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The tomographic Wheeler De Witt equation

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Abstract: Recently quantum cosmology in tomographic representation was considered. as an important tool to analyze the quantum states of the early universe in relation with the subsequent classical evolution. Given a wave function the correspondent tomogram is defined proportional to the square modulus of its fractional Fourier transform. The classical limit obtained by taking the limit $\hbar \to 0$ can be compared with the classical tomogram obtained from the classical hamiltonian formalism. In this paper we show that a set of tomograms can be derived as exact solutions of a third order equation when a de Sitter quantum universe is considered. We finally discuss the limits of this approach to quantum cosmology because the extension to equations with arbitrary potentials can become extremely complicated for potentials given by algebraic polynomials of degree greater than two and therefore also for potentials expressible by arbitrary functions.

Keywords: Quantum cosmology; symplectic tomography; cosmology

0. Introduction

The development that the physical sciences have had in the last century has brought human knowledge to limits never reached before. While the theory of general relativity has led to greater knowledge of the universe on a large scale, quantum mechanics has extended knowledge to extremely small, atomic, nuclear and subnuclear scales. Both have led to a growth in experimental research which has also had an impact on daily life.

However, the researches induced by these two disciplines have not been exhausted. If anything, their study opens up to new questions. Although it is often said that both these two theories are incompatible, in recent decades relativists have posed fundamental questions waiting for answers from quantum theories.

In fact, it is wondered if the microscopic origin of the entropy of black holes can be attributed to quantum entanglement or be described by a model of conforming theory or if there are microscopic entities in analogy with the kinetic theory.

On the other hand, the existence of the predicted singularities in general relativity suggests that the classical theory should be replaced by a quantum theory. In this sense, theories have been developed based on different approaches such as the Hamiltonian approach, loop quantum gravity, string theory.

The study of quantum cosmology plays an important role in these researches. The use of homogeneous models allows to reduce the degrees of freedom to be considered and reduces the different quantum theories of gravitation to theories comparable to quantum mechanics[1][2].

Quantum cosmology makes it possible for these theories to have a contact with experimental reality as, although they differ from quantum mechanics, there is a possibility of understanding some fundamental aspects by comparing them with classical observations. In fact, the classical universe must descend through a transition from the quantum one and bear many early informations with it. And if on the one hand a quantum model must converge to the universe we are observing now, on the other hand a quantum model that undergoes a process of classical transition can provide predictions that can be falsified by observations.

In previous papers we have shown that symplectic tomography can be an interesting investigation tool to understand the causes that led to the classical transition of a quantum universe[3]-[12].

The advantage is that both quantum and classical states can be represented by tomograms. so if they can be compared directly and possibly by taking the classical limit of a quantum tomogram we can see if it converges to a general relativistic classical model or if it introduces some new prediction. We started analyzing the de Sitter models, which are related to the only presence of the constant and the extended to more general models with a generic potential. The final results were obtained in the WKB approximation and showed how some quantum models converged to the classical ones, but also that many other models did not as they introduced interference terms which can be essentially observed in the perturbation spectrum.

We introduced the tomograms as tools, as they can be derived from different representations of the quantum theories by the transforms of the wave function, the Wigner function or the density matrix. In this paper we want to introduce an equation to derive the tomograms equivalent to the Wheeler De Witt equation. We shall introduce it only in the de Sitter case, where a analytic solution can be found, but we will also discuss its possible general extensions. The paper is organized in the following way, In sect. 1 after introducing the problem we resume some properties of the de Sitter universe and its Hamiltonian formulation already given in detail in [10],[11],[12]. In sect. 2, we construct the classical tomogram and discuss some its properties and its relation with the inflationary universe. In sect. 3 we show how the initial conditions of the universe are expressed in tomographic terms starting from the wave functions proposed by Hartle and Hawking, Vilenkin and Linde. We show that these tomograms can be also obtained as the solutions of a third order equation in sect. 4. We add some consideration on the extensibility of this equations to more complicated models in sect. 5 where we draw some conclusions.

1. Symplectic Tomography

The properties of a physical system are deducible from the knowledge of its state. In quantum mechanics, and similarly in quantum cosmology, it is expressed depending on the representation used by a wave function or by a Wigner function, a density matrix or a tomogram .

