

Article

Not peer-reviewed version

---

# Thermal-Condensate Collisional Effects on Atomic Josephson Junction Dynamics

---

[Klejdja Xhani](#) \* and [Nick P. Proukakis](#)

Posted Date: 9 June 2025

doi: 10.20944/preprints202506.0676.v1

Keywords: Bose-Einstein condensates; finite temperature; Josephson effect; vortices; collisions; self-trapping; Zaremba-Nikuni-Griffin model



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a Creative Commons CC BY 4.0 license, which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

## Article

# Thermal-Condensate Collisional Effects on Atomic Josephson Junction Dynamics

Klejdja Xhani <sup>1,\*</sup> , Nick P. Proukakis <sup>2</sup> <sup>1</sup> Institute for Condensed Matter Physics and Complex Systems, DISAT, Politecnico di Torino, I-10129 Torino, Italy.<sup>2</sup> Joint Quantum Centre (JQC) Durham-Newcastle, School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne NE1 7RU, United Kingdom

\* Correspondence: klejdja.xhani@polito.it

**Abstract:** We investigate how collisional interactions between the condensate and the thermal cloud influence the distinct dynamical regimes (Josephson plasma, phase-slip-induced dissipative regime and macroscopic quantum self-trapping) emerging in ultracold atomic Josephson junctions at non-zero subcritical temperatures. Specifically, we discuss how the self-consistent dynamical inclusion of collisional processes facilitating the exchange of particles between the condensate and the thermal cloud impacts both the condensate and the thermal currents, demonstrating that their relative importance depends on the system's dynamical regime. Our study is performed within the full context of the Zaremba-Nikuni-Griffin (ZNG) formalism, which couples a dissipative Gross-Pitaevskii equation for the condensate dynamics to a quantum Boltzmann equation with collisional terms for the thermal cloud. In the Josephson plasma oscillation and vortex-induced dissipative regimes, collisions markedly alter dynamics at intermediate-to-high temperatures, amplifying damping in the condensate imbalance mode and inducing measurable frequency shifts. In the self-trapping regime, collisions destabilize the system even at low temperatures, prompting a transition to Josephson-like dynamics on a temperature-dependent timescale. Our results show the interplay between coherence, dissipation, and thermal effects in a Bose-Einstein condensate at finite temperature, providing a framework for tailoring Josephson junction dynamics in experimentally accessible regimes.

**Keywords:** Bose-Einstein condensates; finite temperature; Josephson effect; vortices; collisions; self-trapping

## 1. Introduction

A Josephson junction consists of a thin barrier or insulator separating two weakly-coupled superfluids [1–18] or superconductors [19–21]. Firstly studied with superconductors [19–21], it has been shown that a current flows across the junction due to tunneling of Cooper pairs from one side to the other of the junction. This current could be direct or alternating in the absence or presence of applied external potential, respectively. Josephson effects have been studied extensively, both theoretically and experimentally, with superfluid helium (e.g. [1–4]), trapped ultracold atomic gases [5–18,22–45] (see also [46–53]), and exciton-polariton condensates [54,55]. In particular, one way to induce the Josephson dynamics is by imposing an initial population imbalance between the two superfluids, which induces an initial acceleration of the superfluid across the junction. Previous studies at  $T = 0$  (e.g. [7,15,23,25,37,40]) have revealed the existence of different main dynamical regimes in an atomic Josephson junction: the Josephson plasma, the dissipative and the self-trapping regime. While individual works had studied transitions between two of those regimes, the 3 dynamical regimes were first clearly unified into a single Josephson phase diagram in our previous work [38]. The first and the latter regime are characterized by coherent dynamics while in the second one, phase slips, well-known from condensed matter systems (see, e.g. [1,56]), lead to the generation of topological defects (the nature of which is set by the underlying system geometry [36–38,57]) and sound waves, with the net effect of inducing dissipation of the Josephson oscillations.

The aim of the present work is to investigate the critical effect that collisions have on these 3 dynamical regimes when a condensate is coupled to a dynamical thermal cloud. Our study focuses on an elongated three-dimensional geometry, inspired by the experiment of [14,15], where the phase-slip was induced by the generation of vortex rings. In the present work we will refer to such dissipative regime as the vortex-induced dissipative regime, in order to distinguish such behaviour from the dissipation of the otherwise coherent dynamics arising purely from thermal effects.

To address this, we will use a fully self-consistent model, known as the Zaremba-Nikuni-Griffin, or 'ZNG', model [58–60]: this describes the atomic dynamics in terms of a dissipative Gross-Pitaevskii equation for the condensate, self-consistently coupled to a quantum Boltzmann equation governing the dynamics of the thermal cloud. Such description includes the full dynamical coupling between the condensate and thermal cloud, e.g. due to their coupled motion, but additionally includes the important effect of collisions which can either perturb the distribution of particles within the thermal cloud, or also transfer particles between the condensate and the thermal cloud. Beyond demonstrating the relevance of our studies [37] to experiments, and identifying the full Josephson junction dynamical regime phase diagram at  $T = 0$  [38], our previous work [39] has carefully characterised the effect of the thermal cloud dynamics on the damping and frequency shift of the main oscillation frequencies of the condensate, thermal and total population imbalance within the Josephson plasma and vortex-induced dissipation regimes: such prior study has been conducted within a simplified form of the ZNG model, in which only the self-consistent dynamical coupling of the condensate and thermal cloud was included, but all perturbing collisional effects were ignored: in other words, this corresponded to a Gross-Pitaevskii equation coupled to a collisionless Boltzmann equation. While this should in principle be a good approximation at very low temperatures, and for relatively short times, collisions are known to affect the overall system dynamics, with their effect increasing with increasing temperature [61,62].

In the present work we extend our earlier analysis in two ways: firstly, we revisit the results of [39], in order to investigate the role of such collisional processes on the system dynamics; secondly, we extend such analysis to the Macroscopic Quantum Self-Trapping regime (not previously studied within such formalism) presenting results for both collisionless and collisional implementations of the model. As such this paper presents a unified view of the role of thermal dissipation across all 3 distinct Josephson Junction dynamical regimes: Josephson Plasma (JP), Vortex-Induced Dissipative (VID) and Macroscopic Quantum Self-Trapping (MQST) regimes.

We note that the finite-temperature decay of the self-trapping regime (already experimentally observed in [11]) has also been previously studied in Ref. [34] in the context of an alternative model, known as the stochastic projected Gross-Pitaevskii equation [63]. Rather than making a clear distinction between condensate and thermal cloud as in ZNG, such model treats the low-lying modes of the system cumulatively as a classical field, while simultaneously ignoring the dynamics of high-lying modes, which are treated as a reservoir yielding a dissipative term and a stochastic noise source to the low-lying modes of the system. The relation between such models has been discussed, e.g. , in [60,64–66]. Importantly for our present work (and leaving aside the comparison of such models) we note that in carrying out their analysis, the work of Ref. [34] fixes the total particle number (up to the imposed energy cutoff) while varying temperature. Given that Josephson junction dynamics (e.g. frequencies) depend on the condensate number, whereas such approximation changes the condensate number with temperature, such work cannot allow for a systematic comparison of the frequency shifts at fixed condensate number.

Our present study focusses on the effects of the collisions on both the condensate and thermal imbalance dynamics, in all three dynamical regimes and for a temperature range  $T \leq 0.67T_c$ , which is sufficiently far from the critical point. The paper is structured as follows. In Sect. 2 we describe the numerical methods employed for this study: in Sect. 3 we show the role of thermal cloud on different dynamical regimes, while in Sect. 4 we describe our main conclusions.

## 2. Methods

We model the system at finite temperature by numerically solving the *full* Zaremba-Nikuni-Griffin (ZNG) formalism [58–60,64], explicitly including self-consistently obtained collisional terms. This model has been successfully implemented for the study of topological defect dynamics [65,67–70], the study of first and second sound in highly elongated trap [71], the study of evaporative cooling [72], collective modes [61,73–75], and mixtures [62]. It consists of separating the system into two components: the condensate wavefunction  $\psi(\mathbf{r}, t)$  and the phase-space distribution function  $f(\mathbf{r}, \mathbf{p}, t)$  representing the thermal cloud. As a result, two coupled equations are solved self-consistently; the dissipative Gross-Pitevskii equation (DGPE) and the Quantum Boltzmann equation (QBE) describing the condensate and thermal cloud dynamics, respectively. The time-dependent DGPE can be written as

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + V_{\text{ext}}(\mathbf{r}) + g(n_{\text{BEC}}(\mathbf{r}, t) + 2n_{\text{th}}(\mathbf{r}, t)) - iR(\mathbf{r}, t) \right] \psi(\mathbf{r}, t) \quad (1)$$

where  $M$  is the particle mass (here  $^6\text{Li}$  molecule),  $V_{\text{ext}}$  the external trapping potential,  $g = 4\pi\hbar^2 a/M$  the interaction strength with  $a$  the  $s$ -wave scattering length between molecules,  $n^{\text{th}}$  the thermal cloud density and  $n^{\text{BEC}} = |\psi|^2$  the condensate density. The rightmost term within the bracket,  $-iR(\mathbf{r}, t)$ , represents the dissipative or source (or ‘growth’) which facilitates the condensate number to change in time due to particle exchange with the thermal cloud.

