Generalized Pareto-Type Distribution and Income Inequality: An Extension of Gibrat's Law

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Article

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Abstract: Motivated by empirical observations, we propose a possible extension of Gibrat's law. By applying it into the random growth theory of income distribution (Gabaix, 2009), we find that the income distribution is described by a generalized Pareto-type distribution (GPD) with three parameters. We observe that there is a parameter η in the GPD that plays a key role in determining the shape of income distribution. By using the Kolmogorov-Smirnov test, we empirically show that, for typical market-economy countries, η is close to 0 significantly, such that the income distribution is characterized by a two-class pattern in which the bottom 90% of the population is approximated by an exponential distribution and the richest 1%~3% is approximated by an asymptotic power law. However, we empirically find that, for China both in planned economy period and in early stages of market reformation (from 1978 to 1990), η is significantly deviated from 0, such that the bottom of the population no longer conforms to an exponential distribution.

Keywords: random growth theory; Kolmogorov forward equation; income distribution; generalized Pareto distribution; Kolmogorov-Smirnov test

1. Introduction

Global income inequality in the 21st century is growing, and top earners are taking hold of a larger and larger fraction of total income [1,2]. Concerning the upper tail of the income distribution, which is referred to as the richest 1%~3% of the population, it has been long known that it is approximated by the Pareto distribution [3,4]. Based on the random growth theory of income distribution (RGTID) [5–9], it has been acknowledged that the Pareto distribution arises because the dynamics of income obey Gibrat's law in the stochastic process. This law was first observed by Gibrat [10] in investigating the growth of firm size. It states that firm growth is a purely random effect, independent of firm size. When this law is applied into the RGTID, the resulting distribution of income is governed by the Pareto law [5,9,11,12]. For this reason, Gibrat's law is widely used to understand the rise in top income inequality. However, the singular focus on the top income class of households overlooks the component of earnings' inequality that is arguably most consequential for the low- and middle-income class of households [13]. The existing empirical observation has shown that [14–18], for market-economy countries, the income structure of the bottom 90% of the population is approximated by an exponential distribution. Therefore, it has been proposed that [14,15] the income distribution is characterized by a two-class pattern, in which the bottom 90% of the population is approximated by an exponential distribution and the richest 1%~3% is approximated by the Pareto distribution. In this paper, we attempt to reproduce such a two-class pattern of income distribution within the framework of the RGTID by using a possible extension of Gibrat's law.

Although Gibrat's law accounts for the emergence of the Pareto distribution, it does not hold exactly. In fact, in the literature of firm size, it has been observed that small firms grow faster than large firms [19,20], and the independence between size and growth becomes clearer as time passes [21]. This suggests that Gibrat's law holds asymptotically for firms above a certain size threshold. Likewise, in the literature of income distribution, Blanchet et al. [22] propose to identify the Pareto distribution of income as an asymptotic law above some high level, while Gabaix et al. [23] consider possible deviations from Gibrat's law so as to explore the extension of the Pareto distribution. By combining these two strands of literature, in this paper we use a possible extension of Gibrat's law to study the income distribution. As with empirical observations in the literature of firm size [19–21],

the extension of Gibrat's law is required to satisfy that the growth rate of a person's income is asymptotically independent of her income above some high level. By applying this extension of Gibrat's law into the RGTID, we find that the resulting distribution of income is denoted by a generalized Pareto-type distribution (GPD) with three parameters. In particular, we observe that there is a key parameter η in the GPD to determine the shape of income distribution. As $\eta = 0$, the GPD becomes an exponential distribution. However, as long as $\eta > 0$, the GPD always has an asymptotic power-law tail (or the Pareto tail) above some high level. This implies that, when η is close to 0, the GPD may produce a two-class pattern of income distribution, in which the bottom of the distribution is approximated by an exponential law and the upper tail of the distribution is approximated by the Pareto law.

