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A Study on Some New Results Arising from (p, q)-Calculus

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Abstract

This paper includes some new investigations and results for *post quantum calculus*, denoted by (p, q)-calculus. A chain rule for (p, q)-derivative is developed. Also, a new (p, q)-analogue of the exponential function is introduced and some its properties including the addition property for (p, q)-exponential functions are investigated. Several useful results involving (p, q)-binomial coefficients and (p, q)-antiderivative are discovered. At the final part of this paper, (p, q)-analogue of some elementary functions including trigonometric functions and hyperbolic functions are considered and some properties and relations among them are analyzed extensively.

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1. Introduction

As q-calculus including q-number with one base q is dealt with by some scientists, (p, q)-calculus including (p, q)-number with independent two variables p and q are firstly considered circa the same time (1991) and independently by Chakrabarti and Jagannathan [4], Brodimas *et al.* [2], Wachs and White [25], and Arik *et al.* [1]. Chakrabarti and Jagannathan [4] introduced the (p, q)-number to generalize or unify several forms of q-oscillator algebras well-known in the physics literature related to the representation theory of single parameter quantum algebras. Brodimas *et al.* [2] introduced (p, q)-number so that (p, q)-algebra can be derived from q-calculus by a Bargmann differential realization of the creation and annihilation operator, the Bose representation of those operators was derived, and (p, q)-Harmonic oscillator was constructed. Wachs and White [25] introduced (p, q)-number in mathematical literature in order to obtain (p, q)-Stirling number which is the generating function the joint distribution of pairs of statistics. Arik *et al.* [1] introduced (p,q)-number to investigate Fibonacci oscillators. Thereby, several physical and mathematical problems lead to the necessity of (p,q)-calculus. Based on the aforementioned papers, many mathematicians and physicists have developed the (p,q)-calculus in many different research areas since 1991. For instance, Burban and Klimyk [3] introduced (p,q)-hypergeometric functions and studied the relations among the basic hypergeometric functions, q-hypergeometric functions, and (p,q)-hypergeometric functions in 1994. Jagannathan [16] considered a more general (p,q)-hypergeometric series as well as Burban's (p,q)-hypergeometric series in [3] and he derived some related preliminary results in 1997. In 2005, Jagannathan and Srinivasa [15] gave a method to embed the q-series into a (p,q)-series, discovered some results corresponding (p,q)-extensions of the known q-identities and they also studied on (p,q)-hypergeometric series, based on the (p,q)-numbers. In 2008, Corcino [7] developed the (p,q)-extension of the binomial coefficients and also he established some properties parallel to those of the ordinary and q-binomial coefficients. Sadjang [22] investigated some properties of the (p,q)-derivative and the (p,q)-integration and presented two appropriate polynomials basis for the (p,q)-derivative, and then he derived various properties of these bases in the year 2013. As an application, he gave two (p,q)-Taylor formulas for polynomials. Furthermore, the fundamental theorem of (p,q)-calculus and the formula of (p,q)-integration by part were given. In the year 2015, Sadjang [23] introduced a new generalization of the Gamma and the Beta functions, calling as (p,q)-Gamma and (p,q)-Beta functions and developed some their properties which reduce to the known results as special cases. Mursaleen *et al.* ([19], [20]) introduced a new analogue of Bernstein operators, calling as (p,q)-Bernstein operators that are a generalization of q-Bernstein operators and they also considered approximation properties based on Korovkin's type approximation theorem of (p,q)-Bernstein operators, and they established some direct theorems in 2015. The (p,q)-analogues of Bernoulli polynomials, Euler polynomials, and Genocchi polynomials were described by Duran *et al.* [9] in early 2016 and the (p,q)-analogues of known earlier formulae were obtained, and relations between the new and old polynomials were investigated by making use of the fermionic p-adic integral over the p-adic number fields. Also in early 2016, a new class of Bernoulli, Euler and Genocchi polynomials based on the theory of (p,q)-calculus were considered by Duran *et al.* [12] and their some properties including addition theorems, difference equations, derivative properties, recurrence relationships were investigated. The (p,q)-extension of Cheon's main result in [5] was acquired, and further, the (p,q)-analogue of the main results given earlier by Srivastava and Pintér in [24] was discovered in [12]. Duran *et al.* [11] researched and gave some connections between the (p,q)-derivative operator and divided differences in 2016. Moreover, in 2016, Duran *et al.* [8] considered an extension of Haar distribution based on (p,q)-numbers. By means of this distribution, the (p,q)-analogue of Volkenborn integration that is a new generalization of q-Volkenborn integration was derived. Some properties of Volkenborn integration based on (p,q)-numbers were investigated. Finally, (p,q)-Bernoulli numbers and polynomials derived from (p,q)-Volkenborn integral were constructed and also some their properties were obtained.

We now review briefly some concepts of (p,q)-calculus.

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We begin with the following notations: $\mathbb{N}$ denotes the set of the natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The twin-basic number or (p,q)-number is defined by, for any number n,

$$[n]_{p,q} := \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$$

which is a natural generalization of the q-number such that

$$[n]_{p,q}\Big|_{p=1} := [n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-2} + q^{n-1}.$$

Note that $[n]_{p,q} = [n]_{q,p}$.

The (p,q)-derivative of a function f with respect to x is defined by

$$D_{p,q;x} f(x) := D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x} \qquad (x \neq 0) \tag{1.1}$$

and $(D_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0. As with the q-derivative and the ordinary derivative, the action of applying the (p,q)-derivative of any function is a linear operator, viz., for any constants a and b,

$$D_{p,q}(af(x) + bg(x)) = aD_{p,q}f(x) + bD_{p,q}g(x).$$

The (p,q)-derivatives of product and the quotient of $f(x)$ and $g(x)$ are given by

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x) &(1.2)\\ &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) &(1.3)\end{aligned}$$

and

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)} &(1.4)\\ &= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. &(1.5)\end{aligned}$$

As well as the formulas (1.4) and (1.5) we may write one more representation in symmetrical form

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{1}{2}\frac{D_{p,q}f(x)(g(px)+g(qx)) - D_{p,q}g(x)(f(px)+f(qx))}{g(px)g(qx)}. \tag{1.6}$$

The formulas in Eqs. (1.4), (1.5) and (1.6) are valid, however one of this forms may be more useful than others under special cases.

