is called *a computing base* if:

i) $1 \in \mathcal{B}$;

ii) Each element of $\mathcal{G}$ is the meaning of a permissible simple word.

The first condition has a rather technical character. It allows us not to consider separately words of a finite length. A given permissible simple word is said to be *finite* if its letters are eventually equal to 1.

In the following, we assume that the considered subset $\mathcal{B}$ is a computing base and all considered words of this alphabet are permissible. In particular, we will omit the word "permissible". It is evident that the linear hull $\langle\mathcal{B}\rangle$ of such a subset $\mathcal{B}$ is dense in $\mathcal{G}$.

Let $\widehat{\mathcal{W}}^1$ denote the set of all simple words, and $(\widehat{\mathcal{W}}, \circ)$ be the free monoid generated by this set. It is called *the monoid of expansions* in the alphabet $\mathcal{B}$. Its elements are finite ordered collections of simple words. If $B = (B_1, \ldots, B_k)$, $k \in \mathrm{N}_0$, is such a collection, then we call the words $B_1, \ldots, B_k$ the *components* of B and B a k-tuple word of the alphabet $\mathcal{B}$. A multituple word is said to be *finite* if all its components are finite. The submonoid of $\widehat{\mathcal{W}}$ consisting of all finite words is denoted by $\widehat{\mathcal{FW}}$. If 1 is an isolated point in $\mathcal{G}$, then all words are finite.

Let now $\hat{\mathrm{N}}$ be the one-point compactification of N, i.e. the set $\hat{\mathrm{N}} = \mathrm{N} \cup \{\infty\}$ endowed with the order topology: each $n \in \mathrm{N}$ is an isolated point in $\hat{\mathrm{N}}$, and the collection of intervals of the form $(n_0, \rightarrow) = \{n \in \hat{\mathrm{N}} : n > n_0\}$, $n_0 \in \mathrm{N}$, is a base of neighborhoods of the element ∞. Assign to each natural k a copy $\hat{\mathrm{N}}_k$ of the space $\hat{\mathrm{N}}$, consider the product $\prod_{k \in \mathrm{N}} \hat{\mathrm{N}}_k$ with the product topology, and denote by $\hat{\mathcal{I}}$ the subspace of this product consisting of all elements with increasing sequences of coordinates where finite coordinates are increasing strictly. *This subspace is compact.*

With each 1-tuple word $B = \{b_i\}$, one can associate a map $\delta_B : \hat{\mathcal{I}} \rightarrow \mathcal{G}$. We also denote it as B if it does not lead to confusion. For $I \neq I_z = (\infty, \ldots)$, this map is defined by the formula $\delta_B(I) = B(I) = \prod b_i$ where i runs finite coordinates of this element $I \in \hat{\mathcal{I}}$. Moreover, set $\delta_B(I_z) = 1$.

Let $\mathcal{I} \subset \hat{\mathcal{I}}$ be the dense subspace consisting of elements with a finite number of finite coordinates (including I_z). We may consider elements of $\mathcal{I}$ as finite subsets of N. Order $\mathcal{I}$ by inclusion. Then the function $\delta_B|_{\mathcal{I}}$ is a net in $\mathcal{G}$. We call it the *net of approximations* corresponding to B. Its limit in $\mathcal{G}$ is the meaning $\hat{\pi}(B)$ of this word, and $\hat{\pi}(B) = B(I_u)$ where $I_u = (1, 2, \ldots)$.

Let now $B = (B_1, \ldots, B_k)$ be any k-tuple word. Order the set $\mathcal{I}^k$ by the relation $\leq$ defined by

$$(I_1', \ldots, I_k') \leq (I_1'', \ldots, I_k'') \quad \text{iff} \quad I_s' \subset I_s'' \quad \text{for all } s = \overline{1, k}.$$

With each its element $I = (I_1, \ldots, I_k)$, one can associate the element $B(I) = B_1(I_1) \diamond \ldots \diamond B_k(I_k) \in \mathcal{G}$. The resulting net is called the *net of approximations* corresponding to this word B. If $g_1, \ldots, g_k$ are the meanings of its components $B_1, \ldots, B_k$, respectively, then the limit of this net is $g = g_1 \diamond \ldots \diamond g_k$. We call it the *meaning* of this word B. This word is called *an expansion of g in the alphabet $\mathcal{B}$*. If a simple word $B = \{b_i\}$ is an expansion of g, then we write sometimes $g = \prod b_i$ or, in the case of additive denotations, $g = \sum b_i$. The net of approximations of the empty word is, by definition, the sequence $1, 1, \ldots$ and its limit is equal to 1. Thus, the following statement is true.

Proposition 1.3. Assigning to each word its meaning, we obtain a surmorphism of monoids $\hat{\pi} : \widehat{\mathcal{W}} \rightarrow \mathcal{G}$, and $\hat{\pi}(\widehat{\mathcal{FW}}) = \langle \mathcal{B} \rangle$. ∎

Remark. The existing numeral systems in commutative monoids (in $(\mathrm{R}, +)$, in $(\mathrm{C}, +)$ etc.) often use expansions of the form $\sum q_i b_i$ with coefficients (digits) q_i from some set of their admissible values. In order to reduce this situation to the above one, it suffices to consider all elements of $\mathcal{G}$ of the form $q_i b_i$ as separate elements of $\mathcal{B}$.

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B) Now, we introduce a topology on the set $\widehat{\mathcal{W}}$. First, we consider the subset $\widehat{\mathcal{W}}^1$ of simple words.

Proposition 1.4. The map δ_B is continuous.

Proof. Let $B = \{b_i\}$, $I = (i_1, i_2, \ldots)$, $b = \delta_B(I)$, and U be a neighborhood of b. Find a neighborhood U' of b and a neighborhood O of 1 so that $U'O \subset U$. If all coordinates of I are finite, let k be a natural such that $b_{i_1} \diamond \ldots \diamond b_{i_k} \in U'$ and $i_{k+1} > n_0$ where n_0 is defined by the neighborhood O as in Definition 1.1. If I has only a finite number of finite coordinates, then $k+1$ is the number of its first infinite coordinate. Let V denote the neighborhood of I consisting of all points whose first k coordinates are equal to $i_1, \ldots, i_k$ and the remaining coordinates are greater than n_0. Then $\delta_B(V) \subset U$. ■

Proposition 1.5. The map $\delta : \widehat{\mathcal{W}}^1 \to \mathcal{G}^{\hat{\mathcal{I}}}$ (the last symbol denotes the set of continuous maps of $\hat{\mathcal{I}}$ into $\mathcal{G}$) defined by $\delta(B) = \delta_B$ is injective.

Proof. Let B' and B'' be different simple words and b_i', b_i'' their different letters. Put $I = (i, \infty, \ldots)$. Then $B'(I) \neq B''(I)$. ■

Consider now the compact-open topology on $G^{\hat{\mathcal{I}}}$, and let $\hat{\tau}$ be its pre-image on the set $\widehat{\mathcal{W}}^1$ under the map δ. With this topology on $\widehat{\mathcal{W}}^1$, the map δ is an embedding, and we will sometimes identify $\widehat{\mathcal{W}}^1$ with its image. The next corollaries follow from Proposition 1.4 and well-known theorems about spaces of maps (see [6], Theorems 3.4.13, 3.4.15, 2.1.6 and 2.6.4).

Corollary 1.6. The space $(\widehat{\mathcal{W}}^1, \hat{\tau})$ is Hausdorff, and if $\mathcal{G}$ is T_3 ($T_{3\frac{1}{2}}$), then it is T_3 (respectively, $T_{3\frac{1}{2}}$), too.

Corollary 1.7. The evaluation map Ω $(\widehat{\mathcal{W}}^1, \hat{\tau}) \times \hat{\mathcal{I}} \to \mathcal{G}$ defined by the formula $\Omega((B, I)) = B(I)$ is continuous.

Let now the topology τ of $\mathcal{G}$ be given by a uniformity with the set of entourages $\{\mathcal{U}\}$. Because of the compactness of $\hat{\mathcal{I}}$, the compact-open topology on $\mathcal{G}^{\hat{\mathcal{I}}}$ coincides with the topology of the uniform convergence induced by $\{\mathcal{U}\}$. For each entourage $\mathcal{U} \in \{\mathcal{U}\}$, denote by $\hat{\mathcal{U}}$ the entourage of the diagonal of the space $\widehat{\mathcal{W}}_1 \times \widehat{\mathcal{W}}_1$ consisting of all pairs (B', B'') of simple words satisfying the condition $|\delta_{B'}(I) - \delta_{B''}(I)| < \mathcal{U}$ (i.e. $(\delta_{B'}(I), \delta_{B''}(I)) \in \mathcal{U}$) for all $I \in \hat{\mathcal{I}}$. Then the uniformity $\{\hat{\mathcal{U}}\}$ on $\widehat{\mathcal{W}}_1$ generates the topology $\hat{\tau}$. This uniformity is said to be induced by $\{\mathcal{U}\}$.

Extend the topology $\hat{\tau}$ (the uniformity $\{\hat{\mathcal{U}}\}$) to the set $\widehat{\mathcal{W}}^k$ of all k-tuple words as the k-th degree of the topology $\hat{\tau}$ (of the uniformity $\{\hat{\mathcal{U}}\}$) on $\widehat{\mathcal{W}}^1$, and to $\widehat{\mathcal{W}}$ as to the sum $\oplus_k \widehat{\mathcal{W}}^k$. The multiplication $\circ$ in $\widehat{\mathcal{W}}$ is continuous (uniformly continuous) in this topology (respectively, in this uniformity).

Proposition 1.8. The homomorphism $\hat{\pi}$ is continuous with respect to the topologies $\hat{\tau}$ and τ. If the monoid $(\mathcal{G}, \diamond, \{\mathcal{U}\})$ is uniform, then $\hat{\pi}$ is uniformly continuous with respect to the uniformities $\{\hat{\mathcal{U}}\}$ and $\{\mathcal{U}\}$.

Proof. The statement follows from properties of the map $\hat{\pi}|_{\widehat{\mathcal{W}}_1}$. ■

Remark.   There exists another method to introduce a topology on $\widehat{\mathcal{W}}^k$. The formula $\delta_B((I_1, \ldots, I_k)) = \delta_{B_1}(I_1) \diamond \ldots \diamond \delta_{B_k}(I_k)$ where $B = (B_1, \ldots, B_k) \in$

$\widehat{\mathcal{W}}^k$, defines the map $\delta_B \colon \hat{\mathcal{I}}^k \to \mathcal{G}$. The function $B \to \delta_B$ is an embedding of the set $\widehat{\mathcal{W}}^k$ into the set of continuous maps $\mathcal{G}^{\hat{\mathcal{I}}^k}$. Then the compact-open topology on this set defines a topology on $\widehat{\mathcal{W}}^k$. One can prove that if $\mathcal{G}$ is a $T_{3\frac{1}{2}}$ space, then this topology on $\widehat{\mathcal{W}}^k$ coincides with the above one. The main part of this proof is similar to the proof of Proposition 3 from [2].

C) In this section, we prove some properties of the topology $\hat{\tau}$. For each simple word B and for each neighborhood O of 1, denote

$$n_O(B) = \min\{n \in \mathrm{N} \colon\ B(I) \in O\ \ \forall\ I = (i_1, \ldots) \in \hat{\mathcal{I}}\ \ with\ \ i_1 > n\}.$$

Lemma 1.9. The function $n_O(B)$ is upper semi-continuous on $\widehat{\mathcal{W}}_1$.

Proof. The subspace $H_n = \{I \in \hat{\mathcal{I}} : i_1 \geq n\}$ is closed and, hence, compact for any n. The subspace $\{B \in \widehat{\mathcal{W}}^1 : n_O(B) < n\}$ consists of 1-tuple words B such that $B(H_n) \subset O$. Therefore, it is open. ∎

Theorem 1.10. (i) Let $\omega(\mathcal{G})$, $\omega(\widehat{\mathcal{W}})$ denote weights of the spaces $\mathcal{G}$ and $\widehat{\mathcal{W}}$. If $\omega(\mathcal{G}) \geq \aleph_0$, then $\omega(\widehat{\mathcal{W}}) \leq \omega(\mathcal{G})$.

(ii) Let $\chi(\mathcal{G})$, $\chi(\widehat{\mathcal{W}})$ denote characters of the spaces $\mathcal{G}$ and $\widehat{\mathcal{W}}$. If $\chi(\mathcal{G}) \geq \aleph_0$, then $\chi(\widehat{\mathcal{W}}) \leq \chi(\mathcal{G})$.

Proof. (i) The weight of the space $\hat{\mathcal{I}}$ is equal $\aleph_0$, and this space is compact. Therefore, the weight of $\mathcal{G}^{\hat{\mathcal{I}}}$ with the compact-open topology does not exceed $\omega(\mathcal{G})$ (see Theorem 3.4.16 from [6]). Hence, $\widehat{\mathcal{W}^1}$, $\widehat{\mathcal{W}^k} = (\widehat{\mathcal{W}^1})^k$ and $\widehat{\mathcal{W}} = \oplus \widehat{\mathcal{W}^k}$ have this property, too, and the first statement is proved.

