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Addendum to "The VIT Transform Approach to Discrete-Time Signals and Linear Time-Varying Systems"

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Abstract: This addendum contains clarifications and a sharpening of some of the results on the VIT transform framework developed in [1]. The focus is on the right-coefficient and left-coefficient forms of the transform, the extraction of a first-order term from a left polynomial fraction, and the application to linear time-varying systems.

Keywords: VIT transform, discrete-time signals, linear time-varying systems

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1. Definition of the VIT transform

Given a real or complex-valued discrete-time signal $x(n)$, where n is the integer-valued time variable, in [1] the VIT transform $X(z,k)$ of $x(n)$ is defined to be the formal power series in z^{-1} given by the right-coefficient form

$$X(z,k) = \sum_{i=0}^{\infty} z^{-i}\, x(k+i), \tag{1}$$

where k is the integer-valued initial time variable. Here z is a symbol or indeterminate. The transform $X(z,k)$ is an element of the set $A\big[[z^{-1}]\big]$ consisting of all formal power series in z^{-1} with coefficients in A , where A is the ring of all functions from the integers into the field of complex numbers with the usual pointwise operations.

It follows from Equation (1) that the VIT transform $X(z,k)$ depends only on the values of the signal $x(n)$ for $n \geq k$. Hence the VIT transform is a one-sided transform. Note that the VIT transform of $x(n)$ is equal to the VIT transform of $x(n)\mathcal{H}(n-k)$, where $\mathcal{H}(n-k)$ is the Heaviside step function defined by $\mathcal{H}(n-k)$ = 1 for $n \geq k$ and $\mathcal{H}(n-k) = 0$ for $n < k$.

In [1], the set $A\big[[z^{-1}]\big]$ is given the structure of a noncommutative ring by defining the usual addition of power series and with multiplication defined by $z^{-(i+j)} = z^{-i}z^{-j}$ and

$$a(k)z^{-i} = z^{-i}a(k+i),\ a(k) \in A, \tag{2}$$

Noncommutativity of the ring $A\big[[z^{-1}]\big]$ is a consequence of the noncommutative multiplication given by Equation (2). Due to this noncommutative multiplication, the elements of $A\big[[z^{-1}]\big]$ are

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sometimes referred to as skew power series, and polynomials in z with coefficients in A and with the multiplication $a(k)z^i = z^i a(k-i)$ are referred to as skew polynomials.

Given a signal $x(n)$, in [2] the generalized z-transform $\hat{x}(z,k)$ of $x(n)$ is defined to be the skew power series $\hat{x}(z,k) = \sum_{i=0}^{\infty} z^{-i} x(i)\delta(k)$, where $\delta(k)$ is the unit pulse located at $k = 0$. The transform $\hat{x}(z,k)$ can be written in the form $\hat{x}(z,k) = \sum_{i=0}^{\infty} z^{-i} x(i+k)\delta(k)$, and thus $\hat{x}(z,k) = X(z,k)\delta(k)$, where $X(z,k)$ is the VIT transform of $x(n)$. In the generalized z-transform, the initial time k is equal to zero; whereas in the VIT transform, the initial time k is a variable ranging over the set of integers.

2. Right-Coefficient and Left-Coefficient Forms of the VIT Transform

Now consider the signal $x(n) = f(n)\mathcal{H}(n-k)$, where $f(n)$ is a real or complex-valued function of n, and the values of $f(n)$ do not depend on the initial time variable k. Then using the definition of multiplication (2), we can write the transform $X(z,k)$ in the left-coefficient form

$$X(z,k) = \sum_{i=0}^{\infty} z^{-i} f(k+i)\mathcal{H}(i) = \sum_{i=0}^{\infty} f(k)\mathcal{H}(i) z^{-i} = \sum_{i=0}^{\infty} f(k) z^{-i}. \tag{3}$$

Since $\sum_{i=0}^{\infty} z^{-i} z = (z-1)^{-1} z$, the left-coefficient form of $X(z,k)$ reduces to $f(k)$ multiplied on the right by the fraction $(z-1)^{-1}z$; that is,

$$X(z,k) = f(k)(z-1)^{-1}z. \tag{4}$$

To check this result, first observe that the inverse VIT transform of $(z-1)^{-1}z$ is equal to $\mathcal{H}(n-k)$. Then by the multiplication by a time function property given in [1], the inverse VIT transform of $f(k)(z-1)^{-1}z$ is equal to $f(n)\mathcal{H}(n-k)$. This verifies the form $f(k)(z-1)^{-1}z$ for the VIT transform of the signal $x(n) = f(n)\mathcal{H}(n-k)$.