The (p,q)-Gauss Binomial formula is defined by

$$\begin{aligned}(x \oplus a)_{p,q}^n &= \begin{cases} (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), & \text{if } n \geq 1\\ 1, & \text{if } n = 0\end{cases}\\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}\end{aligned}$$

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where the notations $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q}$ ((p,q)-Gauss Binomial coefficients) and $[n]_{p,q}!$ ((p,q)-factorial) are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!} \qquad (n \geq k)$$

and

$$[n]_{p,q}! = [n]_{p,q}\,[n-1]_{p,q}\cdots[2]_{p,q}\,[1]_{p,q} \qquad (n \in \mathbb{N}).$$

The (p,q)-exponential functions, $e_{p,q}(x)$ and $E_{p,q}(x)$, are defined by

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!},$$

which hold the basic identity

$$e_{p,q}(x)E_{p,q}(-x) = 1. \tag{1.7}$$

The following (p,q)-derivatives hold true:

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \text{ and } D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{1.8}$$

Note that the (p,q)-derivatives of the (p,q)-exponentials are not precisely themself. However those derivatives are in the similar form of the derivative of classical exponential functions.

A more detailed statement of above, including (p,q)-numbers, is found in [1-4, 7-9, 11, 12, 15, 16, 19, 20, 22, 23, 25].

Taking here $p = 1$, then all notations given in this part reduce to the notations of the usual q-calculus (for details see [6, 10, 14, 17, 18, 21]).

2. Main Results

The formulas (1.2) and (1.3) can be rewritten in explicitly symmetrical form:

$$D_{p,q}\left(f(x)\,g(x)\right) = D_{p,q}f(x)\left(\frac{g(px)+g(qx)}{2}\right) + D_{p,q}g(x)\left(\frac{f(px)+f(qx)}{2}\right). \tag{2.1}$$

More general form of the multiplication rule of (p,q)-derivative is presented with fixed $0 \leq \alpha \leq 1$,

$$D_{p,q}\left(f(x)\,g(x)\right) = \left(\alpha f(qx) + (1-\alpha) f(px)\right) D_{p,q}g(x) + \left(\alpha g(px) + (1-\alpha) g(qx)\right) D_{p,q}f(x).$$

If we choose $\alpha = 1$, $\alpha = 0$ and $\alpha = \frac{1}{2}$, we then get the formulas (1.2), (1.3) and (2.1), respectively.

Let $y = f(x)$ be a injective and surjective mapping. In this case, we have $x = f^{-1}(y)$ where f^{-1} is the inverse function to f. Applying (p,q)-derivative to each side of $x = f^{-1}(y)$ gives

$$\begin{aligned} 1 &= D_{p,q}x = D_{p,q}f^{-1}(y) \\ &= \frac{f^{-1}(y(px)) - f^{-1}(y(qx))}{y(px) - y(qx)} \cdot \frac{y(px) - y(qx)}{px - qx} \\ &= D_{p,q;x}f^{-1}(y(x)) \cdot D_{p,q;x}y(x), \end{aligned}$$

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thus we arrive

$$D_{p,q;x}f^{-1}\left(y\left(x\right)\right)=\frac{1}{D_{p,q;x}y},$$

which is (p,q)-extension of the usual derivative of inverse function f^{-1}.

As it has been given for q-derivative in [17], there doesn't exist a general chain rule for (p,q)-derivatives. That is, if we consider the function $f(u(x))$, where $u=u(x)=\lambda x^{\mu}$ with λ,μ being constants, we have a chain rule as special case:

$$\begin{aligned} D_{p,q}\left[\ f(u(x))\right] &= D_{p,q}\left[f\left(\lambda x^{\mu}\right)\right]=\frac{f(\lambda x^{\mu}p^{\mu})-f(\lambda x^{\mu}q^{\mu})}{x(p-q)} \\ &= \frac{f(\lambda x^{\mu}p^{\mu})-f(\lambda x^{\mu}q^{\mu})}{\lambda x^{\mu}p^{\mu}-\lambda x^{\mu}q^{\mu}}.\frac{\lambda x^{\mu}p^{\mu}-\lambda x^{\mu}q^{\mu}}{x(p-q)} \\ &= \frac{f(up^{\mu})-f(uq^{\mu})}{up^{\mu}-uq^{\mu}}.\frac{u(px)-u(qx)}{x(p-q)}, \end{aligned}$$

which gives

$$D_{p,q}\ f(u(x))=\left(D_{p^{\mu},q^{\mu}}f\right)\left(u(x)\right).D_{p,q}u(x) \tag{2.2}$$

that is (p,q)-extension of the Eq. (1.15) in [17].

Conversely, if we consider the function $u(x)=x^3+x^2$ or $u(x)=\cos x$, the quantity $u(px)$ and $u(qx)$ can not be derived in terms of u in a basic way, and thereby it is impossible to write a general chain rule.

As has been done in [21] for multiple q-calculus, the case $D_{p,q}\varpi\left(x\right)=0$ for any function ϖ if and only if $\varpi\left(px\right)=\varpi\left(qx\right)$ called (p,q)-periodic function. By choosing $p=e^{\ln p}$ and $x=e^{\ln x}=e^{y}$ we can write $\varpi\left(x\right)=\varpi\left(e^{y}\right)\equiv G\left(y\right)$ and

$$\varpi\left(px\right)=G\left(y+\ln p\right).$$

Using condition of (p,q)-periodicity of $F\left(x\right)$, we get

$$G\left(y+\ln p\right)=G\left(y+\ln q\right)$$

and if we denote $y+\ln p=z$, then we realize

$$G\left(z\right)=G\left(z+\ln\frac{q}{p}\right),$$

which implies that $G\left(z\right)$ is standart periodic function

$$G\left(z\right)=G\left(z+t\right)$$

with period $t=\ln\frac{q}{p}=\ln q-\ln p$.

Applying k times the derivative operator $D_{p,q}$ to $(a\ominus x)_{p,q}^{n}$ successively yields to the following proportion.

