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An Approximation to Riemann Hypothesis

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Article

An Approximation to Riemann Hypothesis

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Abstract

In the previous version, there was possibly a issue that in Lemma 2.2 the integrant $|\zeta(s)|^2$ should be rewritten as $\zeta(s)\zeta(2a-s)$ when the integral path is moved from (a) to (c), which has been modified in this version accordingly.

Keywords: Riemann zeta-function; Riemann hypothesis

1. Introduction

Riemann zeta-function $\zeta(s)$ is originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } \operatorname{Re} s > 1.$$

and it also can be expressed as the product form

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s}, \quad \text{for } \operatorname{Re} s > 1.$$

This formula is called Euler's product formula, which indicates the relation between $\zeta(s)$ and prime numbers. About $\zeta(s)$ there is a well-known Riemann hypothesis, states that all the non-trivial zeros of $\zeta(s)$ are on the critical line $\operatorname{Re} s = 1/2$. The researches on the conjecture are no doubt a most time-consuming one in mathematics, refer to see the survey paper [3].

The so-called trivial zeros of $\zeta(s)$ are $s = -2, -4, \dots$, and nontrivial zeros of $\zeta(s)$ are known all in the critical strip $0 \leq \operatorname{Re} s \leq 1$.

Denote by $N(T)$ the number of zeros of $\zeta(\sigma + it)$ in the region $0 \leq \sigma \leq 1, 0 \leq t \leq T$, and by $N_0(T)$ the number of zeros on the critical line $\sigma = 1/2, 0 \leq t \leq T$. Riemann hypothesis is that

$$N_0(T) = N(T).$$

For $N(T)$, it is known that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.1}$$

And for $N_0(T)$, Hardy firstly shown that there are infinity many zeros on the critical line, and then he and Littlewood [5] and Selberg [8] proved that

$$\kappa = \frac{N_0(T)}{N(T)} > 0.$$

Levinson [6] proved

$$\kappa \geq \frac{1}{3}.$$

and then this result has been improved successively. Conrey [2], Feng [4] proved respectively

$$\kappa \geq 0.407, \quad \kappa \geq 0.412.$$

In this paper, we will prove that

Theorem 1.1.

$$N(T) = N_0(T) + \mathcal{E}. \quad (1.2)$$

where $\mathcal{E} \ll T^{1/2}(\log T)^2$.

The main arguments in this paper are based the papers [6],[7] and [1], but instead of using Riemann-Siegel formula, it will be applied an auxiliary function $\omega(s, T_1, T_2)$ defined in Lemma 2.1, which will play a role of mollifier and ferry, it firstly used in [7] but here with a small modification.

In the prior version, possibly there was a issue that in Lemma 2.2 the integrant $|\zeta(s)|^2$ should be rewritten as $\zeta(s)\zeta(2a-s)$ when the integral path is moved from (a) to (c), which has been revised in this version.

2. Some Lemmas

In the following, it will be used an auxiliary function $\omega(s, T_1, T_2)$, which defined as following.

Suppose that $T \leq T_1 \leq T_2 \leq 2T$, $\lambda = T^{1-\epsilon}$, for any $\epsilon > 0$, $s_1 = \lambda + c + iu$, ($c \geq 0$), $s = v + it$, define

$$g(w) = e^\lambda \Gamma(w) \lambda^{-w},$$

and

$$\omega(s, T_1, T_2) = \frac{1}{2\pi} \int_{T_1}^{T_2} g(s_1 - s) du. \quad (2.1)$$

Lemma 1. Let $\sigma = \lambda + c - v$, $\eta = \min\{|x - t| | x \in [T_1, T_2]\}$, there is

$$|\omega(s, T_1, T_2)| \ll \exp(-((c-v)^2 + \eta^2)/2\sigma) \quad (2.2)$$

Suppose $\Delta \geq \Delta_0 = (2\beta\lambda \log T)^{1/2}$, $\beta > 0$, if $t \in [T_1 + \Delta, T_2 - \Delta]$, then

$$|\omega(c + it, T_1, T_2) - 1| \ll T^{-\beta}. \quad (2.3)$$

And if $t \leq T_1 - \Delta$, or $t \geq T_2 + \Delta$, then

$$|\omega(c + it, T_1, T_2)| \ll T^{-\beta}. \quad (2.4)$$

If $|t - u| = o(\sigma^{2/3})$, then

$$\arg g(s_1 - s) = \frac{t - u}{2\sigma} - \frac{(t - u)^3}{6\sigma^2} - \frac{c(t - u)}{\sigma} + O\left(\frac{1}{\sigma}\right). \quad (2.5)$$

