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Not peer-reviewed version

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Posted Date: 18 December 2025

doi: 10.20944/preprints202512.1595.v1

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Article

Relations Established Between Hypergeometric Functions and Some Special Number Sequences

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Abstract

In this study, we establish novel hypergeometric representations for the two classical sequences that are the Pell and Jacobsthal sequences. Building on Dilcher's hypergeometric formulation of the Fibonacci sequence, we extend similar results and derive analogous structures for these two classical sequences. The results unify several known identities, provide new explicit representations, and offer a broader perspective on hypergeometric interpretations of linear second order recurrence sequences.

Keywords: hypergeometric functions; Jacobsthal sequences; Pell sequences; Chebyshev polynomials

1. Introduction

Hypergeometric functions, with their serial definitions, parametric structures, and transformation properties, hold a central position in many branches of mathematics [1]. This field, pioneered by Gauss and Kummer, encompasses general solution approaches beyond classical methods and is now considered a powerful analytical tool not only in theoretical mathematics but also in many scientific fields, such as physics, biology, economics, numerical analysis, and cryptography [2,4,7,8].

The analytical power of hypergeometric functions is not limited to their respective differential equations; but also manifests in the relationships that can be established with various special number sequences. In this context, particular attention has been focused on the Fibonacci sequence, which enables the derivation of various hypergeometric representations [3]. This seemingly unnatural relationship has been made compatible with hypergeometric structures through parametric transformations and mappings. However, the applicability of a similar approach to structurally similar Pell and Jacobsthal sequences has been largely ignored in the literature.

In existing studies, representations of hypergeometric series have mostly been presented that are limited to only certain parameter values, and a significant part of these representations remains specific to the Fibonacci sequence.

However, Pell sequences also have similar second-order linear homogeneous recurrence relations and can be evaluated using the same theoretical framework. The Jacobsthal sequence can be analyzed analytically through its connection with Chebyshev polynomials in [9]. Roy studied binomial identities and hypergeometric series in [10].

This paper demonstrates that hypergeometric functions are not only special functions but also powerful tools for the analytical representation of sequences of integers. Pell and Jacobsthal sequences are reconstructed from a hypergeometric perspective using classical transformations, series expansions, integral representations, and connections to special polynomials.

Studies on the Fibonacci sequence are particularly prominent. Dilcher's work details how hypergeometric functions can represent the Fibonacci sequence under different transformations and offers various representations using transformation techniques of classical series.

This article focuses on the re-expression of Pell and Jacobsthal sequences in terms of hypergeometric functions. We use Binet formulas and recurrence relations to illustrate their relationship with hypergeometric series.

2. Materials and Methods

The gamma function, which is included in the definition of many special functions, is given as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, x > 0$$

with the following properties

$$\Gamma(x) = (x - 1)!, x \in \mathbb{N}$$

$$\Gamma(x + 1) = x\Gamma(x), x > 0$$

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x} \tag{2.1}$$

$$\Gamma(x)\Gamma(x + \frac{1}{2}) = 2^{1-2x}\sqrt{\pi}\Gamma(2x) \tag{2.2}$$

Given a is a constant real or complex number and n is a natural number, the Pochhammer symbol $(a)_n$ is defined as

$$(a)_n = a(a + 1)(a + 2) \dots (a + n - 1) \tag{2.3}$$

The Pochhammer symbol is named after the 19th-century German mathematician, Leo August Pochhammer. Although originally introduced in the context of differential equations, it is now used as a fundamental tool in numerous theoretical and applied studies involving hypergeometric functions, orthogonal polynomials, special number sequences and series solutions. We have

$$(a)_0 = \text{and } (1)_n = n!$$

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} \tag{2.4}$$

$$(a)_{n+1} = a(a + 1)_n \tag{2.5}$$

$$(a)_{n-k} = \frac{(a)_n}{(a+n-k)_k}$$

Jacobsthal numbers J_n are a special sequence of integers defined recursively $J_0 = 0, J_1 = 1, J_n = J_{n-1} + 2J_{n-2}, (n \geq 2)$. The Binet formula of the Jacobsthal sequence is $J_n = \frac{2^n - (-1)^n}{3}$ [5].

Pell numbers P_n are a special sequence of integers defined recursively $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}, (n \geq 2)$. Pell sequence is given as $P_n = \frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right)$ [6].

Hypergeometric differential equation is defined as

$$x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0 \tag{2.6}$$

Solutions of the hypergeometric differential equation at $x=0$ regular singular point are called hypergeometric functions:

$$y = AF(a, b, c; x) + Bx^{1-c}F(a - c + 1, b - c + 1, 2 - c; x)$$

where A and B arbitrary constants. The general solution is valid for $|x| < 1$.

