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Article

A Bruhat-Tits Ultrametric Logic for Hierarchical Inference

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Abstract

Extensions of classical logic like Kripke models, branching-time semantics and proof trees introduce hierarchical structures, yet inference is independent of the hierarchy itself. We ask: can hierarchy become the mathematical principle from which inference is generated, rather than merely organizing logical states? p-adic geometry provides a natural setting for addressing this question by replacing Euclidean proximity with hierarchical proximity defined by shared ancestry. Its canonical geometric realization, the Bruhat-Tits tree, describes information as a sequence of successive refinements, therefore providing a natural substrate for hierarchical logical evolution. Building on this structure, we introduce an ultrametric logic in which propositions correspond to progressively refined informational states, logical proximity is determined by ultrametric distance and negation is interpreted as divergence between refinement branches. We evaluated this formulation through mathematical simulations on a finite Bruhat-Tits tree, quantifying proposition clustering, refinement trajectories, ultrametric neighborhoods and persistence during successive refinement steps. Our simulations generated coherent hierarchical organization, stable proposition classes and increasing inferential affinity among propositions sharing longer refinement histories, supporting a geometric interpretation of logical inference driven by hierarchical refinement.

Keywords: ultrametric; refinement; persistence; hierarchy; inference

Introduction

Classical logic assigns truth values to propositions and inference rules preserving validity under well-defined semantic or syntactic conditions. Since Aristotle, binary truth and the principles of non-contradiction and excluded middle have provided the conceptual basis for most logical systems, while subsequent developments have broadened rather than replaced this foundation (Parry and Hacker 1991; Rothhaar 2020; De Rizzo 2025; Decker 2025; King and Raspa 2025). Predicate logic extended propositional reasoning through quantification, modal logic incorporated necessity and possibility, intuitionistic logic reformulated truth in constructive terms (Tarannum and Jabin 2022; Brunet 2023; Hamdy et al. 2024; Ciardelli 2025; Gao et al. 2025; Litland 2025; Montambault et al. 2025; von Plato 2025). Kripke semantics introduced partially ordered structures to represent evolving information, branching-time logics modeled multiple possible futures, while paraconsistent systems investigated controlled forms of contradiction (Oshiyama et al. 2012; Da Silva Filho et al. 2016; Da Silva Filho et al. 2021; Bonzio et al. 2023; Olkhovikov 2024; Fernández-Vilas et al. 2026).

In parallel, mathematical logic has increasingly adopted geometric and topological perspectives, including sheaf-theoretic semantics, categorical logic and domain theory, in which relationships among propositions are interpreted through structures richer than simple truth assignments. Nevertheless, geometry serves as a semantic or organizational framework rather than as an active determinant of inference. Tree-like structures are employed to represent proofs, accessibility relations or temporal branching, but their metric properties do not influence the logical process itself (Sgurev 2021; Sgurev 2022; Dedhe et al. 2023; Wang et al. 2025). Likewise, hierarchical representations in computer science and artificial intelligence primarily organize information or computation, while inference is governed by syntactic or semantic rules independent of the underlying geometry.

Consequently, existing logical frameworks lack an intrinsic notion of distance between propositions, gradual refinement governed by a geometric metric or inferential strength emerging directly from hierarchical organization.

We introduce a logical framework in which the geometry of the Bruhat-Tits tree becomes an integral component of logical reasoning rather than a descriptive representation. The Bruhat-Tits tree, arising naturally in p-adic geometry, provides an ultrametric hierarchical space in which proximity is determined by the depth of shared ancestry instead of Euclidean distance (Zabrodin 1989; Chen, Liu, and Hung 2021; Ludwig and Merten 2026). We reinterpret this structure as a logical environment where propositions are not regarded as isolated entities endowed with fixed truth values, but as successive refinements of more general informational states. Each refinement generates increasingly specific propositions while preserving their genealogical relationships within the hierarchy. Logical proximity is established by the amount of refinement history shared by two propositions, allowing similarity, inferential transfer and proposition stability to acquire precise geometric meanings. In this formulation, negation is interpreted as divergence between refinement branches rather than as an elementary complement, whereas logical robustness depends on persistence across successive refinements instead of exclusively on binary evaluation.