The rest of the paper is organized as follows. Section 2 introduces a possible extension of Gibrat's law motivated by empirical observations. Section 3 applies this extension of Gibrat's law into the RGTID to derive the income distribution, where we find that the resulting distribution of income is denoted by the GPD with three parameters. Section 4 shows that there is a key parameter η in the GPD to determine the shape of income distribution. In particular, we show that, as $\eta \approx 0$, the GPD reproduces the two-class pattern of income distribution, in which the bottom of the distribution is approximated by an exponential law and the upper tail of the distribution is approximated by the Pareto law. Section 5 employs the Kolmogorov-Smirnov test to examine if the parameter η is close to 0 significantly by using the data from the United States, the United Kingdom, China, and Canada. Section 6 concludes.

2. The extension of Gibrat's law

Gibrat's law is a rule of proportionate growth. If one denotes the size of a firm at the time t by X_t, this law states that the increment of firm size is a linear function of firm size [19–21]

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t X_t \ , \tag{1}$$

where, Δt denotes the increment of time, and the proportional coefficient $\Delta\varepsilon_t$ is assumed to be independent of X_t. In the simplest form, $\Delta\varepsilon_t$ can be considered as a constant, but a more realistic model allows for randomness.

To extend Gibrat's law (1), we write the increment of firm size as a general function of firm size

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t f(X_t), \tag{2}$$

where $f(X_t)$ is assumed to be a smooth function of X_t.

When $f(X_t) = X_t$, equation (2) returns to Gibrat's law (1). Mathematically, the smooth function $f(X_t)$ can be always expanded as a Taylor series:

$$f(X_t) = a_0 + a_1 X_t + a_2 X_t^2 + a_3 X_t^3 + \cdots \tag{3}$$

By using equation (3), equation (2) can be written as

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t(a_0 + a_1 X_t + a_2 X_t^2 + a_3 X_t^3 + \cdots). \tag{4}$$

Using equation (4), the growth rate of firm size, $r_{t+\Delta t}$, is equal to

$$r_{t+\Delta t} = \frac{X_{t+\Delta t} - X_t}{X_t} = \Delta\varepsilon_t \left(\frac{a_0}{X_t} + a_1 + a_2 X_t + a_3 X_t^2 + \cdots\right). \tag{5}$$

Empirical observations show that small firms grow faster than large firms [19,20], and the independence between size and growth becomes clearer as time passes [21]. This can be summarized as a stylized fact as below:

Stylized fact 1: The growth rate of firm size is asymptotically independent of the size above a certain size threshold.

By observing equation (5), we find that $a_2 = a_3 = a_4 = \cdots = 0$ is the unique choice to satisfy the stylized fact 1. In this way, equation (5) reads

$$r_{t+\Delta t} = \frac{X_{t+\Delta t} - X_t}{X_t} = \Delta\varepsilon_t \left(\frac{a_0}{X_t} + a_1\right), \tag{6}$$

which is independent of X_t as $X_t \to \infty$.

Therefore, we assume $a_2 = a_3 = a_4 = \cdots = 0$, by which equation (4) can be written as

$$X_{t+\Delta t} - X_t = \Delta\varepsilon_t\,(a_0 + a_1 X_t). \tag{7}$$

In particular, by equation (6), if $\Delta\varepsilon_t a_0 > 0$ and $\Delta\varepsilon_t a_1 > 0$, then $r_{t+\Delta t}$ is a decreasing function of X_t. This means that, for those firms whose sizes increase with time, small firms grow faster than large firms, as with empirical observations [19,20].

Let us order $\Delta W_t = \Delta\varepsilon_t a_0$ and $\eta = a_1/a_0$. Thus, equation (7) can be rewritten as

$$X_{t+\Delta t} - X_t = (1 + \eta X_t)\,\Delta W_t. \tag{8}$$

Equation (8) is a possible extension of Gibrat's law (1). It asymptotically returns to   Gibrat's law (1) as[1] $X_t \to \infty$.

To obtain the continuous-time form of equation (8), we assume that ΔW_t denotes the Brownian motion with drift, i.e., $\Delta W_t \sim N(\mu\Delta t, \sigma^2\Delta t)$; therefore, as $\Delta t \to dt$, equation (8) yields

$$dX_t = (1 + \eta X_t)(\mu dt + \sigma dZ_t), \tag{9}$$

where μ and σ are two constants, and $dZ_t \sim N(0, dt)$ denotes the standard Brownian motion.