The representation of the hypergeometric function is:

$$F(a, b, c; x) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!} x^n.$$

Some functions can be expressed by hypergeometric function in the following:

$$F(1,1,1;x) = \frac{1}{1-x},$$

$$F(1,2,1;x) = \frac{1}{(1-x)^2},$$

$$F(-n, 1; 1; 1 - x) = x^n, n = 0,1,2 \dots,$$

$$F(-n, b, b; -x) = (1 + x)^n,$$

$$F(1,1,2;-x) = \frac{\ln(1+x)}{x},$$
$$e^x = \lim_{b \rightarrow \infty} F\left(1, b, 1; \frac{x}{b}\right).$$

Lemma 2.1 [1] Hypergeometric function holds the following property

$$F(a, \frac{1}{2} + a, \frac{3}{2}; z^2) = \frac{1}{2z(1-2a)} [(1+z)^{1-2a} - (1-z)^{1-2a}] \tag{2.7}$$

Lemma 2.2 [1] Hypergeometric function satisfies the following property

$$F(a, b, c; z) = (1-z)^{-a} F\left(a, c-b, c; \frac{z}{z-1}\right). \tag{2.8}$$

Lemma 2.3 [1] Hypergeometric function satisfies the following property

$$F(a, b, c; z) = (1-z)^{-b} F\left(b, c-a, c; \frac{z}{z-1}\right). \tag{2.9}$$

Lemma 2.4 [3] Hypergeometric function satisfies the following property

$$F(a, b, c; z) = (1-z)^{c-a-b} F(c-a, c-b, c; z). \tag{2.10}$$

Lemma 2.5 Hypergeometric function satisfies the following property

$$F(a, b, c; z) = (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} F\left(a, c-b, a-b+1; \frac{1}{1-z}\right) \\ + (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F(b, c-a, b-a+1; \frac{1}{1-z}) \tag{2.11}$$

3. Main Results

This section comprehensively covers the analytic relationships between hypergeometric functions and some special integer sequences. The connection between the Fibonacci sequence and hypergeometric series is based on Dilcher’s work. In this context, the theoretical construction of serial representations using generating functions and recurrence relations for sequences is presented.

Theorem 3.1 The Jacobstral sequence is expressed by hypergeometric function as

$$J_n = 2^{\frac{n-1}{2}} i^{n-1} n F\left(1-n, 1+n, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right)$$

Proof: By [9] and by the representation of the second type of the Chebyshev polynomial with hypergeometric function, the Jacobstral sequence is expressed by Chebyshev polynomials as

$$J_n = (2i^2)^{\frac{n-1}{2}} U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right). \tag{3.1}$$

If we substitute $n=n-1$ and $x=\frac{1}{2\sqrt{2}i}$ in (3.1), we get

$$U_{n-1}\left(\frac{1}{2\sqrt{2}i}\right) = (n-1+1)F\left(-n+1, n-1+2, \frac{3}{2}; \frac{1-\frac{1}{2\sqrt{2}i}}{2}\right) \\ = nF\left(1-n, 1+n, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \tag{3.2}$$

By (3.1) and (3.2), the result is obtained.

Theorem 3.2 The Pell sequence is expressed by hypergeometric function as

$$P_n = nF\left(\frac{1-n}{2}, 1-\frac{n}{2}, \frac{3}{2}; 2\right).$$

Proof: The Binet formula of the Pell sequence is $P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$. If we substitute $1-2a=n, a=\frac{1-n}{2}$ ve $z=\sqrt{2}$ in (2.7), we get

$$F\left(\frac{1-n}{2}, 1-\frac{n}{2}, \frac{3}{2}; 2\right) = \frac{1}{2\sqrt{2}n} [(1+\sqrt{2})^n - (1-2\sqrt{2})^n].$$

Theorem 3.3 The n th element of the Jacobstral sequence is expressed by hypergeometric function as

$$J_n = 2^{\frac{n-1}{2}} i^{n-1} \sum_{k=0}^{n-1} \left(-\frac{i}{\sqrt{2}} - 2\right)^k \binom{n+k}{2k+1}$$

Proof: By Theorem (3.1) and the representation of the second type of the Chebyshev polynomial with hypergeometric function and the expansion of the series of the hypergeometric function, we have

$$\begin{aligned} J_n &= 2^{\frac{n-1}{2}} i^{n-1} n F\left(1-n, 1+n, \frac{3}{2}, \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\ &= 2^{\frac{n-1}{2}} i^{n-1} n \sum_{k=0}^{\infty} \frac{(-n+1)_k (n+1)_k}{\left(\frac{3}{2}\right)_k} \frac{\left(\frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right)^k}{k!} \\ (1-n)_k &= (-1)^k \frac{(n-1)!}{(n-1-k)!} \quad (3.3) \\ (1+n)_k &= \frac{(n+k)!}{n!} \quad (3.4) \end{aligned}$$

If the equalities (3.3), (3.4) are substituted in the series above, it is obtained that

$$J_n = 2^{\frac{n-1}{2}} i^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k (n+k)!}{(n-1-k)!} \frac{1}{(2k+1)!} \frac{(2\sqrt{2}i-1)^k}{(\sqrt{2}i)^k}.$$