Note that setting $k = 0$ in the right-coefficient form in Equation (3) results in $\sum_{i=0}^{\infty} z^{-i} f(i)$, which is equal to the formal z-transform of the function $f(n)$. However, setting $k = 0$ in the left-coefficient form in Equation (4) results in $f(0)(z-1)^{-1}z$, which is not equal to the z-transform of $f(n)$. To eliminate this inconsistency, we define the evaluation of the VIT transform $X(z,k)$ at a particular value of k to be the evaluation of the right-coefficient form at that value of k.

It is important to note that if the values of $f(n)$ depend on the initial time variable k, then the form $f(k)(z-1)^{-1}z$ for the VIT transform of $f(n)\mathcal{H}(n-k)$ is not valid. If $f(n)$ does depend on k, then $f(n) = f(n,k)$, and the left-coefficient form of the VIT transform of $x(n) = f(n,k)\mathcal{H}(n-k)$ is

$$X(z,k) = \sum_{i=0}^{\infty} z^{-i} f(k+i,k)\mathcal{H}(i) = \sum_{i=0}^{\infty} f(k,k-i) z^{-i}. \tag{5}$$

Now since $f(n,k)$ varies as k is varied, $f(k,k-i) \neq f(k,k)$, and thus the VIT transform cannot be written in the form $f(k)(z-1)^{-1}z$, where $f(k) = f(k,k)$.

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For an example, let $a(k)$ and $b(k)$ be functions from the integers into the real or complex numbers, and let $a_0(k) = 1, a_i(k) = a(k)a(k+1)\cdots a(k+i-1), i \geq 1$. Consider the signal

$$x(n,k) = a_{n-k}(k)b(k),\ n \geq k, \tag{6}$$

with initial value $x(k,k) = b(k)$. Note that if $a(k)$ is the constant function $a(k) = c$ for all k, where c is a real or complex number, then $a_{n-k} = c^{n-k}$ and $x(n,k) = c^{n-k}b(k),\ n \geq k$. The right-coefficient form of the VIT transform of the signal $x(k,n)$ given by Equation (6) is $X(z,k) = \sum_{i=0}^{\infty} z^{-i}\, a_i(k)b(k)$. This transform can be written in the left polynomial fraction form

$$X(z,k) = \left(z - a(k)\right)^{-1} zb(k) = \left(z - a(k)\right)^{-1}[b(k+1)z]. \tag{7}$$

To verify Equation (7), divide $z - a(k)$ into z using left long division.

The left-coefficient form of the VIT transform of the signal given by Equation (6) is

$$X(z,k) = \sum_{i=0}^{\infty} a_i(k-i)b(k-i)z^{-i} \tag{8}$$

The right side of Equation (8) can be written in the right polynomial fraction form

$$X(z,k) = [b(k)z]\left[z - \frac{b(k-1)}{b(k)}a(k-1)\right]^{-1}, \tag{9}$$

where $\frac{1}{b(k)}$ is viewed as an element of the quotient field $Q(A)$ of A in the case when $b(k)$ has values that are equal to zero. The right polynomial fraction form in Equation (9) can be verified by dividing $z - \frac{b(k-1)}{b(k)}a(k-1)$ into z using right long division. Then combining (7) and (9), we have

$$\left(z - a(k)\right)^{-1}[b(k+1)z] = [b(k)z]\left[z - \frac{b(k-1)}{b(k)}a(k-1)\right]^{-1}. \tag{10}$$

From Equation (10), it is seen that moving $b(k+1)z$ to the left through $\left(z - a(k)\right)^{-1}$ changes the coefficient $a(k)$ in the denominator polynomial to $\frac{b(k-1)}{b(k)}a(k-1)$. This noncommutativity is a fundamental aspect of the VIT transform framework.