Proposition 1. *Let n be a positive integer and $0\leq k\leq n$, we have*

$$D_{p,q}^{k}\frac{1}{\left(x\ominus a\right)_{p,q}^{n}}=\left(-1\right)^{k}q^{\binom{k}{2}}\frac{\left[n+k-1\right]_{p,q}!}{\left[n-1\right]_{p,q}!}\frac{1}{\left(q^{k}x\ominus a\right)_{p,q}^{n+k}}. \tag{2.3}$$

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In a like manner above, we get the following result.

Proposition 2. *Let $n \geq 1$ be an integer and $0 \leq k \leq n$, we have*

$$D_{p,q}^{k}\frac{1}{(a\ominus x)_{p,q}^{n}}=p^{\binom{k}{2}}\frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!}\frac{1}{(a\ominus p^{k}x)_{p,q}^{n+k}}. \tag{2.4}$$

Now, we analyze some properties of the (p,q)-exponential functions. We observe that

$$e_{\frac{1}{p},\frac{1}{q}}(x)=\sum_{n=0}^{\infty}\frac{1}{[n]_{\frac{1}{p},\frac{1}{q}}!}p^{-\binom{n}{2}}x^{n}=\sum_{n=0}^{\infty}\frac{1}{[n]_{p,q}!}q^{\binom{n}{2}}x^{n},$$

thus, we get

$$e_{\frac{1}{p},\frac{1}{q}}(x)=E_{p,q}(x). \tag{2.5}$$

In a like manner, we have

$$E_{\frac{1}{p},\frac{1}{q}}(x)=e_{p,q}(x). \tag{2.6}$$

How about the additive property of the (p,q)-exponential functions? The answer is that the (p,q)-exponential functions have not any additive property. Indeed,

$$\begin{aligned} e_{p,q}(x)E_{p,q}(y) &= \left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{x^{n}}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{y^{n}}{[n]_{p,q}!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}}x^{k}y^{n-k}\right)\frac{1}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty}\frac{(x\oplus y)_{p,q}^{n}}{[n]_{p,q}!}, \end{aligned} \tag{2.7}$$

which is not any form of (p,q)-exponential functions ($e_{p,q}(x)$ and $E_{p,q}(x)$).

We define a new type (p,q)-exponential function as

$$\widetilde{e}_{p,q}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{[n]_{p,q}!}. \tag{2.8}$$

From Eqs. (2.7) and (2.8), we deduce

$$e_{p,q}(x)E_{p,q}(y)=\widetilde{e}_{p,q}(x\oplus y)_{p,q},$$

which can be called addition formula for (p,q)-exponentials. Similar to usual exponential function $e^{x}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}$ and q-exponential function $e_{q}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{[n]_{q}!}$, (p,q)-exponential functions has the following derivative property

$$D_{p,q}\widetilde{e}_{p,q}(x)=\widetilde{e}_{p,q}(x).$$

Note that (p,q)-Pascal rules are given by (*cf.* [7])

$$\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}=p^{k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}+q^{n-k+1}\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q} \tag{2.9}$$

and

$$\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}=q^{k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}+p^{n-k+1}\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q}. \tag{2.10}$$

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Here, we give a theorem about properties of the (p,q)-binomial coefficients.

Theorem 1. *Each (p,q)-binomial coefficient is a polynomial including the parameters p and q of degree $k(n-k)$ whose leading coefficient is 1.*

Proof. This theorem can be proved by the same method in the proof of Corollary 6.1. in [17]. $\square$

Also, the coefficients in the polynomial expression of $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ are symmetric.

Note that the (p,q)-binomial coefficients also have combinatorial interpretations like q-binomial coefficients and usual binomial coefficients.

Definition 1. [22] *The function $F(x)$ is a q-antiderivative of $f(x)$ if $D_{p,q}F(x) = f(x)$. It is shown by*

$$\int f(x)\, d_{p,q}x.$$

The following proposition is a (p,q)-generalization of the Proposition 18.1 in the book [17].

Proposition 3. *Let $0 < q < p \leq 1$. Then, any function $f(x)$ has at most one (p,q)-antiderivative which is continuous at $x = 0$, up to adding a constant.*

Proof. By using the similar proof's technical in [17] for q-antiderivative, this theorem can be proved. $\square$

Let us consider the following formula for the change of variable $u = u(x) = \lambda x^{\mu}$ with λ, μ being constants. If $F(x)$ is a (p,q)-antiderivative of $f(x)$, then we have

$$\int f(u)\, d_{p,q}u = F(u) = F(u(x)).$$

Using the expression (2.7) we have for any $\widehat{p}$ and $\widehat{q}$

$$\begin{aligned} F(u(x)) &= \int D_{\widehat{p},\widehat{q}} F(u(x))\, d_{\widehat{p},\widehat{q}}x \\ &= \int \left(D_{\widehat{p}^{\mu},\widehat{q}^{\mu}} F\right)(u(x)).D_{\widehat{p},\widehat{q}}u(x) d_{\widehat{p},\widehat{q}}x \\ &= \int \left(D_{\widehat{p}^{\mu},\widehat{q}^{\mu}} F\right)(u(x))\, d_{\widehat{p},\widehat{q}}u(x). \end{aligned}$$

If we take $\widehat{p} = p^{1/\mu}$ and $\widehat{q} = q^{1/\mu}$, then we have $D_{\widehat{p}^{\mu},\widehat{q}^{\mu}}F = D_{p,q}F = f$, and hence

$$\int f(u)\, d_{p,q}u = \int f(u(x))\, d_{p^{1/\mu},q^{1/\mu}}u(x).$$

This formula implies that $f(u(x))\, D_{p^{1/\mu},q^{1/\mu}}u(x)$ is one of the (p,q)-antiderivatives of $f(x)$.

The (p,q)-integral is defined (see [22]) by

$$\int f(x)\, d_{p,q}x = (p-q)\sum_{k=0}^{\infty} \frac{q^k x}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right).$$

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Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a formal power series. Applying (p,q)-integral to the both sides of $f(x)$ yields to

$$\int f(x)\, d_{p,q}x = \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{[k+1]_{p,q}} + C$$

where C is a constant.