Equation (9) can be rewritten in the form

$$dX_t = \mu(1 + \eta X_t)dt + \sigma(1 + \eta X_t)dZ_t. \tag{10}$$

Equation (10) is the starting point of this paper. It is easy to check that equation (10) satisfies the asymptotical scale invariance as $X_t \to \infty$. To see this, we order $X_t \to \infty$ so that equation (10) asymptotically yields

$$dX_t = \mu\eta X_t dt + \sigma\eta X_t dZ_t, \tag{11}$$

which is invariant under the scaling change $X_t' \to \lambda X_t$. The scale invariance implies a certain kind of fractal property. Based on the RGTID, it has been known that [5–9,11,12], if the dynamics of income obey the random process (11), the resulting distribution of income is denoted by a power law (or the Pareto law). Jones and Kim [9] have pointed out that top income inequality is well characterized by this fractal property. However, equation (10) indicates that the scale invariance does not hold exactly but only asymptotically, such that the income distribution of top earners should be approximated by an asymptotic power law. Next, we apply equation (10) into the RGTID.

3. The basic model

By using the well-established result in the RGTID literature for generating the income distribution [5,9,11,12], if the dynamics of income x obey the random process (10), the density of the income distribution, $f(x, t)$, satisfies a Kolmogorov forward equation:

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial[\mu(1+\eta x)f(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2[\sigma^2(1+\eta x)^2 f(x,t)]}{\partial x^2}. \tag{12}$$

The derivation of equation (12) can be found in Appendix A.

To obtain the steady-state solution of equation (12), we consider a lower bound on income so that a person cannot go below a given level x_{min}. In real economies, the unemployment compensation plays the role of the lower bound x_{min}. Thus, equation (12) describes a random growth with a reflecting barrier x_{min} [5]. For the steady-state distribution of income, $f(x, t) = f(x)$, we should have $\partial f(x,t)/\partial t = 0$, such that equation (12) yields

$$-\frac{\partial[\mu(1+\eta x)f(x)]}{\partial x} + \frac{1}{2}\frac{\partial^2[\sigma^2(1+\eta x)^2 f(x)]}{\partial x^2} = 0, \tag{13}$$

It is easy to get a solution of equation (13) as below:

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[1] It is easy to check that, as $X_t \to \infty$, $1 + \eta X_t$ yields ηX_t asymptotically. Then, equation (8) can be asymptotically written as $X_{t+\Delta t} - X_t = \eta\Delta W_t X_t$, which is same as equation (1).

$$\begin{cases} f(x) = \frac{c_0}{\sigma^2}(1 + \eta x)^{\frac{2\mu}{\sigma^2\eta}-2}, \\ x \geq x_{min} \end{cases} \tag{14}$$

where c_0 denotes an integral constant.

The existence of the lower bound x_{min} requires the normalization equation:

$$\int_{x_{min}}^{\infty} f(x)dx = 1, \tag{15}$$

which is used to determine the integral constant c_0.

By substituting equation (14) into equation (15) we get the density distribution of income

$$\begin{cases} f(x) = \left(\frac{\eta+\frac{1}{\theta}}{1+\eta x_{min}}\right)\left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-2}, \\ x \geq x_{min} \end{cases} \tag{16}$$

with

$$\theta = -\frac{\sigma^2}{2\mu}, \tag{17}$$

where $\mu < 0$ and $\eta \geq 0$ are used to satisfy the normalization equation (15).

Here, we denote the cumulative distribution of income by

$$F_\eta(X \geq x) = \int_x^{\infty} f(z)dz, \tag{18}$$

where $F_\eta(X \geq x)$ denotes the fraction of population with the income higher than x.

Substituting equation (16) into equation (18) yields

$$F_\eta(X \geq x) = \left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1}, \tag{19}$$

where $x \geq x_{min}$.

Equation (19) is the main result of this paper. It has three parameters to determine the shape of income distribution. Remarkably, equation (19) can be written in the form of the generalized Pareto distribution:

$$F_\eta(X \geq x) = \left[1 + A\left(\frac{x-x_{min}}{B}\right)\right]^{-\frac{1}{A}}, \tag{20}$$

where $A = \frac{\theta\eta}{1+\theta\eta}$ and $B = \frac{\theta(1+\eta x_{min})}{1+\theta\eta}$.