Tomograms are marginal probability functions. They are defined in quantum mechanics as well in classical mechanics. There properties have been thoroughly studied in the last twenty years, they have the important property to be observables and were initially studied, because the permitted the reconstruction of a Wigner function of a quantum system[13][14].

To understand the meaning of a tomogram let us consider a distribution on a one dimensional phase space where al the states are represented by a pair of coordinates (q, p). We consider the projection of this distribution on the q axis i.e we consider a function $W(q)$. If we rotate the coordinate system by the transformations

$$X = \mu q + \nu p \tag{1.1}$$

and

$$P = -\nu q + \mu p \tag{1.2}$$

with $\mu = s\cos\theta$ and $\nu = s^{-1}\sin\theta$, where s is a squeezing factor and θ is the rotation angle of the (q, p) frame, we can take all the projections $\mathcal{W}(X, \mu, \nu)$ of the same distribution on each X axis (where the initial $W(q) = \mathcal{W}(X, 1, 0)$). The set of all these projections is the tomogram[12].

The study of quantum cosmology has the purpose of reconstructing the state of the early universe in the quantum era that can explain its transition to the classical stage. It is therefore necessary to know its initial conditions and its dynamics to understand the cause of this transition. Conversely, it is also possible to consider the possibility reconstruct the initial conditions starting from t the current state of the universe.

Generally an attempt is made to establish the behavior of the physical quantities involved and their different behavior in the quantum sphere compared to the classical one. For example whether a quantum model can avoid the initial singularity or not. However, it is more difficult to establish the relationship between these models and the classical ones that derive from them.

The tomographic approach, instead of studying the relationship between quantum and classical quantities, looks directly at the relationship between quantum states and the classical states of the universe, just by comparing their respective properties.

In previous papers [10][11][12] we used the tomograms in this way, reconstructing them from the quantum wave functions solutions of the Wheeler-De Witt equation and using them as an analysis tool of a quantum system in the sense described above.

In this work we want to show that at least one can find an equation from which the tomogram derives, even if we do not think that it can be considered a definitive result, but a starting point for a subsequent analysis.

In fact in our case we can find an exact solution, similarly an exact solution can be found for the harmonic oscillator problem. But there are few known exact solutions in quantum mechanics. However, the result found can also be considered as an approximation for more general solutions.

We find the equation for a solution that is already known from previous works and is given in the context of a quantum de Sitter universe, that is, a universe with only the cosmological constant λ as source, which we treat for its simplicity both in the classical and in the quantum case.

The de Sitter spacetime in the classical theory is described by the metric

$$ds^2 = -\frac{N^2}{q}d\tau^2 + \frac{q}{1-kr^2}dr^2 + qr^2 d\Omega_2^2 \tag{1.3}$$

where $d\Omega_2^2$ is the metric of a 2-sphere. The choices of $k = \pm 1$ and $k = 0$ correspond to different choices of the spacelike hypersurfaces of the spacetime. This metric represents a vacuum universe endowed only with a constant source and predicts the accelerated expansion,

$$a(\tau) = \exp(\sqrt{\frac{\Lambda}{3}}\tau) \tag{1.4}$$

when we choose $k = 0$. A physical interpretation of the cosmological constant is that it is generated by the vacuum quantum energy, which opens the question of why it is so small with respect to its expectation value according to the accurate analysis put of S. Weinberg[15]. This is called the *cosmological constant problem*.

Besides its mathematical properties, the de Sitter model acquired physical interest when it was realized that it can explain in a general way some important observed properties of the universe through the inflationary model, when $a(\tau)$ has undergone exponential expansion of the type 1.4 originated when the energy of a scalar field in a state of false vacuum prevailed on the other material components of the universe.

