On the practical point of view of option pricing

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September 8, 2022

Abstract

In this note we describe a new approach to the option pricing problem by introducing the notion of the safe (and acceptable) for the writer price of an option, in contrast to the fair price used in the Black-Scholes model. Our starting point is that the option pricing problem is closely related with the hedging problem by practical techniques. Recalling that the Black - Scholes model does not give us the price of the option but the initial value of a replicating portfolio we observe easily that has a serious disadvantage because assumes the building of this replicating portfolio continuously in time and this is a disadvantage of any model that assumes such a construction. Here we study the problem from the practical point of view concerning mainly the over the counter market. This approach is not affected by the number of the underlying assets and is particularly useful for incomplete markets. In the usual Black-Scholes or binomial or some other approaches one assumes that one can invest or borrow at the same risk free rate $r > 0$ which is not true in general. Even if this is the case one can immediately observes that this risk free rate is not a universal constant but is different among different people or institutions. So, the fair price of an option is not so much fair! Moreover, the two sides are not, in general, equivalent against the risk therefore the notion of a fair price has no meaning at all. We also define a variant of the usual binomial model, by estimating safe upward and downward rates u, d, trying to give a cheaper safe or acceptable price for the option.

Keywords Safe price, multi asset options, Bermudan options, incomplete markets
2020 Mathematics Subject Classification 91G20, 91G60
JEL Codes A22, C02, B26

1 Introduction

In this paper we will discuss the option pricing problem concerning the over the counter market. The reason for the existence of these contracts is speculative, both for the seller and the buyer. The buyer, in addition to speculation, may need such a contract for security reasons and she will buy it at the cheapest price on the market. Along with this criterion, however, she should probably consider the credibility of the seller. On the other hand, the seller will have to estimate the price range that such a contract could sell, so that she has a high chance of making a profit. To do this the writer need a mathematical study of this problem.

It should be clear that along with the suggested price given to the writer one should be able to explain how she is covered by this price and what exactly is the risk she undertakes, otherwise has no practical meaning for the writer. Concerning the binomial model for example one can suggest a price explaining

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that with this amount of money the writer can build a replicating portfolio while the various costs are also should be determined. In addition, the writer need to be sure that she can borrow as many shares as she need at any time, although borrowing shares is a way to add risk of bankruptcy! The risk that the writer undertakes in this case is the estimation of the future volatilities of the underlying assets. The writer will make a profit if the future volatilities turns out to be less while she will have a loss if it turns out to be greater than she had predict. This comes from the fact that the bigger the volatility the bigger the possibly payoff and therefore the bigger the contract price should be. Indeed, let us explain this on an one period binomial model for a call option with strike price K and expiration time T. Suppose that the writer's guess are the upward and downward rates d, u with $uS_0 > K$ and $dS_0 < K$. Therefore, if S_0 is the today price of the asset and r the risk - free rate (with $d < e^{rT} < u$) then we have

$$V_0 = e^{-rT}(q(uS_0 - K)^+ + (1-q)(dS_0 - K)^+)$$

with $q = \frac{e^{rT}-d}{u-d}$. Denote by d^*, u^* the future downward and upward rates. If $d = d^*$ and $u = u^*$ then the final value of the portfolio will become exactly the same as the payoff. Consider the case where $u^* > u$, then in the case where the price of the asset jumps to u^*S_0 at time T we have that

$$V_T = au^*S_0 + be^{rT}$$

while the payoff will be $(u^*S_0 - K)^+$, where $a = \frac{(uS_0-K)^+-(dS_0-K)^+}{(u-d)S_0}$ and $b = \frac{(dS_0-K)^+u-(uS_0-K)^+d}{(u-d)e^{rT}}$. In this case we have that $0 < a < 1$. Then it is easy to see that

$$\begin{aligned} V_T - (u^*S_0 - K)^+ &= au^*S_0 + be^{rT} - u^*S_0 + K \\ &= a(u^* - u)S_0 + auS_0 + be^{rT} - (u^* - u)S_0 - (uS_0 - K) \\ &= (u^* - u)S_0(a-1) \\ &< 0 \end{aligned}$$

because $auS_0 + be^{rT} = uS_0 - K$ and $a < 1$. Moreover when $u^* \to \infty$ then the loss goes to infinity as well. In the case where the asset's price moves down to d^*S_0 with $d^* < d$ then the writer has a loss again because

$$V_T - (d^*S_0 - K)^+ \quad = \quad ad^*S_0 + be^{rT} = a(d^* - d)S_0 + adS_0 + be^{rT} = a(d^* - d)S_0 + (dS_0 - K)^+ < 0$$

because $a > 0$.

The same holds for the corresponding put option noting that $-1 < a < 0$ in this case. In the case where $u^* < u$ and $d^* > d$ then we have a profit for the writer. Concerning all the continuous time models that assumes replication continuously in time it should be clear that are not practical, however they may have theoretical significance.

Here we will try to suggest a different and practical way of pricing options giving also the risk that the writer undertakes and how the writer is covered by this price. Our goal here is to define the notion of the safe (and acceptable) price for the writer of an option which, in general, is a multi-asset option with any payoff function, either European or Bermudan type. We distinguish here the options into two classes: those that have unbounded payoffs and those that have bounded payoffs. In the first class belong the call options and at the second class the put options. It is clear that the writer of an option of the first class is at risk of bankruptcy.

2 European Options with unbounded payoffs

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a complete probability space and W_t a one dimensional Wiener process adapted to the filtration $\mathcal{F}_t$.

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Definition 1 *Let P_t be the payoff of the option at time t and let Y be a price of the option at time zero. Then we say that this price is safe for the writer if*

$$\mathbb{P}(P_t \leq Y) \geq p$$

for some $p \in (0,1)$ specified by the writer.