Therefore, we call equation (19) the generalized Pareto-type distribution (GPD).

In particular, if $\eta > 0$, equation (19) always has an asymptotical power-law tail (i.e., the Pareto distribution); that is, as $x \to \infty$, one has[2]

$$F_{\eta>0}(X \geq x) \sim \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1}, \tag{21}$$

where $x_0 = (1 + \eta x_{min})/\eta$.

4. The parameter η

Here, we show that the parameter η in the GPD (19) plays a key role in determining the shape of income distribution. To this end, we observe that, when $\eta = 0$, the GPD (19) becomes an exponential distribution

$$F_{\eta=0}(X \geq x) = \lim_{\eta \to 0} F_\eta(X \geq x) = exp\left(-\frac{x-x_{min}}{\theta}\right), \tag{22}$$

where $x_{min} > 0$.

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[2] As $x \to \infty$, one has $1 + \eta x \sim \eta x$, such that $F_{\eta>0}(X \geq x) \sim \left(\frac{\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} = \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1}$.

The existing empirical observation has shown that [14–18], for market-economy countries, the income distribution is characterized by a two-class pattern, in which the bottom 90% of the population is approximated by the exponential law (22) and the richest 1%~3% is approximated by the power law (21). Next, we show that this empirical observation can be explained by the GPD (19) as long as η is close to zero. To this end, we investigate the conditions that the GPD (19) can be replaced by equations (21) and (22), respectively.

First, we consider $\eta \approx 0$ so that $\eta x \ll 1$. Thus, one has $(1 + \eta x)^{-\frac{1}{\theta\eta}-1} \approx exp(-x/\theta)$. Since $x \geq x_{min}$, one has $\eta x_{min} \ll 1$ to obtain $(1 + \eta x_{min})^{-\frac{1}{\theta\eta}-1} \approx exp(-x_{min}/\theta)$ as well. This means that the GPD (19) for $x_{min} \leq x \ll 1/\eta$ can be replaced by the exponential distribution (22); that is,

$$F_{\eta\approx 0}(X \geq x) = \left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} \approx exp\left(-\frac{x-x_{min}}{\theta}\right), \tag{23}$$

Therefore, we expect that, as long as η is sufficiently close to zero, the exponential law (22) is roughly valid for the majority of the population, in which each person's income is lower than $1/\eta$.

Second, we consider that x is sufficiently large so that $\eta x \gg 1$. In this way, one has

$$F_{\eta\approx 0}(X \geq x) = \left(\frac{1+\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} \approx \left(\frac{\eta x}{1+\eta x_{min}}\right)^{-\frac{1}{\theta\eta}-1} = \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1}, \tag{24}$$

where $x_0 = (1 + \eta x_{min})/\eta$ as denoted by equation (21).

Equation (24) means that the GPD (19) for $x \gg 1/\eta$ can be replaced by the Pareto distribution (21). Thus, we expect that, as long as η is sufficiently close to zero, the Pareto distribution (21) is roughly valid for top earners, in which each person's income is higher than $1/\eta$.

Based on the discussion above, we conclude that, when η is close to zero, the GPD (19) can be approximated by the following two-class pattern

$$F_{\eta\approx 0}(X \geq x) \approx \begin{cases} exp\left(-\frac{x-x_{min}}{\theta}\right) & x_{min} \leq x < x_0 \approx \frac{1}{\eta} \\ \left(\frac{x}{x_0}\right)^{-\frac{1}{\theta\eta}-1} & x > x_0 \approx \frac{1}{\eta} \end{cases}, \tag{25}$$

where $x_0 = (1 + \eta x_{min})/\eta \approx 1/\eta$ arises because $\eta x_{min} \ll 1$.

However, if η is sufficiently larger than 0, then equation (22) may hold no longer for any $x \geq x_{min}$. For example, if $\eta x_{min} \approx 1$, one has neither $(1 + \eta x_{min})^{-\frac{1}{\theta\eta}-1} \approx exp(-x_{min}/\theta)$ nor $(1 + \eta x)^{-\frac{1}{\theta\eta}-1} \approx exp(-x/\theta)$. This implies that, when η is significantly deviated from 0, the bottom of the population cannot be approximated by the exponential distribution (22).