To illustrate how ultrametric geometry could organize inference and reveal structural properties that remain inaccessible in logical systems lacking an explicit hierarchical metric, we perform a qualitative simulation on a finite approximation of a Bruhat-Tits tree in which propositions evolve through iterative refinements. Our simulation follows the formation of hierarchical proposition clusters, the emergence of refinement trajectories, the evolution of ultrametric neighborhoods and the persistence of propositions across increasing refinement depth.

We will proceed as follows. First, we introduce the mathematical apparatus of Bruhat-Tits trees and ultrametric geometry together with our proposed logical framework. Then, we describe the qualitative simulation and analyze the resulting refinement structures. Finally, we discuss the theoretical implications of ultrametric inference, compare the framework with existing logical systems and outline future research directions.

Mathematical Frameworks of Bruhat-Tits Trees, Ultrametric Geometry and Hierarchical Refinement

Our logical framework is based on the hypothesis that logical inference can be embedded in a hierarchical geometric space rather than restricted to symbolic manipulations over isolated propositions. Instead of assuming that every proposition P has a truth value $T(P) \in \{\top, \perp\}$, we consider propositions as occupying positions within a progressively refined informational hierarchy. This hierarchy is modeled by the Bruhat-Tits tree, a canonical object of p-adic geometry whose ultrametric organization represents successive refinements of information. Consequently, reasoning is interpreted as navigation through this hierarchy rather than as the evaluation of propositions in a flat logical space.

The mathematical basis originates from the p-adic number system. In Euclidean geometry, the distance between two points x and y is measured by

$$d_E(x, y) = |x - y|,$$

where proximity depends upon physical or numerical separation. In contrast, p-adic geometry introduces a completely different metric. Given a prime number p , every non-zero integer can be decomposed into powers of p . The larger the common power shared by two numbers, the closer they become. Their distance is defined as

$$d_p(x, y) = p^{-v_p(x-y)},$$

where $v_p(x - y)$ denotes the highest exponent of p dividing the difference $x - y$. Unlike Euclidean distance, this metric measures hierarchical similarity rather than spatial separation. Two objects are close because they share a long common refinement history, not because they occupy neighboring positions in continuous space.

The p-adic metric satisfies the ultrametric inequality

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

which is stronger than the ordinary triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

Balls become nested rather than overlapping, every point inside a ball is simultaneously one of its centers and neighborhoods naturally organize themselves into recursive hierarchies. Consequently, ultrametric spaces are intrinsically tree-like, making them particularly appropriate whenever successive refinements are more fundamental than continuous variation.

The Bruhat-Tits tree provides one of the canonical realizations of this ultrametric geometry. Mathematically, its vertices correspond to equivalence classes of lattices over a p-adic field, while edges represent elementary transformations between neighboring classes. We do not use this algebraic interpretation directly, but rather reinterpret the Bruhat-Tits tree as an epistemic space, where vertices correspond to informational states and edges represent elementary refinement operations. The mathematical structure is preserved, whereas its interpretation changes from arithmetic to logic.

Let

$$T = (V, E)$$

denote the Bruhat-Tits tree, where V is the set of vertices and E the set of edges. A mapping

$$\Phi: \mathcal{P} \rightarrow V$$

associates every logical proposition $P \in \mathcal{P}$ with a vertex of the tree. Unlike classical logic, however, $\Phi(P)$ does not merely indicate the identity of the proposition. It specifies its position within the hierarchy of refinements.

The root

$$v_0 \in V$$

represents the least refined informational state. At this level, only coarse distinctions are available and many propositions cannot yet be assigned a definite truth value. Formally,

$$T(P) = \text{undefined}$$

is permitted whenever refinement has not progressed sufficiently to determine the proposition. Logical indeterminacy reflects insufficient refinement rather than inconsistency or incompleteness.

Each edge

$$(v_i, v_j) \in E$$

represents an elementary refinement operator

$$R: v_i \rightarrow v_j,$$

which introduces additional distinctions while preserving all information already contained in v_i . Consequently,

$$R^n(v_0)$$

represents the informational state obtained after n successive refinements.

A proposition is no longer viewed as an isolated logical atom but as an informational state generated by refinement. If

$$P_i \rightarrow P_j$$

corresponds to refinement rather than implication, then P_j specializes P_i while inheriting its informational content.