In order to treat the quantum model it was proved convenient by J. Louko in [16] to consider the case of compact spatial hypersurfaces $(k = 1)$ with the metric

$$ds^2 = \sigma^2\left[-\frac{N^2}{q}d\tau^2 + q d\Omega_3^2\right] \tag{1.5}$$

where $d\Omega_3^2$ is the metric of a 3-sphere, $q = a^2$ and the lapse is rescaled by a factor $\sqrt{q}$. For numerical reasons we multiplied the metric by a factor $\sigma^2 = 2G/3\pi$

The Einstein equations with cosmological constant

$$G_{ab} + \Lambda g_{ab} = 0 \tag{1.6}$$

become

$$\frac{1}{4}\frac{1}{N^2}\frac{\dot{q}^2}{q}+\frac{1}{q}=\frac{\Lambda}{3} \tag{1.7}$$

and

$$\frac{\ddot{q}}{N^2}+\frac{1}{2}\frac{1}{N^2}\frac{\dot{q}^2}{q}-\frac{\dot{N}}{N^3}\dot{q}+\frac{2}{q}=\frac{4}{3}\Lambda\,. \tag{1.8}$$

We notice that in the gauge $N=\sqrt{q}$ we obtain the classical solution

$$q(t)=\Lambda^{-1}\cosh^2(\Lambda^{1/2}t)\,. \tag{1.9}$$

for the evolution of closed spatial hypersurfaces.

To quantize the gravitational system we use the Hamiltonian formalism. The action, which after integrating the spatial part and eliminating a total derivative with respect to time, is [16]

$$S=\frac{1}{2}\int N\Big(-\frac{\dot{q}^2}{4N}+1-\lambda q\Big)dt \tag{1.10}$$

where λ is now the cosmological constant in Planck units. The coordinates of the phase space are (q,p) with the momentum p defined by

$$p=\frac{\partial L}{\partial \dot{q}}=-\frac{\dot{q}}{4N} \tag{1.11}$$

using the Legendre transform, the action takes the form

$$S=\int(p\dot{q}-N\mathscr{H})dt \tag{1.12}$$

with

$$\mathscr{H}=\frac{1}{2}\Big(-4p^2+\lambda q-1\Big) \tag{1.13}$$

The lapse function N is a Lagrange multiplier, the variation

$$\frac{\delta S}{\delta N}=0 \tag{1.14}$$

implies the Hamiltonian constraint,

$$\frac{1}{2}\Big(-4p^2+\lambda q-1\Big)=0 \tag{1.15}$$

which is equivalent to equation (1.7).

2. The classical tomogram

Let us choose as classical distribution on the phase space [12]

$$f(q,p)=\delta\Big(-4p^2+\lambda q-1\Big). \tag{2.1}$$

where λ is the cosmological constant rescaled in Planck units. We define the classical tomogram by

$$\mathscr{W}(X,\mu,\nu)=\int f(q,p)\delta(X-\mu q-\nu p)dqdp=\int\delta\Big(-4p^2+\lambda q-1\Big)\delta(X-\mu q-\nu p)dqdp\,. \tag{2.2}$$

we find the classical de Sitter tomogram

$$\mathcal{W}(X,\mu,\nu) = \frac{1}{2|\mu|}\frac{1}{\left|\sqrt{\frac{\lambda^2\nu^2}{16\mu^2}+\frac{\lambda X}{\mu}-1}\right|}\,. \tag{2.3}$$

This function is for any μ and ν a marginal probability function, so it must satisfy the condition

$$\int_{-\infty}^{\infty}\mathcal{W}(X,\mu,\nu)dX = 1 \tag{2.4}$$

It is clear that (2.3) is not integrable on an infinite range, then its domain must be restricted to compact support such that the normalization condition (2.4) is satisfied. We choose a closed interval I such that the normalization condition is satisfied. For example if we choose the inferior value of the interval is such that

$$\frac{\lambda^2\nu^2}{16\mu^2}+\frac{\lambda X}{\mu}-1=0 \quad \text{i.e.} \quad X=\frac{\mu}{\lambda}\left(1-\frac{\lambda^2\nu^2}{16\mu^2}\right) \tag{2.5}$$

we see that

$$\int_{-\infty}^{\infty}\mathcal{W}(X,\mu,\nu)dX = \frac{1}{2|\mu|}\int_{\frac{\mu}{\lambda}\left(1-\frac{\lambda^2\nu^2}{16\mu^2}\right)}^{\lambda\mu+\frac{\mu}{\lambda}\left(1-\frac{\lambda^2\nu^2}{16\mu^2}\right)}\frac{dX}{\left|\sqrt{\frac{\lambda^2\nu^2}{16\mu^2}+\frac{\lambda X}{\mu}-1}\right|} = \frac{\mu}{|\mu|}=1\,. \tag{2.6}$$

if $\mu > 0$.