(ii) It suffices to consider the space $\widehat{\mathcal{W}^1}$. For each 1-tuple word B, we will construct a base of this space at the point B whose cardinality does not exceed $\chi(\mathcal{G})$. For that, we consider elements of $\mathcal{G}$ of the form $B(I)$ with $I \in \mathcal{I}$. The set of these elements is countable and contains $B(I_z) = 1$. Select a base at each point $B(I)$, and, for each its element U, consider the open subset $V_U(I) = \{B' \in \widehat{\mathcal{W}}^1 : B'(I) \in U\} \subset \widehat{\mathcal{W}}^1$. Denote by $\{O_\alpha\}$ the selected base at 1 and order it by inclusion. For each O_α, put $W_\alpha = \{B' \in \widehat{\mathcal{W}}^1 : n_{O_\alpha}(B') \leq n_{O_\alpha}(B)\}$. This is an open subset by Lemma 1.9. We will prove that finite intersections of sets of the form $V_U(I)$ and W_α form a base at the point B.

Let $H \subset \hat{\mathcal{I}}$ be a compact set and $U \subset \mathcal{G}$ an open set such that $B(H) \subset U$. It suffices to prove that, for any H and U, the set $(H|U) = \{B' \in \widehat{\mathcal{W}}_1 : B'(H) \subset U\}$ contains one of these finite intersections. First, we find an open subset $U' \subset \mathcal{G}$ and a neighborhood O' of 1 so that the inclusions $U' \diamond O' \subset U$, $B(H) \subset U'$ are true. These exist because of the compactness of the subset $B(H)$. Show now that there is a neighborhood O_α such that if the intersection $B(I) \diamond O_\alpha \cap B(H)$ is non-empty for some element I of $\hat{\mathcal{I}}$, then $B(I) \in U'$. Assume it is false. Then there exist a net $\{b_\alpha\}$ of elements of $B(\hat{\mathcal{I}}) \setminus U'$, a net $\{u_\alpha\}$ with $u_\alpha \in O_\alpha$ and a net $\{h_\alpha\}$ of elements of $B(H)$ such that $b_\alpha \diamond u_\alpha = h_\alpha$. Since $B(H)$ and $B(\hat{\mathcal{I}}) \setminus U'$ are compact, we may assume that the first and the third nets converge to some elements $b \in B(\hat{\mathcal{I}}) \setminus U'$ and

$h \in B(H)$, respectively. The second net converges to 1, whence $b = h$, a contradiction.

We assume that $O_\alpha \subset O'$ and find the number $n_0 = n_{O_\alpha}(B)$. For each $I \in H$, denote by $\nu(I)$ the element from $\hat{\mathcal{I}}$ whose finite coordinates are the coordinates of I which do not exceed n_0. Then the inclusion $B(I) \in B(\nu(I)) \diamond O_\alpha$ is true by the definition of the function n_O, and so the inclusion $B(\nu(I)) \in U'$ is true, too. The set $\{\nu(I) : I \in H\}$ is finite since it can only contain I_z and elements of $\mathcal{I}$ with finite coordinates not exceeding n_0. Denote elements of this set by $I_1, \ldots, I_s$ and consider neighborhoods $U_1, \ldots, U_s$ of the elements $B(I_1), \ldots, B(I_s)$, respectively, belonging to the selected above bases and such that $U_1, \ldots, U_s \subset U'$. If an element B' belongs to the intersection of the sets $V_{U_k}(I_k)$ with $k = \overline{1, s}$ and the set W_α, then $B'(I) \in B'(\nu(I)) \diamond O_\alpha \subset U' \diamond O_\alpha \subset U$ for any $I \in H$. That completes the proof. ∎

Now, we are going to establish a criterion of the compactness for subspaces of $\widehat{\mathcal{W}}^1$ and need some more terminology and denotations for that. Consider $\widehat{\mathcal{W}}^1$ as a subspace of $\mathcal{G}^{\hat{\mathcal{I}}}$. For any I from $\hat{\mathcal{I}}$, denote by γ_I the map $\widehat{\mathcal{W}}^1 \to \mathcal{G}$ defined by the formula $\gamma_I(B) = B(I)$, and by I_k, $k \in N$, the element $(k, \infty, \ldots) \in \hat{\mathcal{I}}$. We say that a subspace $\mathcal{F} \subset \widehat{\mathcal{W}}^1$ is a *subspace of the equiconvergence* if, for any neighborhood O of 1, the function n_O is bounded on $\mathcal{F}$.

Theorem 1.11. Let $\mathcal{G}$ be a T_3-space. A subspace $\mathcal{F} \subset \widehat{\mathcal{W}}^1$ is compact if and only if the following conditions are satisfied:

(i) $\mathcal{F}$ is closed;

(ii) $\mathcal{F}$ is a subspace of the equiconvergence;

(iii) For any $k \in N$, the subspace $\gamma_{I_k}(\mathcal{F}) \subset \mathcal{G}$ consisting of k-th coordinates of elements of $\mathcal{F}$ is closed;

(iv) The closure of the subspace $\Omega(\mathcal{F} \times \hat{\mathcal{I}}) \subset \mathcal{G}$ is compact.

Proof. If $\mathcal{F}$ is compact, then its equiconvergence property follows from Lemma 1.9. The subspace $\Omega(\mathcal{F} \times \hat{\mathcal{I}})$ and all subspaces of the form $\gamma_I(\mathcal{F})$ with $I \in \hat{\mathcal{I}}$ are compact since the maps Ω and γ_I are continuous.

The proof of the sufficiency of conditions (i)–(iv) is based on the Ascoli Theorem (see, for example, [6], Theorem 3.4.20). We need to show that $\mathcal{F}$ is closed in $\mathcal{G}^{\hat{\mathcal{I}}}$ and is an evenly continuous set of maps of $\hat{\mathcal{I}}$ into $\mathcal{G}$.

Suppose a map $\varphi_0 \in \mathcal{G}^{\hat{\mathcal{I}}}$ is a limit point of $\mathcal{F}$ in $\mathcal{G}^{\hat{\mathcal{I}}}$. We will construct a 1-tuple word B so that $\delta_B = \varphi_0$. First, let U be an arbitrary neighborhood of the point $\varphi_0(I_k)$ in $\mathcal{G}$. The set $(I_k|U) = \{\varphi \in \mathcal{G}^{\hat{\mathcal{I}}} : \varphi(I_k) \in U\}$ is a neighborhood of φ_0 and, therefore, has a non-empty intersection with $\mathcal{F}$. Thus, any neighborhood of the point $\varphi_0(I_k)$ contains elements from $\gamma_{I_k}(\mathcal{F})$ and $\varphi_0(I_k) \in \gamma_{I_k}(\mathcal{F}) \subset \mathcal{B}$ since $\gamma_{I_k}(\mathcal{F})$ is closed in $\mathcal{G}$. Denote $\varphi_0(I_k) = b_k$, and let $B = \{b_k\}_{k \in N}$. We will prove that B is the required word.

Consider now an element $I \in \mathcal{I}$ of the form $I = (i_1, \ldots, i_n, \infty, \ldots)$ where n and all i_s belong to N, and let $U_1, \ldots, U_n$ be arbitrary neighborhoods of the elements $b_{i_1}, \ldots, b_{i_n}$ and U an arbitrary neighborhood of $\varphi_0(I)$. Similarly to the above argument, there exists a simple word $B' = \{b'_1, b'_2, \ldots\}$ such that $b'_{i_s} \in U_s$ for all $s = \overline{1, n}$ and $B'(I) \in U$. It now follows from the equality $B'(I) = b'_{i_1} \diamond \ldots \diamond b'_{i_n}$ and the arbitrariness of the neighborhoods $U_1, \ldots, U_n, U$

that $\varphi(I) = b_{i_1} \diamond \ldots \diamond b_{i_n}$.

Finally, let $I = (i_1, i_2, \ldots)$ be an element of $\hat{\mathcal{I}}$ whose all coordinates are finite. Denote $I(n) = (i_1, \ldots, i_n, \infty, \ldots)$. Then $I = \lim I(n)$ and $\varphi_0(I) = \lim b_{i_1} \diamond \ldots \diamond b_{i_n}$ by the continuity of the map φ_0 whence convergence of each infinite product of the form $\prod_s b_{i_s}$ follows.

It remains to verify the condition formulated in Definition 1.1. Let O be some neighborhood of 1, O' be its neighborhood such that $\overline{O'} \subset O$ (the line on top denotes the topological closure in $\mathcal{G}$), and $I = (i_1, \ldots)$ be an element of $\hat{\mathcal{I}}$ such that $i_1 > \max_{B \in \mathcal{F}} n_{O'}(B)$. Similarly to the above argument, in any neighborhood of the point $\varphi_0(I)$, there is a point of the form $B'(I) \in O'$ with $B' \in \mathcal{F}$, and, therefore, $\varphi_0(I) = b_{i_1} \diamond b_{i_2} \diamond \ldots \in \overline{O'} \subset O$. Thus, B is a 1-tuple word, and $\varphi_0(I) = B(I)$ for any $I \in \hat{\mathcal{I}}$, i.e. $\varphi_0 = \delta_B$. It follows now from condition (i) that $\varphi_0 \in \mathcal{F}$.

Show that $\mathcal{F}$ is evenly continuous. For any $g \in \mathcal{G}$, its arbitrary neighborhood U and an arbitrary $I = (i_1, i_2, \ldots) \in \hat{\mathcal{I}}$, we need to find neighborhoods W of I and V of g such that $B(I) \in V$ implies $B(W) \subset U$ for any $B \in \mathcal{F}$. Fix a neighborhood U' of g and a neighborhood O' of 1 so that $U' \diamond O' \subset U$, and let V be a neighborhood of g such that $\overline{V} \subset U'$.

If all coordinates of I are finite, then we put $K_1 = \overline{\Omega(\mathcal{F} \times \hat{\mathcal{I}})} \setminus U'$ and $K_2 = \overline{\gamma_I(\mathcal{F})} \cap \overline{V}$. These sets are compact, and their intersection is empty. Therefore, there exists a neighborhood O'' of 1 such that $K_1 \diamond O'' \cap K_2 = \emptyset$. Indeed, let $\{O_\alpha\}$ be a base of neighborhoods of 1 ordered by inclusion and suppose that, for any neighborhood O_α, there are $x_\alpha \in K_1$, $y_\alpha \in K_2$ and $u_\alpha \in O_\alpha$ such that $x_\alpha \diamond u_\alpha = y_\alpha$. If (x, y) is a limit point of the net $\{x_\alpha, y_\alpha\}$ in the compact set $K_1 \times K_2$, then $x = y$ which is impossible.

Put now $O = O' \cap O''$ if all coordinates of I are finite, and $O = O'$ otherwise. In the first case, let $k + 1$ be the number of the first coordinate of the element I which is greater than $n_O = \max_{B \in \mathcal{F}} n_O(B)$, and, in the second case, it is the number of its first infinite coordinate. Denote by W the neighborhood of I consisting of all points whose first k coordinates are equal to the corresponding coordinates of I and the remaining coordinates are greater than n_O. In the first case, denote $I'_k = (i_1, \ldots, i_k, \infty, \ldots)$.

If $B(I) \in V$ for some $B \in \mathcal{F}$, then $B(I'_k) \in U'$ since $B(I) \in K_2$, $B(I'_k) \in \Omega(\mathcal{F} \times \hat{\mathcal{I}})$ and $B(I) \in B(I'_k) \diamond O''$. Therefore, $B(I') \in B(I'_k) \diamond O \subset U$ for any point $I' \in W$. In the second case, if $B(I) \in V$, then $B(I') \in U$. ∎

D) In this section, we consider the operations of the choice of a subsequence and of the shortening of simple words. We use them in the following.

First, we introduce a structure of a topological monoid on $\hat{\mathcal{I}}$ and consider its action on $\widehat{\mathcal{W}}^1$. For $I = (i_1, i_2, \ldots)$, $K = (k_1, k_2, \ldots) \in \hat{\mathcal{I}}$, we put $I * K = (i_{k_1}, i_{k_2}, \ldots)$. If $k_s = \infty$ for some s, then $i_{k_s} = \infty$.

It is not difficult to prove the next statement.

Lemma 1.12. (i) The set $\hat{\mathcal{I}}$ endowed with the operation $*$ and the above topology is a topological monoid. The element $I_u = (1, 2, 3, \ldots)$ is its identity and the element $I_z = (\infty, \infty, \ldots)$ is its zero.

(ii) Considering elements from $\hat{\mathcal{I}}$ as subsets of $\hat{\mathrm{N}}$, we order $\hat{\mathcal{I}}$ by inclusion:

$I \leq K$ if $I \cup \{\infty\} \subset K \cup \{\infty\}$. Then $I * K \leq I$ for any K. ■

Define now an action of $\hat{\mathcal{I}}$ on $\widehat{\mathcal{W}}_1$. Put $BI = \{b_{i_1}, b_{i_2}, \ldots\}$ for $I = (i_1, i_2, \ldots) \in \hat{\mathcal{I}}$, $B = \{b_1, b_2, \ldots\} \in \widehat{\mathcal{W}}_1$ where $b_{i_s} = 1$ if $i_s = \infty$. Then $BI_u = B$, $BI_z = (1, 1, \ldots)$ and the equalities $\delta_{BI}(K) = \delta_B(I * K)$, $[BI]K = B[I * K]$ are true for any $B \in \widehat{\mathcal{W}}_1$, $I, K \in \hat{\mathcal{I}}$ where the square brackets denote the order of the operations.