Some simple examples of (p,q)-integral are

$$\begin{aligned}
\int (x \oplus y)_{p,q}^{n}\, d_{p,q}x &= \frac{\left(\frac{x}{p} \oplus y\right)_{p,q}^{n+1}}{[n+1]_{p,q}} + C, \\
\int \widetilde{e}_{p,q}(x)\, d_{p,q}x &= \widetilde{e}_{p,q}(x) + C, \\
\int e_{p,q}(x)\, d_{p,q}x &= e_{p,q}\left(\frac{x}{p}\right) + C, \\
\int E_{p,q}(x)\, d_{p,q}x &= E_{p,q}\left(\frac{x}{q}\right) + C.
\end{aligned}$$

3. (p,q)-Trigonometric Functions

The (p,q)-analogues of the sine, cosine, tangent and cotangent functions can be defined in the same manner with their well known Euler expressions by means of the exponential functions.

Definition 2. *Let $i=\sqrt{-1} \in \mathbb{C}$. Then two pairs of (p,q)-trigonometric functions are defined by*

$$\begin{array}{|c|c|}
\hline
\sin_{p,q} x := \dfrac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i}, & \mathrm{SIN}_{p,q} x := \dfrac{E_{p,q}(ix) - E_{p,q}(-ix)}{2i}, \\
\hline
\cos_{p,q} x := \dfrac{e_{p,q}(ix) + e_{p,q}(-ix)}{2}, & \mathrm{COS}_{p,q} x := \dfrac{E_{p,q}(ix) + E_{p,q}(-ix)}{2}, \\
\hline
\tan_{p,q} x := \dfrac{\sin_{p,q} x}{\cos_{p,q} x}, & \mathrm{TAN}_{p,q} x := \dfrac{\mathrm{SIN}_{p,q} x}{\mathrm{COS}_{p,q} x}, \\
\hline
\cot_{p,q} x := \dfrac{\cos_{p,q} x}{\sin_{p,q} x}, & \mathrm{COT}_{p,q} x := \dfrac{\mathrm{COS}_{p,q} x}{\mathrm{SIN}_{p,q} x}. \\
\hline
\end{array} \tag{3.1}$$

Using the identity (1.7), we have

$$\sin_{p,q} x \,\mathrm{SIN}_{p,q} x = -\frac{e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(-ix) - 2}{4}$$

and

$$\cos_{p,q} x \,\mathrm{COS}_{p,q} x = \frac{e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(-ix) + 2}{4}.$$

Thus, we derive the following formula

$$\sin_{p,q} x \,\mathrm{SIN}_{p,q} x + \cos_{p,q} x \,\mathrm{COS}_{p,q} x = 1,$$

which is the (p,q)-analogue of the well-known formula

$$\sin^2 x + \cos^2 x = 1.$$

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By means of the (p,q)-trigonometric functions, the corresponds secant and cosecant functions are described as

$\sec_{p,q} x := \dfrac{1}{\cos_{p,q} x}$,	$\csc_{p,q} x := \dfrac{1}{\sin_{p,q} x}$,
$\mathrm{SEC}_{p,q} x := \dfrac{1}{\mathrm{COS}_{p,q} x}$,	$\mathrm{CSC}_{p,q} x := \dfrac{1}{\mathrm{SIN}_{p,q} x}$.

Note that the (p,q)-tangent and (p,q)-cotangent functions coincide the following equalities

$$\tan_{p,q} x = \mathrm{TAN}_{p,q} x \text{ and } \cot_{p,q} x = \mathrm{COT}_{p,q} x,$$

which are (p,q)-extension of the results in [6]. The two (p,q)-tangent functions are valid, however, one of these functions may be more useful than the other under special cases.

Now let us investigate (p,q)-derivatives of the (p,q)-trigonometric functions. By making use of (1.8), we readily see that

$$\begin{aligned} D_{p,q} \sin_{p,q} x &= D_{p,q}\left(\frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i}\right) \\ &= \frac{D_{p,q}e_{p,q}(ix) - D_{p,q}e_{p,q}(-ix)}{2i} \\ &= \frac{e_{p,q}(px) + e_{p,q}(-px)}{2} = \cos_{p,q}(px). \end{aligned}$$

By the same way, the others are stated as follows.

Theorem 2. *The (p,q)-derivative operator fulfils the following equalities*

$D_{p,q} \sin_{p,q} x = \cos_{p,q}(px)$,	$D_{p,q} \mathrm{SIN}_{p,q} x = \mathrm{COS}_{p,q}(qx)$,
$D_{p,q} \cos_{p,q} x = -\sin_{p,q}(px)$,	$D_{p,q} \mathrm{COS}_{p,q} x = -\mathrm{SIN}_{p,q}(qx)$,
$D_{p,q} \tan_{p,q} x = 1 + \tan_{p,q}(px)\tan_{p,q}(qx)$,	$D_{p,q} \mathrm{TAN}_{p,q} x = 1 + \mathrm{TAN}_{p,q}(px)\mathrm{TAN}_{p,q}(qx)$,
$D_{p,q} \cot_{p,q} x = -\dfrac{\sin^2_{p,q}(px) + \cos^2_{p,q}(px)}{\sin_{p,q}(px)\sin_{p,q}(qx)}$,	$D_{p,q} \mathrm{COT}_{p,q} x = -\dfrac{\mathrm{SIN}^2_{p,q}(qx) + \mathrm{COS}^2_{p,q}(qx)}{\mathrm{SIN}_{p,q}(px)\mathrm{SIN}_{p,q}(qx)}$,
$D_{p,q} \sec_{p,q} x = \sec_{p,q}(qx)\tan_{p,q}(px)$,	$D_{p,q} \mathrm{SEC}_{p,q} x = \mathrm{SEC}_{p,q}(px)\mathrm{TAN}_{p,q}(qx)$,
$D_{p,q} \csc_{p,q} x = -\csc_{p,q}(qx)\cot_{p,q}(px)$,	$D_{p,q} \mathrm{CSC}_{p,q} x = -\mathrm{CSC}_{p,q}(px)\mathrm{COT}_{p,q}(qx)$.