This tomogram is interesting because it represents the state of the universe during inflation. We see that the argument $\frac{\lambda^2\nu^2}{16\mu^2}+\frac{\lambda X}{\mu}-1$ related to the constraint (1.15) takes also values different from zero which shows the presence of perturbations of the spatial curvature. This perturbations reduce almost to zero when the cosmological constant goes to the present value of the order of $10^{-122}\lambda_{Planck}$, and the tomogram can be approximated by

$$\mathcal{W}(X,\mu,\nu) \approx \delta\left(\frac{\lambda}{\mu}X+\frac{\lambda^2\nu^2}{16\mu^2}-1\right), \tag{2.7}$$

which means that the extreme smallness of the cosmological constant implies a very high degree of homogeneity of the classical de Sitter universe.

In a future work we shall study how the cosmological perturbations can modify this tomogram and how to represent in tomographic terms the large scale structure of the universe. Perhaps it may be necessary to loose the restrictions imposed by the cosmological constant taking $I=[X_1,X_2]$ provided that the tomogram is normalized by

$$\bar{\mathcal{W}}(X,\mu,\nu) = \frac{\mathcal{W}(X,\mu,\nu)}{\int_{X_1}^{X_2}\mathcal{W}(X,\mu,\nu)} \tag{2.8}$$

3. The quantum tomograms

The Wheeler-DeWitt equation corresponding to the Hamiltonian (1.13) is [16][17]

$$\left(4\hbar^2\frac{d^2}{dq^2}+\lambda q-1\right)\psi(q)=0\,. \tag{3.1}$$

It can be reduced to the Airy equation [18,19]

$$\frac{d^2\psi(\xi)}{d\xi^2}-\xi\psi(\xi)=0, \tag{3.2}$$

with the change of variable

$$\xi = \frac{1-\lambda q}{(2\hbar\lambda)^{2/3}}\,. \tag{3.3}$$

This equation has two independent solutions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ whose integral representations are respectively [18], [19]

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[i\left(\frac{z^3}{3}+xz\right)\right]dz \tag{3.4}$$

and

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_{0}^{+\infty}\left[\exp\left(-\frac{z^3}{3}+xz\right)+\sin\left(\frac{z^3}{3}+xz\right)\right]dz\,. \tag{3.5}$$

All the solutions of the Airy equation are linear combinations of $\mathrm{Ai}(x)$ and $Bi(x)$.

In quantum mechanics (and in quantum cosmology) the tomogram $\mathcal{W}(X,\mu,\nu)$ is obtained by the square modulus of the fractional Fourier transform of the wave function,

$$\mathcal{W}(X,\mu,\nu) = \frac{1}{2\pi\hbar|\nu|}\left|\int \psi(y)\exp\left[i\left(\frac{\mu}{2\hbar\nu}y^2-\frac{X}{\hbar\nu}y\right)\right]dy\right|^2. \tag{3.6}$$

We can apply this definition to the solutions of eq. (3.1)to obtain the following tomograms,

$$\mathcal{W}_{HH}(X,\mu,\nu) = \frac{\mathrm{a}^2}{2\pi\hbar|\mu|}\left|\mathrm{Ai}\left(\frac{1}{(2\hbar\lambda)^{2/3}}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)\right)\right|^2. \tag{3.7}$$

and

$$\mathcal{W}_{V}(X,\mu,\nu) = \frac{\mathrm{b}^2}{2\pi\hbar|\mu|}\left|\mathrm{Ai}\left(\frac{1}{(2\hbar\lambda)^{2/3}}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)\right)\right.$$
$$\left.+i\mathrm{Bi}\left(\frac{1}{(2\hbar\lambda)^{2/3}}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)\right)\right|^2 \tag{3.8}$$

and

$$\mathcal{W}_{L}(X,\mu,\nu) = \frac{\mathrm{c}^2}{2\pi\hbar|\mu|}\left|\mathrm{Bi}\left(\frac{1}{(2\hbar\lambda)^{2/3}}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)\right)\right|^2 \tag{3.9}$$

which are related to the initial conditions proposed respectively by Hartle-Hawking[20], Vilenkin[21][22] and Linde[23]. We see by inspection that only the Vilenkin tomogram has as classical limit the tomogram (2.3), while the other ones do not have a limit for $\hbar \to 0$. However a deeper examination shows that also taking $\lambda \to 0$ we have the same limit. But as in the present epoch even if very small λ is not zero, we can consider the "classical" Hartle-Hawking and Linde's cosmological models as viable to the description of the universe.