Theorem 1.13. The map $\lambda : \widehat{\mathcal{W}}_1 \times \hat{\mathcal{I}} \to \widehat{\mathcal{W}}_1$ defined by the formula $(B, I) \to BI$ is continuous. If the uniformity $\{\hat{\mathcal{U}}\}$ is given on $\widehat{\mathcal{W}}_1$ as above in section 1.B), then the family of the maps $\{\lambda(\ , I)\}_{I \in \hat{\mathcal{I}}}$ is equiuniformly continuous.

Proof. Fix $B_0 \in \widehat{\mathcal{W}}_1$ and $I_0 \in \hat{\mathcal{I}}$. Let $H \subset \hat{\mathcal{I}}$ be a compact set and $U \subset \mathcal{G}$ an open set such that $\delta_{B_0 I_0}(H) \subset U$. We need to find neighborhoods V and W of B_0 and I_0 which satisfy the inclusion $\delta_{BI}(H) \subset U$ for any $B \in V$ and $I \in W$. The set $I_0 * H$ is compact and $\delta_{B_0}(I_0 * H) \subset U$. Consider the continuous map $\varphi_{B_0} : \hat{\mathcal{I}} \times \hat{\mathcal{I}} \to \mathcal{G}$ defined by the formula $\varphi_{B_0}(I, K) = \delta_{B_0}(I * K)$. The set $\varphi_{B_0}^{-1}(U)$ is open and contains the compact set $I_0 \times H$. Therefore, there is a neighborhood W of the point I_0 such that $\varphi_{B_0}(\overline{W} \times H) \subset U$. Denote by V the neighborhood of B_0 consisting of elements B which satisfy the inclusion $\delta_B(\overline{V} * H) \subset U$. Then, for all $B \in V$ and $I \in W$, we have $\delta_{BI}(H) = \delta_B(I * H) \subset \delta_B(\overline{V} * H) \subset U$, as required.

To prove the second statement, consider an entourage $\mathcal{U}$ of the given uniformity on $\mathcal{G}$ and the corresponding entourage $\hat{\mathcal{U}}$ of the uniformity on $\widehat{\mathcal{W}}^1$. Let B', B'' be 1-tuple words such that $(B', B'') \in \hat{\mathcal{U}}$, i.e. such that $(\delta_{B'}(K), \delta_{B''}(K)) \in \mathcal{U}$ for all $K \in \hat{\mathcal{I}}$. Then $(\delta_{B'I}(K), \delta_{B''I}(K)) = (\delta_{B'}(I * K), \delta_{B''}(I * K)) \in U$ for any $K \in \hat{\mathcal{I}}$ and $(B'I, B''I) \in \hat{\mathcal{U}}$. ■

Corollary 1.14. Let $\{J(n)\}_{n \in \mathbb{N}}$ be a cofinal sequence of elements of $\mathcal{I}$. Then, for any simple word B, the sequence $\{BJ(n)\}$ consists of finite words and converges to B.

Proof. Since $\lim J(n) = I_u$, it suffices to apply the previous theorem. ■

Corollary 1.15. The set $\widehat{\mathcal{FW}}$ of finite words is dense in $\widehat{\mathcal{W}}$.

Corollary 1.16. For each $n \in \mathbb{N}$, denote $I(n) = (1, \ldots, n, \infty, \ldots)$. The map $\lambda(\ , I(n))$ is a retraction onto the subspace consisting of all words of the form $(b_1, \ldots, b_n, 1, \ldots)$. If the topology of $\mathcal{G}$ is $T_{3\frac{1}{2}}$, then this retraction is uniformly continuous with respect to the corresponding uniformity on $\widehat{\mathcal{W}}^1$.

2 The monoids of classes of algebraic and of topological equivalence

A) Here, we define the relation of the algebraic equivalence on $\widehat{\mathcal{W}}$.

First, denote by $\mathcal{R}_a$ the following binary relation on $\widehat{\mathcal{W}}$: *for any B', $B'' \in \widehat{\mathcal{W}}$, $B' \mathcal{R}_a B''$ means that the nets of approximations of B' and B'' have a common cofinal subsequence.*

$B'\mathcal{R}_a B''$ arises as a consequence of finite equalities connecting segments of B' and B'' in this monoid. In particular, for the decimal numeral system in $\mathcal{G} = (\mathrm{R}_0^+, +)$, pairs of expansions possessing this property correspond to different interpretations of the same standard representation of a real number as the sum of a series of the form $\sum n_k 10^k$ with $n_k \in \mathrm{N}_0$: $0,124 = 12 \cdot 10^{-2} + 40 \cdot 10^{-4} = \dots$. But $(0, \overline{9}, 1) \notin \mathcal{R}_a$.

Definition 2.1. The transitive closure of the relation $\mathcal{R}_a$ is called the *relation of the algebraic equivalence.*

We denote this equivalence relation by $\sim_a$, the quotient set $\widehat{\mathcal{W}}/\sim_a$ by $\mathcal{W}'$, the natural map $\widehat{\mathcal{W}} \to \mathcal{W}'$ by α, and $\alpha(\widehat{\mathcal{FW}})$ by $\mathcal{FW}'$.

Proposition 2.2. Any two algebraically equivalent words have equal meanings. Any two finite words with equal meanings are algebraically equivalent.
Proof. The first statement is evident. To prove the second one, let B be a k-tuple finite word with the meaning g. Then $B(I(n)^k) = g$ for all large enough n, whence the statement follows. $\blacksquare$

Corollary 2.3. The map $\hat{\pi}: \widehat{\mathcal{W}} \to \mathcal{G}$ induces a map $\pi': \mathcal{W}' \to \mathcal{G}$; the restriction $\pi'|_{\mathcal{FW}'}$ is injective, and its image is equal to $\langle \mathcal{B} \rangle$.

Proposition 2.4. The relation of the algebraic equivalence is a congruence.
Proof. First, let $B_1' \in \widehat{\mathcal{W}}_{k_1}$, $B_1'' \in \widehat{\mathcal{W}}_{l_1}$, $B_2' \in \widehat{\mathcal{W}}_{k_2}$, and $B_2'' \in \widehat{\mathcal{W}}_{l_2}$ be words such that $B_1'\mathcal{R}_a B_1''$ and $B_2'\mathcal{R}_a B_2''$. Let $\{J_s'(n)\}$, $\{J_s''(n)\}$ be cofinal sequences in $\mathcal{I}^{k_s}$ and $\mathcal{I}^{l_s}$, respectively, such that $B_s'(J_s'(n)) = B_s''(J_s''(n))$ for all $n \in \mathrm{N}$, $s = 1,2$. Denote by $J'(n)$ the element $(J_1'(n), J_2'(n))$ of $\mathcal{I}^{k_1} \times \mathcal{I}^{k_2} \cong \mathcal{I}^{k_1+k_2}$ and by $J''(n)$ the element $(J_1''(n), J_2''(n))$ of $\mathcal{I}^{l_1} \times \mathcal{I}^{l_2} \cong \mathcal{I}^{l_1+l_2}$. Then the sequences $\{J'(n)\}$, $\{J''(n)\}$ are cofinal in $\mathcal{I}^{k_1+k_2}$ and $\mathcal{I}^{l_1+l_2}$, respectively, and $(B_1' \circ B_2')(J'(n)) = B_1'(J_1'(n)) \diamond B_2'(J_2'(n)) = B_1''(J_1''(n)) \diamond B_2''(J_2''(n)) = (B_1'' \circ B_2'')(J''(n))$. Thus, $(B_1' \circ B_2')\mathcal{R}_a(B_1'' \circ B_2'')$.

Let now $B_s' \sim_a B_s''$ and $B_s' = B_s^1, B_s^2, \dots, B_s^{r_s} = B_s''$ be a collection of words such that $B_s^i \mathcal{R}_a B_s^{i+1}$ for all $i = \overline{1, r_s - 1}$ and $s = 1,2$. Then, according to the above argument, every pair of neighboring members of the sequence $B_1' \circ B_2', B_1^2 \circ B_2', \dots, B_1'' \circ B_2', B_1'' \circ B_2^2, \dots, B_1'' \circ B_2''$ belongs to $\mathcal{R}_a$, and it implies that $(B_1' \circ B_2') \sim_a (B_1'' \circ B_2'')$. $\blacksquare$

Denote by $\star$ the multiplication of the quotient monoid structure on $\mathcal{W}'$.

Corollary 2.5. The map π' is a homomorphism of $(\mathcal{W}', \star)$ onto $\mathcal{G}$.

B) Here, we introduce a topology on the set $\mathcal{W}'$. It depends on the type of functions on $\mathcal{G}$ whose values we intend to compute by means of the considered numeral system. Three cases occur in this paper which are not the only possible or mutually exclusive ones. In the case U, $(\mathcal{G}, \diamond, \{\mathcal{U}\})$ is assumed to be a uniform monoid, and we want to study only uniformly continuous functions (with not necessarily T_0 ranges). In the case CR, we assume that the underlying space of $\mathcal{G}$ is a $T_{3\frac{1}{2}}$-space, and we will study its continuous maps into completely regular (not necessarily T_0) spaces. In particular, $\mathcal{G}$ can be a topological group. In the case T, this underlying space is studied as an object of the category TOP of topological spaces and their continuous maps.

It was proved above that, in each of considered cases, the space $\widehat{\mathcal{W}}$ is an object and the map $\hat{\pi}$ is a morphism of the corresponding category (i.e. UNIF for the case U, COMPL.REG. for the case CR and TOP for the case

T).

In each case, we define the topology on $\mathcal{W}'$ so that α is a quotient in this category map. It guarantees that the map π' is its morphism and, in particular, makes $\mathcal{W}'$ a uniform (a uniformisable) space in the case U (respectively, CR) when $\widehat{\mathcal{W}}$ possesses this property.

Denote by $\mathcal{W}'_T$ ($\mathcal{W}'_{CR}$, $\mathcal{W}'_U$) the set $\mathcal{W}'$ endowed with the topology corresponding to the case T (respectively, CR and U). $\mathcal{W}'_U$ is a uniform (not necessary T_0) space, and $\mathcal{W}'_{CR}$ is completely regular.

The chosen topology on $\mathcal{W}'_T$ is equal to the least upper bound of the pre-images of the topologies on all possible topological spaces $\mathcal{M}$ for which there exist a map φ and a continuous map $\hat{\varphi}$ such that the diagram

$$
\begin{array}{ccc}
 & \widehat{\mathcal{W}} & \\
\alpha \downarrow & & \searrow \hat{\varphi} \\
\mathcal{W}'_T & \xrightarrow{\ \varphi\ } & M
\end{array}
$$

commutes. To obtain the topology on $\mathcal{W}'_{CR}$ ($\mathcal{W}'_U$), it is necessary to assume in addition that $\mathcal{M}$ is a completely regular space (respectively, $\mathcal{M}$ is a uniform space and $\hat{\varphi}$ is a uniformly continuous map).

It is evident that in the case, when the monoid $\mathcal{G}$ is uniform, there exist natural bijective continuous maps $\mathcal{W}'_T \to \mathcal{W}'_{CR} \to \mathcal{W}'_U$, and in the case, when only the topology τ is uniform, there exists a natural bijective continuous map $\mathcal{W}'_T \to \mathcal{W}'_{CR}$.

C) Consider now the commutative case.

Proposition 2.6. If the multiplication $\diamond$ in $\mathcal{G}$ is commutative, then there exists a continuous map $\eta\colon \widehat{\mathcal{W}} \to \widehat{\mathcal{W}}^1$ such that $\eta(B) \sim_a B$ for any word $B \in \widehat{\mathcal{W}}$ and, moreover, $\eta(B)$ is a finite word iff B is a finite one. If $\diamond$ is uniformly continuous, then this map η is uniformly continuous, too.

Proof. Let $B = (B_1, \ldots, B_k)$ be a k-tuple word with $B_i = (b_{i1}, b_{i2}, \ldots)$. Consider the 1-tuple word $\eta(B) = B' = (b_{11}, \ldots, b_{k1}, b_{12}, \ldots, b_{k2}, \ldots)$. As above, denote $I(n) = (1, \ldots, n, \infty, \ldots)$ and $I(n)^k = (I(n), \ldots, I(n)) \in \mathcal{I}^k$. Then $B(I(n)^k) = B'(I(nk))$, and the sequences standing in the sides of this equality are cofinal in the corresponding nets of approximations, i.e. $B \sim_a B'$.

Each term of the net of approximations of B' is equal to the product of appropriate terms of the nets of approximations of the components of B. Therefore, the map η is uniformly continuous if $\diamond$ is uniformly continuous.