In practice, the statement "the parameter η is close to 0" may be highly subjective. To address this problem, we use the Kolmogorov-Smirnov test to identify if the parameter η is close to 0 significantly.

## 5. Tests using the data from the US, the UK, China, and Canada

From a statistical perspective, the collection of household income data can be regarded as a sample observed from the income distribution, $F_\eta(X \geq x)$. Therefore, we selected household income data[3] from the latest years available for four representative market-economy countries to test if the parameter η is close to 0 significantly. The four countries include three developed economy (the United States in 2020, the United Kingdom in 2018, and Canada in 2018) and one developing economy (China in 2015) that is at present the world's second largest economy. Figure 1 shows that, for each country, the income distribution of the bottom 90% of the sample (circle) is well approximated by the exponential law (22) (red curve), while the upper tail of the sample is approximated by the power law (21) [i.e., the Pareto distribution] (black curve). By equation (25), this implies that η is close to 0 for four countries.

To strictly identify if the parameter η is close to 0 significantly for four countries, we use the Kolmogorov-Smirnov (KS) test. The hypothesis-testing is written as:

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[3] The data resource can be found in Data Availability Statement.

$$H_0: \eta = 0$$

$$H_1: \eta > 0,$$

where, by equation (22), the null hypothesis $\eta = 0$ indicates that the income distribution is described by the exponential law, in which x_{min} is chosen to be the first quantile of the sample. In our database, the data from United States, the United Kingdom, and China are large samples, each of which includes 99 quantiles. This implies that the KS test is feasible for these three countries. Although the data from Canada only includes 12 quantiles, we still use the KS test to perform a rough examination.

The testing result is listed in Table 1. For the United States in 2020, the United Kingdom in 2018, Canada in 2018, and China in 2015, $\eta = 0$ cannot be rejected at the significance level of 0.1. This means that η is close to 0 significantly. However, η is not equal to zero exactly, because Figure 1 has shown that the income distribution of each country has a power-law tail as described by equation (21).
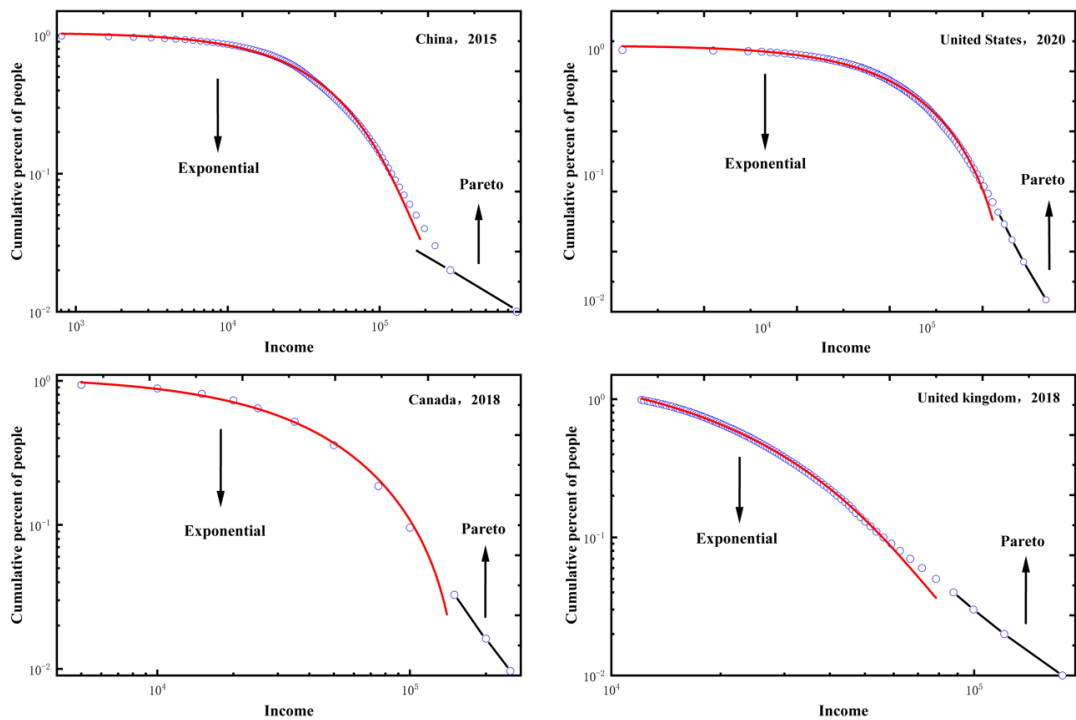
Figure 1. Income distributions of the US, the UK, China, and Canada.