Proof. By Stirling's formula, it has

$$\begin{aligned} \operatorname{Re}(\log \Gamma(\sigma + (u - t)i)) &= \left(\sigma - \frac{1}{2}\right) \frac{\log(\sigma^2 + (u - t)^2)}{2} - \sigma + \frac{1}{2} \log(2\pi) \\ &\quad - (u - t) \arctan\left(\frac{u - t}{\sigma}\right) + O\left(\frac{1}{\sigma}\right) \end{aligned}$$

And

$$\begin{aligned} &\operatorname{Re}(\log(\Gamma(\sigma + (u - t)i))) + \operatorname{Re}(\log(\lambda^{-(\sigma + (u - t)i)})) + \lambda \\ &= -\frac{(c - v)^2}{2\sigma} - \frac{(u - t)^2}{4\sigma^2} - \frac{(u - t)^2}{2\sigma} - \frac{1}{2} \log \sigma + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{\sigma}\right) \end{aligned}$$

Hence,

$$|\omega(s, T_1, T_2)| \leq e^{-(c-v)^2/2\sigma} \frac{1}{\sqrt{2\pi\sigma}} \int_{T_1}^{T_2} e^{-(u-t)^2/2\sigma} du \ll e^{-((c-v)^2+\eta^2)/2\sigma}. \quad \square$$

Besides, it is familiar that

$$e^{-\lambda} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s_1 - s) \lambda^{s-s_1} ds.$$

Hence,

$$1 - \omega(c + it, T_1, T_2) = R_1 + R_2,$$

where

$$R_1 = \frac{e^\lambda}{2\pi} \int_{T_2}^{\infty} \Gamma(\lambda + (u-t)i) \lambda^{-(\lambda+(u-t)i)} du,$$

$$R_2 = \frac{e^\lambda}{2\pi} \int_{-\infty}^{T_1} \Gamma(\lambda + (u-t)i) \lambda^{-(\lambda+(u-t)i)} du,$$

Hence, if $t \in [T_1 + \Delta, T_2 - \Delta]$, then

$$\begin{aligned} |R_1| &\ll \int_{T_2}^{\infty} \left| e^\lambda \Gamma(\lambda + (u-t)i) \lambda^{-(\lambda+(u-t)i)} \right| du \\ &\ll \lambda^{-1/2} \int_{T_2}^{\infty} \exp(-(u-t)^2/2\lambda) du \\ &\ll T^{-\beta}. \end{aligned}$$

and similarly,

$$|R_2| \ll \lambda^{-1/2} \int_{-\infty}^{T_1} \exp(-(u-t)^2/2\lambda) du \ll T^{-\beta}.$$

if $t \leq T_1 - \Delta$, or $t \geq T_2 + \Delta$, then

$$|\omega(c + it, T_1, T_2)| \ll \lambda^{-1/2} \int_{T_1}^{T_2} \exp(-(u-t)^2/2\lambda) du \ll T^{-\beta}.$$

If $|t - u| = o(\sigma^{2/3})$, then

$$\begin{aligned} \operatorname{Im}(\log(\Gamma(s_1 - s) \lambda^{s-s_1})) &= \frac{t-u}{2\sigma} + \frac{(t-u)^3}{3\sigma^2} - \frac{(t-u)^3}{2\sigma^2} - \frac{c(t-u)}{\sigma} + O\left(\frac{1}{\sigma}\right) \\ &= \frac{t-u}{2\sigma} - \frac{(t-u)^3}{6\sigma^2} - \frac{c(t-u)}{\sigma} + O\left(\frac{1}{\sigma}\right). \end{aligned}$$

Lemma 2. Let $L = \log(T/2\pi)$, $G_0(s) = \zeta(s) + \zeta'(s)/L$, and $0 < a \leq 1/2$, $1 < b < \beta - 1$, $u \in [T_1, T_2]$, $s_1 = \lambda + c + iu$, then

$$\begin{aligned} &\frac{1}{2\pi i} \int_{a+(u-\Delta)i}^{a+(u+\Delta)i} g(s_1 - s) G_0(s) G_0(2a - s) ds \\ &= \frac{1}{2\pi i} \int_{b+(u-\Delta)i}^{b+(u+\Delta)i} g(s_1 - s) G_0(s) G_0(2a - s) ds + O(b/T) \end{aligned} \quad (2.6)$$