Verify now that η is continuous if $\diamond$ is only continuous. Let $(K|U)$ be a containing B' element of the subbasis of the topology $\hat{\tau}$ on $\widehat{\mathcal{W}}_1$ defined in section 1.B). Here, K is a compact subset of $\hat{\mathcal{I}}$, U is an open subset of $\mathcal{G}$, and $B'(K) \subset U$. We will find a neighborhood V of B such that $\eta(\tilde{B})(K) \subset U$ for each $\tilde{B} \in V$. First, we select an open subset $U' \subset \mathcal{G}$ and a neighborhood O' of 1 so that $B'(K) \subset U'$ and $U' \diamond O' \subset U$. These exist because of the compactness of the subset $B'(K)$. Similarly to the proof of Theorem 1.10, we find now a neighborhood O of 1 for which the inclusion $B'(I) \in U'$ is true for $I \in \hat{\mathcal{I}}$ if $(B'(I) \diamond O) \cap B'(K) \neq \emptyset$. Denote by $O_1, \ldots, O_k$ neighborhoods

of 1 such that $O_1 \diamond \ldots \diamond O_k \subset O'$ and put $n_i = n_{O_i}(B_i), \ n_0 = n_O(B')$ where $i = \overline{1,k}$. Let n be a natural such that $n \geq n_i$ for $i = \overline{1,k}$ and $kn \geq n_0$. For each $I \in K$, denote by $\nu(I)$ the element from $\mathcal{I}$ whose finite coordinates are equal to those coordinates of I which do not exceed kn. The element $B'(\nu(I))$ belongs to U' since $B'(I) \in (B'(\nu(I)) \diamond O) \cap B'(K)$. There is only a finite number of these elements, and each of them is equal to a product of elements of the form b_{ij} where $i = \overline{1,k}, \ j = \overline{1,n}$. Therefore, there exist neighborhoods V_{ij} of the elements b_{ij} such that if we replace each multiplier of the form b_{ij} of each product of the form $B'(\nu(I))$ with any element $\tilde{b}_{ij}$ from V_{ij}, then all these products $\tilde{B}'(\nu(I))$ will belong to U' as before. For each $i = \overline{1,k}$, the conditions $\tilde{b}_{ij} \in \mathcal{B} \cap V_{ij}$ where $j = \overline{1,n}$ and $n_{O_i}(\tilde{B}_i) \leq n_{O_i}(B_i)$, define a neighborhood V_i of the word B_i. Put $V = V_1 \times \ldots \times V_k$. It is a neighborhood of the word B. For each its element $\tilde{B}$ and each element $I \in K$, we have $\eta(\tilde{B})(I) = \tilde{B}'(I) \in \tilde{B}'(\nu(I)) \diamond O_1 \diamond \ldots \diamond O_k \subset U' \diamond O' \subset U$. ∎

Denote by α^1 the restriction $\alpha|_{\widehat{\mathcal{W}}^1}$.

Proposition 2.7. If the multiplication $\diamond$ is commutative, then, in each of the considered cases, the map α^1 is a quotient in the corresponding category map onto the corresponding space $\mathcal{W}'_T$ (or $\mathcal{W}'_{CR}, \mathcal{W}'_U$).

Proof. According to Proposition 2.6, the quotient map α is the composition of the morphisms $\widehat{\mathcal{W}} \overset{\eta}{\longrightarrow} \widehat{\mathcal{W}}_1 \overset{\alpha^1}{\longrightarrow} \mathcal{W}'$. Hence, α_1 is a quotient map, too. ∎

Remark. Propositions 2.6 and 2.7 show that the method by means of which the algebraic and the topological structures on $\mathcal{W}'_U$ are defined in this paper, gives the same results as the method of the paper [1] does.

D) Consider now the question of the continuity of the operation $\star$.

Proposition 2.8. In the case U, the operation $\star$ is uniformly continuous. In two other considered cases, it is continuous in each argument separately. Thus, in the case U, $(\mathcal{W}'_U, \star)$ is a uniform monoid on a not necessarily T_0 uniform space. In the case CR, $(\mathcal{W}'_{CR}, \star)$ is a semitopological monoid on a completely regular underlying space, and in the case T, $(\mathcal{W}'_T, \star)$ is a semi-topological monoid on a not necessarily T_0 topological space.

Proof. In the case U, the statement follows from the fact that the product of quotient in the category UNIF maps is a quotient in this category map, too (see [7]). In the remaining cases, it follows from Proposition 2.4 and the definition of a quotient map that right (left) translations in $\mathcal{W}'_{CR}$ and $\mathcal{W}'_T$ are continuous since they are continuous in $\widehat{\mathcal{W}}$. ∎

E) We define now the relation of the topological equivalence.

Considering all three cases simultaneously, denote by K any of the symbols U, CR, T. Generally speaking, the topology on $\mathcal{W}'_K$ is not T_0. Consider, in connection with it, the binary relation in the set $\mathcal{W}' \times \mathcal{W}'$ denoted by $\sim_{t_K}$ and called the *relation of the topological equivalence of the type K*.

For elements $B', B'' \in \mathcal{W}'$, we put $B' \sim_{t_K} B''$ if B' and B'' are topologically indistinguishable, i.e. any open subset of $\mathcal{W}'_K$ containing one of B', B'' contains them both.

The relation $\sim_{t_U}$ is the intersection of all entourages of the uniformity on $\mathcal{W}'_U$, and the relation $\sim_{t_{CR}}$ is the intersection of all entourages of all uniformities compatible with the topology on $\mathcal{W}'_{CR}$. Each open set in $\mathcal{W}'_K$

is saturated with respect to the relation $\sim_{t_K}$, and each continuous function on $\mathcal{W}'_K$ into any T_0-space is constant on each class of $\sim_{t_K}$.

Denote now by $\mathcal{W}_K$ the quotient space $\mathcal{W}'_K / \sim_{t_K}$ in the category corresponding to the case K and by θ_K the quotient map $\mathcal{W}'_K \to \mathcal{W}_K$. They are an object and a morphism of this category. $\mathcal{W}_U$ ($\mathcal{W}_{CR}$) is the Hausdorff uniform (respectively, uniformizable) space associated with the initial space $\mathcal{W}'_U$ ($\mathcal{W}'_{CR}$). W_T is the Kolmogorov quotient of W'_T.

In the case K, we will say that *words B_1, B_2 from $\widehat{\mathcal{W}}$ are t_K-a-equivalent if they have the same images by the map* $\theta_K \circ \alpha : \widehat{\mathcal{W}} \to \mathcal{W}_K$. Such words have equal meanings. For any K, this map $\theta_K \circ \alpha$ is quotient in the corresponding category since it is a composition of maps possessing this property.

The map θ_K is open, and its restriction to any subspace of $\mathcal{W}'_K$ is a quotient in this category map, too. The restriction of the map θ_K to the subspace $\mathcal{F}\mathcal{W}'_K$ (i.e. subset $\mathcal{F}\mathcal{W}'$ endowed with the topology of a subspace of $\mathcal{W}'_K$) is bijective since the restriction of the map π'_K to this subspace is bijective and continuous by Corollary 2.4. We denote the image of $\mathcal{F}\mathcal{W}'_K$ by $\mathcal{F}\mathcal{W}_K$. It follows from Corollary 1.15 that it is a dense subset of $\mathcal{W}_K$.

Let $\mathcal{M}$ be a uniform space, and $\phi: \mathcal{M} \to \mathcal{W}'_U$ be a map. This map is uniformly continuous if and only if the map $\theta_U \circ \phi$ is uniformly continuous. Similar statements are true in the cases CR and T.

If $\mathcal{G}$ is a uniform monoid, then there exist three relations of topological equivalence corresponding to the different definitions of the topology on the set $\mathcal{W}'$: $\sim_{t_T}$, $\sim_{t_{CR}}$ and $\sim_{t_U}$, and each of these relations is contained in the next one. Thus, we obtain three quotient spaces connected by continuous surjective maps $\mathcal{W}_T \xrightarrow{\text{tcr}} \mathcal{W}_{CR} \xrightarrow{\text{cru}} \mathcal{W}_U$. If only the topology τ is uniform, then there exist spaces $\mathcal{W}_T$ and $\mathcal{W}_{CR}$ connected by the map tcr.

It easily follows from Proposition 2.8. that *the relation of the topological equivalence of the type K is a congruence in* $(\mathcal{W}', \star)$ *for any K*. We denote by $\bullet$ the result of the transposition of the operation $\star$ onto $\mathcal{W}_K$.

In the case U, its uniform continuity can be proved by the same argument as in Proposition 2.8. In the remaining two cases, this operation is continuous in each argument.

The map π' induces a surmorphism $\pi_K: (\mathcal{W}_K, \bullet) \to \mathcal{G}$ which is continuous in the cases T and CR and is uniformly continuous in the case U.
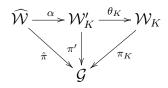
We have proved the following statement.

Proposition 2.9. (i) The set $\mathcal{W}_T$ endowed with the defined above topology and with the multiplication $\bullet$ is a semitopological monoid on a T_0-space, and the map π_T is a continuous surmorphism.

(ii) The set $\mathcal{W}_{CR}$ endowed with the defined above topology and with the multiplication $\bullet$ is a semitopological monoid on a $T_{3\frac{1}{2}}$-space, and the map π_{CR} is a continuous surmorphism.

(iii) The set $\mathcal{W}_U$ endowed with the defined above uniformity and with the multiplication $\bullet$ is a uniform monoid, and the surmorphism π_U is uniformly continuous.

Corollary 2.10. The monoids $\widehat{\mathcal{W}}$, $\mathcal{W}'_K$, $\mathcal{W}_K$ and $\mathcal{G}$ are connected by the following commutative diagram of the defined above continuous (in the cases K=T and K=CR) or uniformly continuous (in the case K=U) homomorphisms:

$$\widehat{\mathcal{W}} \xrightarrow{\alpha} \mathcal{W}'_K \xrightarrow{\theta_K} \mathcal{W}_K$$

$$\hat{\pi} \searrow \quad \downarrow \pi' \quad \swarrow \pi_K$$

$$\mathcal{G}$$

Proof. The statement follows immediately from the definitions of the maps $\hat{\pi}$, π' and π_K. ∎

F) Here, we consider other properties of the monoids $\widehat{\mathcal{W}}$, $\mathcal{W}'_K$ and $\mathcal{W}_K$.
Theorem 2.11. Let $\mathcal{G}_1$, $\mathcal{G}_2$ be Hausdorff topological monoids in the case K=T, or topological monoids on $T_{3\frac{1}{2}}$-spaces in the case K=CR, or uniform monoids in the case K= U and $\mathcal{B}_1$, $\mathcal{B}_2$ arbitrary computing bases in these monoids, respectively. Denote by $\widehat{\mathcal{W}}_i$, $\mathcal{W}'_{iK}$ and $\mathcal{W}_{iK}$ the monoids of $\mathcal{B}_i$-expansions and their classes for $\mathcal{G}_i$, $i = 1$, 2, and K=T, or CR, or U. Let φ be a continuous in the cases K=T and K=CR or uniformly continuous in the case K=U identity preserving homomorphism of $\mathcal{G}_1$ into $\mathcal{G}_2$ such that $\varphi(\mathcal{B}_1) \subset \mathcal{B}_2$.

Then there exist canonical defined continuous in the cases K=T and K=CR and uniformly continuous in the case K=U homomorphisms of monoids $\hat{\psi}$, ψ' and ψ_K such that the diagram

$$
\begin{array}{ccccccc}
\widehat{\mathcal{W}}_1 & \xrightarrow{\alpha_1} & \mathcal{W}'_{1K} & \xrightarrow{\theta_{1K}} & \mathcal{W}_{1K} & \xrightarrow{\pi_{1K}} & \mathcal{G}_1 \\
\downarrow{\hat{\psi}} & & \downarrow{\psi'} & & \downarrow{\psi_K} & & \downarrow{\varphi} \\
\widehat{\mathcal{W}}_2 & \xrightarrow{\alpha_2} & \mathcal{W}'_{2K} & \xrightarrow{\theta_{2K}} & \mathcal{W}_{2K} & \xrightarrow{\pi_{2K}} & \mathcal{G}_2
\end{array}
$$

commutes.
Proof. First, consider the cases T and CR. Define a map $\hat{\psi}_1$ of the set $\widehat{\mathcal{W}}_{11}$ of simple words of the alphabet $\mathcal{B}_1$ into the set $\widehat{\mathcal{W}}_{21}$ of simple words of the alphabet $\mathcal{B}_2$ as follows. For $B = \{b_i\}_{i \in N} \in \widehat{\mathcal{W}}_{11}$ set $\hat{\psi}_1(B) = \{\varphi(b_i)\}_{i \in N}$. It is an element of $\widehat{\mathcal{W}}_{21}$. If B is empty, then $\hat{\psi}_1(B)$ is empty.

Show that $\hat{\psi}_1$ is continuous. Denote by δ_1 (respectively, δ_2) the map $\widehat{\mathcal{W}}_{11} \to \mathcal{G}_1^{\hat{\mathcal{I}}}$ (respectively, $\widehat{\mathcal{W}}_{21} \to \mathcal{G}_2^{\hat{\mathcal{I}}}$) defined as above in section 1.B). Then it is easy to check that $\varphi(\delta_1(B)(I)) = \delta_2(\hat{\psi}_1(B))(I)$ for any $B \in \widehat{\mathcal{W}}_{11}$ and $I \in \hat{\mathcal{I}}$. Consider now the subbasis of the topology on $\widehat{\mathcal{W}}_{21}$ consisting of sets of the form $(K|U) = \{B \in \widehat{\mathcal{W}}_{21} : \delta_2(B)(K) \subset U\}$ where $K \subset \hat{\mathcal{I}}$ is compact and $U \subset \mathcal{G}_2$ is open. Then $\hat{\psi}_1^{-1}((K|U)) = \{B \in \widehat{\mathcal{W}}_{11} : \delta_1(B)(K) \subset \varphi^{-1}(U)\}$ is open in $\widehat{\mathcal{W}}_{11}$ for any K and U such as above. It implies that $\hat{\psi}_1$ is continuous.