Now (p,q)-integration properties of the (p,q)-cosine and (p,q)-sine functions are given as follows.

Theorem 3. *The following (p,q)-integrals are verified with easy computations:*

$\int \sin_{p,q} x d_{p,q}x = -\cos_{p,q}\left(\frac{x}{p}\right) + C$,
$\int \cos_{p,q} x d_{p,q}x = \sin_{p,q}\left(\frac{x}{p}\right) + C$,
$\int \mathrm{SIN}_{p,q} x d_{p,q}x = -\mathrm{COS}_{p,q}\left(\frac{x}{q}\right) + C$,
$\int \mathrm{COS}_{p,q} x d_{p,q}x = \mathrm{SIN}_{p,q}\left(\frac{x}{q}\right) + C$.

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The (p,q)-exponential functions are related to the (p,q)-cosine and (p,q)-sine as

$$\begin{aligned}
e_{p,q}(ix) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} i^n \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} p^{(2n-1)n} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} p^{(2n+1)n} x^{2n+1} \\
&= \cos_{p,q}(x) + i \sin_{p,q}(x)
\end{aligned}$$

and

$$\begin{aligned}
E_{p,q}(ix) &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} i^n \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} q^{(2n-1)n} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} q^{(2n+1)n} x^{2n+1} \\
&= \mathrm{COS}_{p,q}(x) + i\,\mathrm{SIN}_{p,q}(x).
\end{aligned}$$

The following theorem includes a connection with (p,q)-sine and (p,q)-cosine functions.

Theorem 4. *We have*

$$\sin_{p,q} x \,\mathrm{COS}_{p,q} x = \cos_{p,q} x \,\mathrm{SIN}_{p,q} x.$$

The (p,q)-analogues of hyperbolic functions can be defined in the same manner with their well known Euler expressions by means of the exponential functions.

Definition 3. *The (p,q)-hyperbolic functions are described as*

$\sinh_{p,q} x = \dfrac{e_{p,q}(x) - e_{p,q}(-x)}{2}$,	$\mathrm{SINH}_{p,q} x = \dfrac{E_{p,q}(x) - E_{p,q}(-x)}{2}$,
$\cosh_{p,q} x = \dfrac{e_{p,q}(x) + e_{p,q}(-x)}{2}$,	$\mathrm{COSH}_{p,q} x = \dfrac{E_{p,q}(x) + E_{p,q}(-x)}{2}$,
$\tanh_{p,q} x = \dfrac{\sinh_{p,q} x}{\cosh_{p,q} x}$,	$\mathrm{TANH}_{p,q} x = \dfrac{\mathrm{SINH}_{p,q} x}{\mathrm{COSH}_{p,q} x}$,
$\coth_{p,q} x = \dfrac{\cosh_{p,q} x}{\sinh_{p,q} x}$,	$\mathrm{COTH}_{p,q} x = \dfrac{\mathrm{COSH}_{p,q} x}{\mathrm{SINH}_{p,q} x}$.

(3.2)

The following relationhips hold true:

$e_{p,q}(x) = \cosh_{p,q} x + \sinh_{p,q} x$,	$E_{p,q}(x) = \mathrm{COSH}_{p,q} x + \mathrm{SINH}_{p,q} x$.

Using the identity (1.7), we have

$$\sinh_{p,q} x \,\mathrm{SINH}_{p,q} x = \frac{e_{p,q}(x)E_{p,q}(x) + e_{p,q}(-x)E_{p,q}(-x) - 2}{4}$$

and

$$\cosh_{p,q} x \,\mathrm{COSH}_{p,q} x = \frac{e_{p,q}(x)E_{p,q}(x) + e_{p,q}(-x)E_{p,q}(-x) + 2}{4}.$$

Hence, we observe that

$$\cosh_{p,q} x \,\mathrm{COSH}_{p,q} x - \sinh_{p,q} x \,\mathrm{SINH}_{p,q} x = 1,$$

which is the (p,q)-analogue of the well known formula

$$\cosh^2 x - \sinh^2 x = 1.$$

Following the (p,q)-hyperbolic functions, the hyperbolic (p,q)-secant and (p,q)-cosecant functions are defined by

$\operatorname{sech}_{p,q} x := \dfrac{1}{\cosh_{p,q} x}$,	$\operatorname{csch}_{p,q} x := \dfrac{1}{\sinh_{p,q} x}$,
$\operatorname{SECH}_{p,q} x := \dfrac{1}{\operatorname{COSH}_{p,q} x}$,	$\operatorname{CSCH}_{p,q} x := \dfrac{1}{\operatorname{SINH}_{p,q} x}$.

The following theorem consist of the (p,q)-derivative properties of (p,q)-hyperbolic functions.

Theorem 5. *The (p,q)-derivative operator fulfils the following equations*

$D_{p,q} \sinh_{p,q} x = \cosh_{p,q}(px)$,	$D_{p,q} \operatorname{SINH}_{p,q} x = \operatorname{COSH}_{p,q}(qx)$,
$D_{p,q} \cosh_{p,q} x = \sinh_{p,q}(px)$,	$D_{p,q} \operatorname{COSH}_{p,q} x = \operatorname{SINH}_{p,q}(qx)$,
$D_{p,q} \tanh_{p,q} x = 1 - \tanh_{p,q}(px) \tanh_{p,q}(qx)$,	$D_{p,q} \operatorname{TANH}_{p,q} x = 1 - \operatorname{TANH}_{p,q}(px) \operatorname{TANH}_{p,q}(qx)$,
$D_{p,q} \coth_{p,q} x = \dfrac{\sinh^2_{p,q}(px) - \cosh^2_{p,q}(px)}{\sinh_{p,q}(px)\sinh_{p,q}(qx)}$,	$D_{p,q} \operatorname{COTH}_{p,q} x = \dfrac{\operatorname{SINH}^2_{p,q}(qx) - \operatorname{COSH}^2_{p,q}(qx)}{\operatorname{SINH}_{p,q}(px)\operatorname{SINH}_{p,q}(qx)}$,
$D_{p,q} \operatorname{sech}_{p,q} x = -\operatorname{sech}_{p,q}(qx) \tanh_{p,q}(px)$,	$D_{p,q} \operatorname{SECH}_{p,q} x = -\operatorname{SECH}_{p,q}(px) \operatorname{TANH}_{p,q}(qx)$,
$D_{p,q} \operatorname{csch}_{p,q} x = -\operatorname{csch}_{p,q}(qx) \coth_{p,q}(px)$,	$D_{p,q} \operatorname{CSCH}_{p,q} x = -\operatorname{CSCH}_{p,q}(px) \operatorname{COTH}_{p,q}(qx)$.