Proof. Let $\mathcal{B}$ be the rectangle with vertices $a + (u - \Delta)i, a + (u + \Delta)i, b + (u - \Delta)i$ and $b + (u + \Delta)i$. Take the integral $\int_{\mathcal{B}} g(s_1 - s)G(s)G(2a - s)ds$, with the residue theorem, it has

$$\begin{aligned} & \frac{1}{2\pi i} \int_{b+(u-\Delta)i}^{b+(u+\Delta)i} g(s_1 - s)G_0(s)G_0(2a - s)ds \\ &= \frac{1}{2\pi i} \int_{a+(u-\Delta)i}^{a+(u+\Delta)i} g(s_1 - s)G_0(s)G_0(2a - s)ds \\ &+ \frac{1}{2\pi i} \int_{a+(u+\Delta)i}^{b+(u+\Delta)i} g(s_1 - s)G_0(s)G_0(2a - s)ds \\ &+ \frac{1}{2\pi i} \int_{b+(u-\Delta)i}^{a+(u-\Delta)i} g(s_1 - s)G_0(s)G_0(2a - s)ds + R \end{aligned}$$

where R is the residue at the pole $s = 1$, which is easy to know

$$R \ll g(\lambda + c - 1 + iu) \ll O(e^{-c}). \quad \square$$

On the upper side and the lower side, as $|\text{Im}(s_1 - s)| = \Delta$, by Lemma 2.1, it has

$$|g(s_1 - s)| \ll 1/T^\beta$$

Besides, by the functional equation, it is easy to know

$$|G(v + (u \pm \Delta)i)G(2a - (v + (u \pm \Delta)i))| \ll T^b, \quad (a \leq v \leq b).$$

Hence, the two integrals on the upper side and the lower side of $\mathcal{B}$

$$\frac{1}{2\pi i} \int_{a+(u\pm\Delta)i}^{b+(u\pm\Delta)i} g(s_1 - s)G_0(s)G_0(2a - s)ds \ll b/T,$$

and the Lemma is followed.

Lemma 3. Suppose that $0 < \alpha < \beta - 2$. For $r > 0$, let

$$J(r) = \frac{1}{2\pi} \int_{T_1}^{T_2} \int_{u-\Delta}^{u+\Delta} \left(\frac{t}{2\pi}\right)^\alpha e^{-(t-u)^2/2\sigma} \exp\left(it \log\left(\frac{t}{re}\right) - \frac{c(t-u)i}{\sigma}\right) dt du$$

Then for $T_1 - \Delta \leq r \leq T_2 + \Delta$,

$$J(r) \doteq \left(\frac{r}{2\pi}\right)^\alpha r^{1/2} \sigma^{1/2} e^{\pi i/4}. \tag{2.7}$$

And if $r < T_1 - \Delta$, or $r > T_2 + \Delta$, and $\Delta \geq \Delta_0 T^\epsilon$, then

$$J(r) = O(1). \tag{2.8}$$

Proof. Denote by

$$F(u) = \int_{u-\Delta}^{u+\Delta} \left(\frac{t}{2\pi}\right)^\alpha e^{-(t-u)^2/2\sigma} \exp\left(it \log\left(\frac{t}{re}\right) - \frac{c(t-u)i}{\sigma}\right) dt \quad \square$$

And let $t = u(1 + x)$, then

$$F(u) = \left(\frac{u}{2\pi}\right)^\alpha u e^{i u \rho} F_1(u),$$

where $\rho = \log(u/r) - 1$,

$$F_1(u) = \int_{-\Delta/u}^{\Delta/u} \exp(A(x) + B(x)i) dx,$$

$$A(x) = \alpha \log(1+x) - (ux)^2/2\sigma, \quad B(x) = u(1+x) \log(1+x) + uxp - cux/\sigma.$$

By Gauss's integration, it is easy to follow

$$F_1(u) \doteq \frac{1}{u} \frac{2\pi\sigma}{\sqrt{1+\alpha\sigma/u^2-i\sigma/u}} \exp\left(-\frac{(1+\rho-c/\sigma-i\alpha/u)^2u}{2(u/\sigma+\alpha/u-i)}\right)$$

and

$$F(u) \doteq \left(\frac{u}{2\pi}\right)^\alpha e^{iup} \frac{\sqrt{2\pi\sigma}}{\sqrt{1+\alpha\sigma/u^2-i\sigma/u}} \exp\left(-\frac{(\log(u/r)-c/\sigma-i\alpha/u)^2\sigma}{2(1+\alpha\sigma/u^2-i\sigma/u)}\right)$$