The corresponding equation for the thermal cloud is the Quantum Boltzmann equation for the Wigner phase-space distribution function  $f$  in the form:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f - (\nabla_{\mathbf{r}} V_{\text{eff}}^{\text{th}}) \cdot (\nabla_{\mathbf{p}} f) = C_{12}[f, \psi] + C_{22}[f] \quad (2)$$

where  $V_{\text{eff}}^{\text{th}} = V_{\text{ext}} + 2g(n^{\text{BEC}} + n^{\text{th}})$  is the generalized mean-field potential felt by the thermal particles and  $n^{\text{th}}$  the thermal cloud density obtained through the phase-space distribution  $f$  as  $n^{\text{th}}(\mathbf{r}, t) = 1/(2\pi\hbar)^3 \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t)$ . Here, the first collisional term includes the collisions between the thermal and condensate particles which cause particle transfer between the two subsystems, while the second term includes collisions between the thermal particles which re-distribute the thermal atoms between the single-particle states. In the collisional limit considered here, even though the total number is numerically well conserved, the condensate number ( $N_{\text{BEC}}(t)$ ) instead changes in time due to the particle exchange between the condensate and the thermal cloud. This is taken into account in the  $C_{12}$  collisional integral in the Quantum Boltzmann equation; the latter is related to the source term  $-iR$  in the dissipative GPE equation via  $R(\mathbf{r}, t) = (\hbar/2n_{\text{BEC}}) \int d\mathbf{p} / (2\pi\hbar)^3 C_{12}(\mathbf{p}, \mathbf{r}, t)$ . As a consequence, the thermal cloud particle number,  $N^{\text{th}}(t)$ , also changes in time.

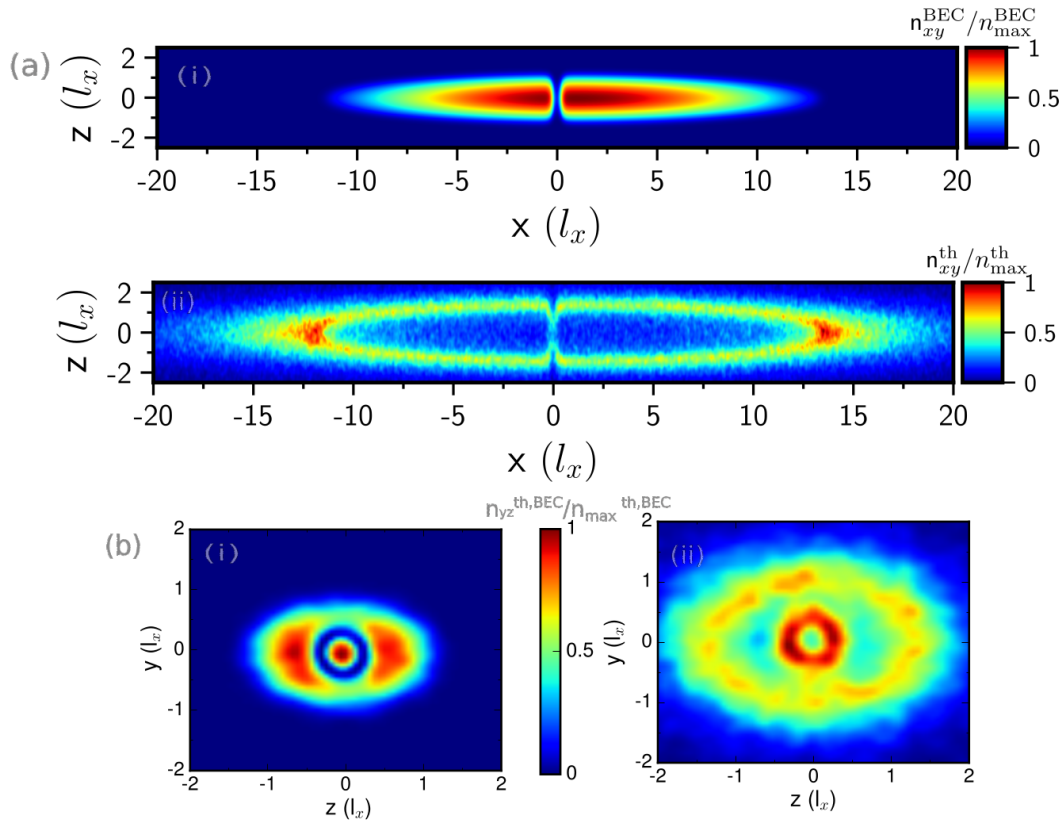
The geometry considered in our study is an anisotropic elongated 3D harmonic trap, with a double-well potential of the form

$$V_{\text{ext}}(x, y, z) = \frac{1}{2} M (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) + V_0 e^{-2x^2/w^2}. \quad (3)$$

The trap frequency values are  $\omega_x = 2\pi \times 15$  Hz,  $\omega_y = 2\pi \times 148$  Hz, and  $\omega_z = 2\pi \times 187.5$  Hz along the  $x$ ,  $y$  and  $z$  directions, respectively, chosen to coincide with the experimental values of Ref. [14,15]. The parameters  $V_0$  and  $w$  are the Gaussian barrier height and  $1/e^2$  width, respectively. Our numerical studies are conducted in a grid of  $[-48, 48]l_x, [-8, 8]l_x, [-8, 8]l_x$  along the  $x$ ,  $y$  and  $z$  directions respectively, where  $l_x = \sqrt{\hbar/M\omega_x}$ , based on  $2048 \times 128 \times 128$  grid points for the thermal cloud. For the less spatially extended condensate, we use a corresponding half grid of size  $[-24, 24]l_x$ , with 1024 grid points along the  $x$  axis. We initially find the equilibrium state by adding the tilted linear potential  $-\epsilon x$  to the  $V_{\text{ext}}(x, y, z)$ . A typical 2D integrated equilibrium profile is shown in Figure 1(a) for the condensate (i) and the thermal cloud (ii). As evident in this plot, the presence of the repulsive



interaction between the condensate and the thermal cloud leads to the thermal particles populating regions of lower condensate density, i.e. at the edges of the condensate, or also in the barrier region.



**Figure 1.** (a) Equilibrium 2D integrated condensate (i) and thermal (ii) cloud density in the xz-plane. (b) The condensate (i) and thermal density along the yz-plane (ii) extracted at the position of the vortex ring along the x-axis at a fixed time during dynamics. In both cases the thermal cloud populates the edges of the condensate, and (b)(ii) shows that the thermal cloud populates also the vortex core, in addition to the edges of the condensate. The data are obtained at  $T = 0.5T_c$  for fixed condensate number  $N_{\text{BEC}} \simeq 5.04 \times 10^4$  and for barrier height  $V_0/\mu(T=0) \simeq 1$  (a) and  $V_0/\mu(T=0) \simeq 0.6$  (b). All densities are scaled to their corresponding maximum values, as indicated by the color bars.

To initiate the dynamics, we instantaneously set the parameter  $\epsilon$  to zero, thus creating an initial population imbalance,  $z_0$ , between the two wells. Our earlier work in the limit of a pure  $T = 0$  condensate revealed that different dynamical regimes emerge depending on the barrier parameters [38]: for relatively small barrier widths or heights the system transits from Josephson plasma (JP) to vortex-induced dissipative regime (VID) with increasing  $z_0$ , as soon as it exceeds a critical value  $z^c$ . In the opposite limit of relatively large barrier height or width the system transits from the Josephson plasma to the macroscopic quantum self-trapping regime (MQST) for  $z_0$  beyond the self-trapping critical value  $z^{st}$ . For our studies here we fix the Gaussian barrier width at  $w = 3.8\zeta$ , where  $\zeta = \hbar/\sqrt{2\mu(T=0)M} = 0.52\mu\text{m}$  denotes the condensate healing length, and change the barrier height. To probe the effect of collisions on these dynamical regimes in the clearest manner, we consider two values for the main barrier height: (i)  $V_0 = 104\hbar v_x \simeq \mu(T=0)$  for the JP-VID transition, and (ii)  $V_0 = 210\hbar v_x \simeq 2\mu(T=0)$  for the JP-MQST transition. For each dynamical regime, when comparing the results at different temperatures the barrier height value,  $V_0$  is kept fixed. In order to avoid the possibility of a transition to a different dynamical regime while increasing temperature [found to occur for fixed total number in [39]], all finite temperature studies presented here are conducted at fixed condensate particle number,  $N^{\text{BEC}} \simeq 5 \times 10^4$ . This means that at different temperatures, the total particle number takes different values, increasing with increasing temperature, in order to keep  $N^{\text{BEC}}$  fixed.

We define the condensate  $z_{\text{BEC}}(t)$ , the thermal cloud  $z_{\text{th}}(t)$  and the total  $z_{\text{tot}}(t)$  population imbalances respectively as:

$$z_{\text{BEC}}(t) = \frac{N_{\text{BEC}}^R(t) - N_{\text{BEC}}^L(t)}{N_{\text{BEC}}(t)} \quad (4)$$

$$z_{\text{th}}(t) = \frac{N_{\text{th}}^R(t) - N_{\text{th}}^L(t)}{N_{\text{th}}(t)} \quad (5)$$

$$z_{\text{tot}}(t) = \frac{N^R(t) - N^L(t)}{N_{\text{tot}}} \quad (6)$$

where  $N_{\text{BEC}}^{R/L}$  and  $N_{\text{th}}^{R/L}$  are the number of the condensate and thermal particles on the right/left sides of the barrier, respectively, while  $N_{\text{BEC}}$  and  $N_{\text{th}}$  are the total thermal and condensate particle number. The presence of a nonzero population imbalance both for the condensate and the thermal particle means that we have a condensate and thermal (or normal) current in the system, with the current extracted as  $I = -(N_{\text{BEC}/\text{th}}/2)(dz_{\text{BEC}/\text{th}}/dt)$  large  $\hbar$ .

While a single numerical run in the absence of collisions generally accurately models the system dynamics, for a systematic monitoring of the results in the collisional case we generally need to consider a number of (typically five) simulations corresponding to the same physical conditions. Although in the JP and VID regimes the collisional results are found to exhibit negligible variation from run to run, in the MQST dynamics the self-trapping decay time is much more sensitive; in the latter case we thus also perform multiple MQST runs even for the collisionless regime (in addition to also doing so for the collisional regime).