In this example, it can be shown that the right polynomial fraction form of the transform $X(z,k)$ can be derived directly from the left polynomial fraction form. This turns out to be true in the general case when $X(z,k) = \mu(z,k)^{-1}\nu(z,k)$, where $\mu(z,k)$ and $\nu(z,k)$ are skew polynomials in z with coefficients in A: Similar to the discussion of the extended right Euclidean algorithm given in [1], the extended left Euclidean algorithm can be used to determine polynomials $\alpha(z,k)$ and $\beta(z,k)$ such that $\mu(z,k)\alpha(z,k) = \nu(z,k)\beta(z,k)$. In general, the coefficients of $\alpha(z,k)$ and $\beta(z,k)$ belong to the quotient field $Q(A)$ of A. Then $\mu(z,k)^{-1}\nu(z,k) = \alpha(z,k)[\beta(z,k)]^{-1}$, and therefore, $\alpha(z,k)[\beta(z,k)]^{-1}$is a right polynomial fraction form of $X(z,k)$.

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3. Extraction of a First-Order Term

Given a skew polynomial $\xi(z,k)$ with coefficients in A and a function $a(k) \in A$, consider the polynomial fraction $\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}$. The extended right Euclidean algorithm can be used to determine a polynomial $\eta(z,k)$ with coefficients in $Q(A)$ and $d(k) \in Q(A)$ such that $\eta(z,k)\left(z-a(k)\right)+d(k)\xi(z,k)=1$. Then

$$\begin{aligned}\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1} &= \left[\eta(z,k)\left(z-a(k)\right)+d(k)\xi(z,k)\right][\xi(z,k)]^{-1}[z-a(k)]^{-1}\\ &= \eta(z,k)\left(z-a(k)\right)[\xi(z,k)]^{-1}[z-a(k)]^{-1}+d(k)[z-a(k)]^{-1}. \quad (11)\end{aligned}$$

If $a(k)$ and the coefficients of $\xi(z,k)$ are constant functions, then $z-a(k)$ commutes with $\xi(z,k)$, that is, $\left(z-a(k)\right)\xi(z,k)=\xi(z,k)\left(z-a(k)\right)$, and $[\xi(z,k)]^{-1}[z-a(k)]^{-1}=[z-a(k)]^{-1}[\xi(z,k)]^{-1}$. In this case, Equation (11) reduces to

$$\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}=\eta(z,k)[\xi(z,k)]^{-1}+d(k)[z-a(k)]^{-1}. \quad (12)$$

Hence the first-order term $d(k)[z-a(k)]^{-1}$is extracted from the fraction $\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}$

If $a(k)$ and/or the coefficients of $\xi(z,k)$ vary as functions of k, then $z-a(k)$ and $\xi(z,k)$ do not commute. In this case, in [1] the extraction of a first-order term from the fraction $\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}$ is approached by first computing $\beta(k) \in Q(A)$ and a polynomial $\varphi(z,k)$ with coefficients in $Q(A)$ such that

$$\left(z-a(k)\right)\xi(z,k)=\varphi(z,k)\left(z-\beta(k)\right). \quad (13)$$

However, it is not necessary to express $\left(z-a(k)\right)\xi(z,k)$ in the form given by Equation (13) in order to extract a first-order term. A sufficient condition for extracting a first-order term is that there exist a polynomial $\rho(z,k)$ with coefficients in $Q(A)$ and $b(k) \in Q(A)$ such that

$$\rho(z,k)\left(z-a(k)\right)+\xi(z,k)b(k)=1. \quad (14)$$

To prove sufficiency, multiple both sides of Equation (14) on the left by $\xi(z,k)^{-1}$ and on the right by $[z-a(k)]^{-1}$. This results in the decomposition

$$\xi(z,k)^{-1}[z-a(k)]^{-1}=\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}=\xi(z,k)^{-1}\rho(z,k)+b(k)[z-a(k)]^{-1}, \quad (15)$$

and thus, the first-order term $b(k)[z-a(k)]^{-1}$ is extracted from the fraction. Also note that the denominators of the terms in the decomposition (15) are equal to the factors $\xi(z,k)$ and $z-a(k)$ comprising the denominator of the fraction $\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}$.