The following theorem includes the (p,q)-integral properties of (p,q)-hyperbolic functions.

Theorem 6. *We have*

$\int \sinh_{p,q}(x)\, d_{p,q}x = \cosh_{p,q}\left(\frac{x}{p}\right) + C$,
$\int \cosh_{p,q}(x)\, d_{p,q}x = \sinh_{p,q}\left(\frac{x}{p}\right) + C$,
$\int \operatorname{SINH}_{p,q}(x)\, d_{p,q}x = \operatorname{COSH}_{p,q}\left(\frac{x}{q}\right) + C$,
$\int \operatorname{COSH}_{p,q}(x)\, d_{p,q}x = \operatorname{SINH}_{p,q}\left(\frac{x}{q}\right) + C$.

The (p,q)-exponential functions are related to the (p,q)- hyperbolic cosine and (p,q)-hyperolic sine by

$$\begin{aligned} e_{p,q}(x) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} p^{(2n-1)n} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} p^{(2n+1)n} x^{2n+1} \\ &= \cosh_{p,q}(x) + \sinh_{p,q}(x) \end{aligned}$$

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and

$$\begin{aligned} E_{p,q}(x) &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} q^{(2n-1)n} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} q^{(2n+1)n} x^{2n+1} \\ &= \mathrm{COSH}_{p,q}\, x + \mathrm{SINH}_{p,q}\, x. \end{aligned}$$

Using the formulas (2.5) and (2.6), we obtain the following results.

Theorem 7. *The following identities are readily attested:*

$\sin_{\frac{1}{p},\frac{1}{q}} x = \mathrm{SIN}_{p,q}\, x,$	$\mathrm{SIN}_{\frac{1}{p},\frac{1}{q}} x = \sin_{p,q}\, x,$
$\cos_{\frac{1}{p},\frac{1}{q}} x = \mathrm{COS}_{p,q}\, x,$	$\mathrm{COS}_{\frac{1}{p},\frac{1}{q}} x = \cos_{p,q}\, x,$
$\tan_{\frac{1}{p},\frac{1}{q}} x = \mathrm{TAN}_{p,q}\, x,$	$\mathrm{TAN}_{\frac{1}{p},\frac{1}{q}} x = \tan_{p,q}\, x,$
$\cot_{\frac{1}{p},\frac{1}{q}} x = \mathrm{COT}_{p,q}\, x,$	$\mathrm{COT}_{\frac{1}{p},\frac{1}{q}} x = \cot_{p,q}\, x,$
$\sec_{\frac{1}{p},\frac{1}{q}} x = \mathrm{SEC}_{p,q}\, x,$	$\mathrm{SEC}_{\frac{1}{p},\frac{1}{q}} x = \sec_{p,q}\, x,$
$\csc_{\frac{1}{p},\frac{1}{q}} x = \mathrm{CSC}_{p,q}\, x,$	$\mathrm{CSC}_{\frac{1}{p},\frac{1}{q}} x = \csc_{p,q}\, x,$
$\sinh_{\frac{1}{p},\frac{1}{q}} x = \mathrm{SINH}_{p,q}\, x,$	$\mathrm{SINH}_{\frac{1}{p},\frac{1}{q}} x = \sinh_{p,q}\, x,$
$\cosh_{\frac{1}{p},\frac{1}{q}} x = \mathrm{COSH}_{p,q}\, x,$	$\mathrm{COSH}_{\frac{1}{p},\frac{1}{q}} x = \cosh_{p,q}\, x,$
$\tanh_{\frac{1}{p},\frac{1}{q}} x = \mathrm{TANH}_{p,q}\, x,$	$\mathrm{TANH}_{\frac{1}{p},\frac{1}{q}} x = \tanh_{p,q}\, x,$
$\coth_{\frac{1}{p},\frac{1}{q}} x = \mathrm{COTH}_{p,q}\, x,$	$\mathrm{COTH}_{\frac{1}{p},\frac{1}{q}} x = \coth_{p,q}\, x,$
$\mathrm{sech}_{\frac{1}{p},\frac{1}{q}} x = \mathrm{SECH}_{p,q}\, x,$	$\mathrm{SECH}_{\frac{1}{p},\frac{1}{q}} x = \mathrm{sech}_{p,q}\, x,$
$\mathrm{csch}_{\frac{1}{p},\frac{1}{q}} x = \mathrm{CSCH}_{p,q}\, x,$	$\mathrm{CSCH}_{\frac{1}{p},\frac{1}{q}} x = \mathrm{csch}_{p,q}\, x.$

In terms of the new (p,q)-exponential function $\widetilde{e}_{p,q}x$, the corresponds trigonometric functions can be defined as

$\widetilde{\sin}_{p,q}x := \dfrac{\widetilde{e}_{p,q}(ix) - \widetilde{e}_{p,q}(-ix)}{2i},$	$\widetilde{\tan}_{p,q}x := \dfrac{\widetilde{\sin}_{p,q}x}{\widetilde{\cos}_{p,q}x},$
$\widetilde{\cos}_{p,q}x := \dfrac{\widetilde{e}_{p,q}(ix) + \widetilde{e}_{p,q}(-ix)}{2},$	$\widetilde{\cot}_{p,q}x := \dfrac{\widetilde{\cos}_{p,q}x}{\widetilde{\sin}_{p,q}x},$
$\widetilde{\sec}_{p,q}x = \dfrac{1}{\widetilde{\cos}_{p,q}x},$	$\widetilde{\csc}_{p,q}x = \dfrac{1}{\widetilde{\sin}_{p,q}x},$