The condensate imbalance data are analyzed by implementing a two-component fit of the form:

$$F(t) = a_J \cos(2\pi\nu_J t + \phi_J) \exp(-\gamma_J t) + a_i \cos(2\pi\nu_i t + \phi_i) \exp(-\gamma_i t) \quad (7)$$

with  $a_{J/i}$  and  $\nu_{J/i}$  being the amplitude and frequency values of each component respectively, while  $\gamma_{J/i}$  are the damping rates, and  $\phi_{J/i}$  the phases of each component.

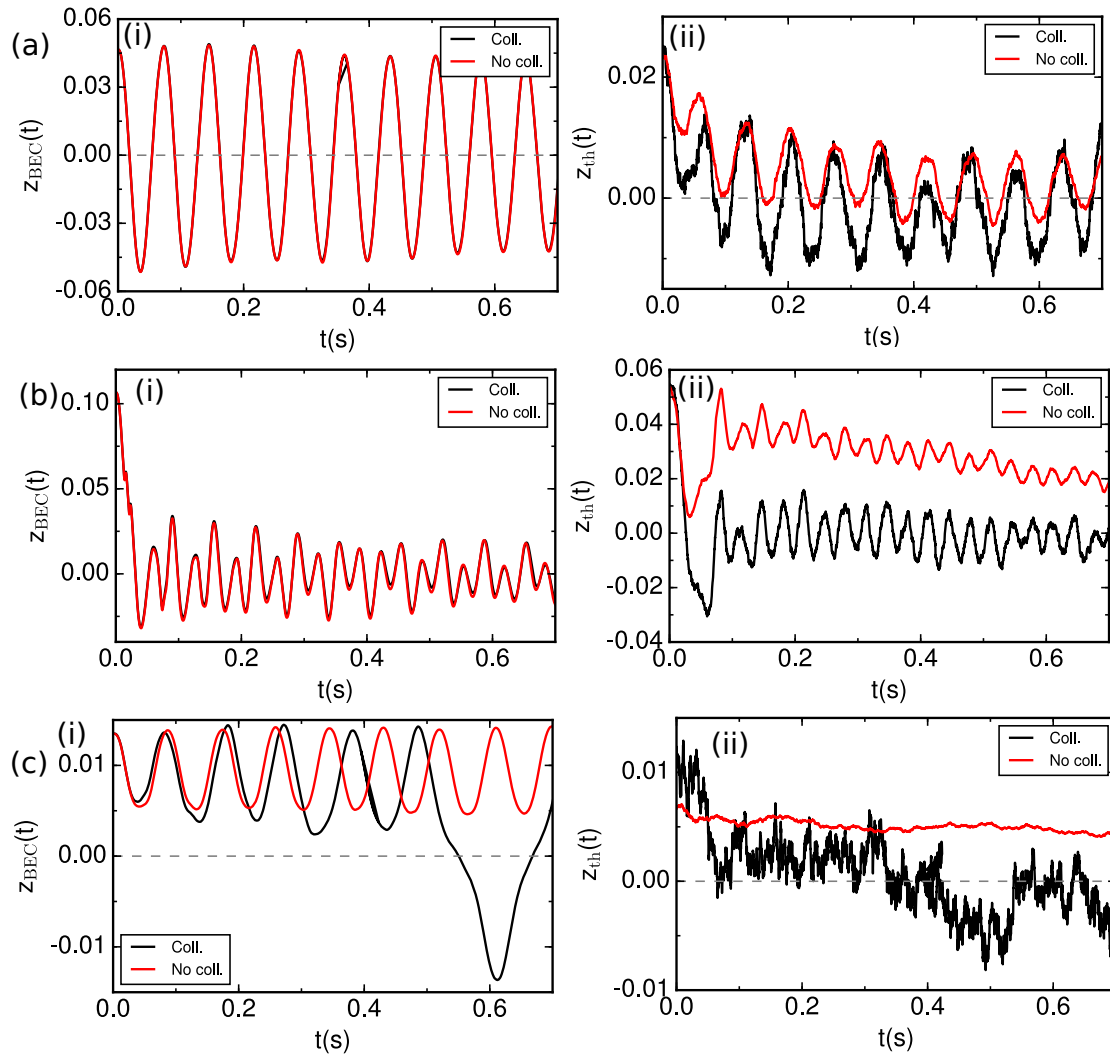
### 3. Results

In this section we will discuss the role of the thermal cloud dynamics and, in particular, the importance of collisions in each of the 3 dynamical Josephson junction regimes. This is shown in Figs. 2-4 for 3 different non-zero temperatures, with each figure showing the dynamics of the condensate (left column) and the thermal cloud (right column) without (red lines) and with (black lines) collisions, and for each of the 3 different regimes – from top to bottom showing Josephson Plasma, Vortex Induced Dissipative and Macroscopic Quantum Self-Trapping regimes. For reference, their corresponding  $T = 0$  dynamics have been previously discussed at length in Ref. [38].

Specifically, at  $T = 0$ , the dynamics in the Josephson plasma regime exhibit coherent oscillations about a zero value with constant amplitude in both the condensate population imbalance (or its time derivative, which gives the current) and the Josephson phase (i.e. the phase difference between the two parts of the condensate). Such oscillations persist at relatively low temperatures  $T \ll T_c$  (where  $T_c$  is the critical temperature), with only very minor damping and collisions having a negligible effect. This can be seen for the specific case of  $T = 0.22T_c$  in Figure 2(a,i).

In the vortex-induced dissipative (VID) regime, when the dynamics following the initial population imbalance exceeds the local speed of sound, the system is known to generate (potentially successive) vortices, each accompanied by a  $2\pi$  phase jump in the relative phase and some sound emission [37]. In the particular geometry, such vortices appear in the form of vortex rings in the transverse plane: such vortex rings can enter and propagate within the bulk condensate and be subsequently observed experimentally, as discussed in [37]. An image of such a vortex ring can be seen in Figure 1(b)

showing both the condensate density containing the vortex ring (i) and the distribution of the thermal cloud (ii) which is maximised within the low-density region of the vortex ring core, and around the condensate. This feature, dominating the dynamics already at  $T = 0$  [38], significantly perturbs the subsequent condensate motion, followed by significantly reduced amplitude sinusoidal oscillations of the imbalance around a zero mean value. We observe similar behavior at  $T = 0.22T_c$  of  $z_{\text{BEC}}(t)$  in both the collisionless and collisional limits [Figure 2(b,i)], with collisions having no significant effect.



**Figure 2.** The temporal evolution of the condensate (i) and thermal (ii) population imbalance in the Josephson 'plasma' regime (JP) (a), in the vortex-induced dissipative regime (VID) (b) and in the macroscopic quantum self-trapping (MQST) regime (c) at  $T = 0.22T_c$ , both in the collisionless (red line) and collisional limit (black line). The data in (a)-(b) are both for barrier height  $V_0 \simeq \mu(T = 0)$  and initial condensate imbalance  $z_{\text{BEC}}(t = 0) < z_{\text{BEC}}^c$  (a) and  $z(t = 0)_{\text{BEC}} > z_{\text{BEC}}^c$  (b) where  $z_{\text{BEC}}^c$  is the critical imbalance defining the transition from JP to VID regime. The data in (c) are for  $V_0 = 2\mu(T = 0)$  and  $z_0 = z_{\text{BEC}}^{\text{ST}}$  where  $z_{\text{BEC}}^{\text{ST}}$  is the initial population imbalance for which the system transits from JP to MQST regime.

In contrast, the macroscopic quantum self-trapping (MQST) regime at  $T = 0$  is characterized by small-amplitude oscillations around a nonzero mean value and  $2\pi$  phase jumps in the relative phase, but without the appearance of propagating vortices and sound waves [38]. In the collisionless limit, the condensate imbalance continues to oscillate around a nonzero mean value at low temperature, as shown in Figure 2(c,i) (red line), with only a very gradual increase in its oscillation amplitude with time. However, when collisions are included, a transition to a different dynamical regime at around  $t = 0.5$ s for our specific parameters (see Figure 2(c,i), black line): such regime sees the imbalance oscillate around a zero time-average value, resembling the anharmonic oscillations regime characteristic of