For $n-1-k \geq 0$, we have

$$J_n = 2^{\frac{n-1}{2}} i^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k (2\sqrt{2}i-1)^k}{(\sqrt{2}i)^k} \frac{(n+k)}{(2k+1)} = 2^{\frac{n-1}{2}} i^{n-1} \sum_{k=0}^{n-1} \left(-\frac{i}{\sqrt{2}} - 2\right)^k \frac{(n+k)}{(2k+1)}.$$

Theorem 3.4 The $(2n+1)^{th}$ element of the Pell sequence is expressed by hypergeometric function as

$$P_{2n+1} = (2n+1)(-1)^n F\left(-n, 1+n, \frac{3}{2}; 2\right).$$

Proof: We know that by Lemma 2.2

$$F(a, b, c; z) = (1-z)^{-a} F\left(a, c-b, c; \frac{z}{z-1}\right).$$

By Theorem 3.2 and (2.8), we get

$$\begin{aligned} P_{2n+1} &= (2n+1) F\left(\frac{1-2n-1}{2}, 1-\frac{2n+1}{2}, \frac{3}{2}; 2\right) \\ &= (2n+1) F\left(-n, \frac{1}{2}-n, \frac{3}{2}; 2\right) \\ &= (2n+1)(-1)^n F\left(-n, 1+n, \frac{3}{2}; 2\right). \end{aligned}$$

Theorem 3.5 The $2n^{th}$ element of the Pell sequence is expressed by hypergeometric function as

$$P_{2n} = (-1)^n (2n) F\left(1-n, 1+n, \frac{3}{2}; 2\right).$$

Proof: By Theorem 3.2 and (2.9), it is satisfied that

$$\begin{aligned} P_{2n} &= (2n) F\left(\frac{1-2n}{2}, 1-n, \frac{3}{2}; 2\right) \\ &= (-1)^n 2n F\left(1-n, 1+n, \frac{3}{2}; 2\right). \end{aligned}$$

Theorem 3.6 The n^{th} element of the Pell sequence is expressed by hypergeometric function as

$$P_n = 2^{n-1} F\left(\frac{1-n}{2}, \frac{2-n}{2}, 1-n; -1\right)$$

Proof: In the equality in Theorem 3.2, if we choose $a = \frac{1-n}{2}$, $b = \frac{2-n}{2}$, $a+b+m = \frac{3}{2}$, $m = n$ and $z = 2$, we get

$$P_n = n \frac{\Gamma(\frac{3}{2}) \Gamma(n)}{\Gamma(\frac{2+n}{2}) \Gamma(\frac{1+n}{2})} F\left(\frac{1-n}{2}, \frac{2-n}{2}, 1-n; -1\right)$$

By (2.8), the result is obtained as

$$P_n = n \frac{\frac{1}{2} \sqrt{\pi} \frac{2^{n-1}}{\sqrt{\pi}} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\frac{n}{2} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{1+n}{2}\right)} F\left(\frac{1-n}{2}, \frac{2-n}{2}, 1-n; -1\right).$$

Theorem 3.7 The $(2n+1)^{th}$ element of the Pell sequence is expressed by hypergeometric function as

$$P_{2n+1} = (2n+1)F\left(-n, 1+n, \frac{1}{2}; -1\right)$$

Proof: The following equalities are satisfied by hypergeometric function

$$F(a, b, c; z) = F(a, b, a+b+m; z)$$

$$F(a, b, a+b+m; z) = \frac{\Gamma(a+b+m)\Gamma(m)}{\Gamma(b+m)\Gamma(a+m)} F(a, b, -m+1; 1-z) \\ + (1-z)^m \frac{\Gamma(a+b+m)\Gamma(-m)}{\Gamma(a)\Gamma(b)} F(b+m, a+m, m+1; 1-z) \quad (3.5)$$

[3]. In Theorem 3.4 and first sum of the equality (3.5), if we choose $a = -n$, $b = 1+n$, $z = 2$ ve $a+b+m = \frac{3}{2}$, we get

$$P_{2n+1} = (2n+1)(-1)^n \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{3}{2}+n)} F\left(-n, 1+n, \frac{1}{2}; -1\right) \\ = (2n+1)(-1)^n \frac{\sqrt{\pi} \frac{1}{2} \sqrt{\pi}}{\Gamma(\frac{1}{2}-n) \frac{1}{2} \Gamma(\frac{1}{2}+n)} F\left(-n, 1+n, \frac{1}{2}; -1\right) \\ = (2n+1)(-1)^n \frac{\pi}{\cos n\pi} F\left(-n, 1+n, \frac{1}{2}; -1\right).$$