If $r \in [T_1 - \Delta, T_2 + \Delta]$, let $u = r(1+x)$, then

$$F(u) \doteq \left(\frac{r}{2\pi}\right)^\alpha \sqrt{2\pi\sigma} \exp\left(-\frac{x^2(\sigma-ri)}{2}\right) (1+x)^\alpha$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_{T_1/r-1}^{T_2/r-1} F(u) r dx &\doteq \frac{r}{2\pi} \left(\frac{r}{2\pi}\right)^\alpha \frac{\sqrt{2\pi\sigma} \cdot \sqrt{\pi}}{\sqrt{(\sigma+\alpha-ri)/2}} \exp\left(\frac{\alpha^2(\sigma+\alpha+ri)}{2(\sigma+\alpha)^2+r^2}\right) \\ &\doteq \left(\frac{r}{2\pi}\right)^\alpha r^{1/2} \sqrt{\sigma} e^{\pi i/4} \end{aligned}$$

If $r < T_1 - \Delta$, or $r > T_2 + \Delta$, and if $\Delta \geq \Delta_0 T^\epsilon$, then

$$\begin{aligned} |F(u)| &\ll T^{\alpha+1/2} \exp\left(-\left(\frac{\Delta}{T} - \frac{c}{\sigma}\right)^2 \frac{\sigma}{2}\right) \\ &\ll T^{\alpha+1/2-\beta} \ll O(1/T). \end{aligned}$$

3. The Proof of Theorem 1.1

Proof. Let $h(s) = \pi^{-s/2} \Gamma(s/2)$, then the functional equation of $\zeta(s)$ can be written as

$$h(s)\zeta(s) = h(1-s)\zeta(1-s) \quad \square \quad (3.1)$$

By Stirling's formula, it has

$$\log h(s) = \frac{1}{2}(s-1) \log \frac{s}{2\pi} - \frac{s}{2} + C_0 + O\left(\frac{1}{s}\right) \quad (3.2)$$

Let $f(s) = \log h(s)$, then

$$f'(s) = \frac{h'(s)}{h(s)} = \frac{1}{2} \log \frac{s}{2\pi} + O\left(\frac{1}{s}\right) \quad (3.3)$$

and for larger t

$$f'(s) + f'(1-s) = \log \frac{t}{2\pi} + O\left(\frac{1}{s}\right) \quad (3.4)$$

Taking logarithm of equation (3.1), and then derivative, it follows

$$h(s)\zeta(s)(f'(s) + f'(1-s)) = -h(s)\zeta'(s) - h(1-s)\zeta'(1-s) \quad (3.5)$$

We note that the right side of (3.5) is a sum of two conjugative complex numbers as $s = 1/2 + it$, so the zeros of the right side of (3.5) occur if and only if

$$\arg(h(s)\zeta'(s)) \equiv \pi/2 \pmod{\pi} \quad (3.6)$$

On the left side of (3.5), clearly, $h(s)$ is never zero, and by (3.4), so these zeros are just the zeros of $\zeta(1/2 + it)$.

Moreover, let $\chi(s) = h(1-s)/h(s)$, then $\zeta(s) = \chi(s)\zeta(1-s)$, and

$$\zeta'(s) = -\chi(s)\{(f'(s) + f'(1-s))\zeta(1-s) + \zeta'(1-s)\} \quad (3.7)$$

By (3.6), the zeros of $\zeta(1/2 + it)$ are the ones

$$\arg(h(1-s)\{(f'(s) + f'(1-s))\zeta(1-s) + \zeta'(1-s)\}) \equiv \pi/2 \pmod{\pi}$$

on $\sigma = 1/2$, equivalently,

$$\arg(h(s)\{(f'(s) + f'(1-s))\zeta(s) + \zeta'(s)\}) \equiv \pi/2 \pmod{\pi} \quad (3.8)$$

on $\sigma = 1/2$. Write $\mathcal{L}(s) = f'(s) + f'(1-s)$, and denote by

$$G(s) = \zeta(s) + \zeta'(s)/\mathcal{L}(s) \quad (3.9)$$

The investigation above means

$$N_0(T) = \frac{1}{\pi} \Delta_0^T \arg(hG(1/2 + it)) \quad (3.10)$$

By(3.2), it can be known that

$$\Delta_0^T \arg(h(1/2 + it)) = \frac{T}{2} \log \frac{T}{2\pi} - \frac{T}{2} + O(\log T) \quad (3.11)$$