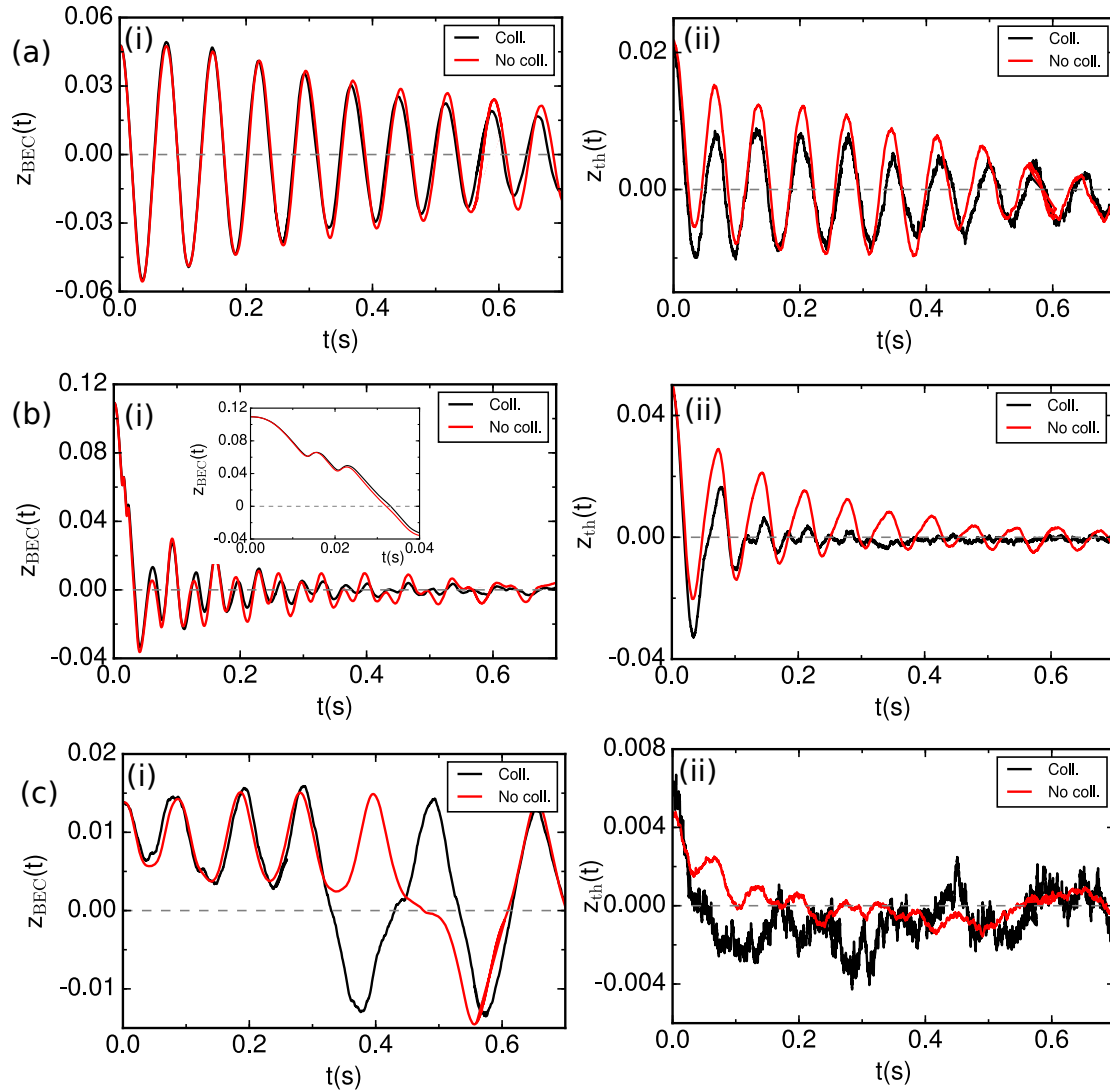
the transition to self-trapping [25]. Thus, inclusion of collisions in the model are in fact essential for accurately describing the self-trapping regime even at low temperatures.

The MQST regime is highly unstable in the presence of a thermal cloud, and its decay time can vary between runs, even in the collisionless limit. Nevertheless, at  $T = 0.22T_c$ , the imbalance  $z_{\text{BEC}}(t)$  in all five collisionless runs conducted did not decay within the maximum explored time interval of 0.72 s. In contrast, in the collisional limit, MQST decayed in all runs, even at this low temperature.

Collisions facilitate thermal equilibration between the two wells across all dynamical regimes, as evidenced in Figure 2(a)-(c)(ii). In both Josephson and dissipative regimes – particularly the latter – the thermal imbalance oscillates around a non-zero mean value under collisionless conditions. Collisions suppress this offset, shifting the equilibrium state toward zero thermal imbalance. This contrasts with previous collisionless studies [39], where low-temperature thermal particles relied solely on incoherent tunneling through the barrier. Our results demonstrate that collisional processes, specifically condensate-thermal particle exchange, enhance such equilibration. The MQST regime presents unique observational challenges due to its minimal initial thermal imbalance ( $z_0^{\text{th}} \simeq 0.01$ ). This small signal amplitude creates a high noise-to-signal ratio, rendering thermal imbalance oscillations nearly indistinguishable despite their physical presence. The suppression of measurable oscillations persists even when accounting for collisional effects in this regime.

Next we consider an intermediate temperature  $T = 70\text{nK} \approx 0.5T_c$ , selected as the limiting value at which the barrier height practically becomes equal to the temperature, i.e.  $V_0 = 104\text{ }h\nu_x \simeq k_B T$ . Figure 3 presents the time evolution of both the condensate imbalance  $z_{\text{BEC}}(t)$  and the thermal imbalance  $z_{\text{th}}(t)$  across all three regimes at this temperature. In the Josephson (Figure 3(a)) and dissipative regimes (Figure 3(b)), the  $z_{\text{BEC}}(t)$  profiles reveal two key features. First, increasing temperature leads to stronger damping of the condensate imbalance oscillations, consistent with previous findings [39]. In fact, the presence of normal current is expected to damp the Josephson oscillations. Second, collisions further enhance this damping effect, particularly in the dissipative regime, as will be discussed in detail later.



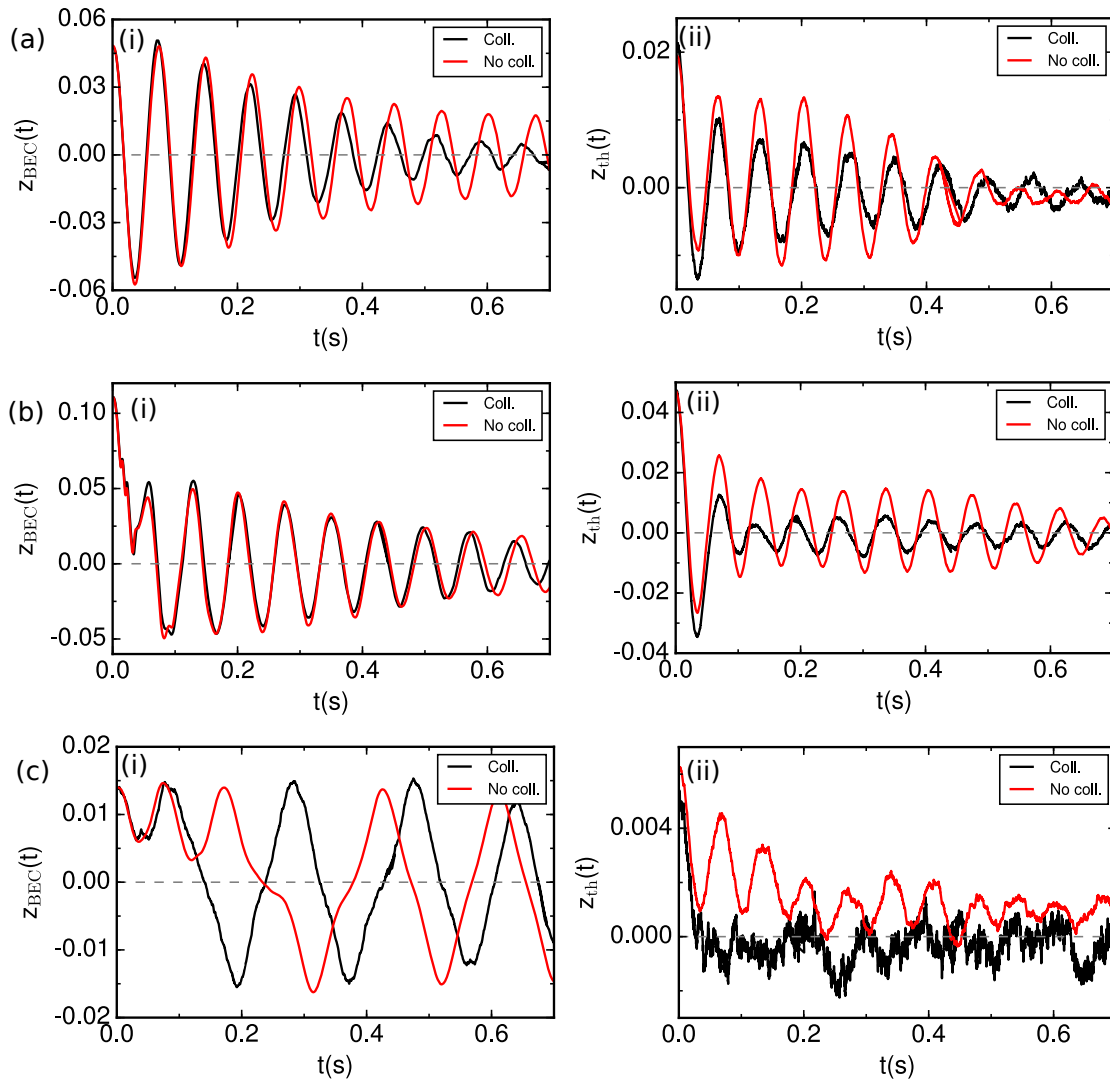


**Figure 3.** Same as in Figure 2, but now for  $T = 0.5T_c \lesssim V_0/k_B$ , marking the highest temperature for which the thermal cloud could not overcome the barrier for the barrier height value in (a) and (b). The inset in (b)-(i) shows a zoomed-in part of the condensate imbalance temporal evolution.

Similarly, collisions significantly reduce the amplitude of  $z_{th}(t)$  oscillations compared to the lower temperature case of  $T = 0.22T_c$ , with the thermal imbalance becoming almost completely damped by  $t = 0.4$  s in the dissipative regime (see Figure 3(b)-(ii)). The inset in Figure 3(b)-(i) shows a zoomed-in view of the early-time evolution of  $z_{BEC}(t)$ , where two distinct ‘dips’ or ‘kinks’ appear even in the collisionless limit. These features are signatures of backflow caused by vortex ring generation, as described in [37]. Notably, collisions have little impact on the timing of these dips, which correspond to the vortex ring entering the local Thomas-Fermi surface, and they slightly influence the decay time (i.e. the time when  $z_{BEC}(t)$  assumes zero value). At this higher  $T$  in the MQST regime, the condensate imbalance decay time and temporal profile differs from run to run and these differences are even more pronounced in the collisional limit (as will be shown later). Therefore, the particular results shown in Figure 3(c) represent just one characteristic example among a range of possible behaviors.