Table 1. Kolmogorov-Smirnov (KS) test for the US, the UK, China, and Canada in the latest years available.

Year		2015	2018	2020
China		0.633	——	——
US	P-	——	——	0.741
UK	value	——	0.970	——
Canada	(KS)	——	0.593	——

China is a special sample that has undergone the transition from a centrally planned economy to a market economy. Here, we selected the data[4] from China in 1978, 1980, 1990, and 2000 to check if the parameter η is close to 0 significantly in the early stages of market-oriented economic

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[4] The data resource can be found in Data Availability Statement.

reformation. As shown in Table 2, $\eta = 0$ is rejected at the significance level of 0.01 in 1978, 1980, and 1990, while it cannot be rejected in 2000. This suggests that, for China both in planned economy period and in early stages of market reformation (from 1978 to 1990), η is significantly deviated from 0, such that the bottom of the population no longer conforms to an exponential distribution. The KS test results are supported by the data fitting in Figure 2, which has shown how the bottom 90% of the population in China deviates from an exponential distribution of income (red curve) during from 1978 to 1990. In Appendix B, we further discuss the economic implication of the parameter η. Within the framework of the random growth theory of income distribution (RGTID), we find that the parameter η may have the significance of characterizing the inequality of earning opportunity. Ideally, $\eta = 0$ corresponds to the equality of earning opportunity. From this sense, the test result that $\eta = 0$ is rejected for China in 1978, 1980, and 1990 implies that the equality of earning opportunity is broken significantly both in planned economy period and in early stages of market reformation.

However, for all the years, the income distribution always has a power-law tail (black line) as shown in Figure 2. This empirical observation agrees with equation (21).
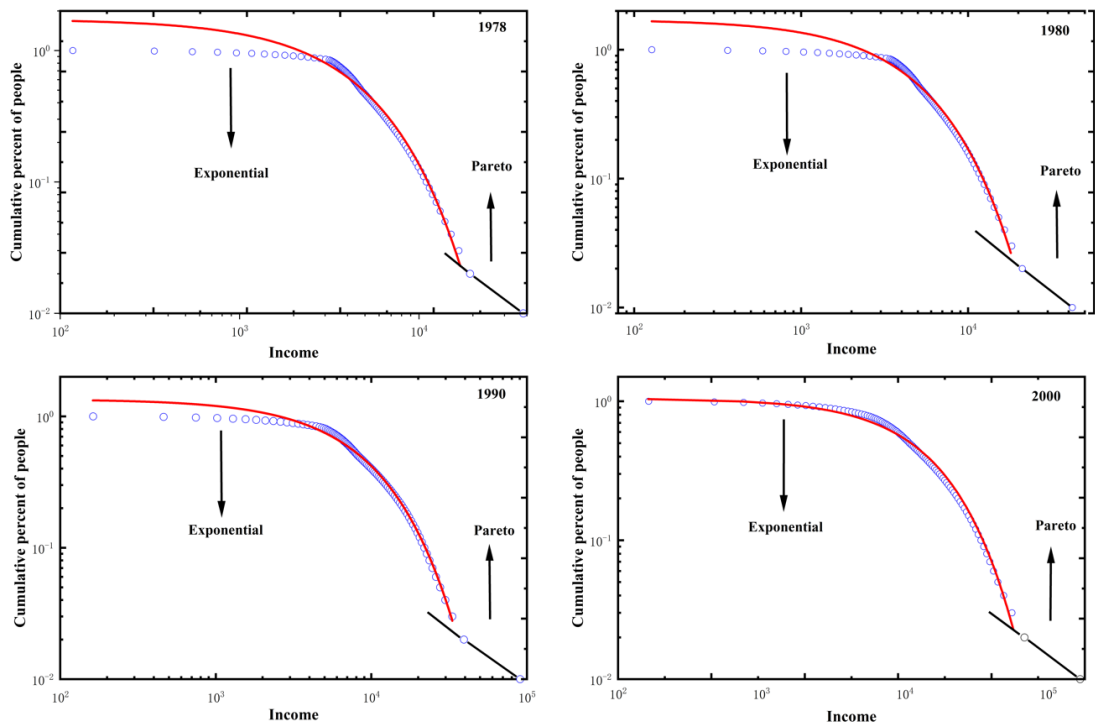
Figure 2. Income distribution of China during from 1978 to 2000.