Now, it follows from the definitions of the monoid $\widehat{\mathcal{W}}$ in section 1.A) and its topology in section 1.B) that the map $\hat{\psi}_1$ can be uniquelly extended to a continuous homomorphism of $\widehat{\mathcal{W}}_1$ into $\widehat{\mathcal{W}}_2$. We denote this extension by $\hat{\psi}$.

Prove that $\hat{\psi}$ takes algebraically equivalent elements into algebraically equivalent ones. Let $B = (B_1, \ldots, B_k)$ be an arbitrary k-tuple word from $\widehat{\mathcal{W}}_1$. First, we show that the net of approximations of the word $\hat{\psi}(B) \in \widehat{\mathcal{W}}_2$ is the image under φ of the net of approximations of B. In particular, it implies that the map φ takes the meaning of B into the meaning of $\hat{\psi}(B)$, i.e. $\varphi \circ \hat{\pi}_1 = \hat{\pi}_2 \circ \hat{\psi}$. Indeed, we have for any $I = (I_1, \ldots, I_k) \in \mathcal{I}^k$: $\hat{\psi}(B)(I) = \hat{\psi}_1(B_1)(I_1) \diamond \ldots \diamond \hat{\psi}_1(B_k)(I_k) = \varphi(B_1(I_1)) \diamond \ldots \diamond \varphi(B_k(I_k)) = \varphi(B_1(I_1)) \diamond \ldots \diamond \varphi(B_k(I_k)) = \varphi(B(I))$. For arbitrary B', $B'' \in \widehat{\mathcal{W}}_1$, $B'\mathcal{R}_a B''$ means that the nets of approximations of these elements have a common cofinal subsequence. The image under φ of this subsequence is the common cofinal subsequence of the nets of approximations of the elements $\hat{\psi}(B')$ and $\hat{\psi}(B'')$. Hence, $\hat{\psi}(B')\mathcal{R}_a\hat{\psi}(B'')$. (We denote equally the relations $\mathcal{R}_a$, $\sim_a$, $\sim_t$ in the monoids of expansions and of classes of expansions corresponding to the given monoids $\mathcal{G}_1$ and $\mathcal{G}_2$ and their computing bases.) Therefore, $B' \sim_a B''$ implies $\hat{\psi}(B') \sim_a \hat{\psi}(B'')$.

Thus, there exists an only map ψ' such that $\psi' \circ \alpha_1 = \alpha_2 \circ \hat{\psi}$. Here, K=T or CR, and α_1, α_2 are quotient maps in the corresponding categories. Hence, ψ' is continuous.

Let now B' and B'' are elements of $\mathcal{W}'_{1K}$ such that $B' \sim_{t_K} B''$, i.e. any open subset of $\mathcal{W}'_{1K}$ containing one of B', B'' contains them both. The proved continuity of ψ' implies that $\psi'(B') \sim_{t_K} \psi'(B'')$. Therefore, there exists an only map ψ_K such that $\psi_K \circ \theta_{1K} = \theta_{2K} \circ \psi'$. This map is continuous since θ_{1K} and θ_{2K} are quotient maps in the corresponding categories.

The equality $\varphi \circ \pi_{1K} = \pi_{2K} \circ \psi_K$ follows from $\varphi \circ \hat{\pi}_1 = \hat{\pi}_2 \circ \hat{\psi}$.

Consider now the case U. It suffices to prove that the map $\hat{\psi}$ is uniformly continuous. Let the topologies τ_1 on $\mathcal{G}_1$ and τ_2 on $\mathcal{G}_2$ are given by uniformities with bases of entourages $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively, and $\hat{U}''$ be an entourage of the corresponding uniformity on $\widehat{\mathcal{W}}_2$. We may assume that it has the form

$$\hat{U}'' = \{(B', B'') \in \widehat{\mathcal{W}}_2 \times \widehat{\mathcal{W}}_2 : |B'(I) - B''(I)| < U'' \text{ for any } I \in \hat{\mathcal{I}}\}$$

where $U'' \in \mathcal{U}_2$. There exists an entourage U' of the considered uniformity on $\mathcal{G}_1$ such that $(\varphi \times \varphi)(U') \subset U''$. Denote by $\hat{U}'$ the corresponding entourage of the considered uniformity on $\widehat{\mathcal{W}}_1$. Then it is not difficult to verify that $(\hat{\psi} \times \hat{\psi})(\hat{U}') \subset \hat{U}''$. ∎

Corollary 2.12. In the case T, denote by $\mathcal{C}_1$ the category whose objects are pairs $(\mathcal{G}, \mathcal{B})$ where $\mathcal{G}$ is a Hausdorff topological monoid and $\mathcal{B}$ is a computing base in $\mathcal{G}$, and whose morphisms are continuous identity preserving homomorphisms $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ such that $\varphi(\mathcal{B}_1) \subset \mathcal{B}_2$, and by $\mathcal{C}_2$ the category of Hausdorff topological monoids. Then the function $P(\varphi) = \hat{\psi}$ is a functor from $\mathcal{C}_1$ into $\mathcal{C}_2$. Similar statements are true for functions $P(\varphi) = \psi'$ and $P(\varphi) = \psi$ in the case T and for all three functions in the cases CR and U.

Theorem 2.13. Any identity which is true for one of the monoids $\mathcal{G}$, $\mathcal{W}'$ and $\mathcal{W}_K$ where K is U, CR or T, is also true for the others.

Proof. Let $x_1 \ldots x_p = y_1 \ldots y_q$ be an identity in $\mathcal{G}$ written in some alphabet $\mathcal{A}$. First, we will prove that it is true in $\mathcal{W}'$. Consider some maps $\varphi \colon \mathcal{A} \to \mathcal{W}$ and $\hat{\varphi} \colon \mathcal{A} \to \widehat{\mathcal{W}}$ such that $\varphi = \alpha \circ \hat{\varphi}$. Denote $\hat{\varphi}(x_i) = B_i' \in \widehat{\mathcal{W}}^{k_i}$, $\hat{\varphi}(y_j) = B_j'' \in \widehat{\mathcal{W}}^{l_j}$ where $i = \overline{1, p}$, $j = \overline{1, q}$. Put $k = \sum k_i$, $l = \sum l_j$ and denote $I^s(n) = (I(n), \ldots, I(n)) \in \mathcal{I}^s$ for any $s \in N$ where $I(n) = (1, \ldots, n, \infty, \ldots)$ as above. Then $(B_1' \circ \ldots \circ B_p')(I^k(n)) = B_1'(I^{k_1}(n)) \diamond \ldots \diamond B_p'(I^{k_p}(n)) = B_1''(I^{l_1}(n)) \diamond \ldots \diamond B_q''(I^{l_q}(n)) = (B_1'' \circ \ldots \circ B_q'')(I^l(n))$ since the considered identity is true in $\langle \mathcal{B} \rangle$. Sequences $\{I^k(n)\}$, $\{I^l(n)\}$ are cofinal in $\mathcal{I}^k$ and $\mathcal{I}^l$, respectively. Therefore, $(B_1' \circ \ldots \circ B_n') \mathcal{R}_a (B_1'' \circ \ldots \circ B_m'')$ and $\varphi(x_1) \star \ldots \star \varphi(x_n) = \varphi(y_1) \star \ldots \star \varphi(y_m)$.

It is evident that if an identity is true in $\mathcal{W}'$, then it is true in $\mathcal{W}_K$, too.

Consider now an identity in $\mathcal{W}_K$. It is also true in the submonoid $\mathcal{FW}_K$. The restriction of the map π_K to this submonoid is an isomorphism onto $\langle \mathcal{B} \rangle$. This is a dense submonoid of $\mathcal{G}$. Therefore, the considered identity is also true in this initial monoid. ∎

We turn now to the commutative case and prove a statement which can serve as one of bases for the construction of positional numeral systems.

Assume that $\mathcal{B}$ is linearly ordered so that each simple word contains the greatest in this word letter. Denote this order by $\geq$. *A simple word* $B = \{b_1, b_2, \ldots\}$ *is said to be ordered if* $b_j \leq b_i$ *follows from* $j > i$ *for any indices* i, j.

Theorem 2.14. In the cases K= CR and K=U, let $\mathcal{G}$ be commutative and $\mathcal{B}$ linearly ordered as above. Then any word from $\widehat{\mathcal{W}}$ is t_K-a-equivalent to an ordered one.

Proof. Here, we will use additive denotations: $+$ instead of $\diamond$ and 0 instead of 1. Then $0 \in \mathcal{B}$, and we observe that, when adding or deleting zeros from a certain word, we obtain an algebraically equivalent word. Therefore, we may assume with no loss of generality that the initial word $B = \{b_1, b_2, \ldots\}$ is simple (see Proposition 2.6), infinite (see Proposition 2.2) and does not contain zeros.

Then it follows from Definition 1.1 that the number of occurrences of any letter in this word B is finite. Therefore, we can place its letters in the decreasing order. First, we prove that the obtained word $B' = \{b_1', b_2', \ldots\}$ is permissible. Let O be an arbitrary neighborhood of 0 and n a natural such that the sum $\sum_{i \in I} b_i$ converges and its value belongs to O for any finite or infinite strictly increasing sequence of natural numbers $I = (i_1, i_2, \ldots)$ with $i_1 > n$. Denote by n' a natural such that $b_i' \notin \{b_1, \ldots, b_n\}$ for any $i > n'$, and let $r_n(B)$ be the remainder of the word B after deletion of its first n letters and $r_{n'}(B')$ the similar remainder of B'. Then the number of occurrences in $r_{n'}(B')$ of any its letter is equal to the number of its occurrences in $r_n(B)$. Hence, the value of any sum of the form $\sum_{i \in I} b_i'$ where $I = (i_1, i_2, \ldots)$ is a finite or infinite strictly increasing sequence of natural numbers with $i_1 > n'$, belongs to O.

It now suffices to prove that any neighborhood of B' contains a word which is algebraically equivalent to B. Indeed, it implies that any neighborhood of $\alpha(B')$ in $\mathcal{W}'_{CR}$ (and also in $\mathcal{W}'_U$) contains $\alpha(B)$, and then they are topologically equivalent since $\mathcal{W}'_{CR}$ and $\mathcal{W}'_U$ are completely regular.

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Thus, consider an arbitrary neighborhood of B'. We may assume that it has the form $\bigcap_{l=1}^{k} (H_l|U_l)$ where all H_l are compact subsets of $\hat{\mathcal{I}}$, all U_l are open subsets of $\mathcal{G}$ and $(H_l|U_l)$ denotes, as above, the set of all simple words $\tilde{B}$ such that $\delta_{\tilde{B}}(H_l) \subset U_l$. Since the map $\delta_{B'}$ is continuous by Proposition 1.3, the subset $\delta_{B'}(H_l)$ of U_l is compact. Therefore, there exist open sets $U_l' \subset U_l$ and neighborhoods O_l of 0 such that $\delta_{B'}(H_l) \subset U_l'$ and $O_l + U_l' \subset U_l$, $l = \overline{1, k}$. Let O be the intersection of all O_l.

Denote now by k_m the number of the first places of the word B' which are occupied by its first m different letters together with their repetitions. As above, $I(k_m)$ is the sequence of naturals from 1 to k_m which is supplemented by symbols ∞. It is an element from $\mathcal{I}$, and the sequence $I(k_1)$, $I(k_2)$, ... is cofinal in $\mathcal{I}$. Then, by Corollary 1.14, $\{B'I(k_m)\}_{m \in \mathbb{N}}$ is a sequence of simple finite words converging to B'. Therefore, $B'I(k_m) \in \bigcap_{i=1}^{k} (H_i|U_i')$ for all sufficiently large m.

Each transposition of a finite number of letters of any word leads to an algebraically equivalent one. Therefore, for any m, there exists a word which is algebraically equivalent to B, begins with the first k_m letters of the word B' (or $B'I(k_m)$) and whose order of the remaining letters repeats their order in B. We denote it by $B^m = \{b_1^m, b_2^m, \ldots\}$ and show that if m is sufficiently large, then this word belongs to the considered neighborhood of B'.

In order to prove this statement, select n for this neighborhood O of zero as above. For all sufficiently large m, all letters $b_1, \ldots, b_n$ from B are contained among the first k_m letters of the word B^m. Therefore, all letters of the k_m-th remainder of this word belong to the n-th remainder of B and are disposed there in the same order. Hence, the value of any sum of the form $\sum_{i \in I} b_i^m$ where $I = (i_1, i_2, \ldots)$ is a finite or infinite strictly increasing sequence of natural numbers with $i_1 > k_m$, belongs to O.

Let now $I = (i_1, i_2, \ldots) \in H_l$ for some $l = \overline{1, k}$. Denote by I' the subset of I consisting of all i_j with $i_j \leq k_m$ and set $I'' = I \setminus I'$. Then $\delta_{B_m}(I) = \delta_{B'I(k_m)}(I) + \sum_{i \in I''} b_i^m \in U_l' + O \subset U_l$ and, therefore, $B^m \in (H_l|U_l)$ for all $l = \overline{1, k}$ and for all sufficiently large m. ∎

3 The concept of a numeral system

A) We can now define the concept of a numeral system.