Figure 4 instead shows the profiles of  $z_{BEC}(t)$  and  $z_{th}(t)$  at  $T = 0.58T_c$ , for which  $T > V_0$ . We have previously shown [39] that such a condition implies the thermal cloud can now pass over the barrier, thus able to execute its own independent oscillations. This new motion can in turn significantly affect the condensate dynamics. In the Josephson plasma regime (Figure 4(a)), collisions cause significantly increased damping of both (i) the condensate imbalance  $z_{BEC}(t)$  and (ii) the thermal imbalance  $z_{th}(t)$  compared to the collisionless case. Interestingly, in the dissipative regime (Figure 4(b)), the condensate

imbalance starts oscillating with a single dominant frequency, driven by the oscillating thermal component. Thus, in the collisionless limit and in the VID regime, the second component disappears from the condensate imbalance spectrum at temperature  $T_1 = V_0/k_B$ . In this regime, collisions have only a minor effect on  $z_{\text{BEC}}$  (Figure 4(b)-(i)), while they strongly influence the thermal imbalance  $z_{\text{th}}$  (Figure 4(b)-(ii)). The MQST regime is included as an example of possible imbalance profiles, but it is important to note that these profiles can vary substantially from run to run.



**Figure 4.** Same as in Figs. 2-3, but now for  $T = 0.58T_c > V_0/k_B$ , such that the thermal cloud can now overcome the barrier on its own for parameters in (a)-(b). As such, it suppresses the condensates' sound excitations, while also allowing the thermal cloud to drive the condensate.

### 3.1. Effect of Collisions on the Frequency and Damping of Dominant Modes in the Josephson Plasma and Vortex-Induced Dissipative Regimes

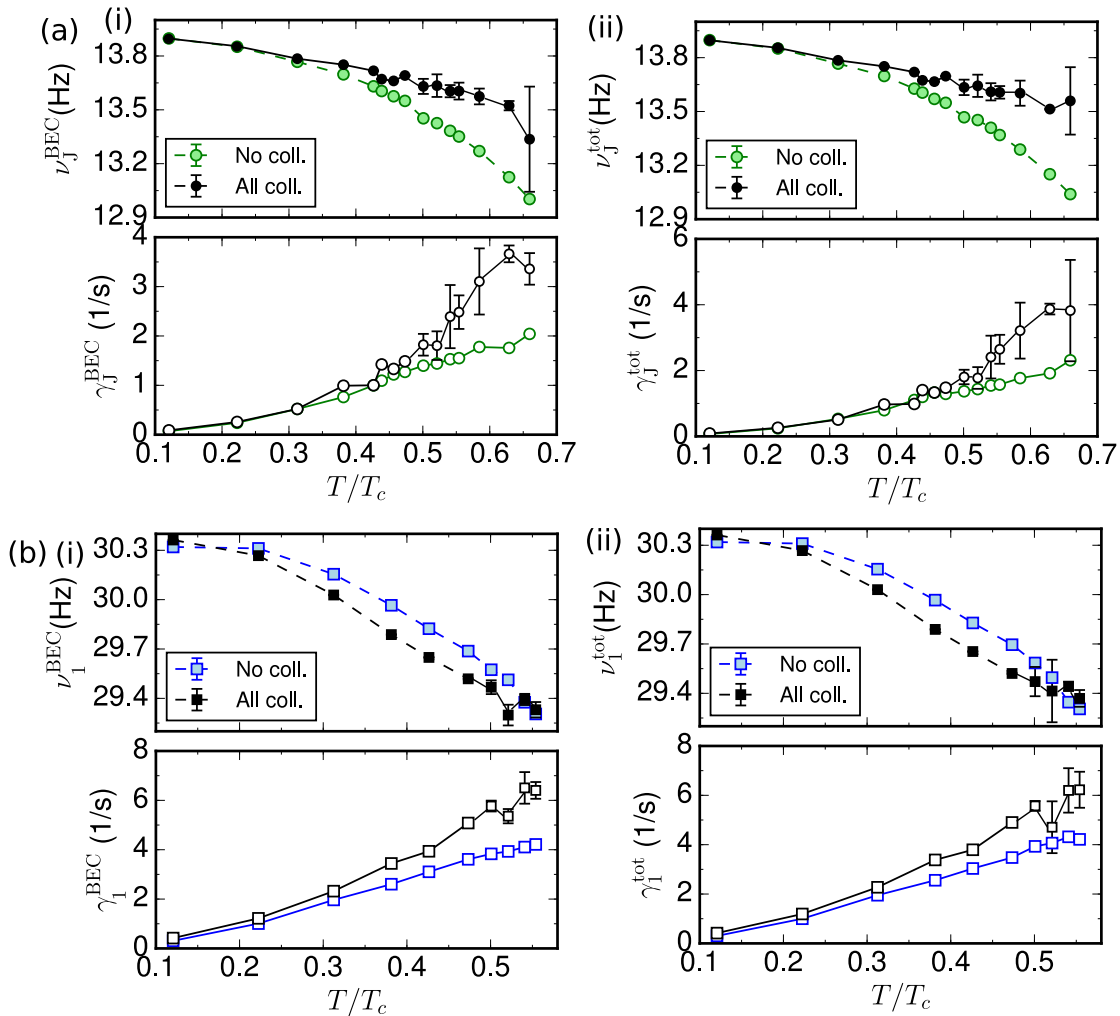
For each of the Josephson and dissipative regime, we fit the  $z_{\text{BEC}}(t)$  and  $z_{\text{tot}}(t)$  at each of the five runs with two-component frequency fit and take their mean value as best estimate of our results, with the error bars indicating the maximum deviation.

Consistent with the collisionless results of Ref. [39], the dominant frequencies of both JP and VID at  $T \lesssim V_0/k_B \approx 0.5T_c$  correspond to the Josephson plasma frequency  $\nu_j^{\text{BEC}}$  and the frequency  $\nu_1^{\text{BEC}} \approx 2\nu_j^{\text{BEC}}$  (attributed to second-order term in the tunneling hamiltonian). While the Josephson frequency completely dominates the BEC dynamics in the JP regime, the opposite is true in the VID regime, due to the significant sound energy emitted [38]. At the other extreme, for  $T \approx 0.58T_c > V_0/k_B$ , the thermal cloud starts exhibiting its own oscillations over the barrier: such motion emerges in

a pronounced way in both dynamical regimes, with the sound waves becoming simultaneously significantly suppressed. In such a limit, we previously found [39] that the system dynamics can once again be described by two dominant frequencies, only now these are the Josephson frequency and a frequency associated with the thermal cloud motion, at a frequency approaching the trap frequency.

It is thus natural to ask how this picture is affected by the presence of collisions: The first observation to make here is that the addition of collisions appears to slightly shift the temperature at which the transition between the low and high temperature behaviours emerges to slightly lower values.

The results of the main component frequency value and damping rates as a function of temperature are shown in Figure 5. In particular, Figure 5(a) shows the dominant frequency value (upper subplots) and damping rates (lower subplots) of the condensate and total imbalance in the Josephson ‘plasma’ regime. In both cases the main frequency has a similar value and close to the expected Josephson plasma frequency; this is why it is called  $\nu_J^{\text{BEC}}$  and  $\nu_J^{\text{tot}}$ . Figure 5(b) instead shows the corresponding plot for the dissipative regime where now the dominant frequency for  $0 < T \leq 0.58T_c$  becomes  $\nu_1^{\text{BEC}}$  for the condensate imbalance and  $\nu_1^{\text{tot}}$  for the total imbalance. In the collisionless limit, the increased temperature shifts towards lower values both the condensate and total imbalance dominant frequency while enhancing their damping due to repulsive interaction with thermal component as in Ref. [39].

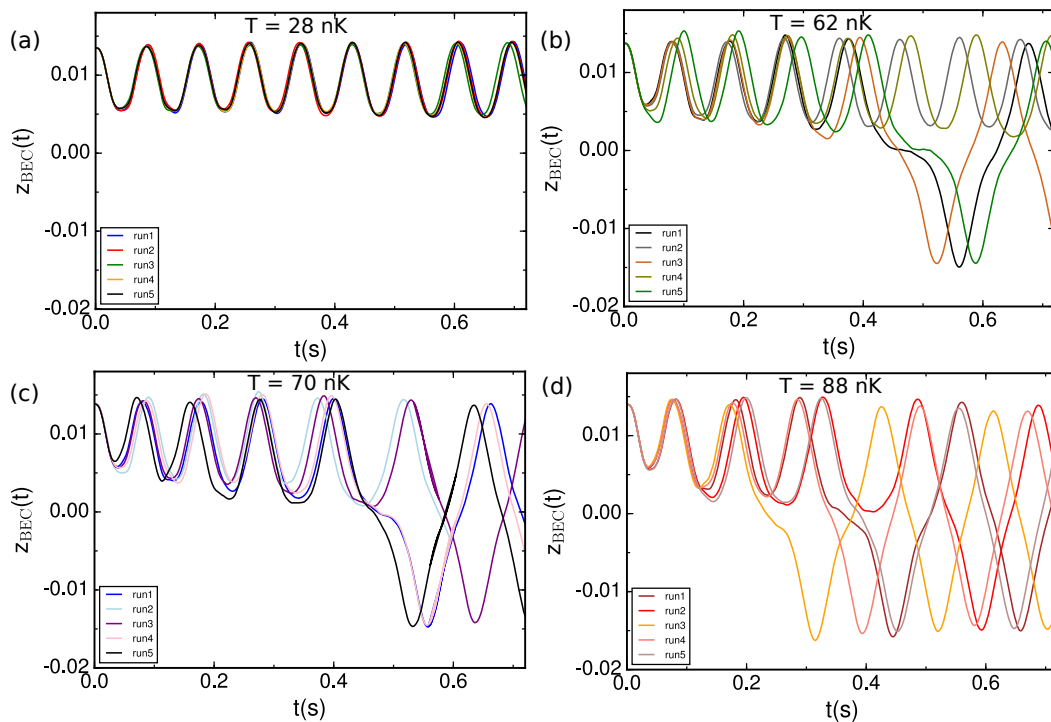


**Figure 5.** The dominant frequency profile as a function of the temperature and the corresponding damping rates in the Josephson (a) and dissipative regime (b) in both the collisionless and collisional limit. In (a) the mean values shown in the collisional case are the results from five runs.