4. The tomographic Wheeler-De Witt equation

We show now that it is possible to obtain these solutions directly from an equation. This equation is obtained by introducing the correspondences between operators,

$$q \to -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+i\hbar\frac{\nu}{2}\frac{\partial}{\partial X} \tag{4.1}$$

and

$$\frac{d}{dq} \to \frac{\mu}{2}\frac{\partial}{\partial X}-\frac{i}{\hbar}\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu} \tag{4.2}$$

which applied to (3.1)

$$\left(4\hbar^2\frac{d^2}{dq^2}+\lambda q-1\right)\psi(q)=0 \tag{4.3}$$

they give the following equation

$$\left(\hbar^2\mu^2\frac{\partial^2}{\partial X^2}-4i\hbar\mu\frac{\partial}{\partial\nu}-4\left(\frac{\partial}{\partial X}\right)^{-2}\frac{\partial^2}{\partial\nu^2}\right. \tag{4.4}$$

$$\left.-\lambda\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+i\frac{\hbar\lambda\nu}{2}\frac{\partial}{\partial X}-1\right)\Psi(X,\mu,\nu)=0 \tag{4.5}$$

To solve this equation we first separate the real and imaginary parts

$$4\mu\frac{\partial\Psi}{\partial\nu}=\frac{\lambda\nu}{8\mu} \tag{4.6}$$

$$\hbar^2\mu^2\frac{\partial^2\Psi}{\partial X^2}-4\left(\frac{\partial}{X}\right)^{-2}\frac{\partial^2\Psi}{\partial\nu^2}-\lambda\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial\Psi}{\partial\mu}-\Psi=0 \tag{4.7}$$

Deriving eq. (4.6) by $\partial/\partial\nu$ and substituting $\partial^2\Psi/\partial\nu^2$ in (4.7) we finally obtain the following equation

$$\hbar^2\mu^2\frac{\partial^2\Psi}{\partial X^2}-\left(1+\left(\frac{\lambda\nu}{4\mu}\right)^2\right)\Psi-\lambda\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial\Psi}{\partial\mu}-\frac{\lambda}{2\mu}\left(\frac{\partial}{\partial X}\right)^{-1}\Psi=0 \tag{4.8}$$

This equation can be solved [12] and its solutions are the fractional Fourier transforms of the wave functions with the only difference that their argument is

$$\left(\frac{1}{(\hbar\lambda)^{2/3}}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)\right) \tag{4.9}$$

instead of

$$\left(\frac{1}{(2\hbar\lambda)^{2/3}}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)\right). \tag{4.10}$$

Equation (4.8) is an integro-differential equation due to the presence of the $(\partial/\partial X)^{-1}$ operator. We notice that deriving this equation by $(\partial/\partial X)$, we can obtain a third order differential equation. on the other side we know that the squares of the Airy functions i.e. $\Phi(x)=\mathrm{Ai}^2(x)$ $\Phi(x)=\mathrm{Ai}(x)\mathrm{Bi}(x)$ and $\Phi(x)=Bi^2(x)$, are solutions of the third order equation

$$\Phi'''(x)-4x\Phi'(x)-2\Phi(x)=0\,. \tag{4.11}$$

We therefore want to proof that the equation

$$\hbar^2\mu^2\frac{\partial^3\Psi}{\partial X^3}-\frac{\lambda}{2}\Psi-\left(\frac{\lambda\nu}{4\mu}\right)^2\frac{\partial\Psi}{\partial X}-\lambda\frac{\partial\Psi}{\partial\mu}-\frac{\partial\Psi}{\partial X}=0 \tag{4.12}$$

obtained deriving eq. (4.8) by $(\partial/\partial X)$ is an equation whose solutions are the tomograms (3.7),(3.8) and (3.9).

