Short Note

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Short Note

Another Disproof of the Riemann Hypothesis

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Abstract: In this note we give a second disproof of the Riemann hypothesis by Littlewood's oscillatory theorem and a result of Fujii.

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1. Introduction

The infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = \sigma + it$ is a complex number, converges for $\sigma > 1$. The Riemann zeta function is its meromorphic continuation to the whole complex plane. The Riemann hypothesis asserts that all zeros in the strip $0 < \sigma < 1$ satisfy $\sigma = 1/2$. For the basic theory of the Riemann zeta function one may refer to [1,4,5,8].

In a recent preprint [9], the author proposed a disproof of the Riemann hypothesis by considering an integral equivalence of Hu [3]. In this short note we give another disproof by Littlewood's oscillatory theorem and a result of Fujii [2].

Theorem 1. *The Riemann hypothesis is false.*

2. Proof of Theorem 1

Let $\Lambda(n)$ be the von Mangoldt function and let Ω denote the big Omega notation. The following is Littlewood's oscillatory theorem.

Theorem 2 ([7])**.** *As $x \to \infty$,*

$$\sum_{n \le x} \Lambda(n) = x + \Omega\left(\sqrt{x} \log\log\log x\right). \tag{1}$$

Let

$$r_2(n) = \sum_{\ell + m = n} \Lambda(\ell)\Lambda(m).$$

In his study of the Goldbach conjecture Fujii proved the following result.

Theorem 3 ([2])**.** *Suppose the Riemann hypothesis is true, then*

$$\sum_{n \le x} r_2(n) = \frac{1}{2}x^2 + O(x^{3/2}). \tag{2}$$

We now return to the proof of Theorem 1.

Proof of Theorem 1. It follows readily from (1) that as $x \to 1^-$,

$$f(x) := \sum_{n=1}^{\infty} \Lambda(n) x^n = \frac{1}{1-x} + \Omega\left(\frac{1}{\sqrt{1-x}} \log\log\log \frac{1}{1-x}\right), \tag{3}$$

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and thus as $x \to 1^-$,

$$f(x)^2 = \sum_{n=1}^{\infty} \left(\sum_{\ell+m=n} \Lambda(\ell)\Lambda(m) \right) x^n \tag{4}$$

$$= \sum_{n=1}^{\infty} r_2(n) x^n \tag{5}$$

$$= \frac{1}{(1-x)^2} + \Omega\left(\frac{1}{(1-x)^{3/2}} \log\log\log \frac{1}{1-x} \right). \tag{6}$$

Then by the well known Hardy-Littlewood tauberian theorem [6] we deduce that

$$\sum_{n \le x} r_2(n) = \frac{1}{2}x^2 + \Omega(x^{3/2} \log\log\log x), \tag{7}$$

which contradicts (2). □

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