The computation of $b(k)$ that satisfies Equation (14) can be carried out by evaluating both sides of Equation (14) at $z^i=a_i(k)$, where $a_0(k)=1, a_i(k)=a(k)a(k+1)\cdots a(k+i-1), i\geq 1$. First, it follows from the results in [3] that the evaluation at $z^i=a_i(k)$ of a skew polynomial $\gamma(z,k)$, with coefficients written on the left, is equal to the remainder after dividing $\gamma(z,k)$ on the right by $z-a(k)$. If $\gamma(z,k)$ has $z-a(k)$ as a right factor, the remainder after dividing by $z-a(k)$ on the right is equal to zero. Hence, in this case, the evaluation of $\gamma(z,k)$ at $z^i=a_i(k)$ is

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equal to zero. Finally, evaluation is an additive operation; that is, the evaluation of the sum of two skew polynomials is equal to the sum of the evaluations of the two polynomials.

Now suppose that $\xi(z,k) = z^N + \sum_{i=0}^{N-1} \xi_i(k) z^i, \xi_i(k) \in A$. Then

$$\xi(z,k)b(k) = b(k+N)z^N + \sum_{i=0}^{N-1} \xi_i(k) b(k+i) z^i,$$

and evaluating both sides of Equation (14) at $z^i = a_i(k)$ gives

$$b(k+N)a_N(k) + \sum_{i=0}^{N-1} \xi_i(k) b(k+i)\, a_i(k) = 1 \tag{16}$$

Thus, the solution $b(k)$ to Equation (14) satisfies the Nth-order linear time-varying difference equation (16). Note that if $a(k) = a$ and $\xi_i(k) = \xi_i$ for all integers k, then $b(k)$ is also a constant and is equal to $\left[a^N + \sum_{i=0}^{N-1} \xi_i\, a^i\right]^{-1}$.

Equation (16) specifies $b(k)$ for all k ranging over the set of integers. Given a fixed integer k_0, $b(k)$ can be computed iteratively for $k \geq k_0$ by solving Equation (16) with initial values $b(k_0 - i), i = 1,2,\ldots,N$. Since the values of $b(k)$ for $k \geq k_0$ depend on the initial values, there is no unique solution for $b(k)$ for $k \geq k_0$. If $b(k)$ is approximately constant over every $(N+1)$-step interval $k, k+1, \ldots, k+N$, then $b(k)$ can be approximated by

$$b(k) = [a_N(k) + \sum_{i=0}^{N-1} \xi_i(k) a_i(k)]^{-1}.$$

Once $b(k)$ has been computed for some desired range of k, the polynomial $\rho(z,k)$ in Equation (14) can be determined by equating the coefficients of like powers of z in the left and right sides of Equation (14), with the coefficients of polynomials written on the left of the z^i. The details are omitted.

Given $\eta(z,k) = \sum_{i=0}^{M} \eta_i(k) z^i, \eta_i(k) \in A, M \leq N$, we shall now extract a first-order term from the left polynomial fraction $\left[\left(z - a(k)\right)\xi(z,k)\right]^{-1} \eta(z,k)$. First, define $\hat{\eta}(z,k) = \sum_{i=0}^{M} \eta_i(k-i) z^i$. Then from the results in [3], the remainder $r(k)$ after dividing $\eta(z,k)$ on the left by $z - a(k)$ is equal to $\hat{\eta}(z,k)$ evaluated at $z^i = \hat{a}_i(k)$, where $\hat{a}_i(k) = a(k)a(k-1)\cdots a(k-i+1), i \geq 1, \hat{a}_0(k) = 1$. Thus,

$$(z - a(k))^{-1} \eta(z,k) = q(z,k) + \left(z - a(k)\right)^{-1} r(k), \tag{17}$$

where $q(z,k)$ is a polynomial in z with coefficients in $Q(A)$ and

$$r(k) = \sum_{i=0}^{M} \eta_i(k-i) \hat{a}_i(k). \tag{18}$$

Multiplying both sides of Equation (15) on the right by $\eta(z,k)$ gives

$$\left[\left(z - a(k)\right)\xi(z,k)\right]^{-1} \eta(z,k) = \xi(z,k)^{-1} \rho(z,k) \eta(z,k) + b(k)[z - a(k)]^{-1} \eta(z,k). \tag{19}$$

Inserting Equation (17) into Equation (19) yields

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$$\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}\eta(z,k)=\xi(z,k)^{-1}\rho(z,k)\eta(z,k)+ b(k)q(z,k)+b(k)[z-a(k)]^{-1}r(k). \quad (20)$$