In order to define such a safe price we have to make some assumptions. Let us denote by P_T the payoff of the option at the expiration time T and by P_0 the payoff of the option today which was started before T years.

Assumption 2 (Call like options) *For a given $L \geq 0$ we suppose that there exists some $m, \sigma \in \mathbb{R}_+$ such that*

$$P_T = \max(X_T, L)$$

where $X_t = P_0 + mt + \sigma W_t$ and $P_0 \geq L$.

Note that we have not assumed anything about the underlying assets but only for the process of the payoffs. We have supposed that $X_t = P_0 + mt + \sigma W_t$ in order to make our calculations easier. Of course one can assume that

$$X_t = P_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s$$

for suitable chosen functions f, g. In some contracts maybe it is better to assume that $P_T = \max\{X_T - K, L\}$ with $X_t = P_0 + \int_0^t (m_1 + m_2 X_s)ds + \int_0^t \sigma X_s dW_s$ or for some lookback options that $X_t = P_0 + mt + \sigma \sup_{0 \leq s \leq t} W_s$ for suitable chosen m_1, m_2, σ, K.

Theorem 3 *Given some $p \in (0,1)$ and under Assumption 2, the price $Y = P_0 + mT + z_p \sigma \sqrt{T}$ is a safe price for the writer, where z_p is such that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_p} e^{-\frac{r^2}{2}} dr = p$.*

Proof. Under assumption 2 we can write, noting that $Y \geq L$ because $m, \sigma \in \mathbb{R}_+$ and $P_0 \geq L$,

$$\begin{aligned}
\mathbb{P}(P_T \leq Y) &= \mathbb{P}(\max(P_0 + mT + \sigma W_T, L) \leq Y) \\
&= \mathbb{P}(P_0 + mT + \sigma W_T \leq P_0 + mT + z_p \sigma \sqrt{T}) \\
&= \mathbb{P}\left(W_T \leq z_p \sqrt{T}\right) \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{z_p \sqrt{T}} e^{-\frac{r^2}{2T}} dr \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_p} e^{-\frac{r^2}{2}} dr \\
&= p
\end{aligned}$$

□

We can estimate m and σ using historical data concerning only the payoffs at the past and not the prices of the assets.

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For barrier call like options one can assume that $P_T = \max\{X_T, K\}\mathbb{I}_A$ for suitable chosen event A (see for example [13]). Then it follows that the safe price is exactly as before because $\mathbb{P}(P_T > Y) \leq \mathbb{P}(\max\{X_T, K\} > Y) = 1 - p$ noting that $Y > 0$.

Denoting by Π the profit of the writer in this case we can easily see that $\mathbb{P}\left(\Pi > z_p\sigma\sqrt{T}\right) = 1/2$ and of course $\mathbb{P}(\Pi > 0) = p$.

In the case where the writer can invest in a risk free asset with interest rate r then she can sell the option at the price $e^{-rT}Y$.

We will now study the same situation under a slightly different assumption.

Assumption 4 *We suppose that there exists some $m, \sigma \in \mathbb{R}_+$ and $L_1, L_2 \geq 0$ such that*

$$P_T = \max(X_T - L_2, L_1)$$

where $X_t = (P_0 + L_2) + m\int_0^t X_s ds + \sigma\int_0^t X_s dW_s = (P_0 + L_2)e^{\sigma W_t + (m-\sigma^2/2)t}$.

Theorem 5 *Given some $p \in (0,1)$ and under Assumption 4, the price*

$$Y = (P_0 + L_2 + \varepsilon)e^{\sigma z\sqrt{T} + (m-\sigma^2/2)T} - L_2$$

is a safe price for the writer for any $z \geq z_p$ and $\varepsilon \geq 0$ so that $Y > L_1$ where z_p is such that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_p} e^{-\frac{r^2}{2}}dr = p$.

Proof. Because $Y > L_1$ and $z \geq z_p$ we have

$$\begin{aligned}
\mathbb{P}(P_T \leq Y) &= \mathbb{P}\left(P_T \leq (P_0 + L_2 + \varepsilon)e^{\sigma z\sqrt{T} + (m-\sigma^2/2)T} - L_2\right) \\
&\geq \mathbb{P}\left(P_T \leq (P_0 + L_2)e^{\sigma z\sqrt{T} + (m-\sigma^2/2)T} - L_2\right) \\
&\geq p
\end{aligned}$$

□

In the case where $L_2 = P_0 = 0$ then we should choose some strictly positive ε at the above theorem. An obvious example to apply theorem 5 is a call option with strike price K. Then we choose $L_1 = 0$ and $L_2 = K$ while P_0 is the today payoff of a call option starting T years ago.

Example 6 *Consider a contract which expires at $T > 0$ and gives the following payoff*

$$P_T = \max\{S_1, \cdots, S_n, K\}$$

where $S_1, \cdots, S_n$ are the values of n assets at the time T and K a fixed amount of money.

In this case we propose to find the $m \geq 0$ and $\sigma > 0$ so that the random variable

$$P_T = \max\{P_0 + mT + \sigma W_T, K\}$$

fits as much as possible into historical data (see for example [11]), where P_0 is the today payoff. Then one assumes that the random variable will behave similarly to the past so a safe price, according to assumption 2, is

$$Y = P_0 + mT + z_p\sigma\sqrt{T}$$

We can also assume that

$$P_T = \max\{X_T, K\}$$

where

$$X_t = P_0 + \int_0^t (a_1 + a_2 X_s) ds + \int_0^t (a_3 + a_4 X_s) dW_s$$

So in this case one should find the $a_1, a_2, a_3, a_4 \in \mathbb{R}_+$ *so that* P_T *fits as much as possible into historical data. In general, one can assume that*

$$X_t = P_0 + \int_0^t f(s, X_s) ds + \int_0^t g(s, X_s) dW_s$$

for some suitable chosen functions f, g *trying to fit* P_T *as much as possible into historical data. In order to find the probability density function of* X_T *maybe we should engaged the associated Fokker - Planck equation which is a partial differential equation.*