For each element $b \in \langle \mathcal{B} \rangle$, denote by $\beta_K(b)$ the class of its finite expansions from $\mathcal{W}_K$. The map β_K is a monomorphism with the dense image $\mathcal{FW}_K$. The composition $\pi_K \circ \beta_K$ is the identity map, and the restriction $\pi_K|$ of the map π_K to $\beta_K(\langle \mathcal{B} \rangle)$ is bijective and continuous in the cases T and CR and uniformly continuous in the case U.

Definition 3.1. A $\mathcal{B}$-*numeral system of the type K* where $K = T$, CR or U, in $\mathcal{G}$ is a section of the map $\pi_K \colon \mathcal{W}_K \to \mathcal{G}$ (i.e. it is a map $\phi \colon \mathcal{G} \to \mathcal{W}_K$ such that $\pi_K \circ \phi = \mathrm{id}$) which coincides with the map β_K on $\langle \mathcal{B} \rangle$.

Remark. Elements of $\mathcal{W}$ are classes of generalized words of the alphabet $\mathcal{B}$. A numeral system ϕ assigns such a class to each element of $\mathcal{G}$. The topology of the subspace $\phi(\mathcal{G})$ and its algebraic structure have to simulate the

corresponding structures of $\mathcal{G}$. Naturally, the class $\phi(b)$ corresponding to an element $b \in \mathcal{B}$ has to contain the word $\{b, 1, 1, \ldots\}$. Therefore, if $b_1, \ldots, b_n$ are elements of $\mathcal{B}$, then the class $\phi(b_1 \diamond \ldots \diamond b_n)$ has to contain the word $\{b_1, 1, 1, \ldots\} \circ \ldots \circ \{b_n, 1, 1, \ldots\}$ and is equal to $\beta(b_1 \diamond \ldots \diamond b_n)$.

The continuity of a given numeral system permits to regulate the error of calculations. If such a numeral system exists, then it is unique since $\langle \mathcal{B} \rangle$ is dense in $\mathcal{G}$.
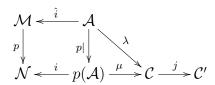
Proposition 3.2. For $K = T$ (respectively, CR, U), if there exists a continuous (respectively, continuous, uniformly continuous) section of the map π_K, then each of the maps $\hat{\pi}$, π'_K, π_K is quotient in the category TOP (respectively, COMPL.REG, UNIF).

Proof. Let ϕ be this section. It is a homeomorphism of $\mathcal{G}$ onto the subspace $\mathrm{Im}\,\phi$ of the corresponding space $\mathcal{W}_K$ in the cases $K = T$ and $K = CR$ and an isomorphism of uniform spaces $\mathcal{G}$ and $\mathrm{Im}\,\phi$ in the case $K = U$. The map $\phi \circ \pi_K$ is a retraction of the space $\mathcal{W}_K$ on this subspace. Hence, it is quotient in the corresponding category. The maps $\pi'_K = \pi_K \circ \theta_K$ and $\hat{\pi} = \pi'_K \circ \alpha$ are compositions of quotient maps. Therefore, they are quotient maps, too. ■

We will now turn to the main theorem of this paper. For that, we need to introduce some more terminology and to prove several lemmas. Let $\mathcal{M}$ be a topological space, $\mathcal{R} \subset \mathcal{M} \times \mathcal{M}$ a binary relation, and $\mathcal{A}$ a dense subset of $\mathcal{M}$. This subset is said to be *dense in* $\mathcal{R}$ if $(\mathcal{A} \times \mathcal{A}) \cap \mathcal{R}$ is dense in $\mathcal{R}$. In other words, if, for any $m', m'' \in \mathcal{M}$ with $m'\mathcal{R}m''$ and for any neighborhoods V' of m' and V'' of m'', there exist $a', a'' \in \mathcal{A}$ such that $a' \in V'$, $a'' \in V''$ and $a'\mathcal{R}a''$.

Lemma 3.3. Let $\mathcal{M}$ and $\mathcal{N}$ be uniform spaces and $p\colon \mathcal{M} \to \mathcal{N}$ a quotient in the category UNIF map. Denote $\mathcal{P} = \{(m', m'') \in \mathcal{M} \times \mathcal{M} : p(m') = p(m'')\}$, and let $\mathcal{R}$ be a binary relation in $\mathcal{M}$ such that its transitive closure coincides with $\mathcal{P}$. If $\mathcal{A}$ is a dense subset of $\mathcal{M}$ which is also dense in $\mathcal{R}$, then the restriction $p|$ of this map p to $\mathcal{A}$ (considering as a map onto $p(\mathcal{A})$) is a quotient in this category map, too.

Proof. Let $\mathcal{C}$ be an arbitrary uniform space, $\mathcal{C}'$ its completion, and λ, μ be maps such that λ is uniformly continuous and the diagram

$$
\begin{array}{ccc}
\mathcal{M} & \xleftarrow{\;\hat{i}\;} & \mathcal{A} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p|} \quad \searrow{\scriptstyle \lambda} \\
\mathcal{N} & \xleftarrow{\;i\;} p(\mathcal{A}) & \xrightarrow{\;\mu\;} \mathcal{C} \xrightarrow{\;j\;} \mathcal{C}'
\end{array}
$$

commutes. Here, i, $\hat{i}$ are the canonical embeddings of subspaces and j a canonical map of $\mathcal{C}$ into its completion. If $\mathcal{C}$ is Hausdorff, then this map is an embedding, too. We have to prove that μ is uniformly continuous.

The map $j \circ \lambda$ can be extended to a uniformly continuous map $\hat{\lambda}\colon \mathcal{M} \to \mathcal{C}'$. Show that equivalent with respect to $\mathcal{P}$ elements of $\mathcal{M}$ have the same images by $\hat{\lambda}$. If $m'\mathcal{P}m''$, then we can assume that $m'\mathcal{R}m''$. Therefore, there exist convergent respectively to m' and m'' nets $\{a'_\alpha\}$, $\{a''_\alpha\}$ consisting of elements of $\mathcal{A}$ and satisfying the condition $a'_\alpha \mathcal{R} a''_\alpha$ for all α. Then $a'_\alpha \mathcal{P} a''_\alpha$, $\lambda(a'_\alpha) = \lambda(a''_\alpha)$ for all α and, therefore, $\hat{\lambda}(m') = \hat{\lambda}(m'')$. Hence, there exists a map $\hat{\mu}\colon \mathcal{N} \to \mathcal{C}'$

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extending $j \circ \mu$ and such that $\hat{\lambda} = \hat{\mu} \circ p$. The map p is quotient in the category UNIF, and so $\hat{\mu}$ and its restriction μ are uniformly continuous. ∎

Lemma 3.4. In the case U, the restriction $\alpha|$ of the map $\alpha \colon \widehat{\mathcal{W}} \to \mathcal{W}'_U$ to the subspace $\widehat{\mathcal{FW}}$ (considered as a map onto $\mathcal{FW}'_U$) is a quotient map in the category UNIF.

Proof. The subspace of finite words $\widehat{\mathcal{FW}}$ is dense in $(\widehat{\mathcal{W}}, \hat{\tau})$. Show that it is dense in the binary relation $\mathcal{R}_a$. Consider any words B', B'' satisfying the condition $B'\mathcal{R}_a B''$ and construct sequences of finite words $\{B'(n)\}$, $\{B''(n)\}$, $n \in \mathrm{N}$, convergent to B' and B'', respectively, such that $B'(n)\mathcal{R}_a B''(n)$ for all n. Let B' be a k-tuple word, B'' a l-tuple word and $\{I'(n)\}$, $\{I''(n)\}$, $n \in \mathrm{N}$, cofinal sequences in $\mathcal{I}^k$ and $\mathcal{I}^l$, respectively, such that $B'(I'(n)) = B''(I''(n))$ for all n. Denote by $(B'_1, \ldots, B'_k)$, $(B''_1, \ldots, B''_l)$ collections of components of the words B' and B'' and by $(I'_1(n), \ldots, I'_k(n))$, $(I''_1(n), \ldots, I''_l(n))$ collections of coordinates of $I'(n)$ and $I''(n)$. For each $n \in \mathrm{N}$, it is evident that the limits of the finite words $B'(n) = (B'_1 I'_1(n), \ldots, B'_k I'_k(n))$ and $B''(n) = (B''_1 I''_1(n), \ldots, B''_l I''_l(n))$ coincide and, moreover, $B'_s = \lim B'_s I_s(n)$ and $B''_r = \lim B''_r I''_r(n)$ for all $s = \overline{1, k}$, $r = \overline{1, l}$ by Corollary 1.14. It remains to apply the previous lemma. ∎

Lemma 3.5. In the case CR, suppose that $(\widehat{\mathcal{W}}, \hat{\tau})$ and $\mathcal{W}'_{CR}$ are endowed with their fine uniformities, i.e. with the finest uniformities which are compatible with their topologies, and $\widehat{\mathcal{FW}}$ and $\mathcal{FW}'_{CR}$ are endowed with the uniformities of their subspaces. Then the restriction $\alpha|$ of the map α to $\widehat{\mathcal{FW}}$ (considered as a map onto $\mathcal{FW}'_{CR}$) is a quotient in the category UNIF map.

Proof. Since α is quotient in the category COMPL.REG (see the definition of the topology on $\mathcal{W}'_{CR}$ in section 2.B)) and $\widehat{\mathcal{W}}$ and $\mathcal{W}'_{CR}$ are considered with their fine uniformities, then the map α is also quotient in the category UNIF. Now, we can apply Lemma 3.3 together with the proved above fact that $\widehat{\mathcal{FW}}$ is dense in $\mathcal{R}_a$. ∎

Lemma 3.6. In the case U, if the multiplication $\diamond$ in $\mathcal{G}$ is commutative, and in the case CR, if $\diamond$ is commutative and, moreover, $\widehat{\mathcal{W}}^1$ and $\mathcal{W}'_{CR}$ are endowed with their fine uniformities and their subsets $\widehat{\mathcal{FW}}^1$ (consisting of finite simple words) and $\mathcal{FW}'_{CR}$ are endowed with the uniformities of their subspaces, then the restriction $\alpha_1|$ of the map $\alpha_1 \colon \widehat{\mathcal{W}}^1 \to \mathcal{W}'_U$ (respectively, $\alpha_1 \colon \widehat{\mathcal{W}}^1 \to \mathcal{W}'_{CR}$) to $\widehat{\mathcal{FW}}_1$ (considered as a map onto $\mathcal{FW}'_U$ or, respectively, on $\mathcal{FW}'_{CR}$) is a quotient in the category UNIF map.

Proof. By Proposition 2.7, α_1 is a quotient in the corresponding category map, and we can use the arguments of lemmas 3.4 and 3.5. ∎

We use the following terminology below. Let $\mathcal{C}$ be a completely regular space in the case CR or a uniform space in the case U and $\lambda \colon \langle \mathcal{B} \rangle \to \mathcal{C}$ be a map. We say that this map is *extendible to a continuous (respectively, uniformly continuous) map of the space $\widehat{\mathcal{W}}$ ($\widehat{\mathcal{W}}^1$)* (with the topology (respectively, with the uniformity) defined in section 1.B)), if there exists a map $\mu \colon \widehat{\mathcal{W}} \to \mathcal{C}$ ($\mu \colon \widehat{\mathcal{W}}^1 \to \mathcal{C}$) possessing this property of the continuity and satisfying the condition $\mu|_{\widehat{\mathcal{FW}}} = \lambda \circ \hat{\pi}|_{\widehat{\mathcal{FW}}}$ ($\mu|_{\widehat{\mathcal{FW}}^1} = \lambda \circ \hat{\pi}|_{\widehat{\mathcal{FW}}^1}$).

Theorem 3.7. The following statements are equivalent in the cases CR and U:

(i) There exists a continuous (in the case CR) or uniformly continuous (in the case U) $\mathcal{B}$-numeral system of the corresponding type;

(ii) The corresponding to the considered case map π_K is an isomorphism of topological (respectively, uniform) monoids $(\mathcal{W}_K, \bullet)$ and $\mathcal{G}$;

(iii) For any completely regular in the case CR or uniform in the case U space $\mathcal{C}$, each map $\lambda \colon \langle \mathcal{B} \rangle \to \mathcal{C}$ which is extendible to a continuous in the case CR or uniformly continuous in the case U map of the space $\widehat{\mathcal{W}}$, is itself continuous (respectively, uniformly continuous) and can be extended to a map of $\mathcal{G}$ possessing the same continuity property.

Additionally, if the multiplication in $\mathcal{G}$ is commutative, then properties (i)–(iii) are equivalent to the next one:

(iv) For any completely regular in the case CR or uniform in the case U space $\mathcal{C}$, each map $\lambda \colon \langle \mathcal{B} \rangle \to \mathcal{C}$ which is extendible to a continuous in the case CR or uniformly continuous in the case U map of the space $\widehat{\mathcal{W}}^1$, is itself continuous (respectively, uniformly continuous) and can be extended to a map of $\mathcal{G}$ possessing the same continuity property.