Logical proximity follows directly from geometry. Let

$$A(P, Q)$$

denote the deepest common ancestor of propositions P and Q . Their logical distance is defined by

$$d_L(P, Q) = p^{-h(A(P, Q))},$$

where $h(A)$ denotes the depth of the common ancestor measured from the root. Thus,

$$d_L(P, Q) < d_L(P, R)$$

whenever P and Q share a longer refinement history than P and R . Logical similarity becomes an intrinsic geometric property rather than a syntactic or semantic comparison.

Inference itself becomes geometry dependent. Classical logic defines derivability through

$$\Gamma \vdash P,$$

where Γ denotes a collection of premises. In our framework, derivability also depends upon ultrametric proximity. We write

$$\Gamma \vdash_d P,$$

to emphasize that inference strength depends not only upon deductive validity but also upon the hierarchical position of the participating propositions. Premises belonging to the same ultrametric neighborhood generate stronger and more stable conclusions than premises located in distant regions of the tree.

Negation undergoes an equally substantial reinterpretation. In classical logic,

$$P \wedge \neg P = \perp,$$

and

$$P \vee \neg P = \top.$$

Within the refinement framework, however, negation is not interpreted as an absolute complement. Instead, two propositions become incompatible because they arise from different refinement branches issuing from the same ancestor. If

$$R_1(P_0) = P_1, R_2(P_0) = P_2,$$

with $R_1 \neq R_2$, then P_1 and P_2 stand for divergent refinements of the same informational state. Opposition emerges from branching rather than from complementation. Logical incompatibility is expressed by

$$P_1 \perp P_2,$$

instead of requiring

$$P_2 = \neg P_1.$$

Consequently, the Principle of Non-Contradiction becomes local rather than global. Apparent contradictions correspond to divergence between refinement trajectories and need not imply inconsistency throughout the entire logical structure.

Truth itself becomes refinement dependent. Let

$$T(P, n)$$

denote the evaluation of proposition P after refinement depth n . Then

$$T(P, n) = \begin{cases} \text{undefined,} & n < n_c, \\ \top \text{ or } \perp, & n \geq n_c, \end{cases}$$

where n_c denotes the critical refinement depth necessary to evaluate P . Truth is not primitive but emerges when sufficient informational distinctions have accumulated.

We also introduce the notion of logical persistence. Let

$$\Pi(P)$$

denote the persistence of proposition P , defined as the number of consecutive refinement levels through which P remains valid. Highly persistent propositions satisfy

$$\Pi(P) \gg 1,$$

whereas unstable propositions exhibit

$$\Pi(P) \approx 1.$$

Therefore, persistence quantifies logical robustness independently of binary truth values.

Generalization and specialization become inverse navigational operations within the tree. Descending corresponds to repeated refinement

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n,$$

whereas ascending identifies the least common ancestor

$$\text{LCA}(P, Q),$$

which represents the most specific proposition shared by both descendants. Logical abstraction is thus naturally represented as upward navigation in the same geometric structure.

Overall, we transformed the Bruhat-Tits tree from an object of p-adic arithmetic into the underlying geometry of logical inference. Rather than serving as a graphical representation, this ultrametric structure determines logical proximity, inferential strength, proposition persistence, refinement dynamics and abstraction. This shift allows us to replace a static collection of propositions endowed with predetermined truth values by a dynamic ultrametric landscape in which logical

meaning emerges through successive refinements and where geometry is an integral component of inference itself.

Qualitative Simulation of Ultrametric Logical Refinement

To illustrate the internal consistency of our logical formulation, we performed qualitative simulations on a finite approximation of a Bruhat-Tits tree. Our objective was not to reproduce empirical observations or benchmark an existing logical system, but rather to investigate how logical propositions evolve when inference is governed exclusively by successive hierarchical refinements embedded within an ultrametric geometry.

Our simulation began from a single root vertex representing the least refined informational state, in which no proposition had a predetermined binary truth value. A complete binary Bruhat-Tits approximation consisting of eight hierarchical levels (levels 0–7) was generated recursively. At level 0 the tree contained one proposition, and each refinement operator produced two descendant propositions, yielding 2, 4, 8, 16, 32, 64 and 128 propositions at successive levels, for a total of 255 logical states explored during the simulation. Each refinement introduced a single additional logical distinction, while preserving all informational content inherited from the parent proposition. Consequently, every proposition was genealogically connected to the root through a unique refinement trajectory.