Table 2. Kolmogorov-Smirnov (KS) test for China in the early stages of market-oriented economic reformation.

Year		1978	1980	1990	2000
China	P-value (KS)	0.000	0.000	0.008	0.658

6. Concluding remarks

Motivated by empirical observations, we propose a possible extension of Gibrat's law, which requires that the growth rate of firm size is asymptotically independent of the size above a certain size threshold. By applying this extension of Gibrat's law into the RGTID, we find that the income distribution is described by a generalized Pareto-type distribution (GPD) with three parameters. In particular, we observe that there is a key parameter η in the GPD to determine the shape of income distribution. As $\eta = 0$, the GPD becomes an exponential distribution. However, as long as $\eta > 0$, the

GPD always has an asymptotic power-law tail (or the Pareto tail) above some high level. This implies that, when η is close to 0, the GPD may produce a two-class pattern of income distribution, in which the bottom of the distribution is approximated by an exponential law and the upper tail of the distribution is approximated by the Pareto law.

By using the Kolmogorov-Smirnov test, we empirically show that $\eta = 0$ cannot be rejected at the significance level of 0.1 for typical market-economy countries such as the United States, the United Kingdom, Canada, and China (after 2000), while income distributions of these countries have a power-law tail without exception. This suggests that η is close to 0 significantly for these market-economy countries, such that the income distribution is characterized by a two-class pattern, in which the bottom 90% of the population is approximated by an exponential distribution and the richest 1%~3% of the population is approximated by the Pareto distribution. However, we empirically find that $\eta = 0$ is rejected at the significance level of 0.01 for China during from 1978 to 1990. This suggests that, for China both in planned economy period and in early stages of market reformation, η is significantly deviated from 0, such that the bottom of the population no longer conforms to an exponential distribution.

Appendixes

Appendix A: Derivation of equation (12)

Let us consider the Ito process

$$dX_t = \mathcal{A}(t, X_t)dt + \mathcal{B}(t, X_t)dZ_t, \tag{A.1}$$

where $\mathcal{A}(t, X_t)$ and $\mathcal{B}(t, X_t)$ are two functions of X_t and t, and dZ_t denotes the standard Brownian motion.

By using the theory of stochastic differential equations, it has been known that, when the variable x evolves according to the random process (A.1), the density distribution of x satisfies the Kolmogorov forward equation:

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial[\mathcal{A}(t,x)f(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2[\mathcal{B}(t,x)^2 f(x,t)]}{\partial x^2}. \tag{A.2}$$

The derivation of equation (A.2) can be found in any book on stochastic calculus, e.g., see page 282 in [24] or page 50 in [25].

Comparing equations (10) and (A.1), one has

$$\mathcal{A}(t, X_t) = \mu(1 + \eta X_t), \tag{A.3}$$

and

$$\mathcal{B}(t, X_t) = \sigma(1 + \eta X_t). \tag{A.4}$$

Substituting equations (A.3) and (A.4) into equation (A.2) yields equation (12).

Appendix B: Economic implication of the parameter η

Here, we provide a possible economic implication for the parameter η within the framework of the random growth theory of income distribution (RGTID). In the setting of random growth process (10), the resulting distribution of income is characterized by the random variable X with cumulative distribution function, $F_\eta(X \geq x)$. This setting means that the distribution of income, $F_\eta(X \geq x)$, can be understood as a probability distribution[5]; that is, the probability of acquiring the income x for a person. Given this understanding, we propose a *special* definition for identifying the equality of earning opportunity. To this end, let us calculate the following conditional probability:

$$F_\eta(X \geq x + y | X \geq y) = \frac{F_\eta(\{X \geq x+y\} \cap \{X \geq y\})}{F_\eta(X \geq y)} = \frac{F_\eta(X \geq x+y)}{F_\eta(X \geq y)}, \tag{B.1}$$

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[5] This understanding was widely used in the literature of income distribution [9,22].

which denotes the probability of acquiring the earnings x for a person given that she has earned the income y.