Finally, since $M \leq N$, the fraction $\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}\eta(z,k)$ is strictly proper, and since $b(k)[z-a(k)]^{-1}r(k)$ is also strictly proper, there exists a polynomial $\pi(z,k)$ such that

$$\xi(z,k)^{-1}\rho(z,k)\eta(z,k)+b(k)q(z,k)=\xi(z,k)^{-1}\pi(z,k). \quad (21)$$

Inserting Equation (21) into Equation (20) completes the proof of the following result:

Theorem 1. Given $a(k) \in A, \xi(z,k)=z^N+\sum_{i=0}^{N-1}\xi_i(k)z^i, \xi_i(k) \in A$, and $\eta(z,k)=\sum_{i=0}^{M}\eta_i(k)z^i, \eta_i(k) \in A, M \leq N$, the left polynomial fraction $\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}\eta(z,k)$ has the decomposition

$$\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}\eta(z,k)=\xi(z,k)^{-1}\pi(z,k)+b(k)[z-a(k)]^{-1}r(k), \quad (22)$$

where $b(k)$ is the solution to the Nth-order linear time-varying difference equation (16), and $r(k)$ is given by Equation (18).

Let $x(n,k)$ denote the signal $x(n,k)=a_{n-k-1}(k), n > k$, with initial value $x(k,k)=0$. Then the inverse VIT transform of the first-order term $b(k)[z-a(k)]^{-1}r(k)$ in the decomposition (22) is equal to $b(n)x(n,k)r(k)$. Hence, the inverse transform of the first-order term is a scaling of $x(n,k)$ by $b(n)$ in the time variable n, and a scaling of $x(n,k)$ by $r(k)$ in the initial time variable k. Also note that if the skew polynomial $\xi(z,k)$ has the factorization $\xi(z,k)=(z-e(k))\theta(z,k)$, where $e(k) \in A$, and $\theta(k,z)$ is a polynomial with coefficients in A, the above procedure can be repeated to extract a first-order term from $[(z-e(k))\theta(z,k)]^{-1}$. Continuing this process will yield a decomposition of the fraction $\left[\left(z-a(k)\right)\xi(z,k)\right]^{-1}\eta(z,k)$ given in terms of a sum of first-order terms.

4. Application to Linear Time-Varying Systems

Consider a causal linear time-varying discrete-time system with input $u(n)$ and resulting output response $y(n)$. It is assumed that the input is applied beginning at time k, and is zero before time k. Here k is the initial time which is allowed to vary over the set of integers. Then the input can be expressed in the form $u(n)=u(n)\mathcal{H}(n-k)$, which shows that $u(n)$ depends on k, so we shall write the input as $u(n,k)$. For example, let $\delta(n-k)$ denote the unit pulse defined by $\delta(n-k)=1, n=k, \delta(n-k)=0, n \neq k$. Then the input $u(n,k)=\delta(n-k)$ is the unit pulse applied at the initial time k. The output response of the system to the unit-pulse $\delta(n-k)$ with zero initial conditions is the unit-pulse response function $h(n,k)$. By casualty, $h(n,k)$ is equal to zero when $n<k$.

Now consider the input $u(n,k)\mathcal{H}(n-k)$ which can be written in the form

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$u(n,k)\mathcal{H}(n-k) = \sum_{r=k}^{\infty} u(r,k)\,\delta(n-r)$. By definition of $h(n,k)$, the response to $\delta(n-r)$ is $h(n,r)$. Then by linearity, the output response to the input $u(n,k)\mathcal{H}(n-k)$ with zero initial energy at time k is given by

$$y(n,k) = \sum_{r=k}^{\infty} h(n,r)u(r,k). \tag{23}$$

This is the input/output relationship of the system when the input $u(n,k)\mathcal{H}(n-k)$ depends on both the current time n and the initial time k.

In [2], the transfer function $H(z,k)$ of the system given by Equation (23) is defined to be the skew power series

$$H(z,k) = \sum_{i=0}^{\infty} z^{-i}h(k+i,k), \tag{24}$$

and in the case when $u(n,k) = u(n)\mathcal{H}(n)$, it is proved that $\hat{y}(z,k) = H(z,k)\hat{u}(z,k)$, where $\hat{u}(z,k)$ and $\hat{y}(z,k)$ are the generalized z-transforms of the input and output, respectively. Here the output $y(n)$ is the response of the system to the input $u(n)$ applied beginning at the initial time $n = 0$.