The payoff of this option is not bounded from above but the writer can buy some call options of the underlying assets with the same expiration time and for suitable strike prices in order to bound from above the payoff. Unfortunately this is not the case for all options with unbounded payoffs. For example consider an option written on one underlying asset with payoff $P_T = \max\{\max_{0 \leq t \leq T} S_t - K, 0\}$*, that is call on maximum option. In this case there is not a way to buy some call options in order to bound from above the payoff. One idea, although, is to buy a series of call options with expiration times* $t_1 < t_2 < \cdots < T$ *so to minimize as much as possible the risk of bankruptcy.* □

3 Bermudan Options with unbounded payoffs

Let an option which expires at the time T and suppose that the buyer of the option can exercise it at a set of times which is a subset of $[0, T]$. We denote by E the exercise set. Denote by Y_t the safe price of the option if it was expired at time t. Then a safe price for the writer of the option is

$$Y = \sup_{t \in E} Y_t = Y_T$$

The above safe price includes the case of the American type option, where $E = [0, T]$, and of course of the European option, where $E = \{T\}$.

The writer, pricing in this way a contract, takes the risk of guessing the parameters m, σ of X_t while the buyer transfer this risk to the writer. Moreover, we have to note here that selling at a lower price one should have a way to eliminate this extra risk, i.e. by constructing a suitable replicating portfolio, investing at a risk free asset or another type of investment. Selling at a lower price without having a practical way to eliminate this extra risk has no meaning for the writer. Indeed, under assumption 2, selling at a lower price than $V_0 + mT$ (concerning European options for example), without constructing a replicating portfolio, the probability of loss is greater than $1/2$.

If there is a possibility of infinity loss (like in a call option), even if the writer sell the option at the safe price, she should try to minimize the loss and the risk of guessing the parameters of X_t constructing a suitable portfolio containing calls of the underlying assets, among others (see for example [1]).

 doi:10.20944/preprints202208.0284.v2

4 Options with bounded payoffs

Let an option of European type that expires at time $T > 0$. Suppose that the payoff of the option is bounded above by K so in this case the buyer obviously hopes to buy the contract at a price lower than K.

Definition 7 *We say that the price Y is acceptable by the writer if*

$$\mathbb{P}(P_T \leq Y) > 1/2$$

Assumption 8 (Put like options) *Given some $L_1, L_2 \in \mathbb{R}_+$ with $L_1 < L_2$ we suppose that there exists some $\sigma \in \mathbb{R}_+$ and $m \in \mathbb{R}$ such that*

$$P_t = \max\{L_2 - X_t, L_1\}$$

with

$$X_t = (L_2 - P_0) + \int_0^t mX_s ds + \int_0^T \sigma X_s dW_s = (L_2 - P_0)e^{\sigma W_t + (m-\sigma^2/2)t}$$

and $L_1 \leq P_0 < L_2$.

We have assumed that $X_t = (L_2 - P_0)e^{\sigma W_t+(m-\sigma^2/2)t}$ in order to make our calculations easier. Of course one can suppose that

$$X_t = (L_2 - P_0) + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s$$

for suitable chosen functions f, g.

In this case the writer can sell this option without having the risk of bankruptcy. Therefore she can choose to sell it at a lower price than the safe price.

Theorem 9 *Under assumption 8, any price Y such that*

$$\max\{L_2 - (L_2 - P_0)e^{(m-\sigma^2/2)T}, L_1\} < Y < L_2$$

is acceptable by the writer.

Proof. Indeed, we can write noting that $Y \geq L_1$,

$$\begin{aligned}
\mathbb{P}(P_T \leq Y) &= \mathbb{P}(L_2 - X_T \leq Y) \\
&= \mathbb{P}(X_T \geq L_2 - Y) \\
&= \frac{1}{\sqrt{2\pi T}} \int_z^\infty e^{-\frac{r^2}{2T}} dr
\end{aligned}$$

where

$$z = \frac{\ln \frac{L_2-Y}{L_2-P_0} + (\sigma^2/2 - m)T}{\sigma}$$

Therefore, in order Y to be an acceptable by the writer price is enough to choose any Y such that

$$\max\{L_2 - (L_2 - P_0)e^{(m-\sigma^2/2)T}, L_1\} < Y < L_2$$

Note also that $\mathbb{P}(P_T \leq Y) = \frac{1}{\sqrt{2\pi T}} \int_z^\infty e^{-\frac{r^2}{2T}} dr$ for $z = \frac{\ln \frac{L_2-Y}{L_2-P_0} + (\sigma^2/2-m)T}{\sigma}$ where Y is the chosen price, that is we can easily compute the above probability. □

 doi:10.20944/preprints202208.0284.v2

Remark 10 *Note that as* $\sigma \to \infty$ *we have that* $Y \to L_2$ *and that if the writer sell at the price* $Y = \max\{L_2 - (L_2 - P_0)e^{(m-\sigma^2/2)T}, L_1\}$ *then* $\mathbb{P}(P_T \leq Y) = 1/2$. □

For barrier put like options one can assume that $P_T = \max\{L_2 - X_T, L_1\}\mathbb{I}_A$ for suitable chosen event A. Then it follows that the acceptable price is exactly as before because $\mathbb{P}(P_T > Y) \leq \mathbb{P}(\max\{L_2 - X_T, L_1\} > Y) = 1 - p$ noting that $Y > 0$.

Next we consider barrier like options.

Assumption 11 (Barrier like options) *Given some* $L_1, L_2 \in \mathbb{R}_+$ *with* $L_1 < L_2$ *we suppose that there exists some* $m, \sigma \in \mathbb{R}_+$ *such that*

$$P_t = \max\{\min\{L_2, X_t\}, L_1\}$$

with $X_t = P_0 + mt + \sigma W_t$.