So, the main task to determine $N_0(T)$ is to calculate $\Delta_0^T \arg(G(1/2 + it))$. Let $L = \log(T/2\pi)$, $U \leq T$, and let D be the rectangle with the vertices $1/2 + iT$, $c + iT$, $c + i(T + U)$, $1/2 + i(T + U)$, ($c \geq 3$). First of all, we might as well assume there are no zeros of $G(s)$ on the boundary of D , then by the principle of argument, the change of $\arg G(s)$ around D is equal to 2π times $N_G(D)$, the number of zeros of $G(s)$ in D .

On the right side of D

$$|G(c + it) - 1| \leq \sum_{n \geq 2} n^{-c} + O(1/L) \leq 1/3$$

so, $\arg G(s)$ change less than π . On the lower side and the upper side of D , by a known result [9, §9.4], a extension of Jessen's theorem, taking account on the order of $G(s)$, we can know that $\arg G(s) = O(L)$ as $0 < \sigma \leq 3$, and $\arg G(\sigma + it) = O(2^{-\sigma})$ as $\sigma \geq 3$, hence, for any $0 \leq b \leq c$, it has

$$\int_b^c \arg G(\sigma + iT) d\sigma, \int_b^c \arg G(\sigma + i(T + U)) d\sigma \ll O(L) \quad (3.12)$$

So,

$$\Delta_T^{T+U} \arg(G(1/2 + it)) = -2\pi N_G(D) + O(\log T) \quad (3.13)$$

Now the work is turned into to evaluate $N_G(D)$.

Let $1/2 - a = O(1/L)$, and $\mathcal{C}$ be the rectangle with vertices $a + iT$, $c + iT$, $c + i(T + U)$, $a + i(T + U)$. Taking the integral $\int_{\mathcal{C}} \log G(s) ds$, by the Littlewood's Lemma [9, §9.9], it has

$$\begin{aligned} & \int_T^{T+U} \log |G(a + it)| dt - \int_T^{T+U} \log |G(c + it)| dt + \int_a^c \arg G(\sigma + i(T + U)) d\sigma \\ & - \int_a^c \arg G(\sigma + iT) d\sigma = 2\pi \sum \text{dist} \end{aligned} \quad (3.14)$$

where $\sum \text{dist}$ is the sum of the distances of the zeros of $G(s)$ from the left.

By (3.9), it is easy to know

$$\int_T^{T+U} \log G(c+it) dt = \int_T^{T+U} \log \zeta(c+it) dt + O(1/L)$$

and it is familiar that

$$\log \zeta(s) = \sum_n \frac{-\Lambda(n)}{n^s \log n}$$

So

$$\int_T^{T+U} \log |G(c+it)| dt \ll 1.$$

With (3.12), the rest is to calculate the first integral of (3.14).

By the concavity of logarithm, it has

$$\begin{aligned} \int_T^{T+U} \log |G(a+it)| dt &= \frac{1}{2} \int_T^{T+U} \log |G(a+it)|^2 dt \\ &\leq \frac{1}{2} U \log \left(\frac{1}{U} \int_T^{T+U} |G(a+it)|^2 dt \right) \end{aligned} \quad (3.15)$$

At first, we simplify $G(s)$ as

$$G_0(s) = \zeta(s) + \frac{\zeta'(s)}{L}.$$

Then

$$\begin{aligned} G(s) &= G_0(s) + E(s). \\ E(s) &= \left(\frac{1}{\mathcal{L}(s)} - \frac{1}{L} \right) \zeta'(s) \ll \frac{1}{L^3} \zeta'(s). \end{aligned}$$

And

$$\begin{aligned} \int_{T_1}^{T_2} |G(a+it)|^2 dt &= \int_{T_1}^{T_2} |G_0(a+it)|^2 dt + 2\text{Re} \int_{T_1}^{T_2} G_0(a+it) E(a-it) dt \\ &\quad + \int_{T_1}^{T_2} |E(a+it)|^2 dt \end{aligned} \quad (3.16)$$