The transfer function $H(z,k)$ defined by Equation (24) is equal to the VIT transform of the unit-pulse response function $h(n,k)$, and as proved in [1], the VIT transform $Y(z,k)$ of the output $y(n,k)$ resulting from the input $u(n,k)$ is given by the product

$$Y(z,k) = H(z,k)U(z,k), \tag{25}$$

where $U(z,k)$ is the VIT transform of the input $u(n,k)$. Here the output $y(n,k)$ is the response of the system to the input $u(n,k)$ applied beginning at the initial time $n = k$, where k varies over the set of integers. Hence, the VIT transform framework captures the dependency of the output response on the time when the input is applied, which is a key aspect of time-varying systems.

Note that the left-coefficient form of $H(z,k)$ is given by

$$H(z,k) = \sum_{i=0}^{\infty} h(k,k-i)z^{-i}. \tag{26}$$

When z is viewed as a complex variable, $H(z,k)$ defined by Equation (26) is equal to the ordinary z-transform of $h(k,k-n)$, which is the definition of the transfer function given in [4]. Thus, the transfer function has the same form in both the z-transform approach developed in [4] and the VIT transform approach developed in [1]. However, the two approaches differ significantly since the skew ring framework in [1] is based on the noncommutative multiplication $a(k)z^{i} = z^{-i}a(k-i)$, $a(k) \in A$, whereas there is no noncommutative multiplication in the z-transform framework. It is a consequence of the noncommutative multiplication in the ring framework that the VIT transform of the output is equal to the product of $H(z,k)$ with the VIT transform of the input; whereas the z-transform of the output is not equal in general to the product of $H(z,k)$ with the z-transform of the input.

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From the results in [1] and [2], when the system is given by the input/output difference equation

$$y(n+N,k)+\sum_{i=0}^{N-1}\xi_i(n)y(n+i,k)=\sum_{i=0}^{M}\nu_i(n)u(n+i,k),\ \ \xi_i(n),\nu_i(n)\in A, \tag{27}$$

where $M\le N$, the transfer function $H(z,k)$ in the skew ring framework has the left polynomial fraction form given by

$$H(z,k)=\left[z^N+\sum_{i=0}^{N-1}\xi_i(k)z^i\right]^{-1}\left[\sum_{i=0}^{M}\nu_i(k)z^i\right]. \tag{28}$$

Let $\xi(z,k)=z^N+\sum_{i=0}^{N-1}\xi_i(k)z^i$ and $\nu(z,k)=\sum_{i=0}^{M}\nu_i(k)z^i$. Then the VIT transform $Y(z,k)$ of the output of the system defined by the difference equation (27) is equal to $\xi(z,k)^{-1}\nu(z,k)U(z,k)$.

One consequence of the form $\xi(z,k)^{-1}\nu(z,k)U(z,k)$ for $Y(z,k)$ is that it leads directly to the inverse system. To show this, multiply both sides of $Y(z,k)=\xi(z,k)^{-1}\nu(z,k)U(z,k)$ on the left by $\xi(z,k)$, and then multiply the result on the left again by $\nu(z,k)^{-1}$. This yields

$$\nu(z,k)^{-1}\xi(z,k)Y(z,k)=U(z,k). \tag{29}$$

From Equation (29), it is seen that $\nu(z,k)^{-1}[\xi(z,k)]$ is the transfer function of the inverse system; that is, this system reproduces the input $u(n,k)$ from the output $y(n,k)$. However, if $M<N$, then dividing $\nu(z,k)$ into $\xi(z,k)$ using left long division will result in a term of the form $w(k)z^{N-M}$, where $w(k)\in A$ if $\nu_M(k)\neq 0$ for all integers k. As a result, the inverse system defined by Equation (29) is not causal. To achieve causality, multiply both sides of Equation (29) on the left by $z^{-(N-M)}$which gives

$$z^{-(N-M)}[\nu(z,k)]^{-1}\xi(z,k)Y(z,k)=[\nu(z,k)z^{N-M}]^{-1}\xi(z,k)Y(z,k)=z^{-(N-M)}U(z,k).$$

From the shifting property of the VIT transform given in [1], the inverse transform of $z^{-(N-M)}U(z,k)$ is equal to $u(n-(N-M),k)$. Hence, the causal inverse system with transfer function $[\nu(z,k)z^{N-M}]^{-1}\xi(z,k)$ reproduces a time delayed version of $u(n,k)$ from the output $y(n,k)$.