Theorem 12 *Under assumption 11, the price* $Y = P_0 + mT + z\sigma\sqrt{T}$ *for any* z *with* $0 < z < \frac{L_2 - P_0 - mT}{\sigma\sqrt{T}}$ *is acceptable by the writer.*

Proof. Indeed, since $P_0 \geq L_1$ and $m, \sigma \in \mathbb{R}_+$ then $L_1 \leq Y < L_2$ so we can write

$$\begin{aligned}
\mathbb{P}(P_T \leq Y) &= \mathbb{P}(\max\{\min\{L_2, X_T\}, L_1\} \leq Y) \\
&= \mathbb{P}(X_T \leq Y) \\
&= \mathbb{P}(P_0 + mt + \sigma W_t \leq P_0 + mT + z\sigma\sqrt{T}) \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{z\sqrt{T}} e^{-\frac{r^2}{2T}} dr \\
&> 1/2
\end{aligned}$$

□

Remark 13 *We can assume that* $X_t = P_0 + m\int_0^t X_s ds + \sigma \int_0^t X_s dW_s$ *for some* $m, \sigma \in \mathbb{R}_+$ *and* P_0 *is the today payoff. Then the price*

$$Y = P_0 e^{\sigma z_p \sqrt{T} + (m - \sigma^2/2)T}$$

is an acceptable by the writer price for suitable chosen z_p *as before.* □

In the case where the writer can invest in a risk free asset with interest rate r then she can sell the option at the price $e^{-rT}Y$. It is easy to extend all the above for Bermudan type options.

Remark 14 *A slightly different way is to assume that the price of any asset follows a geometric Brownian motion, i.e.*

$$S_i(t) = S_i(0) + m_i \int_0^t S_i(r)dr + \sigma_i \int_0^t S_i(r)dW_r^i$$

for some $m, \sigma \in \mathbb{R}_+$ *and* W^i *independent Brownian motions.*

Then, as before, we can estimate the $\bar{Y}_i$ *and* $\underline{Y}_i$ *such that*

$$\mathbb{P}(S_i(T) > \bar{Y}_i) \leq p \text{ and } \mathbb{P}(S_i(T) < \underline{Y}_i) \leq p$$

For call like options one needs the $\bar{Y}_i$ *prices and for put like options the* $\underline{Y}_i$ *prices. At the options written on several assets, some of them behave like call like and some other like put like options. In order to determine a safe price* P_Y *for the contract one simply evaluate the payoff concerning the* $\bar{Y}_i$ *and* $\underline{Y}_i$ *values. The above idea is simpler from the statistical point of view however it seems difficult to estimate the probability* $\mathbb{P}(P_T \leq P_Y)$ *in contrast with the previous method.* □

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5 From unbounded to bounded payoffs

The options that has unbounded payoffs can be divided into two subclasses. Those that can be bounded buying some call options and those that can not be bounded. The general idea is to buy some call options in order to transfer the whole risk (or even a part) of bankruptcy to the derivative market. Even if someone wants to sell a call option that does not appear in the derivative market, for example if the expiration date is different, should try to transfer the risk by buying some call options with the nearest expiration date.

Consider for example the option written on two underlying assets with payoff $X = \max\{S_1 - S_2, 0\}$ where S_1, S_2 are the prices of the two underlying assets with expiration time T. The writer can buy a call option with the same expiration date and for suitable chosen strike price K with underlying asset the S_1. The writer can think this option as a put like option (with $L_2 = K$ and $L_1 = 0$) which its payoff is bounded from above so she can price it as we have described. The final price may be $Y + C(S_1, K, T)$ where $C(S_1, K, T)$ is the price of the call option.

Indeed, choose some $K > 0$ to be the strike price of the call option. The amount of money that the writer will pay is $\hat{P}_T = \max\{P_T - C(S_1, K, T), 0\}$ which is such that $\hat{P}_T < K$. At this point we may assume that

$$\hat{P}_T = \max\{K - X_T, 0\}$$

with

$$X_t = (K - \hat{P}_0) + \int_0^t mX_s ds + \int_0^T \sigma X_s dW_s = (K - \hat{P}_0)e^{\sigma W_t + (m-\sigma^2/2)t}$$

where P_T and $C(S_1, K, T)$ are the payoffs of the option and the call option. Moreover P_0 is the payoff today and $C(S_1, K, 0)$ is the payoff of the call option today which has started T years before, i.e. equals to $(S_0 - K)^+$. Therefore we have to estimate the $\sigma \in \mathbb{R}_+$ and $m \in \mathbb{R}$ such that this model fits as much as possible to the historical data. Any price $Y + C(S_1, K, T)$ with

$$\max\{K - (K - \hat{P}_0)e^{(m-\sigma^2/2)T}, 0\} < Y$$

is acceptable by the writer, i.e. $\mathbb{P}(P_T < Y + C(S_1, K, T)) > 1/2$, and moreover the risk of bankruptcy is zero because the writer has transfer this risk (to the derivative market) buying the call option. If the writer want to be more competitive she can find the best K that she should choose using historical data. In this contract the possible loss for the writer is less than $K - Y$ while the possible profit is less than Y. On the other hand, the possible loss for the buyer is less than $Y + C(S_1, K, T)$ while the possible profit is unbounded from above, and the difference is paid by the call option.