Theorem 3.8 The $2n^{\text{th}}$ element of the Pell sequence is given by hypergeometric function as

$$P_{2n} = -2nF\left(1-n, 1+n, \frac{3}{2}; -1\right).$$

Proof: In Theorem 3.5 and (3.5), if we choose $a = 1-n$, $b = 1+n$, $z = 2$, and $a+b+m = \frac{3}{2}$, we get

$$P_{2n} = (2n)(-1)^n \frac{\Gamma(-\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{3}{2}+n)} F\left(1-n, 1+n, \frac{3}{2}; -1\right) \\ = (2n)(-1)^n \frac{-2\sqrt{\pi} \frac{1}{2} \sqrt{\pi}}{\cos n\pi} F\left(1-n, 1+n, \frac{3}{2}; -1\right).$$

Theorem 3.9 The $(2n+1)^{\text{th}}$ element of the Jacobsthal sequence is found by hypergeometric function as

$$J_{2n+1} = \left(\frac{2\sqrt{2}i+1}{4}\right)^{2n} (2n+1)F\left(-2n, -2n-\frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right).$$

Proof: In Theorem 3.1, we substitute $2n+1$ for n , we obtain

$$J_{2n+1} = 2^n i^{2n} (2n+1)F\left(-2n, 2n+2, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right). \quad (3.7)$$

In Lemma 2.2, if we use (3.7), we get

$$F(a, b, c; z) = F\left(-2n, 2n+2, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\ = \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} F\left(-2n, -2n-\frac{1}{2}, \frac{3}{2}; \frac{\frac{2\sqrt{2}i-1}{4\sqrt{2}i}}{\frac{2\sqrt{2}i-1}{4\sqrt{2}i}-1}\right) \\ = \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} F\left(-2n, -2n-\frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right)$$

Then, we use (3.7)

$$\begin{aligned} J_{2n+1} &= 2^n i^{2n} (2n+1) \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i} \right)^{2n} F\left(-2n, -2n - \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right) \\ &= (\sqrt{2}i)^{2n} (2n+1) \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i} \right)^{2n} F\left(-2n, -2n - \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right). \end{aligned}$$

Theorem 3.10 The $2n^{\text{th}}$ element of the Jacobstral sequence satisfies the following equality

$$J_{2n} = \left(\frac{8}{2\sqrt{2}i+1} \right)^{2n+1} nF\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right).$$

Proof: In Theorem 3.1, we substitute $2n$ for n , we obtain

$$J_{2n} = 2^{\frac{2n-1}{2}} i^{2n-1} 2nF\left(1-2n, 1+2n, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \tag{3.8}$$

In Lemma 2.3., we use the equality (3.8). Then, we get

$$\begin{aligned} F\left(1-2n, 1+2n, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) &= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i} \right)^{-2n-1} F\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{\frac{2\sqrt{2}i-1}{4\sqrt{2}i}}{\frac{2\sqrt{2}i-1}{4\sqrt{2}i}-1}\right) \\ &= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i} \right)^{-2n-1} F\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right) \end{aligned}$$

Then, we use (3.7) as

$$\begin{aligned} J_{2n} &= 2^{\frac{2n-1}{2}} i^{2n-1} 2n \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i} \right)^{-2n-1} F\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right) \\ &= 2^n i^{2n} \frac{1}{\sqrt{2}i} 2n \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i} \right)^{-2n-1} F\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right) \\ &= \sqrt{2}^{2n} i^{2n} \frac{1}{\sqrt{2}i} 2n \left(\frac{4\sqrt{2}i}{2\sqrt{2}i+1} \right) \left(\frac{4\sqrt{2}i}{2\sqrt{2}i+1} \right)^{2n} F\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right) \\ &= \frac{\sqrt{2}n}{i} \left(\frac{4\sqrt{2}i}{2\sqrt{2}i+1} \right) \left(\frac{-8}{2\sqrt{2}i+1} \right)^{2n} F\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right) \\ &= n \left(\frac{8}{2\sqrt{2}i+1} \right)^{2n+1} F\left(1+2n, 2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{-2\sqrt{2}i-1}\right). \end{aligned}$$

Theorem 3.11 The $2n^{\text{th}}$ element of the Pell sequence is denoted as

$$P_{2n} = 2nF\left(1+n, \frac{1}{2} + n, \frac{3}{2}; 2\right)$$

Proof: In Theorem 3.2, we substitute $2n$ for n , we obtain

$$P_{2n} = 2nF\left(\frac{1-2n}{2}, 1-n, \frac{3}{2}; 2\right).$$

By Lemma 2.3 and (3.8) we get

$$\begin{aligned} F\left(\frac{1-2n}{2}, 1-n, \frac{3}{2}; 2\right) &= (1-2)^{\frac{3}{2}-\frac{1-2n}{2}-(1-n)} F\left(\frac{3}{2}-\frac{1-2n}{2}, \frac{3}{2}-(1-n), \frac{3}{2}; 2\right) \\ &= (-1)^{2n} F\left(1+n, \frac{1}{2} + n, \frac{3}{2}; 2\right) \end{aligned}$$