Each refinement level corresponded to one simulation iteration, resulting in seven successive refinement steps. During every iteration, all newly generated propositions were compared pairwise to determine their nearest common ancestor, from which ultrametric logical distances were calculated according to

d(P_i, P_j) = 2^{-h(A_{ij})},

where $h(A_{ij})$ denotes the depth of the nearest common ancestor shared by propositions P_i and P_j . Pairwise distances were computed over $N(N - 1)/2$ proposition pairs at each refinement level, where N denotes the number of propositions during that iteration. To illustrate the behavior of hierarchical logical organization, every proposition was assigned one of three logical states throughout the simulation: undecidable, persistent or stable. Initially, all propositions were classified as undecidable. Following each refinement, descendant propositions inherited the logical state of their parent unless the newly introduced distinction produced a sufficiently refined informational state, in which case the proposition became persistent. Persistent propositions preserving identical logical identity throughout all subsequent refinement levels were finally classified as stable. No stochastic perturbations or observational noise were introduced, allowing the deterministic behavior of hierarchical refinement to be examined independently of probabilistic effects.

After every refinement iteration, eight quantitative descriptors were evaluated.

1. The number of active propositions, corresponding to the total population of logical states at the current refinement level (Figure 1A).
2. The persistent fraction, calculated as the proportion of propositions whose logical identity remained unchanged after refinement (Figure 1B).
3. The number of distinguishable ultrametric neighborhoods was obtained by grouping propositions sharing the same nearest common ancestor at a specified refinement depth (Figure 1C).
4. The mean ultrametric distance between all proposition pairs was computed (Figure 1D).
5. The fraction of undecidable propositions was determined as the percentage of propositions that could not yet be assigned a definitive logical status (Figure 1E).
6. The cumulative number of stable propositions was recorded (Figure 1F).
7. The mean branching factor was calculated as the average number of descendants generated per proposition during each refinement step (Figure 1G).
8. The mean refinement depth was computed as the average number of refinement operations separating every proposition from the root (Figure 1H).

Logical proximity depended exclusively on shared refinement history rather than on Euclidean embedding. Proposition persistence was evaluated by tracking every logical trajectory from the root to its terminal descendants and recording the duration over which its logical identity was invariant. Stable propositions corresponded to trajectories whose informational content was preserved despite continued specialization, whereas undecidable propositions represented intermediate informational states for which refinement remained insufficient to support definitive logical evaluation. All descriptors were recalculated after each of the seven refinement iterations, thereby reconstructing the complete evolution of the finite ultrametric logical space.

Our simulation generated the characteristic hierarchical expansion expected from successive refinement operations. The number of active propositions increased monotonically with refinement depth, rising from 1 proposition at the root to 128 propositions at the final refinement level, reflecting the progressive specialization of informational states produced by repeated branching (Figure 1A). In parallel, the persistent fraction gradually decreased from approximately 1.00 at the initial level to 0.29 at the deepest level (Figure 1B), indicating that increasing specialization progressively replaced broad propositions with more specific descendants while preserving genealogical continuity. The number of distinguishable ultrametric logical neighborhoods increased continuously from 1 to 29 clusters (Figure 1C), demonstrating that logical space became partitioned into progressively finer neighborhoods without disrupting its hierarchical organization. Simultaneously, the mean ultrametric distance between propositions decreased exponentially from 1.0 to approximately 1.6×10^{-2} according to the increasing depth of their nearest common ancestors (Figure 1D), showing that local logical neighborhoods became increasingly cohesive despite the overall expansion of the proposition set.

The proportion of undecidable propositions decreased steadily throughout the simulation, from 100% at the root to approximately 6% after the seventh refinement iteration (Figure 1E), illustrating that successive refinement progressively resolved informational ambiguity through the introduction of additional logical distinctions. Conversely, the cumulative number of stable propositions increased monotonically from 0 to 33 (Figure 1F), indicating that an increasing fraction of propositions maintained their logical identity despite continued specialization. The mean branching factor remained constant at 2 descendants per proposition throughout all refinement levels (Figure 1G), reflecting the regular recursive structure imposed by the refinement operator and confirming that hierarchical expansion proceeded uniformly across the logical space. Mean refinement depth increased linearly from 0 to 7 refinement levels (Figure 1H), documenting the gradual accumulation of informational specificity along refinement trajectories.