In the presence of collisions, two main effects are noted. Firstly, we observe that the inclusion of collisions affects differently the dominant frequency in the Josephson and dissipative regime both for the condensate and total imbalance. In fact, in the former case collisions soften the effect of the decrease of  $\nu_j^{\text{BEC}}$  and  $\nu_j^{\text{tot}}$  with temperature. For  $T < 0.5T_c$  the damping rate in the collisional limit has similar value to the collisionless limit but when the thermal cloud starts having more of an effect and exhibiting its own motion, i.e. for  $T > 0.5T_c = V_0/k_B$ , the damping rate  $\gamma_j^{\text{BEC}}$  and  $\gamma_j^{\text{tot}}$  becomes larger in the collisional limit. In the dissipative regime instead, the collisions makes the  $\nu_1^{\text{BEC}}$  (i) and  $\nu_1^{\text{tot}}$  (ii) achieving smaller values with respect to the collisionless limit (for the same  $T/T_c$ ) but with a profile similar to the collisionless limit. Moreover, the damping rates  $\gamma_1^{\text{BEC}}$  and  $\gamma_1^{\text{tot}}$  are larger in the collisional limit almost the entire range of  $T$ .

### 3.2. Role of Collisions on the Macroscopic Quantum Self-Trapping

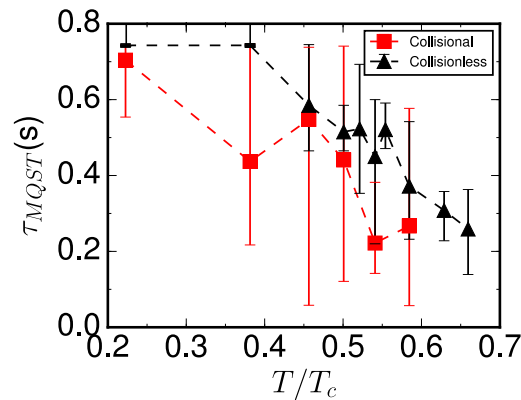
Next we consider the thermally-induced dissipation of Macroscopic Quantum Self-Trapping. The presence of the thermal component induces a normal Ohmic current, i.e. a normal conductance [34] in the system by causing dissipation and decay of the condensate imbalance or current oscillations in the self-trapping regime. This decay time can vary significantly from run to run, even in the collisionless limit, as illustrated in Figure 6 for four different temperatures: 28 nK ( $0.22T_c$ ), 62 nK ( $0.44T_c$ ), 70 nK ( $0.5T_c$ ), and 88 nK ( $0.58T_c$ ). At the lowest temperature (28 nK), all runs remain in the self-trapping regime within our probed temporal regime, as shown in Figure 6(a). However, as the temperature increases, the influence of the thermal cloud becomes more pronounced, leading to a decay of the self-trapping regime. This behaviour is evident at higher temperatures, where the imbalance begins to oscillate around zero, indicating a transition to a different dynamical regime.



**Figure 6.** The temporal profile of the condensate imbalance in the collisionless limit at four different temperatures and for five independent numerical runs. The data are for  $V_0 = 210\hbar\omega_x$  and  $N_{\text{BEC}}(t=0) = 50400$ .

To quantify this decay process within our simulated results, we extract the decay time from each run and calculate the mean value and associated error bars for both the collisional and collisionless limits, with the results presented in Figure 7. In the collisionless regime, we find that for temperatures below  $0.4T_c$ , the self-trapping state remains stable throughout the observed time interval (0.72 s). As the temperature increases, the self-trapping regime decays progressively earlier, with the decay time

decreasing with temperature. In contrast, in the collisional regime, self-trapping is unstable across the entire temperature range, with a decay time which is shorter than the corresponding time in the collisionless regime. These findings highlight the significant impact of temperature and interactions on the stability of the self-trapping regime.



**Figure 7.** The profile of the mean decay time of the condensate imbalance in the collisionless and collisional limit as a function of the temperature. The data are for  $V_0 = 210\hbar\omega_x$  and  $N_{\text{BEC}}(t = 0) = 50400$ .

#### 4. Conclusions

We investigated the impact of thermal-condensate collisions on the dynamical regimes of a bosonic Josephson junction by numerically solving the full Zaremba-Nikuni-Griffin (ZNG) model, which accounts self-consistently for all collisional processes at finite temperatures. We analyzed how temperature influences the frequency and damping of the dominant oscillation modes of both the condensate and total current in the Josephson plasma (JP) and vortex-induced dissipative (VID) regimes, as well as the decay time in the macroscopic quantum self-trapping (MQST) regime. Our results show that the decay time in the MQST regime is highly sensitive to collisions even at low temperatures, decreasing as temperature rises. In contrast, collisions in the JP and VID regimes primarily increase damping of the oscillation frequency. Notably, collisions cause a shift toward higher values of oscillation frequency in these regimes. Overall, our study demonstrates that collisions start playing a crucial role in describing the JP and VID regimes at intermediate to high temperatures, while they are already significant for the MQST regime at low temperatures. It would be interesting to compare our predictions against well-calibrated experimental data in the future.

**Author Contributions:** Conceptualization of the study K.X. and N.P.P and numerical simulations and analysis K.X. All authors contributed to the interpretation of the results and to the writing of the manuscript. All authors have read and agreed to the published version of the manuscript.

**Data Availability Statement:** Data presented in this paper are available from the corresponding author upon reasonable request.

**Conflicts of Interest:** The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

#### Abbreviations

The following abbreviations are used in this manuscript:



BEC	Bose-Einstein condensate
GPE	Gross-Pitaevskii equation
ZNG	Zaremba-Nikuni-Griffin
JP	Josephson Plasma
VID	Vortex-Induced Dissipation
MQST	Macroscopic Quantum Self-Trapping

## References

1. Avenel, O.; Varoquaux, E. Josephson effect and quantum phase slippage in superfluids. *Phys. Rev. Lett.* **1988**, *60*, 416–419. <https://doi.org/10.1103/PhysRevLett.60.416>.
2. Davis, J.C.; Packard, R.E. Superfluid  $^3\text{He}$  Josephson weak links. *Rev. Mod. Phys.* **2002**, *74*, 741–773. <https://doi.org/10.1103/RevModPhys.74.741>.
3. Sato, Y.; Hoskinson, E.; Packard, R. Josephson Effects in Superfluid Helium. In *Fundamentals and Frontiers of the Josephson Effect*; Tafuri, F., Ed.; Springer, Cham, 2019; Vol. 286.
4. Hoskinson, E.; Sato, Y.; Hahn, I.; Packard, R.E. Transition from phase slips to the Josephson effect in a superfluid  $4\text{He}$  weak link. *Nat. Phys.* **2006**, *2*, 23–26.
5. Anderson, B.P.; Kasevich, M. Macroscopic quantum interference from atomic tunnel arrays. *Science* **1998**, *282*, 1686.
6. Cataliotti, F.S.; Burger, S.; Fort, C.; Maddaloni, P.; Minardi, F.; Trombettoni, A.; Smerzi, A.; Inguscio, M. Josephson Junction Arrays with Bose-Einstein Condensates. *Science* **2001**, *293*, 843–846. <https://doi.org/10.1126/science.1062612>.
7. Albiez, M.; Gati, R.; Fölling, J.; Hunsmann, S.; Cristiani, M.; Oberthaler, M.K. Direct Observation of Tunneling and Nonlinear Self-Trapping in a Single Bosonic Josephson Junction. *Phys. Rev. Lett.* **2005**, *95*, 010402. <https://doi.org/10.1103/PhysRevLett.95.010402>.
8. Anker, T.; Albiez, M.; Gati, R.; Hunsmann, S.; Eiermann, B.; Trombettoni, A.; Oberthaler, M.K. Nonlinear Self-Trapping of Matter Waves in Periodic Potentials. *Phys. Rev. Lett.* **2005**, *94*, 020403. <https://doi.org/10.1103/PhysRevLett.94.020403>.
9. Schumm, T.; Hofferberth, S.; Andersson, L.M.; Wildermuth, S.; Groth, S.; Bar-Joseph, I.; Schmiedmayer, J.; Krüger, P. Matter-wave interferometry in a double well on an atom chip. *Nature Physics* **2005**, *1*, 57–62. <https://doi.org/10.1038/nphys125>.
10. Shin, Y.; Saba, M.; Pasquini, T.A.; Ketterle, W.; Pritchard, D.E.; Leanhardt, A.E. Atom Interferometry with Bose-Einstein Condensates in a Double-Well Potential. *Phys. Rev. Lett.* **2004**, *92*, 050405. <https://doi.org/10.1103/PhysRevLett.92.050405>.
11. Levy, S.; Lahoud, E.; Shomroni, I.; Steinhauer, J. The a.c. and d.c. Josephson effects in a Bose-Einstein condensate. *Nature* **2007**, *449*, 579–583. <https://doi.org/10.1038/nature06186>.
12. Spagnolli, G.; Semeghini, G.; Masi, L.; Ferioli, G.; Trenkwalder, A.; Coop, S.; Landini, M.; Pezzè, L.; Modugno, G.; Inguscio, M.; et al. Crossing Over from Attractive to Repulsive Interactions in a Tunneling Bosonic Josephson Junction. *Phys. Rev. Lett.* **2017**, *118*, 230403. <https://doi.org/10.1103/PhysRevLett.118.230403>.
13. LeBlanc, L.J.; Bardon, A.B.; McKeever, J.; Extavour, M.H.T.; Jervis, D.; Thywissen, J.H.; Piazza, F.; Smerzi, A. Dynamics of a Tunable Superfluid Junction. *Phys. Rev. Lett.* **2011**, *106*, 025302. <https://doi.org/10.1103/PhysRevLett.106.025302>.
14. Valtolina, G.; Burchianti, A.; Amico, A.; Neri, E.; Xhani, K.; Seman, J.A.; Trombettoni, A.; Smerzi, A.; Zaccanti, M.; Inguscio, M.; et al. Josephson effect in fermionic superfluids across the BEC-BCS crossover. *Science* **2015**, *350*, 1505–1508. <https://doi.org/10.1126/science.aac9725>.
15. Burchianti, A.; Scazza, F.; Amico, A.; Valtolina, G.; Seman, J.A.; Fort, C.; Zaccanti, M.; Inguscio, M.; Roati, G. Connecting Dissipation and Phase Slips in a Josephson Junction between Fermionic Superfluids. *Phys. Rev. Lett.* **2018**, *120*, 025302. <https://doi.org/10.1103/PhysRevLett.120.025302>.
16. Kwon, W.J.; Del Pace, G.; Panza, R.; Inguscio, M.; Zwerger, W.; Zaccanti, M.; Scazza, F.; Roati, G. Strongly correlated superfluid order parameters from dc Josephson supercurrents. *Science* **2020**, *369*, 84–88. <https://doi.org/10.1126/science.aaz2463>.
17. Luick, N.; Sobirey, L.; Bohlen, M.; Singh, V.P.; Mathey, L.; Lompe, T.; Moritz, H. An ideal Josephson junction in an ultracold two-dimensional Fermi gas. *Science* **2020**, *369*, 89–91. <https://doi.org/10.1126/science.aaz2342>.
18. Del Pace, G.; Kwon, W.J.; Zaccanti, M.; Roati, G.; Scazza, F. Tunneling Transport of Unitary Fermions across the Superfluid Transition. *Phys. Rev. Lett.* **2021**, *126*, 055301. <https://doi.org/10.1103/PhysRevLett.126.055301>.