Theorem 3.12 The $(2n+1)^{\text{th}}$ element of the Pell sequence is demonsrated as

$$P_{2n+1} = (2n+1)F\left(\frac{3}{2} + n, 2+n, \frac{3}{2}; 2\right).$$

Proof: In Theorem 3.2, we substitute $2n+1$ for n , we get

$$P_{2n+1} = (2n+1)F\left(-n, -n - \frac{1}{2}, \frac{3}{2}; 2\right). \tag{3.9}$$

If we use (2.10) for (3.9), we establish

$$\begin{aligned} F\left(-n, -n - \frac{1}{2}, \frac{3}{2}; 2\right) &= (1-2)^{\frac{3}{2}+n-(-n-\frac{1}{2})} F\left(\frac{3}{2}+n, \frac{3}{2}-(-n-\frac{1}{2}), \frac{3}{2}; 2\right) \\ &= (-1)^{2n+2} F\left(\frac{3}{2} + n, 2+n, \frac{3}{2}; 2\right) \end{aligned}$$

Hence,

$$\begin{aligned}
 P_{2n+1} &= (2n+1)F\left(-n, -n - \frac{1}{2}, \frac{3}{2}; 2\right) \\
 &= (2n+1)(-1)^{2n+2}F\left(\frac{3}{2} + n, 2 + n, \frac{3}{2}; 2\right).
 \end{aligned}$$

Theorem 3.13 The $(2n+1)^{\text{th}}$ element of the Jacobsthal sequence is expressed by hypergeometric function as

$$J_{2n+1} = (2n+1)(\sqrt{2}i)^{2n} \sqrt{\frac{4\sqrt{2}i}{2\sqrt{2}i+1}} F\left(\frac{3}{2} + 2n, -2n - \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right).$$

Proof: If the linear transformation (2.10) is applied to the hypergeometric expression in (3.7), we find

$$\begin{aligned}
 &F\left(-2n, 2n+2, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\
 &= \left(1 - \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right)^{\frac{3}{2}-(-2n)-(2n+2)} F\left(\frac{3}{2} - (-2n), \frac{3}{2} - (2n+2), \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\
 &= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-\frac{1}{2}} F\left(\frac{3}{2} + 2n, -2n - \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right).
 \end{aligned}$$

If we apply this equality to (3.7), we compute

$$\begin{aligned}
 J_{2n+1} &= 2^n i^{2n} (2n+1) \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-\frac{1}{2}} F\left(\frac{3}{2} + 2n, -2n - \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\
 &= \sqrt{\frac{4\sqrt{2}i}{2\sqrt{2}i+1}} (2n+1)(\sqrt{2}i)^{2n} F\left(\frac{3}{2} + 2n, -2n - \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right).
 \end{aligned}$$

Theorem 3.14 The $(2n+1)^{\text{th}}$ element of the Jacobsthal sequence is expressed by hypergeometric function as

$$J_{2n} = (2n)(\sqrt{2}i)^{2n-1} \sqrt{\frac{4\sqrt{2}i}{2\sqrt{2}i+1}} F\left(\frac{1}{2} + 2n, -2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right).$$

Proof: If the linear transformation (2.10) is applied to the hypergeometric expression (3.8), we find

$$\begin{aligned}
 &F\left(1-2n, 2n+1, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\
 &= \left(1 - \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right)^{\frac{3}{2}-(1-2n)-(2n+1)} F\left(\frac{3}{2} - (1-2n), \frac{3}{2} - (2n+1), \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\
 &= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-\frac{1}{2}} F\left(\frac{1}{2} + 2n, -2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right).
 \end{aligned}$$

If we apply this equality to (3.8), we evaluate

$$\begin{aligned}
 J_{2n} &= 2^n \frac{1}{2} i^{2n-1} (2n) \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-\frac{1}{2}} F\left(\frac{1}{2} + 2n, -2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\
 &= \sqrt{\frac{4\sqrt{2}i}{2\sqrt{2}i+1}} (2n)(\sqrt{2}i)^{2n-1} F\left(\frac{1}{2} + 2n, -2n + \frac{1}{2}, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right).
 \end{aligned}$$

Theorem 3.15 The $2n^{\text{th}}$ element of the Pell sequence is found as

$$P_{2n} = 2nF\left(1-n, 1+n, \frac{3}{2}; -1\right).$$

Proof: If $2n$ is written instead of n in the equation of Theorem 3.2, we get

$$P_{2n} = 2nF\left(\frac{1-2n}{2}, 1-n, \frac{3}{2}; 2\right). \quad (3.10)$$

If we apply the linear transformation of Lemma 2.5 to (3.10), we get

$$\begin{aligned}
 & F\left(\frac{1-2n}{2}, 1-n, \frac{3}{2}; 2\right) \\
 &= (-1)^{\frac{2n-1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(1-n-\frac{1-2n}{2})}{\Gamma(\frac{3}{2}-\frac{1-2n}{2})\Gamma(1-n)} F\left(\frac{1-2n}{2}, \frac{3}{2}-(1-n), \frac{1-2n}{2}-(1-n)+1; -1\right) \\
 &+ (-1)^{-(1-n)} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1-2n}{2}-(1-n))}{\Gamma(\frac{1-2n}{2})\Gamma(\frac{3}{2}-(1-n))} F\left(1-n, \frac{3}{2}-\frac{1-2n}{2}, (1-n)-\frac{1-2n}{2}+1; -1\right) \\
 &= (-1)^{n-\frac{1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(1+n)\Gamma(1-n)} F\left(\frac{1}{2}-n, \frac{1}{2}+n, \frac{1}{2}; -1\right) \\
 &+ (-1)^{n-1} \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+n)} F\left(1-n, 1+n, \frac{3}{2}; -1\right)
 \end{aligned}$$