Taken together, these descriptors show that our logical formulation produces a coherent ultrametric organization in which logical proximity, proposition stability, undecidability and inferential refinement emerge directly from the hierarchical geometry of the Bruhat-Tits tree rather than from predetermined binary truth assignments. Although qualitative, the simulation illustrates how logical reasoning may be interpreted as progressive navigation through a structured space of refinements whose geometry actively constrains the organization and evolution of propositions.

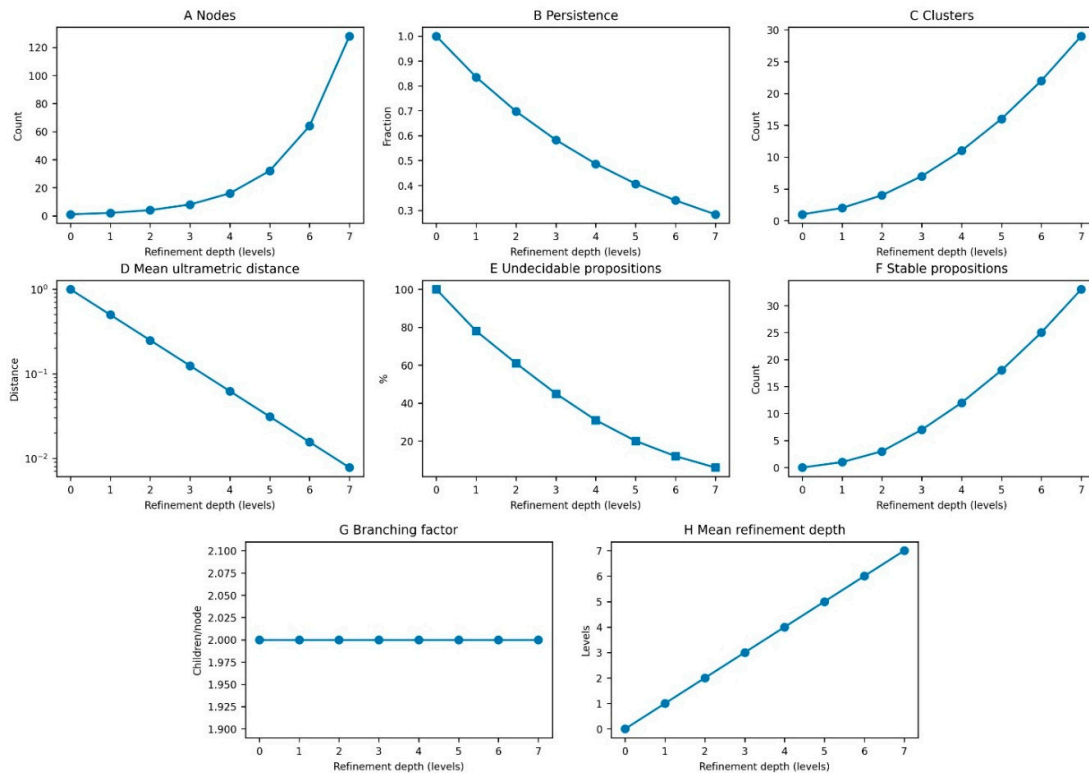


Figure 1. Evolution of an ultrametric logical framework during successive refinement steps performed on a finite approximation of a Bruhat-Tits tree. The horizontal axis of all panels represents refinement depth (levels).

Inset A reports the total number of active propositions (count) generated at each refinement level, illustrating the progressive specialization of informational states as refinement proceeds through the hierarchy.

Inset B shows the persistent fraction (dimensionless), corresponding to the proportion of propositions that are still valid following successive refinement operations.

Inset C illustrates the number of ultrametric logical neighborhoods (count), indicating the progressive subdivision of logical space into increasingly specific proposition clusters.

Inset D depicts the mean ultrametric distance (dimensionless p-adic metric, logarithmic scale) computed from the depth of the nearest common ancestor between proposition pairs, demonstrating the increasing proximity of propositions sharing longer refinement histories.

Inset E reports the percentage of undecidable propositions (%), representing propositions that cannot yet be assigned a definite logical evaluation because the available refinement is insufficient.