Definition B.1 (Equality of earning opportunity): *If the probability of acquiring the earnings x for a person is denoted by $F_\eta(X \geq x)$, the earning opportunity is equal to everyone as long as $F_\eta(X \geq x + y | X \geq y)$ is independent of y for any x and y.*

To understand the implication of the Definition B.1, we substitute equation (19) into equation (B.1) to obtain

$$F_\eta(X \geq x + y | X \geq y) = \left[\frac{1+\eta(x+y)}{1+\eta y}\right]^{-\frac{1}{\theta\eta}-1}. \tag{B.2}$$

When $\eta = 0$, equation (B.2) becomes

$$F_{\eta=0}(X \geq x + y | X \geq y) = \lim_{\eta \to 0} F_\eta(X \geq x + y | X \geq y) = exp\left(-\frac{x}{\theta}\right), \tag{B.3}$$

which means that a person's probability of acquiring the future earnings x is irrelevant with her past earnings y. In other words, the probability of acquiring the earnings x is equal to everyone.

However, when $\eta > 0$, equation (B.2) can be written as

$$F_{\eta>0}(X \geq x + y | X \geq y) = \left[1 + \frac{\eta x}{1+\eta y}\right]^{-\frac{1}{\theta\eta}-1}, \tag{B.4}$$

which is a monotonically increasing function of y. This means that, when $\eta > 0$, a person's probability of acquiring the future earnings x is positively proportional to her past earnings y. This is the embodiment of the Matthew effect [26] in income accumulation.

Based on equations (B.3) and (B.4), by the Definition B.1 we identify the parameter η as an index of characterizing the inequality of earning opportunity. That is, $\eta = 0$ indicates the equality of earning opportunity, while $\eta > 0$ indicates equal opportunity breaking. It should, however, be clarified that the *Definition B.1 is only applied to the RGTID, where the income distribution is ideally equivalent to a probability distribution of income*. It does not mean "equality of earning opportunity" in general sense. In fact, identifying the equality of earning opportunity in general sense is a complicated issue. For example, earnings mobility should be considered [27]. From this sense, identifying $\eta = 0$ as the equality of earning opportunity in strict sense is naive. It holds strictly if and only if the income distribution is ideally equivalent to a probability distribution of income.

By using the Definition B.1, it is possible to provide an explanation of why the richest 1%~3% of the population is taking hold of a larger and larger fraction of total income (in sharp contrast to the bottom 90% of the population) to aggravate global income inequality as reported by the literature [28]. First, we show that, if $\eta \neq 0$, the earning opportunity of top earners may be improved radically. To do so, by equation (B.4) we observe that $F_\eta(X \geq x + y | X \geq y)$ is a monotonically increasing function of y as long as $\eta > 0$. In particular, as $y \to \infty$, by equation (B.4) one has

$$\lim_{y \to \infty} F_{\eta>0}(X \geq x + y | X \geq y) = 1, \tag{B.5}$$

which means that, if an earner's past income y is sufficiently large, then she would has the potential to win any earnings x in future.

Second, our empirical observation in the section 5 has implied that $\eta \approx 0$ for typical market-economy countries. In this case, by equation (B.4), $F_\eta(X \geq x + y | X \geq y)$ is approximately independent of y as long as $\eta y \ll 1$. This means that, when $\eta \approx 0$, the earners[6] in the bottom of the population have roughly equal probabilities denoted by equation (B.3) to win the earnings x. However, when y is sufficiently large, by equation (B.5) the earning probabilities of top earners may be improved radically even if $\eta \approx 0$. This in turn sharply enlarges the gap of earning opportunity between top earners and the bottom of the population.

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[6] The statement "earners in the bottom of the population" means that their incomes in the past are much less than $1/\eta$.

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