The option that its payoff is $P_T = \max\{S_1, S_2, \cdots, S_d, K\}$ can be bounded buying suitable call options $C(S_1, K_1, T), \cdots, C(S_d, K_d, T)$. The writer can price this option as a barrier like option. Indeed, denote by $\hat{P}_T = \max\{P_T - \sum_{i=1}^d C(S_i, K_i, T), 0\}$ the amount of money that the writer will pay. Denoting by $K_{max} = \max\{K_1, \cdots, K_d\}$ we may assume that

$$\hat{P}_T = \min\{\max\{X_T, 0\}, K_{max}\}$$

where

$$X_t = \hat{P}_0 + mt + \sigma W_t$$

for suitable chosen $m, \sigma \in \mathbb{R}_+$. The price

$$Y = \hat{P}_0 + mT + z\sigma\sqrt{T} + \sum_{i=1}^d C(S_i, K_i, T)$$

for any $z > 0$ is acceptable by the writer, i.e. $\mathbb{P}(P_T \leq Y) > 1/2$, and moreover the risk of bankruptcy is zero.

At the second class belongs the options like the call on maximum options in which one can not buy suitable call options in order to bound from above the payoff. In this contract, the possible loss of the writer is unbounded while the possible profit is the price of the contract. On the other hand, the possible loss for the buyer is the price of the contract while the possible profit is unbounded. Therefore, the two sides are not equivalent against the risk of bankruptcy. The writer can buy a series of call options at time zero in order to minimize as possible the risk of bankruptcy. Let's discretize the time interval $[0, T]$ into N subintervals with $\delta = t_{i+1} - t_i$. We assume that we have estimated some $m, \sigma \in \mathbb{R}_+$ such that

$$P_T = \max\{X_T, 0\}$$

where

$$X_t = P_0 + mt + \sigma \sup_{0 \leq s \leq t} W_s$$

Then it holds that

$$\begin{aligned}
\mathbb{P}(P_{t_1} > K) &= \mathbb{P}(X_{t_1} > K) \\
&= \mathbb{P}\left(M_{t_1} > \frac{K - P_0 - mt_1}{\sigma}\right) \\
&= \frac{2}{\sqrt{2\pi}} \int_{\frac{K-P_0-mt_1}{\sigma\sqrt{t_1}}}^{\infty} e^{-r^2/2} dr
\end{aligned}$$

where $M_t = \sup_{0 \leq s \leq t} W_s$. We have used the fact that $\mathbb{P}(M_t > x) = 2\mathbb{P}(W_t > x)$. Therefore we can choose suitable K, δ (note that $\delta = t_1 - t_0 = t_1$) so that $\mathbb{P}(X_{t_1} > K) = 1 - p$ with $p \in (0, 1)$ chosen by the writer. In practice the time subintervals may are not uniformly chosen depending on what call options can someone buy from the derivative market. So the price of this contract can be

$$Y + \sum_{k=1}^{N} C(K, t_k)$$

where Y is the safe price and $C(K, t_i)$ is the call option with strike price K and expiration date t_i. The idea is that the probability that the payoff exceed K at each time interval is close to zero and if this will be the case the call option pays the difference that exists at exactly the time t_i. Unfortunately, the risk of bankruptcy for the writer has not been eliminated while the possible profit remains bounded. On the other hand, the possible profit for the buyer is unbounded and the possible loss is bounded. If someone want to use the binomial model in order to price this contract she will have to use it for several periods and therefore the various costs has to be estimated as well. The possible loss is (again) unbounded for the writer because the future volatility can be bigger than the estimated.

6 Binomial Model

The disadvantage of the safe price as we have defined it above is that it may be too expensive. We can try to use the binomial model in order to find a cheaper safe price for the option. One can find in [6] the description of the binomial model with simple mathematical arguments, although rigorous, for one underlying asset while in [5] we prove various relations such as put - call parity using again simple mathematical induction. Here, we will use the binomial model to construct a replicating portfolio for a contract written on several assets, in general.

We will try to construct a portfolio that contains the d assets and also an amount of cash $b \in \mathbb{R}$ which the writer can invest or borrow in a risk free asset with interest rate r (we may assume, if this the case, that $r = 0$). In general, the investment interest rate is different from the borrowing rate. We can assume that the writer has the ability to place or withdraw any amount of money at any time she wants on a risk-free asset at an interest rate r. Then at time $t_0 = 0$ we have

$$V_0 = a_1^0 S_1^0 + +a_2^0 S_2^0 + + \cdots + a_d^0 S_d^0 + b_0$$

where $S_1^0, \cdots S_d^0$ are the prices of the assets today and b_0 the amount of cash. So, the mathematical problem is: minimize V_0 for $a_1^0, \cdots, a_d^0 \in \mathbb{R}_+$ and $b_0 \in \mathbb{R}$ (or $a_1^0, \cdots, a_d^0, b_0 \in \mathbb{R}$ if the writer has the ability and want to borrow assets) so that $V_T \geq P_T$ where P_T is the payoff at time T. We can also construct a replicating portfolio as above by minimizing the variance of the portfolio.

It is well known that in order to use the binomial model with N periods we have to predict the (future) volatilities σ of each asset using historical data. We discretize uniformly the time interval $[0, T]$ into N subintervals with $\delta = t_{i+1} - t_i$ and we find the historical upward and downwards rates u, d for each asset. We denote by u_i the historical upward rate for asset S from time t_{i-1} to t_i and we determine the σ_i so that $u_i = e^{\sigma_i \sqrt{\delta}}$. We suppose that the downward rates d_i are such that $d_i = 1/u_i$, therefore the past volatilities σ_i are such that

$$\sigma_i = \begin{cases} \frac{\ln S_{i+1} - \ln S_i}{\sqrt{\delta}}, & \text{if } S_{i+1} > S_i \\ \\ \frac{\ln S_i - \ln S_{i+1}}{\sqrt{\delta}}, & \text{if } S_{i+1} \leq S_i \end{cases}$$