By Cauchy's inequality

$$\int_{T_1}^{T_2} G_0(a+it) E(a-it) dt \leq \left(\int_{T_1}^{T_2} |G_0(a+it)|^2 dt \int_{T_1}^{T_2} |E(a+it)|^2 dt \right)^{1/2}$$

The third integral in the right side of (3.16) is much smaller than the first one, which will be actually calculated later, hence

$$\int_{T_1}^{T_2} |G(a+it)|^2 dt = (1 + \epsilon) \int_{T_1}^{T_2} |G_0(a+it)|^2 dt.$$

Let

$$\omega_1(s) = \frac{1}{2\pi i} \int_{|u-t| \leq \Delta} g(s_1 - s) ds_1. \quad (3.17)$$

From Lemma 2.1, it is known that $\omega_1(s)$ is the domination of $\omega(s, T_1, T_2)$, and which is a positive real number apart from a small error term.

Moreover, let

$$\phi(s) = \omega_1^{1/2}(s). \quad (3.18)$$

Then it has

$$\arg \phi(v + T_1 i) = o(1), \quad \arg \phi(v + T_2 i) = o(1).$$

And

$$\int_a^c \arg \phi(v + T_1 i) dv \leq o(c), \quad \int_a^c \arg \phi(v + T_2 i) dv \leq o(c). \quad (3.19)$$

Moreover, by Lemma 2.1, there is

$$\int_{T_1}^{T_2} \log |\phi(c + it)| dt \ll T^{-b}. \quad (3.20)$$

(3.19) and (3.20) indicate that function $\phi(s)$ may be used as a mollifier.

Let

$$\mathcal{G}(s) = G_0(s)\phi(s). \quad (3.21)$$

In the following we will replace $G(s)$ by $\mathcal{G}(s)$, and let

$$\mathcal{I} = \int_{T_1}^{T_2} |\mathcal{G}(s)|^2 dt \quad (3.22)$$

Then,

$$\begin{aligned} \mathcal{I} &= \int_{T_1}^{T_2} |G_0(s)|^2 \omega_1(s) dt \\ &= \frac{1}{2\pi} \int_{T_1}^{T_2} \int_{t-\Delta}^{t+\Delta} g(s_1 - s) du |G_0(a + it)|^2 dt \\ &= \frac{1}{2\pi} \int_{T_1}^{T_2} \int_{u-\Delta}^{u+\Delta} g(s_1 - s) |G_0(a + it)|^2 dt du + R_1 + R_2 - R_3 - R_4 \end{aligned}$$

where

$$\begin{aligned} R_1 &= \frac{1}{2\pi} \int_{T_1}^{T_1+\Delta} \int_{u-\Delta}^{T_1} g(s_1 - s) du |G_0(a + it)|^2 dt, \\ R_2 &= \frac{1}{2\pi} \int_{T_2-\Delta}^{T_2} \int_{u+\Delta}^{T_1} g(s_1 - s) du |G_0(a + it)|^2 dt, \\ R_3 &= \frac{1}{2\pi} \int_{T_1-\Delta}^{T_1} \int_{T_1}^{u+\Delta} g(s_1 - s) du |G_0(a + it)|^2 dt, \\ R_4 &= \frac{1}{2\pi} \int_{T_2}^{T_2+\Delta} \int_{u-\Delta}^{T_2} g(s_1 - s) du |G_0(a + it)|^2 dt. \end{aligned}$$

By the mean-value theorems (cf. [9, Ch.7]), it has

$$R_1 \ll \int_{T_1}^{T_1+\Delta} |G_0(a + it)|^2 dt \ll \Delta L,$$

similarly,

$$R_i \ll \Delta L, \quad i = 2, 3, 4.$$

Then, by Lemma 2.2, it has

$$\mathcal{I} = \frac{1}{2\pi i} \int_{T_1}^{T_2} \int_{b+(u-\Delta)i}^{b+(u+\Delta)i} g(s_1 - s) G(s) G(2a - s) ds + O(\Delta L)$$

Moreover, by the functional equation of $\zeta(z)$, it has

$$\zeta(2a - (b + it)) = \chi(2a - (b + it))\zeta(1 + b - 2a - it)$$

$$\zeta'(2a - (b + it)) = -\chi(2a - (b + it))(L\zeta(1 + b - 2a + it)) + \zeta'(1 + b - 2a + it)$$

and

$$G(2a - (b + it)) = -\chi(2a - (b + it))\frac{\zeta'(1 + b - 2a + it)}{L}$$

and it is easy to know

$$\chi(2a - (b + it)) = t^{1/2+b-2a} \exp\left(-\frac{\pi i}{4} + it \log\left(\frac{t}{2\pi e}\right)\right)$$