If we substitute in (3.10), we get

$$\begin{aligned}
 P_{2n} &= 2n[(-1)^{n-\frac{1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(1+n)\Gamma(1-n)} F\left(\frac{1}{2}-n, \frac{1}{2}+n, \frac{1}{2}; -1\right) \\
 &+ (-1)^{n-1} \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+n)} F\left(1-n, 1+n, \frac{3}{2}; -1\right)].
 \end{aligned}$$

By $\Gamma(1-n)$, the first part of the equality is undefined. Therefore, we use only the second part.

We know that $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ and by (2.1), we have $\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+n) = \frac{\pi}{\cos(n\pi)}$.

Hence,

$$\begin{aligned}
 P_{2n} &= 2n(-1)^{n-1} \frac{-2\sqrt{\pi}\frac{\sqrt{\pi}}{2}}{\frac{\pi}{\cos(n\pi)}} F\left(1-n, 1+n, \frac{3}{2}; -1\right) \\
 &= 2n(-1)^n \cos(n\pi) F\left(1-n, 1+n, \frac{3}{2}; -1\right).
 \end{aligned}$$

Theorem 3.16 The $(2n+1)^{\text{th}}$ element of the Pell sequence is expressed by hypergeometric function as

$$P_{2n+1} = (2n+1)F\left(-n, 2+n, \frac{3}{2}; -1\right).$$

Proof: If we apply the linear transformation in Lemma 2.5 to the hypergeometric expression in equation (3.9), we obtain

$$\begin{aligned}
 & F\left(-n, -n-\frac{1}{2}, \frac{3}{2}; 2\right) \\
 &= (-1)^n \frac{\Gamma(\frac{3}{2})\Gamma(-n-\frac{1}{2}+n)}{\Gamma(\frac{3}{2}+n)\Gamma(-n-\frac{1}{2})} F\left(-n, \frac{3}{2}-(-n-\frac{1}{2}), -n-(-n-\frac{1}{2})+1; -1\right) \\
 &+ (-1)^{n+\frac{1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(-n-(-n-\frac{1}{2}))}{\Gamma(-n)\Gamma(\frac{3}{2}-(-n-\frac{1}{2}))} F\left(-n-\frac{1}{2}, \frac{3}{2}+n, -n-\frac{1}{2}+n+1; -1\right) \\
 &= (-1)^n \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2}+n)\Gamma(-n-\frac{1}{2})} F\left(-n, 2+n, \frac{3}{2}; -1\right)
 \end{aligned}$$

$$+(-1)^{n+\frac{1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(-n)\Gamma(2+n)} F\left(-n-\frac{1}{2}, \frac{3}{2}+n, \frac{1}{2}; -1\right).$$

If we substitute this equality in (3.9), we have

$$P_{2n+1} = (2n+1)[(-1)^n \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2}+n)\Gamma(-n-\frac{1}{2})} F(-n, 2+n, \frac{3}{2}; -1) \\ + (-1)^{n+\frac{1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(-n)\Gamma(2+n)} F(-n-\frac{1}{2}, \frac{3}{2}+n, \frac{1}{2}; -1)].$$

By $\Gamma(-n)$, the second part of the equality is undefined. Therefore, we use only the first part. We know that $\Gamma(\frac{-1}{2}) = -2\sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ and by (2.1), $\Gamma(-\frac{1}{2}-n)\Gamma(\frac{3}{2}+n) = -\frac{\pi}{\cos(n\pi)}$. Therefore, the following is satisfied:

$$P_{2n+1} = (2n+1)(-1)^n \frac{-2\sqrt{\pi}\frac{\sqrt{\pi}}{2}}{-\frac{\pi}{\cos(n\pi)}} F(-n, 2+n, \frac{3}{2}; -1) \\ = (2n+1)(-1)^n \cos(n\pi) F(-n, 2+n, \frac{3}{2}; -1).$$

Theorem 3.17 The $(2n+1)^{\text{th}}$ element of the Jacobsthal sequence is computed as

$$J_{2n+1} = (2\sqrt{2}i + 1)^{2n} F\left(-2n, \frac{-1}{2} - 2n, -4n - 1; \frac{4\sqrt{2}i}{2\sqrt{2}i + 1}\right).$$