We seek for a solution of this equation of the form

$$\Psi(z)=\frac{1}{|\mu|}\Phi(z)=\begin{cases}\frac{1}{2\pi\mu}\Phi(z), & \text{if } \mu>0\\ -\frac{1}{2\pi\mu}\Phi(z), & \text{if } \mu<0\end{cases} \tag{4.13}$$

where z is the argument the function is a function of (X,μ,ν,λ).

In both cases the equation for $\Phi(z)$ is

$$\hbar^2\mu^2\frac{\partial^3\Phi}{\partial X^3}+\frac{\lambda}{2}\Phi-\left(\frac{\lambda\nu}{4\mu}\right)^2\frac{\partial\Phi}{\partial X}-\lambda\frac{\partial\Phi}{\partial\mu}-\frac{\partial\Phi}{\partial X}=0 \tag{4.14}$$

It is straightforward to see that if we take

$$z=\frac{1-\frac{\lambda X}{\mu}-\left(\frac{\lambda\nu}{4\mu}\right)^2}{(2\hbar\lambda)^{2/3}} \tag{4.15}$$

and substitute

$$\frac{\partial}{\partial X}=\frac{\partial z}{\partial X}\frac{d}{dz}=-\frac{1}{(2\hbar\lambda)^{2/3}}\frac{\lambda}{\mu}\frac{d}{dz} \tag{4.16}$$

and

$$\frac{\partial}{\partial\mu}=\frac{\partial z}{\partial\mu}\frac{d}{dz}=\frac{1}{(2\hbar\lambda)^{2/3}}\left(\frac{\lambda X}{\mu^2}+2\left(\frac{\lambda\nu}{4\mu}\right)^2\right)\frac{d}{dz} \tag{4.17}$$

one finds finally the equation

$$\frac{d^3\Phi}{dz^3}-4z\frac{d\Phi}{dz}-2\Phi=0 \tag{4.18}$$

which coincides with equation (4.11). So we have verified that this is the equation for the tomogram in the particular case with a potential

$$V(q)=\lambda q-1. \tag{4.19}$$

5. Discussion and conclusions

In quantum mechanics there are few exact solutions to the Schrödinger equations. Very often one has to resort to various approximation techniques such as the WKB method or perturbation theory must be used to calculate the different states of a system. These techniques have also been applied to the Wheeler De Witt equation as well in many works of quantum cosmology.

In this paper we show an attempt to find an equation that gives us a tomogram as a solution without having to resort to the Radon transform or the Fractional Fourier transform that is connected to it. The result is positive as far as we consider a de Sitter universe. There are also examples in which one can obtain solutions with a square potential like in the case of the harmonic oscillator (see [3][13][25]). But for algebraic potential for higher power law apparently the order of the equation should rise and in many cases studying the problem should become prohibitive without applying any approximation techniques, not to mention problems with a generic potential.

However the situation does not appear as desperate as it may seem because on the one hand we have already seen in a previous work how to find solutions in WKB approximations, on the other hand the Airy equation can be seen as an approximation of a second order equation

$$y''+V(x)y=0 \tag{5.1}$$

when we take the expansion of $V(x)$ in the neighborhood of x_0

$$y''+\left[V(x_0)+(x-x_0)V'(x_0)\right]=0\,. \tag{5.2}$$

On the other hand it can be seen that the approximation by means of the Airy functions is uniform and therefore better than the WKB approximation.

In many cases this type of approximation allows to understand the physical aspects of a problem for a more complicated system, such as the case of a scalar field that varies in a negligible way (see for example [2]) or simply a typical wave function for the universe is a of the simpler kind [26]. Finally we can extend these models where matter fields act as relational time, the consequence will be a modification of equation 4.6 and consequently of the resulting tomogram.