19. Josephson, B.D. Possible new effects in superconductive tunnelling. *Phys. Lett.* **1962**, *1*, 251 – 253. [https://doi.org/https://doi.org/10.1016/0031-9163\(62\)91369-0](https://doi.org/https://doi.org/10.1016/0031-9163(62)91369-0).
20. Anderson, P.W.; Rowell, J.M. Probable Observation of the Josephson Superconducting Tunneling Effect. *Phys. Rev. Lett.* **1963**, *10*, 230–232. <https://doi.org/10.1103/PhysRevLett.10.230>.
21. Barone, A.; Paterno, G. *Physics and Applications of the Josephson Effect*; John Wiley: New York, 1982.
22. Milburn, G.J.; Corney, J.; Wright, E.M.; Walls, D.F. Quantum dynamics of an atomic Bose-Einstein condensate in a double-well potential. *Phys. Rev. A* **1997**, *55*, 4318–4324. <https://doi.org/10.1103/PhysRevA.55.4318>.
23. Smerzi, A.; Fantoni, S.; Giovanazzi, S.; Shenoy, S.R. Quantum Coherent Atomic Tunneling between Two Trapped Bose-Einstein Condensates. *Phys. Rev. Lett.* **1997**, *79*, 4950–4953. <https://doi.org/10.1103/PhysRevLett.79.4950>.
24. Zapata, I.; Sols, F.; Leggett, A.J. Josephson effect between trapped Bose-Einstein condensates. *Phys. Rev. A* **1998**, *57*, R28–R31. <https://doi.org/10.1103/PhysRevA.57.R28>.
25. Raghavan, S.; Smerzi, A.; Fantoni, S.; Shenoy, S.R. Coherent oscillations between two weakly coupled Bose-Einstein condensates: Josephson effects,  $\pi$  oscillations, and macroscopic quantum self-trapping. *Phys. Rev. A* **1999**, *59*, 620–633. <https://doi.org/10.1103/PhysRevA.59.620>.
26. Ruostekoski, J.; Walls, D.F. Bose-Einstein condensate in a double-well potential as an open quantum system. *Phys. Rev. A* **1998**, *58*, R50–R53. <https://doi.org/10.1103/PhysRevA.58.R50>.
27. Meier, F.; Zwerger, W. Josephson tunneling between weakly interacting Bose-Einstein condensates. *Phys. Rev. A* **2001**, *64*, 033610. <https://doi.org/10.1103/PhysRevA.64.033610>.
28. Paraoanu, G.S.; Kohler, S.; Sols, F.; Leggett, A.J. The Josephson plasmon as a Bogoliubov quasiparticle. *Journal of Physics B: Atomic, Molecular and Optical Physics* **2001**, *34*, 4689–4696. <https://doi.org/10.1088/0953-4075/34/23/313>.
29. Kohler, S.; Sols, F. Chemical potential standard for atomic Bose–Einstein condensates. *New Journal of Physics* **2003**, *5*, 94. <https://doi.org/10.1088/1367-2630/5/1/394>.
30. Sakellari, E.; Leadbeater, M.; Kylstra, N.J.; Adams, C.S. Josephson spectroscopy of a dilute Bose-Einstein condensate in a double-well potential. *Phys. Rev. A* **2002**, *66*, 033612. <https://doi.org/10.1103/PhysRevA.66.033612>.
31. Sakellari, E.; Proukakis, N.P.; Leadbeater, M.; Adams, C.S. Josephson tunnelling of a phase-imprinted Bose–Einstein condensate in a time-dependent double-well potential. *New Journal of Physics* **2004**, *6*, 42–42. <https://doi.org/10.1088/1367-2630/6/1/042>.
32. Zou, P.; Dalfovo, F. Josephson oscillations and self-trapping of superfluid fermions in a double-well potential. *J Low Temp. Phys.* **2014**, *177*, 240.
33. Abad, M.; Guilleumas, M.; Mayol, R.; Piazza, F.; Jezek, D. M.; Smerzi, A.. Phase slips and vortex dynamics in Josephson oscillations between Bose-Einstein condensates. *EPL* **2015**, *109*, 40005. <https://doi.org/10.1209/0295-5075/109/40005>.
34. Bidasyuk, Y.M.; Weyrauch, M.; Momme, M.; Prikhodko, O.O. Finite-temperature dynamics of a bosonic Josephson junction. *J. Phys. B* **2018**, *51*, 205301. <https://doi.org/10.1088/1361-6455/aae022>.
35. Polo, J.; Ahufinger, V.; Hekking, F.W.J.; Minguzzi, A. Damping of Josephson Oscillations in Strongly Correlated One-Dimensional Atomic Gases. *Phys. Rev. Lett.* **2018**, *121*, 090404. <https://doi.org/10.1103/PhysRevLett.121.090404>.
36. Polo, J.; Dubessy, R.; Pedri, P.; Perrin, H.; Minguzzi, A. Oscillations and Decay of Superfluid Currents in a One-Dimensional Bose Gas on a Ring. *Phys. Rev. Lett.* **2019**, *123*, 195301. <https://doi.org/10.1103/PhysRevLett.123.195301>.
37. Xhani, K.; Neri, E.; Galantucci, L.; Scazza, F.; Burchianti, A.; Lee, K.L.; Barenghi, C.F.; Trombettoni, A.; Inguscio, M.; Zaccanti, M.; et al. Critical Transport and Vortex Dynamics in a Thin Atomic Josephson Junction. *Phys. Rev. Lett.* **2020**, *124*, 045301. <https://doi.org/10.1103/PhysRevLett.124.045301>.
38. Xhani, K.; Galantucci, L.; Barenghi, C.F.; Roati, G.; Trombettoni, A.; Proukakis, N.P. Dynamical phase diagram of ultracold Josephson junctions. *New Journal of Physics* **2020**, *22*, 123006. <https://doi.org/10.1088/1367-2630/abc8e4>.
39. Xhani, K.; Proukakis, N.P. Dissipation in a finite-temperature atomic Josephson junction. *Phys. Rev. Res.* **2022**, *4*, 033205. <https://doi.org/10.1103/PhysRevResearch.4.033205>.
40. Wlazłowski, G.; Xhani, K.; Tylutki, M.; Proukakis, N.P.; Magierski, P. Dissipation Mechanisms in Fermionic Josephson Junction. *Phys. Rev. Lett.* **2023**, *130*, 023003. <https://doi.org/10.1103/PhysRevLett.130.023003>.
41. Furutani, K.; Tempere, J.; Salasnich, L. Quantum effective action for the bosonic Josephson junction. *Phys. Rev. B* **2022**, *105*, 134510. <https://doi.org/10.1103/PhysRevB.105.134510>.