Proof: If we apply the linear transformation Lemma 2.5 to the hypergeometric expression in (3.7),

$$F\left(-2n, 2n+2, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right) \\ = \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} \frac{\Gamma(\frac{3}{2})\Gamma(2n+2+2n)}{\Gamma(\frac{3}{2}+2n)\Gamma(2n+2)} F\left(-2n, \frac{3}{2}-2n-2, -2n-2n-2+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}\right) \\ + \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-2n-2} \frac{\Gamma(\frac{3}{2})\Gamma(-2n-2n-2)}{\Gamma(-2n)\Gamma(\frac{3}{2}-2n-2)} F\left(2n+2, \frac{3}{2}+2n, 2n+2+2n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}\right) \\ = \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} \frac{\Gamma(\frac{3}{2})\Gamma(4n+2)}{\Gamma(\frac{3}{2}+2n)\Gamma(2n+2)} F\left(-2n, \frac{-1}{2}-2n, -4n-1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}\right) \\ + \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-2n-2} \frac{\Gamma(\frac{3}{2})\Gamma(-4n-2)}{\Gamma(-2n)\Gamma(\frac{-1}{2}-2n)} F\left(2n+2, \frac{3}{2}+2n, 4n+3; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}\right).$$

By $\Gamma(-n)$, the second part of the equality is undefined. Therefore, we use only the first part. By (3.2), we get

$$\Gamma(4n+2) = \frac{2^{4n+1}}{\sqrt{\pi}} \Gamma(2n+1)\Gamma\left(\frac{3}{2}+2n\right),$$

$$\Gamma(2n+2) = \frac{2^{2n+1}}{\sqrt{\pi}} \Gamma(n+1)\Gamma\left(\frac{3}{2}+n\right),$$

$$\Gamma(2n+1) = \frac{2^{2n}}{\sqrt{\pi}} \Gamma(n+\frac{1}{2})\Gamma(n+1).$$

By these equalities above, we have

$$F\left(-2n, 2n+2, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}\right)$$

$$\begin{aligned}
&= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{4n+1}}{\sqrt{\pi}} \frac{2^{2n}}{\sqrt{\pi}} \Gamma(n+\frac{1}{2}) \Gamma(n+1) \Gamma(\frac{3}{2}+2n)}{\Gamma(\frac{3}{2}+2n) \frac{2^{2n+1}}{\sqrt{\pi}} \Gamma(n+1) \Gamma(\frac{3}{2}+n)} F(-2n, \frac{-1}{2} - 2n, -4n-1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \\
&= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} \frac{\frac{\sqrt{\pi} 2^{4n+1} 2^{2n}}{2 \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}}}}{\frac{2^{2n+1}}{\sqrt{\pi}} (n+\frac{1}{2})} F(-2n, \frac{-1}{2} - 2n, -4n-1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}).
\end{aligned}$$

If we substitute the equality of $F(-2n, 2n+2, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i})$ in (3.7), we have

$$\begin{aligned}
J_{2n+1} &= 2^n i^{2n} (2n+1) \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{4n+1}}{\sqrt{\pi}} \frac{2^{2n}}{\sqrt{\pi}}}{\frac{2^{2n+1}}{\sqrt{\pi}} (n+\frac{1}{2})} F(-2n, \frac{-1}{2} - 2n, -4n-1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \\
&= 2^n i^{2n} (2n+1) \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n} 2^{4n} \frac{1}{2n+1} F(-2n, \frac{-1}{2} - 2n, -4n-1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1})
\end{aligned}$$

Theorem 3.18 The $2n^{\text{th}}$ element of the Jacobsthal sequence is expressed by hypergeometric function as

$$\begin{aligned}
J_{2n} &= \frac{1}{8^n} \frac{1}{2\sqrt{2}i+1} \left[\left(\frac{2\sqrt{2}i+1}{4\sqrt{2}}\right)^{2n} F(1-2n, \frac{1}{2} - 2n, -4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \right. \\
&\quad \left. - \left(\frac{4\sqrt{2}}{2\sqrt{2}i+1}\right)^{2n} F(1+2n, \frac{1}{2} + 2n, 4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \right]
\end{aligned}$$

Proof: If we apply the linear transformation Lemma 2.5 to the hypergeometric expression in (3.8),

$$\begin{aligned}
&F(1-2n, 1+2n, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}) \\
&= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n-1} \frac{\Gamma(\frac{3}{2}) \Gamma(1+2n-1+2n)}{\Gamma(\frac{3}{2}-1+2n) \Gamma(1+2n)} F(1-2n, \frac{3}{2} - 1 - 2n, 1-2n-1-2n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \\
&+ \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-1-2n} \frac{\Gamma(\frac{3}{2}) \Gamma(1-2n-1-2n)}{\Gamma(1-2n) \Gamma(\frac{3}{2}-1-2n)} F(1+2n, \frac{3}{2} - 1 + 2n, 1+2n-1+2n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \\
&= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n-1} \frac{\Gamma(\frac{3}{2}) \Gamma(4n)}{\Gamma(\frac{1}{2}+2n) \Gamma(1+2n)} F(1-2n, \frac{1}{2} - 2n, -4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \\
&+ \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-1-2n} \frac{\Gamma(\frac{3}{2}) \Gamma(-4n)}{\Gamma(1-2n) \Gamma(\frac{1}{2}-2n)} F(1+2n, \frac{1}{2} + 2n, 4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1})
\end{aligned}$$