Hence,

$$\mathcal{I} = I_1 + I_2 + O(\Delta L).$$

where

$$I_1 = -\frac{1}{2\pi} \int_{T_1}^{T_2} \int_{u-\Delta}^{u+\Delta} g(s_1 - s) \chi(2a - (b + it)) \zeta(b + it) \frac{\zeta'(1 + b - 2a + it)}{L} dt du$$

$$I_2 = -\frac{1}{2\pi} \int_{T_1}^{T_2} \int_{u-\Delta}^{u+\Delta} g(s_1 - s) \chi(2a - (b + it)) \frac{\zeta'(b + it)}{L} \frac{\zeta'(1 + b - 2a + it)}{L} dt du$$

Expanding $\zeta(s)$ and $\zeta'(s)$ as Dirichlet's series, and by the Lemma 2.1, it has

$$I_1 = e^{-\frac{(c-b)^2}{2\sigma}} \frac{1}{L} \sum_{y \geq 1} \frac{\log y}{y^{b'}} \sum_{x \geq 1} \frac{1}{x^b} J(2\pi xy),$$

$$I_2 = e^{-\frac{(c-b)^2}{2\sigma}} \frac{1}{L^2} \sum_{y \geq 1} \frac{\log y}{y^{b'}} \sum_{x \geq 1} \frac{\log x}{x^b} J(2\pi xy).$$

where $b' = b + 1 - 2a$, and $J(r)$ is same as in Lemma 2.3.

Dividing I_1 and I_2 into three parts respectively,

$$I_1 = I_{1,1} + I_{1,2} + I_{1,3} = \sum_{2\pi xy < T_1 - \Delta} + \sum_{T_1 - \Delta \leq 2\pi xy \leq T_2 + \Delta} + \sum_{2\pi xy > T_2 + \Delta},$$

$$I_2 = I_{2,1} + I_{2,2} + I_{2,3} = \sum_{2\pi xy < T_1 - \Delta} + \sum_{T_1 - \Delta \leq 2\pi xy \leq T_2 + \Delta} + \sum_{2\pi xy > T_2 + \Delta}.$$

Then by Lemmas 1,2,3,¹

$$I_{1,1} \ll \frac{1}{L} \sum_{2\pi xy \leq T_1 - \Delta} \frac{\log y}{x^b y^{b'}} = o(1)$$

$$I_{1,3} \ll \frac{1}{L} \sum_{2\pi xy > T_2 + \Delta} \frac{\log y}{x^b y^{b'}} = o(1)$$

$$I_{2,1} \ll \frac{1}{L^2} \sum_{2\pi xy \leq T_1 - \Delta} \frac{\log x \log y}{x^b y^{b'}} = o(1)$$

$$I_{2,3} \ll \frac{1}{L^2} \sum_{2\pi xy > T_2 + \Delta} \frac{\log x \log y}{x^b y^{b'}} = o(1)$$

¹ In Lemma 2.3, for the simplicity, $\arg(f)$ is only taken $c(u - t)/\sigma$

and

$$I_{1,2} = e^{-\frac{(c-b)^2}{2\sigma}} \frac{1}{L} \sum_{T_1 - \Delta \leq 2\pi xy \leq T_2 + \Delta} x^{1-2a} \log y,$$

$$I_{2,2} = e^{-\frac{(c-b)^2}{2\sigma}} \frac{-1}{L^2} \sum_{T_1 - \Delta \leq 2\pi xy \leq T_2 + \Delta} x^{1-2a} \log x \log y.$$

Let

$$H(n) = \frac{1}{L} \sum_{1 \leq xy \leq n} x^{1-2a} \log y - \frac{1}{L^2} \sum_{1 \leq xy \leq n} x^{1-2a} \log x \log y$$

It is easy to follow that

$$H(n) \doteq \frac{n}{L^2(1-a)} \left(\frac{n^{1-2a}}{(1-2a)^3} - \frac{\log^2 n}{2(1-2a)} - \frac{\log n}{(1-2a)^2} - \frac{1}{(1-2a)^3} \right)$$