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which satisfy

$D_{p,q}\widetilde{\sin}_{p,q}x=\widetilde{\cos}_{p,q}x,$
$D_{p,q}\widetilde{\cos}_{p,q}x=-\widetilde{\sin}_{p,q}x,$
$D_{p,q}\widetilde{\tan}_{p,q}x=\dfrac{\widetilde{\cos}_{p,q}x+\widetilde{\tan}_{p,q}(px)\widetilde{\sin}_{p,q}x}{\widetilde{\cos}_{p,q}(qx)},$
$D_{p,q}\widetilde{\cot}_{p,q}x=-\dfrac{\widetilde{\sin}_{p,q}x+\widetilde{\cot}_{p,q}(px)\widetilde{\cos}_{p,q}x}{\widetilde{\sin}_{p,q}(qx)},$
$D_{p,q}\widetilde{\sec}_{p,q}x=\dfrac{\widetilde{\sin}_{p,q}x}{\widetilde{\cos}_{p,q}(px)\widetilde{\cos}_{p,q}(qx)},$
$D_{p,q}\widetilde{\csc}_{p,q}x=-\dfrac{\widetilde{\cos}_{p,q}x}{\widetilde{\sin}_{p,q}(px)\widetilde{\sin}_{p,q}(qx)}.$

The (p,q)-exponential function $\widetilde{e}_{p,q}x$ are correlated with the new (p,q)-cosine and (p,q)-sine functions as

$$\begin{aligned}\widetilde{e}_{p,q}(ix) &= \sum_{n=0}^{\infty} i^n \frac{x^n}{[n]_{p,q}!}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{[2n]_{p,q}!}x^{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n}{[2n+1]_{p,q}!}x^{2n+1}\\ &= \widetilde{\cos}_{p,q}(x)+i\widetilde{\sin}_{p,q}(x).\end{aligned}$$

In terms of the new(p,q)-exponential fuction $\widetilde{e}_{p,q}x$, the correspond hyperbolic functions are described as

$\widetilde{\sinh}_{p,q}x:=\dfrac{\widetilde{e}_{p,q}(x)-\widetilde{e}_{p,q}(-x)}{2},$	$\widetilde{\tanh}_{p,q}x:=\dfrac{\widetilde{\sinh}_{p,q}x}{\widetilde{\cosh}_{p,q}x},$
$\widetilde{\cosh}_{p,q}x:=\dfrac{\widetilde{e}_{p,q}(x)+\widetilde{e}_{p,q}(-x)}{2},$	$\widetilde{\coth}_{p,q}x:=\dfrac{\widetilde{\cosh}_{p,q}x}{\widetilde{\sinh}_{p,q}x},$
$\widetilde{\mathrm{sech}}_{p,q}x=\dfrac{1}{\widetilde{\cosh}_{p,q}x},$	$\widetilde{\mathrm{csch}}_{p,q}x=\dfrac{1}{\widetilde{\sinh}_{p,q}x},$

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which fulfil

$D_{p,q}\widetilde{\sinh}_{p,q}x = \widetilde{\cosh}_{p,q}x,$
$D_{p,q}\widetilde{\cosh}_{p,q}x = \widetilde{\sinh}_{p,q}x,$
$D_{p,q}\widetilde{\tanh}_{p,q}x = \dfrac{\widetilde{\cosh}_{p,q}x + \widetilde{\tanh}_{p,q}(px)\,\widetilde{\sinh}_{p,q}x}{\widetilde{\cosh}_{p,q}(qx)},$
$D_{p,q}\widetilde{\coth}_{p,q}x = -\dfrac{\widetilde{\sinh}_{p,q}x - \widetilde{\coth}_{p,q}(px)\,\widetilde{\cosh}_{p,q}x}{\widetilde{\sinh}_{p,q}(qx)},$
$D_{p,q}\widetilde{\operatorname{sech}}_{p,q}x = -\dfrac{\widetilde{\sinh}_{p,q}x}{\widetilde{\cosh}_{p,q}(px)\,\widetilde{\cosh}_{p,q}(qx)},$
$D_{p,q}\widetilde{\operatorname{csch}}_{p,q}x = -\dfrac{\widetilde{\cosh}_{p,q}x}{\widetilde{\sinh}_{p,q}(px)\,\widetilde{\sinh}_{p,q}(qx)}.$

Theorem 8. *The following (p,q)-integrals are valid:*

$\int \widetilde{\sin}_{p,q}x d_{p,q}x = -\widetilde{\cos}_{p,q}x + C,$
$\int \widetilde{\cos}_{p,q}x d_{p,q}x = \widetilde{\sin}_{p,q}x + C,$
$\int \widetilde{\sinh}_{p,q}x d_{p,q}x = \widetilde{\cosh}_{p,q}x + C,$
$\int \widetilde{\cosh}_{p,q}x d_{p,q}x = \widetilde{\sinh}_{p,q}x + C.$

New (p,q)-trigonometric functions can be expressed by earlier defined (p,q)-trigonometric functions as follows.

Theorem 9. *The following equalities*

$\widetilde{\sin}_{p,q}(x \oplus x)_{p,q} = \sin_{p,q}x\,\mathrm{COS}_{p,q}x + \cos_{p,q}x\,\mathrm{SIN}_{p,q}x,$
$\widetilde{\sin}_{p,q}(x \ominus x)_{p,q} = \sin_{p,q}x\,\mathrm{COS}_{p,q}x - \cos_{p,q}x\,\mathrm{SIN}_{p,q}x,$
$\widetilde{\cos}_{p,q}(x \oplus x)_{p,q} = \cos_{p,q}x\,\mathrm{COS}_{p,q}x - \sin_{p,q}x\,\mathrm{SIN}_{p,q}x,$
$\widetilde{\cos}_{p,q}(x \ominus x)_{p,q} = \cos_{p,q}x\,\mathrm{COS}_{p,q}x + \sin_{p,q}x\,\mathrm{SIN}_{p,q}x,$
$\widetilde{\sinh}_{p,q}(x \oplus x)_{p,q} = \sinh_{p,q}x\,\mathrm{COSH}_{p,q}x + \cosh_{p,q}x\,\mathrm{SINH}_{p,q}x,$
$\widetilde{\sinh}_{p,q}(x \ominus x)_{p,q} = \sinh_{p,q}x\,\mathrm{COSH}_{p,q}x - \cosh_{p,q}x\,\mathrm{SINH}_{p,q}x,$
$\widetilde{\cosh}_{p,q}(x \oplus x)_{p,q} = \cosh_{p,q}x\,\mathrm{COSH}_{p,q}x + \sinh_{p,q}x\,\mathrm{SINH}_{p,q}x,$
$\widetilde{\cosh}_{p,q}(x \ominus x)_{p,q} = \cosh_{p,q}x\,\mathrm{COSH}_{p,q}x - \sinh_{p,q}x\,\mathrm{SINH}_{p,q}x$

are true.