Now suppose that the system defined by the difference equation (27) is asymptotically stable as defined in [1]. Using the VIT transform framework, we shall determine the steady-state output response of the system to the input $u(n,k)=a_{n-k}(k)=a(k)a(k+1)\cdots a(n-1), n>k$, with initial value $u(k,k)=a_0(k)=1$ Here $a(k)$ is an element of A with the condition that $a_{n-k}(k)$ does not converge to zero as $n\to\infty$. Note that $u(n+1,k)=a(n)u(n,k)$, and the VIT transform of $u(n,k)$ is equal to $(z-a(k))^{-1}z$. Then the VIT transform $Y(z,k)$ of the output response is

$$Y(z,k)=\xi(z,k)^{-1}\nu(z,k)(z-a(k))^{-1}z. \tag{30}$$

Dividing $\nu(z,k)$ on the right by $z-a(k)$ gives

$$\nu(z,k)\big(z-a(k)\big)^{-1}=\theta(z,k)+s(k)\big(z-a(k)\big)^{-1}. \tag{31}$$

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where $\theta(z,k)$ is a polynomial in z with coefficients in $Q(A)$, and the remainder $s(k)$ is equal to the evaluation of $v(z,k)$ at $z^i = a_i(k)$; that is, $s(k) = \sum_{i=0}^{M} v_i(k)a_i(k)$. Then inserting (31) into (30) results in

$$Y(z,k) = \xi(z,k)^{-1}\theta(z,k)z + \xi(z,k)^{-1}\, s(k)\big(z - a(k)\big)^{-1} z. \quad (32)$$

The inverse VIT transform of the first term on the right side of Equation (32) decays to zero as $n \to \infty$ since the system is asymptotically stable.

As in the above constructions leading to the proof of Theorem 1, it is possible to determine a polynomial $\tau(z,k)$ with coefficients in $Q(A)$ and $p(k) \in Q(A)$ such that

$$\tau(z,k)\big(z - a(k)\big) + \xi(z,k)p(k) = s(k), \quad (33)$$

Evaluating Equation (33) at $z^i = a_i(k)$ results in the following difference equation for $p(k)$:

$$p(k+N)a_N(k) + \sum_{i=0}^{N-1} \xi_i(k)a_i(k)p(k+i) = s(k) = \sum_{i=0}^{M} v_i(k)a_i(k). \quad (34)$$

Then multiplying both sides of Equation (33) on the left by $\xi(z,k)^{-1}$ and on the right by $\big(z - a(k)\big)^{-1}$, we have that the second term on the right side of Equation (32) has the decomposition

$$\xi(z,k)^{-1}s(k)\big(z - a(k)\big)^{-1} z = \xi(z,k)^{-1}\tau(z,k)z + p(k)\big(z - a(k)\big)^{-1} z. \quad (35)$$

The inverse VIT transform of the first term on the right side of Equation (35) decays to zero as $n \to \infty$. Hence, the steady-state response to the input $u(n,k) = a_{n-k}(k), k \geq n$, is equal to the inverse VIT transform of $p(k)\big(z - a(k)\big)^{-1} z$, which is equal to $p(n)a_{n-k}(k), k \geq n$. This proves the following result.

Theorem 2. Suppose that the system defined by the input/output difference equation (27) is asymptotically stable. Then the steady-state response to the input $u(n,k) = a_{n-k}(k), k \geq n$, $a(n) \in A$, is equal to $p(n)u(n,k)$, where $p(n)$ is the solution to the difference equation (34) with $k = n$.

By Theorem 2, the steady-state output response of an asymptotically stable system to the input $u(n,k) = a_{n-k}(k), k \geq n$, is equal to a scaling of the input by the time function $p(n)$. It is interesting to note that the expression for $p(n)$ given by Equation (34) can be generated directly from the input/output difference equation by inserting $u(n,k) = a_{n-k}(k)$ and $y(n,k) = p(n)a_{n-k}(k)$ into Equation (27) and solving for $p(n)$. The VIT transform framework as utilized here verifies that this solution to the input/output difference equation is in fact the steady-state response in the case when the system is asymptotically stable.

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