Let $1 - 2a = 1/L$, then

$$H(n) \doteq (e - 2.5)nL$$

Take $c = b + (2\sigma \log((e - 2.5)L/2\pi))^{1/2}$, $T_1 = T$, $T_2 = T + U$, then it has

$$\mathcal{I} = U + O(\Delta)$$

and

$$\int_T^{T+U} |\mathcal{G}(a + it)|^2 dt = U + O(\Delta).$$

Let $U = T$, by (3.15), it follows

$$\begin{aligned} \int_T^{T+U} \log |\mathcal{G}(a + it)| dt &\leq \frac{T}{2} \log \left(1 + \frac{O(\Delta)}{T} \right) \\ &\leq O(\Delta). \end{aligned} \quad (3.23)$$

With (3.12), (3.14), (3.19), (3.20) and (3.23), and recall that $a = 1/2 - 1/L$, it follows

$$2\pi N_G(D) \leq \frac{O(\Delta)}{1/2 - a} \ll T^{1/2} L^2.$$

i.e.

$$\Delta_{2T}^T \arg G(1/2 + it) \leq O(T^{1/2} L^2).$$

and

$$(N(2T) - N(T)) - (N_0(2T) - N_0(T)) \leq O(T^{1/2} L^2).$$

Then let T be $T/2^k$, $1 \leq k \leq \log_2(T)$, and summing. This proves Theorem 1.1 in the case that there are no zeros of $G(s)$ on the boundary of D .

For the rest case, let N_1 and N_2 be the numbers of zeros of $G(s)$ on the left side of D , $\sigma = 1/2$, and in D with $\sigma > 1/2$, respectively. Indent the left side of D with small semicircles with centers at the zeros and lying in $\sigma \geq 1/2$. Let N'_1 be the number of distinct zeros in the N_1 zeros. Let V_j be the variation in $\arg G$ in the j th interval between the successive semicircles. Then by the principle of argument, it has

$$\sum_j V_j - \pi N_1 = 2\pi N_2 + O(L), \quad (3.24)$$

Let W_j be the variation of argument of

$$h(s)(f'(s) + f'(1-s))G(s)$$

in the j th interval, where W_j is taken for increasing t , while V_j is taken for decreasing t . With (3.2) and (3.24), it has

$$\begin{aligned}\sum_j W_j &= \operatorname{Im}(f)|_T^{T+U} - \sum_j V_j \\ &= \operatorname{Im}(f)|_T^{T+U} - (2\pi N_2 + \pi N_1) + O(L)\end{aligned}\quad (3.25)$$

By (3.8), in the j th open interval, the number of zeros of $\zeta(1/2 + it)$ is at least

$$(W_j/\pi) - 1.$$

and in all the open intervals, the number of zeros is at least

$$\begin{aligned}\frac{1}{\pi} \sum_j W_j - N'_1 - 1 &= \frac{1}{\pi} \operatorname{Im}(f)|_T^{T+U} - (2N_2 + N_1) - N'_1 - 1 + O(L) \\ &= \frac{1}{\pi} \operatorname{Im}(f)|_T^{T+U} - 2N_G(D) + N_1 - N'_1 + O(L)\end{aligned}\quad (3.26)$$

Moreover, by (3.7), we can know that on the side $\sigma = 1/2$, a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater, so there are $N_1 + N'_1$ such zeros of $\zeta(1/2 + it)$, adding to (3.26), in total, it has

$$N_0(T+U) - N_0(T) \geq \frac{1}{\pi} \operatorname{Im}(f)|_T^{T+U} - 2N_G(D) + 2N_1 + O(L).$$

By (3.11), we can know

$$\frac{1}{\pi} \operatorname{Im}(f)|_T^{T+U} = N(T+U) - N(T) + O(L).$$

i.e.

$$(N(T+U) - N(T)) - (N_0(T+U) - N_0(T)) \leq O(T^{1/2}L^2).$$

Besides, we know that on the critical line a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater. Hence

$$\sum (m-1) \leq N_G(D).$$

where sum is over the distinct zeros of $\zeta(s)$ on the left side of D , m is the multiplicity of a zero.

And so,

$$\sum_{m \geq 2} m \leq 2N_G(D) \leq O(T^{1/2}L^2). \quad (3.27)$$

This means that the non-trivial zeros of $\zeta(s)$ are all on the critical line, and all are simple, with at most $O(T^{1/2}L^2)$ ones excepted.

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