Remark to statement (iii). If $\lambda|$ can be extended to a continuous (respectively, uniformly continuous) map λ of $\mathcal{G}$ into $\mathcal{C}$, then this extension is unique since $\langle \mathcal{B} \rangle$ is dense in $\mathcal{G}$. Moreover, $\lambda \circ \hat{\pi}$ is an extension of $\lambda|$ on $\widehat{\mathcal{W}}$ possessing the same continuity property.

Proof. Consider the case U. The implication (ii) $\Rightarrow$ (i) is evident since the map $\pi_U'^{-1}$ is a uniformly continuous numeral system.

(ii) $\Leftrightarrow$ (iii). First, we will show that the statement (iii) is true if and only if the restriction $\hat{\pi}|$ of the map $\hat{\pi}$ to the subspace $\widehat{\mathcal{FW}}$ is quotient in the category UNIF. Assume that (iii) is true and consider the diagram from the proof of Lemma 3.3 with $\mathcal{M} = \widehat{\mathcal{W}}$, $\mathcal{A} = \widehat{\mathcal{FW}}$, $\mathcal{N} = \mathcal{G}$, $p = \hat{\pi}$. Then $p(\mathcal{A}) = \langle \mathcal{B} \rangle$. Let λ be uniformly continuous. It can be extended to a uniformly continuous map $\hat{\lambda} \colon \widehat{\mathcal{W}} \to \mathcal{C}'$, and it implies that the map $j \circ \mu$ can be extended to a uniformly continuous map of $\widehat{\mathcal{W}}$. Hence, it is uniformly continuous. Therefore, μ is uniformly continuous, too, and $\hat{\pi}|$ is quotient.

Conversely, let $\hat{\pi}|$ be a quotient map in UNIF and μ a map which can be extended to a uniformly continuous map of $\widehat{\mathcal{W}}$. Denote by $\hat{\lambda}$ such an extension. Then λ and, hence, μ are uniformly continuous. Therefore, μ can be extended to a uniformly continuous map $\hat{\mu} \colon \mathcal{G} \to \mathcal{C}'$. We have now two uniformly continuous maps of $\widehat{\mathcal{W}}$ into $\mathcal{C}'$: $j \circ \hat{\lambda}$ and $\hat{\mu} \circ \hat{\pi}$. They coincide on the dense set $\widehat{\mathcal{FW}}$ and that's why coincide everywhere. Therefore, $\operatorname{Im} \hat{\mu} \subset j(\mathcal{C})$ and the map μ is uniformly continuously extended onto $\mathcal{G}$.

We show now that the statement (ii) is true if and only if the restriction $\pi_U|$ of the map π_U to the subspace $\mathcal{FW}_U$ of classes of finite expansions is an isomorphism of uniform spaces. The necessity is evident. To prove the sufficiency, assume that $\pi_U|$ is an isomorphism. Denote by $\tilde{\mathcal{W}}_U$ and $\tilde{\mathcal{G}}$ the completions of the uniform spaces $\mathcal{W}_U$ and $\mathcal{G}$, respectively. Since $\mathcal{W}_U$ and $\mathcal{G}$ are Hausdorff, we can consider them as subspaces of $\tilde{\mathcal{W}}_U$ and $\tilde{\mathcal{G}}$, respectively.

The map π_U can be extended to a uniformly continuous map $\tilde{\pi}_U \colon \tilde{\mathcal{W}}_U \to \tilde{\mathcal{G}}$. The subspace $\langle \mathcal{B} \rangle$ is dense in $\tilde{\mathcal{G}}$, and $\beta = (\pi_U|)^{-1}$ is its uniformly continuous map into $\tilde{\mathcal{W}}_U$. It can be extended to a uniformly continuous map $g \colon \tilde{\mathcal{G}} \to \tilde{\mathcal{W}}_U$. The map $g \circ \tilde{\pi}_U$ is uniformly continuous, and its restriction to $\mathcal{FW}_U$ is an identical map. Hence, $g \circ \tilde{\pi}_U$ is identical, and π_U is an isomorphism between uniform spaces $\mathcal{W}_U$ and $\mathcal{G}$. Moreover, π_U is a homomorphism, and (ii) is true.

Prove now that $\hat{\pi}|$ is a quotient map in UNIF if and only if the map $\pi_U|$ is an isomorphism of uniform spaces. First, let $\pi_U|$ be an isomorphism. The equality $\hat{\pi} = \pi_U \circ \theta_U \circ \alpha$ implies $\hat{\pi}| = \pi_U| \circ \theta_U| \circ \alpha|$ where $\alpha|$ is the restriction of the map α to $\widehat{\mathcal{FW}}$ and $\theta_U|$ is the restriction of the map θ_U to $\mathcal{FW}'_U$. The map $\alpha|$ is quotient by Lemma 3.4 (in the case CR, it will be necessary to use Lemma 3.5). So the map $\theta_U|$ is quotient as well as any restriction of θ_U to a subspace. Hence, the map $\hat{\pi}|$ is quotient as a composition of quotient maps.

Conversely, suppose the map $\hat{\pi}|$ is quotient in UNIF. Then $\pi_U|$ is quotient, too. However, it is also bijective, since the restriction $\pi'|$ of the map π' to $\mathcal{FW}'$ is bijective by Corollary 2.3. Hence, $\pi_U|$ is an isomorphism of uniform spaces.

(ii) $\Leftrightarrow$ (iv) if the multiplication $\diamond$ in $\mathcal{G}$ is commutative. Similarly to the above argument, we can prove that (iv) is true if and only if the restriction $\hat{\pi}_1|$ of the map $\hat{\pi}$ to the subspace $\widehat{\mathcal{FW}}^1$ of finite simple words is a quotient map in the category UNIF. The equality $\hat{\pi}_1| = \pi_U| \circ \theta_U| \circ \alpha_1|$ is true, and the map $\alpha_1|$ is quotient in the category UNIF by Lemma 3.6. Now, it is easy to verify that the map $\hat{\pi}_1|$ is quotient if and only if the map $\pi_U|$ is an isomorphism of uniform spaces.

(i) $\Rightarrow$ (ii). Let $\varphi \colon \mathcal{G} \to \mathcal{W}_U$ be a uniformly continuous $\mathcal{B}$-numeral system. Its restriction to $\langle \mathcal{B} \rangle$ is back to $\pi_U|$. Hence, $\varphi \circ \pi_U$ is identical, and π_U is an isomorphism of uniform monoids.

For the case U, the proof is complete. To obtain the proof for the case CR, endow the spaces $\mathcal{G}$, $\widehat{\mathcal{W}}$, $\mathcal{W}'_{CR}$, $\mathcal{W}_{CR}$ with their fine uniformities and the spaces $\langle \mathcal{B} \rangle$, $\widehat{\mathcal{FW}}$, $\mathcal{FW}'_{CR}$, $\mathcal{FW}_{CR}$ with the uniformities of their subspaces. Then $\mathcal{W}_{CR}$ becomes the Hausdorff uniform space associated with the uniform space $\mathcal{W}'_{CR}$. Any continuous map of any uniformizable space endowed with its fine uniformity into a uniform space is uniformly continuous. Using these facts, it is possible to verify that the above argument is also true in the case CR. ∎

The next statement supplements the proved theorem.

Proposition 3.8. In the case U (CR, T), let $\hat{\phi}$ be a uniformly continuous (respectively, continuous, continuous) map of $\widehat{\mathcal{W}}$ into a Hausdorff uniform (respectively, $T_{3\frac{1}{2}}$, Hausdorff) space $\mathcal{C}$ such that $\hat{\phi}(B_1) = \hat{\phi}(B_2)$ for any finite words B_1 and B_2 with a common meaning. Then there exists a morphism of the corresponding category $\phi : \mathcal{W}_K \to \mathcal{C}$ such that $\hat{\phi} = \phi \circ \theta_K \circ \alpha$.

Proof. It follows from our assumption that $\hat{\phi}(B_1) = \hat{\phi}(B_2)$ if $B_1 \mathcal{R}_a B_2$ is true. Hence, the map $\hat{\phi}$ is constant on each class of algebraic equivalence, and so it generates some map $\phi' : \mathcal{W}'_K \to \mathcal{C}$. The definition of the topology on $\mathcal{W}'_K$ implies that this map is a morphism of the corresponding category.

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In particular, it is continuous. Therefore, it is constant on every class of topological equivalence and induces some map $\phi : \mathcal{W}_K \to \mathcal{C}$. This is the required map. By the definition of the topology on $\mathcal{W}_K$, it is a morphism of the corresponding category. ∎

We establish now the following sufficient condition of the existence of a uniformly continuous numeral system.

Proposition 3.9. The map π_U is an isomorphism of uniform monoids $\mathcal{W}_U$ and $\mathcal{G}$, and there exists a uniformly continuous $\mathcal{B}$-number system if, for each entourage $\mathcal{U}$ of the given uniformity on $\mathcal{G}$, there exists its entourage $\mathcal{V}$ such that, for each elements g', $g'' \in \langle \mathcal{B} \rangle$ with $(g', g'') \in \mathcal{V}$, there exist their finite simple expansions B', B'' with $(B', B'') \in \hat{\mathcal{U}}$ where $\hat{\mathcal{U}}$ is the corresponding to $\mathcal{U}$ entourage of the uniformity on $\widehat{\mathcal{W}}_1$ which was defined in section 1.B).

Proof. Let $\mathcal{O}$ be an entourage of the uniformity on $\mathcal{W}_U$ and $\mathcal{U}$ an entourage of the uniformity on $\mathcal{G}$ such that $(\theta_U \circ \alpha) \times (\theta_U \circ \alpha)(\hat{\mathcal{U}}) \subset \mathcal{O}$. This $\mathcal{U}$ exists for an arbitrary $\mathcal{O}$ since the map $\theta_U \circ \alpha$ of $\widehat{\mathcal{W}}_1$ into $\mathcal{W}_U$ is uniformly continuous. If $(g', g'') \in \mathcal{V}$, then $(\beta_U(g'), \beta_U(g'')) = ((\theta_U \circ \alpha)(B'), (\theta_U \circ \alpha)(B'')) \in \mathcal{O}$. Hence, the map $\pi_U|^{-1} = \beta_U$ is uniformly continuous, and $\pi_U|$ is an isomorphism of uniform monoids. Now, the statement follows from the proof of the preceding theorem. ∎

4   Examples

A) The first example is especially important since we are showing that standard numeral systems in $(\mathrm{R}_0^+, +)$ (endowed with the usual topology) can be obtained by means of the above construction. For that, we are considering the variable-length numeral system which was defined and studied in Chapter IV of "General topology" of Bourbaki (see [4]). It generalizes classical numeral systems such as the binary one etc.

First, let $\mathcal{B} = \{b_i\}_0^\infty$ where $b_0 = 1$ and $b_{i-1}/b_i \in \mathrm{N} \setminus \{1\}$ for all $i \in \mathrm{N}$. In particular, it is proved in [4] that any positive number can be written as the value of a finite or infinite sum of the form $\sum n_k \, b_{i_k}$ where $k, n_k \in \mathrm{N}$ and the following conditions are satisfied: *if this sum consists of k_0 addends ($k_0 \in \mathrm{N}$ or $k_0 = \infty$), then $0 \le i_k < i_{k+1}$ for every $1 \le k \le k_0 - 1$ and $0 < n_k < b_{i_k-1}/b_{i_k}$ for every $k \le k_0$ such that $i_k \ge 1$.*

We call such sums *standard*. Each positive number has exactly one infinite standard expansion. If it has also a finite expansion, then any sufficiently distant remainder of its standard infinite expansion has the form $\sum_{i \ge k}(b_{i-1}/b_i - 1)b_i$.

To use denotations of the preceding sections, *add the element $b_{-1} = 0$ to $\mathcal{B}$*. By Proposition 2.6, any permissible word of the alphabet $\mathcal{B}$ is algebraically equivalent to a simple one. In this section, we will often use the additive form of the recording of simple words, i.e. we will write $\sum b_i$ instead of $\{b_i\}_{i \in N}$. For each k, if such a sum contains n_k consecutive addends $b_{i_k} \ne 0$, then we will write it as $\sum n_k \, b_{i_k}$. This is a one-to-one correspondence between simple words and convergent series with addends from $\mathcal{B}$. For any non-zero $b \in \mathcal{B}$, the number of its occurrences in such a series is finite. Finite words correspond

to finite sums where the infinite number of zeros is added. Simple words with the same non-zero addends are algebraically equivalent. Operating with series, we will sometimes use the term "the value of the sum" instead of "the meaning of the word". Every standard sum corresponds to a permissible word which is also said to be standard.

First of all, we prove that any simple word is algebraically equivalent to a standard one. If this word B is finite, then it can be easily transformed to a standard one with the same meaning. Suppose now that the word B is infinite. Writing it as a sum, place brackets in it in the following way. In the first brackets, we only put the first term of B. We denote its number in $\mathcal{B}$ by i_1. Assume now that the k-th brackets are already placed, the last their term is the term of the considered sum with the number r_k and the greatest among the numbers of elements of $\mathcal{B}$ included in these brackets is i_k. Then $(k+1)$-th brackets begin with the term with the number $r_k + 1$ and end with the term with the number r_{k+1} satisfying the following conditions:

i) the sequence $\{[S_m/b_{i_k}]\}_m$ where S_m is the m-th partial sum of B and the symbol $[\,]$ denotes the whole part of a number, is stationary for $m \geq r_{k+1}$;

ii) if i_{k+1} is the greatest among indices of elements from $\mathcal{B}$ that occur in these brackets, then $i_{k+1} > i_k$.