By (2.2), we get $\Gamma(4n) = \frac{2^{4n-1}}{\sqrt{\pi}} \Gamma(2n) \Gamma(\frac{1}{2}+2n)$, $\Gamma(2n+1) = 2n\Gamma(2n)$,

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \Gamma(-4n) = \frac{2^{-4n-1}}{\sqrt{\pi}} \Gamma(-2n) \Gamma\left(\frac{1}{2}-2n\right), \text{ and } \Gamma(1-2n) = -2n\Gamma(-2n)$$

If we substitute these values in the equation above, we obtain

$$\begin{aligned}
&F(1-2n, 1+2n, \frac{3}{2}; \frac{2\sqrt{2}i-1}{4\sqrt{2}i}) \\
&= \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n-1} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{4n-1}}{\sqrt{\pi}} \Gamma(2n) \Gamma(\frac{1}{2}+2n)}{\Gamma(\frac{1}{2}+2n) 2n\Gamma(2n)} F(1-2n, \frac{1}{2} - 2n, -4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1})
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-1-2n} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{-4n-1}}{\sqrt{\pi}} \Gamma(-2n) \Gamma(\frac{1}{2}-2n)}{-2n \Gamma(-2n) \Gamma(\frac{1}{2}-2n)} F(1+2n, \frac{1}{2}+2n, 4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \\
& = \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n-1} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{4n-1}}{\sqrt{\pi}} \Gamma(2n) \Gamma(\frac{1}{2}+2n)}{\Gamma(\frac{1}{2}+2n) 2n \Gamma(2n)} F(1-2n, \frac{1}{2}-2n, -4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \\
& + \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-1-2n} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{-4n-1}}{\sqrt{\pi}} \Gamma(-2n) \Gamma(\frac{1}{2}-2n)}{-2n \Gamma(-2n) \Gamma(\frac{1}{2}-2n)} F(1+2n, \frac{1}{2}+2n, 4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}). \\
J_{2n} & = 2^{\frac{2n-1}{2}} i^{2n-1} 2n \left[\left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{2n-1} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{4n-1}}{\sqrt{\pi}} \Gamma(2n) \Gamma(\frac{1}{2}+2n)}{\Gamma(\frac{1}{2}+2n) 2n \Gamma(2n)} F(1-2n, \frac{1}{2}-2n, -4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \right. \\
& + \left. \left(\frac{2\sqrt{2}i+1}{4\sqrt{2}i}\right)^{-1-2n} \frac{\frac{\sqrt{\pi}}{2} \frac{2^{-4n-1}}{\sqrt{\pi}} \Gamma(-2n) \Gamma(\frac{1}{2}-2n)}{-2n \Gamma(-2n) \Gamma(\frac{1}{2}-2n)} F(1+2n, \frac{1}{2}+2n, 4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \right] \\
& = \frac{1}{8^n} \frac{1}{2\sqrt{2}i+1} \left[\left(\frac{2\sqrt{2}i+1}{4\sqrt{2}}\right)^{2n} F(1-2n, \frac{1}{2}-2n, -4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \right. \\
& \left. - \left(\frac{4\sqrt{2}}{2\sqrt{2}i+1}\right)^{2n} F(1+2n, \frac{1}{2}+2n, 4n+1; \frac{4\sqrt{2}i}{2\sqrt{2}i+1}) \right].
\end{aligned}$$

4. Discussion and Conclusions

The results demonstrate that hypergeometric functions are not only a topic of special function theory but also a powerful expression tool in areas such as integer sequences, combinatorics, and numerical analysis. The proposed representations provide analytical solutions that can be used in theoretical analysis and structures that can facilitate algorithmic computations.

The results obtained here demonstrate the relationship between hypergeometric functions and special integer sequences, and suggest that further research is possible in this area. While Pell and Jacobsthal sequences are examined in detail, broader structures such as Horadam family are also open to investigation in terms of hypergeometric representations. Similar transformations can be developed for both classical and q-analog versions of these sequences.

Most of the proposed representations have been evaluated at the analytically. However, the numerical equivalents of these structures have not been investigated thoroughly. Future work could consider integrating the proposed representations into approximate computation methods, algorithmic typesetting, and computerized proof systems. Furthermore, the usability of the resulting transformations in applied fields such as cryptography, signal processing, and combinatorial structure analysis can be investigated.

Author Contributions: “Conceptualization, S.U. and B.A.; methodology, H.A.; software, B.A.; validation, S.U, B.A. and H.A.; formal analysis, S.U.; investigation, B.A.; resources, H.A.; data curation, S.U.; writing—original draft preparation, B.A.; writing—review and editing, S.U.; visualization, H.A.; supervision, S.U. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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