42. Bardin, A.; Lorenzi, F.; Salasnich, L. Quantum fluctuations in atomic Josephson junctions: the role of dimensionality. *New Journal of Physics* **2024**, *26*, 013021. <https://doi.org/10.1088/1367-2630/ad127b>.
43. Ji, S.C.; Schweigler, T.; Tajik, M.; Cataldini, F.; Sabino, J.a.; Møller, F.S.; Erne, S.; Schmiedmayer, J. Floquet Engineering a Bosonic Josephson Junction. *Phys. Rev. Lett.* **2022**, *129*, 080402. <https://doi.org/10.1103/PhysRevLett.129.080402>.
44. Pace, G.D.; Hernández-Rajkov, D.; Singh, V.P.; Grani, N.; Fernández, M.F.; Nesti, G.; Seman, J.A.; Inguscio, M.; Amico, L.; Roati, G. Shapiro steps in strongly-interacting Fermi gases, 2024, [[arXiv:cond-mat.quant-gas/2409.03448](https://arxiv.org/abs/cond-mat.quant-gas/2409.03448)].
45. Bernhart, E.; Röhrle, M.; Singh, V.P.; Mathey, L.; Amico, L.; Ott, H. Observation of Shapiro steps in an ultracold atomic Josephson junction, 2024, [[arXiv:cond-mat.quant-gas/2409.03340](https://arxiv.org/abs/cond-mat.quant-gas/2409.03340)].
46. Gauthier, G.; Szigeti, S.S.; Reeves, M.T.; Baker, M.; Bell, T.A.; Rubinsztein-Dunlop, H.; Davis, M.J.; Neely, T.W. An atomtronic oscillator circuit for quantum gases. *arXiv e-prints* **2019**, p. arXiv:1903.04086, [[arXiv:cond-mat.quant-gas/1903.04086](https://arxiv.org/abs/cond-mat.quant-gas/1903.04086)].
47. Pigneur, M.; Berrada, T.; Bonneau, M.; Schumm, T.; Demler, E.; Schmiedmayer, J. Relaxation to a Phase-Locked Equilibrium State in a One-Dimensional Bosonic Josephson Junction. *Phys. Rev. Lett.* **2018**, *120*, 173601. <https://doi.org/10.1103/PhysRevLett.120.173601>.
48. Furutani, K.; Salasnich, L. Interaction-induced dissipative quantum phase transition in a head-to-tail atomic Josephson junction. *Phys. Rev. B* **2024**, *110*, L140503. <https://doi.org/10.1103/PhysRevB.110.L140503>.
49. Saha, A.K.; Dubessy, R. Dynamical phase diagram of a one-dimensional Bose gas in a box with a tunable weak link: From Bose-Josephson oscillations to shock waves. *Phys. Rev. A* **2021**, *104*, 023316. <https://doi.org/10.1103/PhysRevA.104.023316>.
50. Momme, M.R.; Bidasyuk, Y.M.; Weyrauch, M. Collective excitations and tunneling dynamics in long bosonic Josephson junctions. *Phys. Rev. A* **2019**, *100*, 033601. <https://doi.org/10.1103/PhysRevA.100.033601>.
51. Labouvie, R.; Santra, B.; Heun, S.; Wimberger, S.; Ott, H. Negative Differential Conductivity in an Interacting Quantum Gas. *Phys. Rev. Lett.* **2015**, *115*, 050601. <https://doi.org/10.1103/PhysRevLett.115.050601>.
52. Begg, S.E.; Davis, M.J.; Reeves, M.T. Nonequilibrium Transport in a Superfluid Josephson Junction Chain: Is There Negative Differential Conductivity? *Phys. Rev. Lett.* **2024**, *132*, 103402. <https://doi.org/10.1103/PhysRevLett.132.103402>.
53. Saha, A.K.; Ray, D.S.; Deb, B. Phase diffusion and fluctuations in a dissipative Bose-Josephson junction. *Phys. Rev. E* **2023**, *107*, 034141. <https://doi.org/10.1103/PhysRevE.107.034141>.
54. Lagoudakis, K.G.; Pietka, B.; Wouters, M.; André, R.; Deveaud-Plédran, B. Coherent Oscillations in an Exciton-Polariton Josephson Junction. *Phys. Rev. Lett.* **2010**, *105*, 120403. <https://doi.org/10.1103/PhysRevLett.105.120403>.
55. Adiyatullin, A.F.; Anderson, M.D.; Flayac, H.; Portella-Oberli, M.T.; Jabeen, F.; Ouellet-Plamondon, C.; Sallen, G.C.; Deveaud, B. Periodic squeezing in a polariton Josephson junction. *Nature Communications* **2017**, *8*, 1329. <https://doi.org/10.1038/s41467-017-01331-8>.
56. D'Errico, C.; Abbate, S.S.; Modugno, G. Quantum transport in ultracold atoms. *Philos. Trans. Royal Soc. A* **2017**, *375*, 20160425.
57. Wright, K.C.; Blakestad, R.B.; Lobb, C.J.; Phillips, W.D.; Campbell, G.K. Driving Phase Slips in a Superfluid Atom Circuit with a Rotating Weak Link. *Phys. Rev. Lett.* **2013**, *110*, 025302. <https://doi.org/10.1103/PhysRevLett.110.025302>.
58. Griffin, A.; Nikuni, T.; Zaremba, E. *Bose-Condensed Gases at Finite Temperatures*; Cambridge University Press, 2009. <https://doi.org/10.1017/CBO9780511575150>.
59. Jackson, B.; Zaremba, E. Modeling Bose-Einstein condensed gases at finite temperatures with N-body simulations. *Phys. Rev. A* **2002**, *66*, 033606. <https://doi.org/10.1103/PhysRevA.66.033606>.
60. Proukakis, N.P.; Jackson, B. Finite-temperature models of Bose-Einstein condensation. *J. Phys. B: At. Mol. Opt. Phys.* **2008**, *41*, 203002. <https://doi.org/10.1088/0953-4075/41/20/203002>.
61. Lee, K.L.; Proukakis, N.P. Non-equilibrium atomic condensates and mixtures: collective modes, condensate growth and thermalisation. *J. Phys. B: At. Mol. Opt. Phys.* **2016**, *49*, 214003. <https://doi.org/10.1088/0953-4075/49/21/214003>.
62. Lee, K.L.; Jørgensen, N.B.; Liu, I.K.; Wacker, L.; Arlt, J.J.; Proukakis, N.P. Phase separation and dynamics of two-component Bose-Einstein condensates. *Phys. Rev. A* **2016**, *94*, 013602. <https://doi.org/10.1103/PhysRevA.94.013602>.

63. Blakie, P.B.; Bradley, A.S.; Davis, M.J.; Ballagh, R.J.; Gardiner, C.W. Dynamics and statistical mechanics of ultra-cold Bose gases using c-field techniques. *Advances in Physics* **2008**, *57*, 363–455, [arXiv:cond-mat.stat-mech/0809.1487]. <https://doi.org/10.1080/00018730802564254>.
64. Proukakis, N.; Gardiner, S.; Davis, M.; Szymańska, M., Eds. *Quantum Gases: Finite Temperature and Non-Equilibrium Dynamics*; Imperial College Press: London, United Kingdom, 2013.
65. Rooney, S.J.; Allen, A.J.; Zülicke, U.; Proukakis, N.P.; Bradley, A.S. Reservoir interactions of a vortex in a trapped three-dimensional Bose-Einstein condensate. *Phys. Rev. A* **2016**, *93*, 063603. <https://doi.org/10.1103/PhysRevA.93.063603>.
66. Proukakis, N.P.; Rigopoulos, G.; Soto, A. Self-Consistent Stochastic Finite-Temperature Modelling: Ultra-cold Bose Gases with Local (s-wave) and Long-Range (Dipolar) Interactions, 2024, [arXiv:cond-mat.quant-gas/2407.20178].
67. Jackson, B.; Proukakis, N.P.; Barenghi, C.F.; Zaremba, E. Finite-temperature vortex dynamics in Bose-Einstein condensates. *Phys. Rev. A* **2009**, *79*, 053615. <https://doi.org/10.1103/PhysRevA.79.053615>.
68. Jackson, B.; Proukakis, N.P.; Barenghi, C.F. Dark-soliton dynamics in Bose-Einstein condensates at finite temperature. *Phys. Rev. A* **2007**, *75*, 051601. <https://doi.org/10.1103/PhysRevA.75.051601>.
69. Allen, A.J.; Zaremba, E.; Barenghi, C.F.; Proukakis, N.P. Observable vortex properties in finite-temperature Bose gases. *Phys. Rev. A* **2013**, *87*, 013630. <https://doi.org/10.1103/PhysRevA.87.013630>.
70. Allen, A.J.; Zuccher, S.; Caliari, M.; Proukakis, N.P.; Parker, N.G.; Barenghi, C.F. Vortex reconnections in atomic condensates at finite temperature. *Phys. Rev. A* **2014**, *90*, 013601. <https://doi.org/10.1103/PhysRevA.90.013601>.
71. Arahata, E.; Nikuni, T. Propagation of first and second sound in a highly elongated trapped Bose-condensed gas at finite temperatures. *Phys. Rev. A* **2013**, *87*, 033610. <https://doi.org/10.1103/PhysRevA.87.033610>.
72. Märkle, J.; Allen, A.J.; Federsel, P.; Jetter, B.; Günther, A.; Fortágh, J.; Proukakis, N.P.; Judd, T.E. Evaporative cooling of cold atoms at surfaces. *Phys. Rev. A* **2014**, *90*, 023614. <https://doi.org/10.1103/PhysRevA.90.023614>.
73. Jackson, B.; Zaremba, E. Quadrupole Collective Modes in Trapped Finite-Temperature Bose-Einstein Condensates. *Phys. Rev. Lett.* **2002**, *88*, 180402. <https://doi.org/10.1103/PhysRevLett.88.180402>.
74. Jackson, B.; Zaremba, E. Finite-Temperature Simulations of the Scissors Mode in Bose-Einstein Condensed Gases. *Phys. Rev. Lett.* **2001**, *87*, 100404. <https://doi.org/10.1103/PhysRevLett.87.100404>.
75. Straatsma, C.J.E.; Colussi, V.E.; Davis, M.J.; Lobser, D.S.; Holland, M.J.; Anderson, D.Z.; Lewandowski, H.J.; Cornell, E.A. Collapse and revival of the monopole mode of a degenerate Bose gas in an isotropic harmonic trap. *Phys. Rev. A* **2016**, *94*, 043640. <https://doi.org/10.1103/PhysRevA.94.043640>.

**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.