Write now the content of the k-th brackets, $k \geq 2$, as $p_k b_{i_{k-1}} + q_k b_{i_k}$ where $p_k,\ q_k \in \mathrm{N}_0$ and $q_k b_{i_k} < b_{i_{k-1}}$. This word is algebraically equivalent to the initial one and can be written as $\sum(q_k + p_{k+1})b_{i_k}$ where $q_1 = 1$ and $(q_{k+1}+p_{k+2})b_{i_{k+1}} < b_{i_k}$ for $k \geq 1$ since the above sequence is stationary. Now, we transform each expression $(q_k + p_{k+1})b_{i_k}$ in $\sum_{i_{k-1}+1}^{i_k} n_i b_i$ where $i_0 = -1$, $n_i \in \mathrm{N}_0$ and $n_i < b_{i-1}/b_i$ for all $i \in \mathrm{N}$. To obtain the required standard sum such that the corresponding word is algebraically equivalent to B, we have to omit each addend $n_i b_i$ with $n_i = 0$. The unique expansion of zero is the empty one.

Finite expansions with a common sum are algebraically equivalent by Proposition 2.3. An infinite expansion cannot be algebraically equivalent to a finite one because each term of its net of approximations is less than its meaning, whereas each sufficiently distant term of the net of approximations of any finite expansion is equal to its meaning.

Algebraic structure on $\mathcal{W}'$ is defined by the following rules: the sum of two classes of finite expansions is a class of finite expansions, the sum of two classes of infinite expansions or a class of finite expansions with a class of infinite expansions is a class of infinite expansions. It is clear that the monoid $(\mathcal{W}', \star)$ is not cancellable, and so *the statement of Theorem 2.11 is not applicable to conditional identities.*

Describe now the topologies of the spaces $\mathcal{W}'_T$, $\mathcal{W}'_{CR}$ and $\mathcal{W}'_U$. By Proposition 2.7, they are quotient spaces of the space $\widehat{\mathcal{W}}_1$ in the corresponding categories. Therefore, we will only consider simple words.

First, we will prove that if a number a has both the class of finite expansions and the class of infinite ones, then the first of them is not closed and the second one is closed in $\mathcal{W}'_T$. Indeed, let $\sum n_i b_i$ be the standard infinite expansion of this number a. Then there exists $i_0 \in \mathrm{N}$ such that $n_i = b_{i-1}/b_i - 1$ for $i \geq i_0$. Any neighborhood of this expansion contains a finite word. To find it, it is enough to add unity to one of these coefficients and to put all

subsequent terms equal to zero.

Consider now the subset of $\widehat{\mathcal{W}}_1$ consisting of all simple finite expansions of this number a and all simple expansions with the sums from interval $]a, a+\epsilon[$ where $\epsilon > 0$. We will show that this subset is open. It is saturated by the relation of the algebraic equivalence, hence, its image in $\mathcal{W}'_T$ is open, too. Let B be a finite expansion of a. Denote by i the greatest among indices of elements of $\mathcal{B}$ that occur in B, and consider the neighborhood of B in $\widehat{\mathcal{W}}_1$ consisting of all words whose letters differ from the corresponding non-zero letters of B by less than $1/2\ b_i$ and whose meaning is less than $a + \epsilon$. The letters of these words corresponding to non-zero letters of B coincide with them. The meanings of these words cannot be less than a. If such a meaning is equal to a, then the remaining letters of this word are equal to zero, i.e. it is finite.

Let now $U \subset \mathcal{W}'_T$ be a neighborhood of the class of infinite expansions of a. We will show that it contains all classes of the algebraic equivalence with the sums from some neighborhood of a. Indeed, the pre-image of U in $\widehat{\mathcal{W}}_1$ contains the standard infinite expansion B of a and, therefore, all words B' satisfying the inequality $|B'(I) - B(I)| < \epsilon$ for some $\epsilon > 0$ and any $I \in \hat{\mathcal{I}}$. Let $b \in]a, a+\epsilon[$ and $\sum n_i b_i$ be the standard expansion of the number $b - a$. Writing B in the form of a series and adding to each group of its equal to b_i addends n_i more such addends, we obtain an expansion B' of the number b. Each term of B does not exceed the corresponding term of B' and, hence, $0 \le B'(I) - B(I) \le B'(I_u) - B(I_u) = b - a < \epsilon$. Now, let i_0 be a number such that the expansion B contains some non-zero basis element $b_{i_0} < \epsilon$ with non-zero coefficient, and $b \in]a - b_{i_0}, a[$. Denote the standard expansion of the number $b - a + b_{i_0}$ by $\sum n_i b_i$. Delete one equal to b_{i_0} addend of the expansion B and add all addends of the expansion $\sum n_i b_i$. The resulting expansion B' of b satisfies the inequality $|B'(I) - B(I)| < \epsilon$ for all $I \in \hat{\mathcal{I}}$, too. Thus, the neighborhood U contains all classes of infinite expansions of all numbers from the interval $]a - b_{i_0}, a+\epsilon[$ and, hence, also all classes of their finite expansions.

The proved property implies that the restriction of the map π' to the submonoid of $(\mathcal{W}'_T, \star)$ consisting of classes of infinite expansions is a homeomorphism onto R_0^+. Moreover, it is a homomorphism onto $(\mathrm{R}_0^+, +)$. Hence, these monoids are topologically isomorphic.
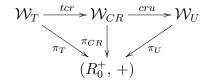
Note that the first part of the above argument does not use that the expansion B is infinite. It implies that any neighborhood of the class of finite expansions of the number a contains all classes of expansions with the sums from some interval of the form $]a, a+\epsilon[$.

Therefore, a base of the space $\mathcal{W}'_T$ is completely described. *This space satisfies the separation axiom T_0 but does not satisfy the axiom T_1.* In particular, it implies that $(\mathcal{W}'_T, \star)$ and $(\mathcal{W}_T, \bullet)$ are topologically isomorphic.

Prove now that π_U is an isomorphism of the uniform monoids $\mathcal{W}_U$ and $(\mathrm{R}_0^+, +)$. It is sufficiently to show that Proposition 3.9 can be used. Let $g', g'' \in \langle \mathcal{B} \rangle$, $|g'' - g'| < \delta$, and $g' < g''$. Construct such finite simple expansions B' and B'' of g' and g'' that $|B''(I) - B'(I)| < \delta$ for each $I \in \hat{\mathcal{I}}$. Let $\tilde{B}'$ and $\tilde{B}''$ be any finite simple expansions of g' and $g'' - g'$. Denote by B' the expansion of g' whose terms are alternately terms of the expansion $\tilde{B}'$

and zeros and by B'' the expansion of g'' whose terms are alternately terms of the expansions $\tilde{B}'$ and $\tilde{B}''$. One can verify that the previous inequalities are true.

It remains to consider the topology of $\mathcal{W}_{CR}$. The diagram of continuous surjective maps (see section 2.E)

$$\mathcal{W}_T \xrightarrow{\ tcr\ } \mathcal{W}_{CR} \xrightarrow{\ cru\ } \mathcal{W}_U$$
$$\pi_T \searrow \quad \pi_{CR}\downarrow \quad \swarrow \pi_U$$
$$(R_0^+,\ +)$$

commutes, and $\mathcal{W}_{CR}$ is a $T_{3\frac{1}{2}}$-space. Therefore, the map tcr identifies classes of finite words with corresponding classes of infinite words having the same meanings. Hence, it follows from the above properties of the maps π_T and π_U that π_{CR} is an isomorphism of topological monoids, too.

Thus, a continuous (uniformly continuous) $\mathcal{B}$-numeral system exists in the case CR (respectively, U) and assigns to each element of R_0^+ the class of its standard expansion. It does not exist in the case T although $(R_0^+,\ +)$ is topologically isomorphic to a submonoid of $\mathcal{W}_T$.

B) We consider now the computing base $\mathcal{B}$ in $(R;\ +)$ endowed with the usual topology, consisting of zero, of all numbers 2^{-n} with $n \in N_0$ and of some irrational negative a. We will prove that monoid $(\mathcal{W}_{CR}, \bullet)$ is topologically isomorphic to $(R_0^+,\ +) \oplus (N_0,\ +)$, and the map π_{CR} can be defined by the formula $(x,n) \to x + na$ where $x \in R_0^+$, $n \in N_0$. Therefore, π_{CR} is not an isomorphism, and for this computing base there are no continuous numeral systems of the type CR.

As above, we will only consider simple words. Let B be such a word. Denote by $n(B)$ the number of occurrences of a in it, and consider the entourage of the diagonal of $\widehat{\mathcal{W}}_1 \times \widehat{\mathcal{W}}_1$ consisting of pairs $(B',\ B'')$ of words satisfying the condition $|B'(I) - B''(I)| < |a|$ for all $I \in \hat{\mathcal{I}}$. It can only contain pairs with the same n. Moreover, such a pair belongs to this entourage iff these words contain all letters a on the same places. Hence, the space $\widehat{\mathcal{W}}_1$ is the sum of its subspaces $\widehat{\mathcal{W}}_1(n)$, $n \in N_0$, consisting of words with a fixed n, and each of these subspaces is the sum of its subspaces consisting of words containing all letters a on the same places. Each change of places of letters a gives a homeomorphism of such subspaces taking each word to an algebraically equivalent one.

Words with different n cannot be algebraically equivalent, since all distant enough terms of their nets of approximations are different. It implies that $\mathcal{W}'_{CR}$ is the sum of its subspaces $\mathcal{W}'_{CR}(n) = \widehat{\mathcal{W}}_1(n)/ \sim_a$ endowed with the quotient in the category COMPL.REG topology.

The corresponding numbers n are summed up by addition of elements of $\mathcal{W}'_{CR}$. Hence, $\mathcal{W}'_{CR}(0)$ is a submonoid in $\mathcal{W}'_{CR}$. Now, it follows from the argument of the previous example that this submonoid consists of classes of expansions of the usual binary numeral system, and, therefore, it is topologically isomorphic to $(R_0^+,\ +)$.

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Let now B be a word not containing a. Placing n letters a as its initial terms, we obtain some word B'. It is evident that the map $B \to B'$ is a homeomorphism of the subspace $\widehat{\mathcal{W}}_1(0)$ onto the summand S_n of $\widehat{\mathcal{W}}_1(n)$ consisting of words containing n letters a as their initial terms. Words B_1, B_2 not containing a are algebraically equivalent if and only if the corresponding them words B_1', B_2' possess this property.

For any n, each word from $\widehat{\mathcal{W}}_1(n)$ is algebraically equivalent to some word B whose $n(B) = n$ first letters are a. Therefore, the quotient spaces $\oplus_n(S_n/ \sim_a) = (\oplus_n S_n)/ \sim_a$ and $(\oplus_n \widehat{\mathcal{W}}_1(n))/ \sim_a = \mathcal{W}'_{CR}$ are canonically homeomorphic. The above described translation makes each space $S_n/ \sim_a$ canonically homeomorphic to $S_0/ \sim_a = \mathcal{W}'_{CR}(0) \approx \mathrm{R}_0^+$, and this homeomorphism can be defined by the formula $x \to x + na$ since this translation adds na to the meanings of all words from $S_0 = \widehat{\mathcal{W}}_1(0)$. Moreover, the relation of the topological equivalence is trivial, since $\mathcal{W}'_{CR}$ is already a T_0-space, and therefore $\mathcal{W}'_{CR} = \mathcal{W}_{CR}$. It completes the proof.

References

1. Knuth, D. *The Art of Computer Programming,* 3rd Ed, Volume 2, 194-213.

2. Duda, Jarek, *Complex base numeral systems*, arXiv: math/0712.1309v3 [math. DS], 24 Feb **2008**.

3. Jamil, T., *The complex binary number system*, IEEE Potentials, **2002**, **20**, **5**, 39-41.

4. Khmelnik, S. I., *Coding of Complex Numbers and Vectors*, Mathematics in Computers, Israel, **2004**, 121 pp. (in Russian).

5. Markovskii, A. D., *Parallel algorithms of division on the base of multiplicative expansions of real numbers*, publ. of the Institute for Space Research of the Soviet Academy of Sciences, **1990**, 1-50, (in Russian).

6. —, *Computing bases for the main number systems*, Obozr. Prikl. Prom. Mat., **2012**, **19**, **3**, (in Russian).

7. Averbukh, B. G., *Numeral systems in abelian topological groups*, Fundamentalnaja i prikladnaja matematika, **2000**, **6**, **2** , 319–356, (in Russian).

8. —, *Numeral systems in topological n-semigroups* , in the periodical collection "Abelian groups and modules", **2000**, **15**, Tomsk, 6–16, (in Russian)

9. —, *Formalization of the concept of a numeral system* , Uspekhi Mat. Nauk, **1999**, **54**, **1**, 253–254, (in Russian).

10. Engelking, R., *General Topology*, Heldermann Verlag, Berlin, **1989**, xiii+529 pp.

11. Hušek, M. and Rice, M., *Productivity of coreflective subcategories of uniform spaces*, Gen. Top. Appl., **1978**, **9**, 295–306.

12. Bourbaki, N., *Elements of Mathematics. General Topology, Chapters 1-4,* Springer Verlag, Berlin **1989**, 373-377.