The following intriguing identities between (p,q)-trigonometric and (p,q)-hyperbolic functions hold true.

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Theorem 10. *We have*

$\sinh_{p,q} x = -i \sin_{p,q}(ix)$ $\mathrm{SINH}_{p,q} x = -i\, \mathrm{SIN}_{p,q}(ix)$ $\widetilde{\sinh}_{p,q} x = -i\widetilde{\sin}_{p,q}(ix)$	$q \to p = 1$	$\sinh x = -i \sin(ix),$
$\cosh_{p,q} x = \cos_{p,q}(ix)$ $\mathrm{COSH}_{p,q} x = \mathrm{COS}_{p,q}(ix)$ $\widetilde{\cosh}_{p,q} x = \widetilde{\cos}_{p,q}(ix)$	$q \to p = 1$	$\cosh x = \cos(ix),$
$\tanh_{p,q} x = -i \tan_{p,q}(ix)$ $\mathrm{TANH}_{p,q} x = -i\, \mathrm{TAN}_{p,q}(ix)$ $\widetilde{\tanh}_{p,q} x = -i\widetilde{\tan}_{p,q}(ix)$	$q \to p = 1$	$\tanh x = -i \tan(ix),$
$\coth_{p,q} x = i \cot_{p,q}(ix)$ $\mathrm{COTH}_{p,q} x = i\, \mathrm{COT}_{p,q}(ix)$ $\widetilde{\coth}_{p,q} x = i\widetilde{\cot}_{p,q}(ix)$	$q \to p = 1$	$\coth x = i \cot(ix).$

Here the (p,q)-trigonometric functions and the (p,q)-hypergeometric functions are examined whether these functions are odd functions or even functions.

Theorem 11. *We have*

$\sin_{p,q}(-x) = -\sin_{p,q} x,$	$\mathrm{SIN}_{p,q}(-x) = -\mathrm{SIN}_{p,q} x,$	$\widetilde{\sin}_{p,q}(-x) = -\widetilde{\sin}_{p,q} x,$
$\cos_{p,q}(-x) = \cos_{p,q} x,$	$\mathrm{COS}_{p,q}(-x) = \mathrm{COS}_{p,q} x,$	$\widetilde{\cos}_{p,q}(-x) = \widetilde{\cos}_{p,q} x,$
$\tan_{p,q}(-x) = -\tan_{p,q} x,$	$\mathrm{TAN}_{p,q}(-x) = -\mathrm{TAN}_{p,q} x,$	$\widetilde{\tan}_{p,q}(-x) = -\widetilde{\tan}_{p,q} x,$
$\cot_{p,q}(-x) = -\cot_{p,q} x,$	$\mathrm{COT}_{p,q}(-x) = -\mathrm{COT}_{p,q} x,$	$\widetilde{\cot}_{p,q}(-x) = -\widetilde{\cot}_{p,q} x,$
$\sec_{p,q}(-x) = \sec_{p,q} x,$	$\mathrm{SEC}_{p,q}(-x) = \mathrm{SEC}_{p,q} x,$	$\widetilde{\sec}_{p,q}(-x) = \widetilde{\sec}_{p,q} x,$
$\csc_{p,q}(-x) = -\csc_{p,q} x,$	$\mathrm{CSC}_{p,q}(-x) = -\mathrm{CSC}_{p,q} x,$	$\widetilde{\csc}_{p,q}(-x) = -\widetilde{\csc}_{p,q} x,$
$\sinh_{p,q}(-x) = -\sinh_{p,q} x,$	$\mathrm{SINH}_{p,q}(-x) = -\mathrm{SINH}_{p,q} x,$	$\widetilde{\sinh}_{p,q}(-x) = -\widetilde{\sinh}_{p,q} x,$
$\cosh_{p,q}(-x) = \cosh_{p,q} x,$	$\mathrm{COSH}_{p,q}(-x) = \mathrm{COSH}_{p,q} x,$	$\widetilde{\cosh}_{p,q}(-x) = \widetilde{\cosh}_{p,q} x,$
$\tanh_{p,q}(-x) = -\tanh_{p,q} x,$	$\mathrm{TANH}_{p,q}(-x) = -\mathrm{TANH}_{p,q} x,$	$\widetilde{\tanh}_{p,q}(-x) = -\widetilde{\tanh}_{p,q} x,$
$\coth_{p,q}(-x) = -\coth_{p,q} x,$	$\mathrm{COTH}_{p,q}(-x) = -\mathrm{COTH}_{p,q} x,$	$\widetilde{\coth}_{p,q}(-x) = -\widetilde{\coth}_{p,q} x,$
$\mathrm{sech}_{p,q}(-x) = \mathrm{sech}_{p,q} x,$	$\mathrm{SECH}_{p,q}(-x) = \mathrm{SECH}_{p,q} x,$	$\widetilde{\mathrm{sech}}_{p,q}(-x) = \widetilde{\mathrm{sech}}_{p,q} x,$
$\mathrm{csch}_{p,q}(-x) = -\mathrm{csch}_{p,q} x,$	$\mathrm{CSCH}_{p,q}(-x) = -\mathrm{CSCH}_{p,q} x,$	$\widetilde{\mathrm{csch}}_{p,q}(-x) = -\widetilde{\mathrm{csch}}_{p,q} x.$

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