where S_i is the asset price at the past time t_i. It is clear that the σ_i are different at different time intervals. Therefore we have to decide which σ should we use for the asset S in order our calculations be safe. To do that, one idea is to assume that $\frac{\sigma_{i+1}}{\sigma_i} = e^{s(W_{t_{i+1}} - W_{t_i}) + (m - s^2/2)\delta}$ and find the best $m \geq 0$ and $s > 0$ such that fits as much as possible to the historical data. This assumption is equivalent to the assumption that σ is a stochastic process that satisfies the following stochastic differential equation

$$\sigma(t) = \sigma(a) + \int_a^t m\sigma(r)dr + \int_a^t s\sigma(r)dW_r$$

We can choose then as $\sigma^{t_{i+1}} = \sigma^{t_i} e^{z_p s \sqrt{\delta} + (m - s^2/2)\delta}$ (where z_p is as before) to be the future volatility of the asset S from t_i to t_{i+1} period of the binomial model, where $i = 0, \cdots, N$, that is we use different volatility at each period of the binomial model. Note that $\mathbb{P}(\widehat{\sigma}^{t_1} > \sigma^{t_1}) \leq 1 - p$ where $\widehat{\sigma}^{t_1}$ is the future volatility of the asset S at the time interval $(0, t_1)$. For example, assuming that $m = 0$ and choosing $z_p = 0$, we have that $\sigma^{t_i} = \sigma^{t_0}$ for all i where σ^{t_0} is known. We compute also all the relevant costs of the construction of this replicating portfolio adding them to the initial value of the portfolio. Therefore we can sell the contract at this price or higher! After that we can decide whether to actually build this replicated portfolio or not. If not, then it is equivalent to selling the contract at the safe price for properly chosen p. But if we build it, there is always the possibility that our assumptions and estimates will be wrong and in that case the various costs will also change for better or worse.

Another point of view for using the binomial model is to assume that the price of the asset S follows the following stochastic differential equation

$$S_t = S_0 + m \int_0^t S_r dr + \sigma \int_0^t S_r dW_r = S_0 e^{\sigma W_t + (m - \sigma^2/2)t}$$

for some $\sigma > 0$ and $m \geq 0$. We discretize the time interval $[0, T]$ into N subintervals so that $\delta = t_{i+1} - t_i$ for $i = 0, \cdots, N$. Then at times t_n and t_{n+1} we have that

$$\frac{S_{n+1}}{S_n} = U_{n,n+1} = e^{\sigma(W_{n+1}-W_n)+(m-\sigma^2/2)\delta}$$

where $S_n := S_{t_n}$ and $W_n := W_{t_n}$. Therefore we can choose as upward and downward rates u, d for the binomial model to be the following

$$u = e^{\sigma z_p \sqrt{\delta}+(m-\sigma^2/2)\delta}, \quad d = e^{-\sigma z_p \sqrt{\delta}+(m-\sigma^2/2)\delta}$$

for suitable chosen z_p. This comes from the fact that

$$\mathbb{P}(d < U_{n,n+1} < u) = \frac{1}{\sqrt{2\pi\delta}} \int_{-z_p\sqrt{\delta}}^{z_p\sqrt{\delta}} e^{-\frac{r^2}{2\delta}} dr$$

So, for a given $p \in (0, 1)$ we choose a suitable z_p so that

$$\frac{1}{\sqrt{2\pi\delta}} \int_{-z_p\sqrt{\delta}}^{z_p\sqrt{\delta}} e^{-\frac{r^2}{2\delta}} dr = p$$

We can choose also z_1, z_2 such that $z_1 \geq z_p$, $z_2 \geq z_p$, $u > 1$ and

$$\begin{aligned} &u = e^{\sigma z_1 \sqrt{\delta}+(m-\sigma^2/2)\delta} \\ &d = e^{-\sigma z_2 \sqrt{\delta}+(m-\sigma^2/2)\delta} \\ &ud = 1 \\ &\mathbb{P}(d < U_{n,n+1} < u) \geq p \end{aligned}$$

With a choice like that we are safe enough to assume that the future rates will be inside the interval (d, u). Let us call this price the safe (or acceptable) price via replication.

Consider for example a contract written on one asset and the binomial model in one period. Denoting by Π the profit of the writer we have

$$\mathbb{P}(\Pi > 0) = \mathbb{P}(d < U_{0,1} < u) \geq p$$

In the case where the contract is written on two or more assets and the binomial model is used for two or more periods it is not so obvious how to calculate the probability $\mathbb{P}(\Pi > 0)$.

In the binomial context the notion of the fair value has a practical meaning if the writer and the buyer agrees on the interest rate r, the future upward and downward rates u, d and the number of periods N. The problem of finding the arbitrage free price is a case by case problem and it has a practical meaning if the intensity of competition is high.

One can insert also the covariances of the assets accordingly (see for example [2]) arriving maybe at a cheaper safe price, but then the writer has more unknown factors to guess.

The above approach can be easily extended for the multi-period binomial model and for Bermudan type of options of any kind. One disadvantage of this point of view is that in every step we have to solve a minimization problem that may have computational cost.

Selling at this price the writer takes the risk of guessing the volatilities (and maybe other factors) while the buyer transfer this risk to the writer. Note also that the writer have a way to eliminate the extra risk,

 doi:10.20944/preprints202208.0284.v2

selling at a lower price than the safe price that we have described at the previous section, by constructing a suitable replicating portfolio as described above.

In practice, if someone prices in this way, should also take into account other factors such as dividends, transaction costs, etc. otherwise this view has no practical significance. This makes the mathematical problem intractable and is a disadvantage of all pricing methods that use replication techniques. Even worse, the risk of guessing the future d, u can not be hedged by the above construction. Thus, from a practical point of view, the first approach is simpler and safer!