References

1. D. L. Wiltshire, An Introduction to quantum cosmology, gr-qc/0101003.
2. J. J. Halliwell, Introductory Lectures On Quantum Cosmology, In *Jerusalem 1989, Proceedings, Quantum cosmology and baby universes* 159-243 and MIT Cambridge - CTP-1845 (90,rec.May) 108 p [arXiv:0909.2566 [gr-qc]].
3. V. I. Manko, G. Marmo and C. Stornaiolo, Radon transform of Wheeler-DeWitt equation and tomography of quantum states of the universe, *Gen. Rel. Grav.* **37** (2005) 99
4. V. I. Man'ko, G. Marmo and C. Stornaiolo, Cosmological dynamics in tomographic probability representation, *Gen. Rel. Grav.* **37** (2005) 2003 [gr-qc/0412091].
5. V. I. Man'ko, G. Marmo and C. Stornaiolo, Tomographic entropy and cosmology, *Gen. Rel. Grav.* **40** (2008) 1449 [gr-qc/0612073].
6. S. Capozziello, V. I. Man'ko, G. Marmo and C. Stornaiolo, Tomographic Representation of Minisuperspace Quantum Cosmology and Noether Symmetries, *Gen. Rel. Grav.* **40** (2008) 2627
7. S. Capozziello, V. I. Man'ko, G. Marmo and C. Stornaiolo, A Tomographic description for classical and quantum cosmological perturbations, *Phys. Scripta* **80** (2009) 045901
8. C. Stornaiolo, Tomographic Represention of Quantum and Classical Cosmology, *Astrophys. Space Sci. Proc.* **38** (2014) 3, 211.
9. C. Stornaiolo, Tomographic cosmology, *Phys. Scripta* **90** (2015) 7, 074032.
10. C. Stornaiolo, Tomographic analysis of the De Sitter model in quantum and classical cosmology, *Int. J. Geom. Meth. Mod. Phys.* **16** (2018), 1950012
11. C. Stornaiolo, Tomographic analysis of quantum and classical de Sitter cosmological models, *Int. J. Mod. Phys. D* **28** (2019), 2040009
12. C. Stornaiolo, Int. J. Geom. Meth. Mod. Phys. **17** (2020) no.11, 2050167 doi:10.1142/S0219887820501674 [arXiv:2007.03726 [gr-qc]].
13. V. I. Man'ko, Conventional quantum mechanics without wave function and density matrix, *AIP Conf. Proc.* **464** (1999) no.1, 191 doi:10.1063/1.58229
14. A. Ibort, V. I. Man'ko, G. Marmo, A. Simoni and F. Ventriglia, An Introduction to the tomographic picture of quantum mechanics, *Phys. Scripta* **79** (2009), 065013 doi:10.1088/0031-8949/79/06/065013
15. S. Weinberg, The Cosmological Constant Problem, *Rev. Mod. Phys*. **61** (1989) 1.
16. J. Louko, Propagation Amplitude in Homogeneous Quantum Cosmology, *Class. Quant. Grav.* **4** (1987), 581-593 doi:10.1088/0264-9381/4/3/016
17. J. J. Halliwell and J. Louko, Steepest Descent Contours in the Path Integral Approach to Quantum Cosmology. 1. The De Sitter Minisuperspace Model, *Phys. Rev. D* **39** (1989) 2206. doi:10.1103/PhysRevD.39.2206
18. Abramowitz, M. and Stegun, I. (1965). *Handbook of Mathematical Functions*, Dover Publications, New York.
19. O. Vallee, M. Soares, *Airy Functions and Applications to Physics*, Imperial College Press, (2004)
20. J. B. Hartle and S. W. Hawking, Wave Function of the Universe, *Adv. Ser. Astrophys. Cosmol.* **3** (1987), 174-189
21. A. Vilenkin, Boundary Conditions in Quantum Cosmology, *Phys. Rev. D* **33** (1986), 3560
22. A. Vilenkin, Quantum Cosmology and the Initial State of the Universe, *Phys. Rev. D* **37** (1988), 888
23. A. D. Linde, Quantum Creation of the Inflationary Universe, *Lett. Nuovo Cim.* **39** (1984), 401-405
24. J. B. Hartle, The Quantum mechanics of cosmology, In *Jerusalem 1989, Proceedings, Quantum cosmology and baby universes* 65-157.
25. V. I. Man'ko "Classical formulation of quantum mechanics" J. Russ. Laser Res., **17**, 1996, 579-584
26. G. W. Gibbons and L. P. Grishchuk, "What Is a Typical Wave Function for the Universe?," Nucl. Phys. B **313** (1989), 736-748 doi:10.1016/0550-3213(89)90405-7
27. Lapshinskii V G and Rubakov V A Quantum gravitation: quantization of the Friedmann model Theor. Math. Phys. **33**, (1977) 1076
28. Lemos N A Radiation-dominated quantum Friedmann models J. Math. Phys. **37**, (1996) ,1449