We can use the binomial model in order to find the acceptable or the safe prices of a contract. Assume that there exists some m_i, σ_i such that

$$S_i(t) = S_i(0) + \int_0^t m_i S_i(s) ds + \int_0^t \sigma_i S_i(s) dW_s$$

where S_i is the i asset. We discretize the time interval $[0, T]$ into N subintervals as before. Then it holds that

$$\mathbb{P}\left(\frac{S_i(t_{n+1})}{S_i(t_n)} > 1\right) = \mathbb{P}\left(\sigma_i(W_{n+1} - W_n) > (\sigma_i^2/2 - m_i)\delta\right) = \frac{1}{\sqrt{2\pi\delta}} \int_{\frac{(\sigma_i^2/2 - m_i)\delta}{\sigma_i}}^{\infty} e^{-\frac{r^2}{2\delta}} dr =: q_i$$

We can assume as before that

$$u_i = e^{\sigma_i z_1 \sqrt{\delta} + (m_i - \sigma_i^2/2)\delta}, \quad d_i = e^{-\sigma_i z_2 \sqrt{\delta} + (m_i - \sigma_i^2/2)\delta}$$

concerning the asset S_i. Therefore, at time T we have M numbers of possibly payoffs together with their probabilities. We sort these numbers from minimum to maximum and therefore we have the couples $(P_1, w_1), (P_2, w_2), \cdots, (P_N, w_n)$ where P_i is a payoff and w_i its probability. If we want to find an acceptable price for this contract we choose some k so that $\sum_{i=k+1}^N w_i < 1/2$ and therefore all the prices $P_k, P_{k+1}, \cdots$ are acceptable. If we want to find a safe price we choose some p close to one and then a suitable k so that $\sum_{i=k+1}^N w_i < 1 - p$. Therefore, in this case all the prices $P_k, P_{k+1}, \cdots$ are safe. Let us call this price the binomial safe (or acceptable) price.

So, we have introduced in this section the safe (or acceptable) price by replication and the binomial safe (or acceptable) price. Note that at the second method we do not have to worry about costs because we do not build any replicating portfolio. However, in both methods we assume that there exists some m_i, σ_i such that

$$S_i(t) = S_i(0) + \int_0^t m_i S_i(s) ds + \int_0^t \sigma_i S_i(s) dW_s$$

This is the only assumption that we make in these two methods.

Example 15 *Let the contract that pays $P_T = \max\{S_1^T - S_2^T, 0\}$ with S_1, S_2 the two underlying assets. We can find the safe price by replication as we have described above using the one period binomial model. We should compute the d_i, u_i in order to get the safe price but there are two reasons for bankruptcy. The first one is that the price of S_1 can be any bigger than we have estimated and the second reason for bankruptcy is that we have to borrow some shares of S_2. Therefore we have to buy some call options regarding the assets S_1, S_2 for suitable strike prices K_1, K_2 and the same expiration date as our contract. We compute the minimum price V_0 of the replicating portfolio such that*

$$V_0 = aS_1^0 + bS_2^0 + c$$

 doi:10.20944/preprints202208.0284.v2

and $V_T \geq P_T$, where P_T are all the possible payoffs. i.e. $P_T^{u_1,u_2}, P_T^{u_1,d_2}, \cdots$ That is we have to minimize V_0 for $a, b, c \in \mathbb{R}$ assuming that we can invest or borrow at a risk free asset with interest rate r. Next we buy $1 - a > 0$ calls for S_1 and $-b > 0$ calls for S_2 and therefore the risk of bankruptcy is zero. The price of the contract will be the initial price of the replicating portfolio plus the call options plus the various costs assuming that the writer can borrow S_2 assets. Finally, in order to decide the price of the contract, we compare with the safe price that we have described before in which we should also buy a call option per share for S_1. Note that pricing by replication requires us to buy $1-a > 0$ calls of S_1 and $-b > 0$ calls for S_2 while pricing without replication we should buy one call for S_1. The disadvantage of the safe price by replication at this example is that in the case where the S_1 moves down to $d^* S_1^0$ with $d^* < d_1$ and S_2 moves up to $u^* S_2^0$ with $u^* > u_2$ there will be a loss for the writer (however bounded because of the call option for S_2) while at the safe price (i.e. without replication) there will be clearly a profit because the payoff is zero (or very close)! Intuitively thinking, it seems that there is no need to work on several periods concerning the binomial model at this example in contrast with the look-back options. □

Remark **16 (Risk of Bankruptcy)** *At the options with unbounded payoff the writer will always has the risk of bankruptcy, either by selling the option at the safe price either by constructing a replicating portfolio. In this situation the writer should also construct a portfolio containing some call options of the underlying assets but this is a case by case problem and can not be solved for general options with unbounded payoffs.*

For options that the payoffs are bounded from above the situation is different. Selling at the acceptable price the writer has no risk of bankruptcy. In the contrary, if the writer sell the option at a price that assumes construction of a replicating portfolio that contain shares of the assets and no call options then the risk of bankruptcy appears if she borrows some shares! This will be the case if the share price rise faster than expected. In order to eliminate that risk the writer may has to buy some call options as well!

This is a serious disadvantage of the methods that assumes replication because if one sum up the various costs and the call options that may have to buy then the total cost maybe is bigger than that of the acceptable or even the safe price! Unfortunately there is also a disadvantage even in the case where the replicating portfolio does not contain borrowed shares. Consider for example a call like option and the case where the price of the underlying asset falls so that the payoff equals to zero. Therefore, also the replicating portfolio have lower value or even negative depending on the behavior of the volatility. This means that the seller's profit shrinks or even worse becomes a loss. In the contrary, selling at the safe or acceptable price there will be a profit in this case. □

7 Conclusion

In this note we have try to give some practical ways on the option pricing problem. We have given the notion of the safe (or acceptable) price for the writer of an option, concerning mainly the over the counter market. One advantage of this point of view is that we do not need a risk free asset with a specified interest r. In the usual binomial or Black-Scholes or some other models one assumes that we can borrow or invest at the risk free asset with the same rate r which is not true in general. Even if this is the case one can immediately observes that this risk free rate is not a universal constant but is different among different people or institutions. So, the fair price of an option is not so much fair!

Another advantage in our approach is that the number of the underlying assets does not affect in our calculations because we study only the payoffs of the option at the past. We can try also the binomial model as we have described above in order to find a cheaper safe or acceptable price for the option.

We will summarize our thoughts by a hypothetical example. Let $p \in (1/2, 1)$ chosen by the writer and let a multi asset contract, in general, with X as a safe (or acceptable) price included the call options that the writer should buy. Let also that the safe (or acceptable) price via replication is Y including the various cost and call options concerning the same p. If $X \leq Y$ then the writer seems that should sell at the price X or higher, i.e. she does not need to built a replicating portfolio. If $X \geq Y$ then the writer should examine the possibility to sell at the price Y if she wants to be more competitive, but taking into account all the disadvantages of building the replicating portfolio. Denoting by Π_s and Π_r the profits of the writer selling at the safe price or selling at the safe price via replication respectively, one may use the criterion

$$\mathbb{P}(\Pi_s > 0) \geq \mathbb{P}(\Pi_r > 0)$$

in order to decide which method is more likely to be profitable. As we have seen the first probability is quite easy to compute in contrast to the second one. We can do the same for the binomial safe price and the corresponding profit Π_b.

We should remind here that the probability $\mathbb{P}(\Pi_s > 0) = p$ while the estimate of the probabilities $\mathbb{P}(\Pi_r > 0)$ and $\mathbb{P}(\Pi_b > 0)$ seems harder to achieve.

The above theory can be used also by the buyer of a contract. Suppose that someone wants to buy a contract for speculation reasons. Then she wants to know the probability of profit buying at a specific price Y. For example, she want to buy a contract written on two assets with payoff $P_T = \max\{S_1 - S_2, 0\}$. One way to estimate the probability of profit is to assume that $P_T = \max\{X_T, 0\}$ where $X_t = P_0 + mt + \sigma W_t$ for some $m, \sigma \in \mathbb{R}_+$. Then $\mathbb{P}(P_T \leq Y) = p$ with p such that $\int_{z_p}^{\infty} e^{-\frac{r^2}{2}} dr = p$, given the price Y and assuming that $Y = P_0 + mT + \sigma z_p \sqrt{T}$. That is, given the price Y one can easily find the z_p and then the probability p, therefore the probability of profit is $1 - p$. In fact, if the buyer can use a risk - free rate r_b then she should compute the probability $\mathbb{P}(P_T > e^{r_b T} Y)$.

Note that this estimate is independent of the way that the writer has computed the price Y, i.e. the writer may has used replication methods, hedging techniques or not. The risk - free rate that the writer use for the computation of the price plays a very important role here. The bigger the rate the smaller the price of the contract and this fact can drive both the seller and the buyer of the contract to a profit. That is, denoting the profit of the writer by Π_w and the profit of the buyer by Π_b, we may have that $\mathbb{P}(\Pi_w > 0) + \mathbb{P}(\Pi_b > 0) > 1$! This is one more evidence that the notion of the fair price has no meaning. The above can be true even if the risk - free rate is zero. Let us give a specific example. Assume that the underlying asset follows the following stochastic differential equation

$$S_t = 3.2 + 0.76 \int_0^t S_r dr + 0.62 \int_0^t S_r dW_r, \quad [0, T]$$

with $T = 10$ days. Let a call option with strike price $K = 3.1$ that expires at 10 days from now. The writer choose to price it via replication using the one period binomial model and choosing as $u = 1.1$ and $d = 0.93$. With such a choice the writer has a probability of profit $p = 0.6$, that is $\mathbb{P}(\Pi_w > 0) = \mathbb{P}(d \leq U_{0,1} \leq u) = 0.6$ where $U_{0,1} = \frac{S_T}{S_0}$, and the price of the option is $Y = 0.17$ using zero risk - free rate. The buyer want to estimate the probability of profit buying at this price and assumes that the underlying asset follows the above differential equation. Then the payoff is $P_T = \max\{S_T - K, 0\}$ and it follows that $\mathbb{P}(\Pi_b > 0) = \mathbb{P}(P_T > 0.17) > 0.46$. Note that the writer here is exposed to a risk of bankruptcy while the buyer has not such a risk, therefore the writer has every reason to ask for more!

Consider the case of the put on minimum option which has as payoff the $P_T = \max(K - \min(S_1, \cdots, S_d), 0)$. In this case in order to construct a replicating portfolio one should borrow assets of $S_1, \cdots, S_d$. Then she should also buy a call per borrowed share in order to bound the possible loss. The price of the contract should be the initial value of the replicating portfolio plus the call options plus the various costs. In this case the writer seems to should shell at the acceptable price (i.e. without replication) in which she should not

 doi:10.20944/preprints202208.0284.v2

have to borrow assets and the possible loss is again bounded (by K). This is also true for a put option written on one underlying asset. Suppose that the underlying asset price follows the same stochastic differential equation as above and the strike price is $K = 3.3$. The writer compute the acceptable price by replication using the one period binomial model with $p = 0.5$ and the initial value of the replicating portfolio is 0.16. Then she compute the acceptable price without replication with $p = 0.5$ and the price is 0.05. If the writer shell at the acceptable price via replication and build the replicating portfolio she will borrow some shares of the underlying asset and therefore she should buy also one call per share! If the writer shell at the price 0.05 without replication then both the writer and the buyer have the same probability of profit, i.e. $p = 0.5$ and both are equivalent against the risk. Therefore, in the case where the two sides are equivalent against the risk the notion of the fair price has a meaning. In the above case this fair price is the acceptable price without replication!

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