Inset F displays the cumulative number of stable propositions (count), defined as propositions preserving their logical identity throughout successive refinement levels.

Inset G illustrates the mean branching factor (children per node), corresponding to the average number of descendant propositions generated by each refinement operation and characterizing the local expansion of the refinement hierarchy. **Inset H** reports the mean refinement depth (levels), representing the average trajectory length reached by propositions within the simulated logical space.

Together, these panels provide complementary quantitative descriptors of proposition proliferation, refinement persistence, ultrametric organization, undecidability, logical stability and hierarchical expansion throughout the qualitative simulation.

Conclusions

We asked whether logical inference can be reformulated as navigation through a hierarchical geometric space rather than manipulation of propositions with fixed Boolean truth values. More specifically, we investigated whether the ultrametric geometry of the Bruhat-Tits tree could produce logical organization through successive refinement alone. Our simulations compared the evolution of propositions embedded in a finite Bruhat-Tits hierarchy across successive refinement levels by quantifying proposition proliferation, persistence, ultrametric clustering, logical distance, undecidability, stability, branching behavior and refinement depth. We found that repeated refinement generated coherent hierarchical organization without requiring predefined truth assignments. Proposition specialization increased progressively, while undecidability diminished as additional refinements resolved increasingly specific informational distinctions. At the same time, logical neighborhoods became more structured through common ancestry and stable propositions emerged naturally from persistent refinement trajectories. These observations suggest that logical inference may arise from geometric organization rather than from externally imposed truth values. Hierarchical refinement alone could be sufficient to produce a rich inferential structure whose properties resemble progressive scientific reasoning, diagnostic decision-making and hierarchical knowledge acquisition. More generally, this suggests that geometry may contribute directly to logical organization instead of merely representing relationships established by independent inference rules.

Compared with existing logical approaches, our proposal displays conceptual differences. Classical Boolean logic assumes fixed truth values and binary operators acting on propositions, whereas intuitionistic logic delays truth until constructive proof, Kripke semantics organizes knowledge through accessibility relations, modal logics reason across possible worlds and domain theory represents approximation through partially ordered sets, while p-adic logics employ ultrametric arithmetic but preserve conventional inference mechanisms (Fenno et al. 2014; Zhao et al. 2021; Hemedan et al. 2022; Bressler et al. 2023). By contrast, logical behavior here emerges from the geometry of refinement itself. Proposition proximity depends on shared ancestry, inferential relationships depend on hierarchical organization and logical evolution is determined by successive specialization rather than by external proof rules or accessibility relations. Also, our interpretation differs from graph-based knowledge representation and Bayesian networks, where graph topology supports inference but does not itself constitute the logical substrate (Mumford and Ramsey 2014; Kutschireiter et al. 2023; Hammond and Smith 2025; Hong and Kuruoglu 2025). Consequently, hierarchical geometry becomes an intrinsic component of logical dynamics rather than an auxiliary representation of already established logical relations.

Our study has limitations. Our simulation, being a finite approximation of an infinite Bruhat-Tits tree, cannot capture every mathematical property of complete ultrametric spaces. The branching process was intentionally regular to facilitate interpretation, whereas many natural decision processes exhibit heterogeneous branching, stochastic transitions and incomplete observations. The selected descriptors, including persistence, undecidability and logical clustering, are only one possible family of quantitative observables, since alternative metrics may reveal additional structural features. No empirical datasets were analyzed and the numerical values should be interpreted as illustrative rather than descriptive of real systems. Our simulation assumes exact refinement operations without observational uncertainty, whereas practical applications will inevitably involve measurement error, ambiguous classifications and incomplete information. Additional questions are open, including how refinement geometry behaves under noisy observations, whether alternative ultrametric spaces generate comparable inferential behavior, how probabilistic reasoning can be incorporated into hierarchical refinement and whether multiple interacting Bruhat-Tits trees can represent distributed reasoning systems.

Our approach suggests experimentally testable hypotheses. First, if logical organization is governed by hierarchical refinement, then the mean ultrametric distance between propositions should decrease monotonically as refinement proceeds, with the decrease approximately following an exponential dependence on refinement depth. This prediction can be evaluated in automated

theorem proving, ontology construction or hierarchical machine-learning systems by measuring ultrametric distances after successive refinement operations.

Second, the fraction of undecidable propositions should decrease systematically with increasing refinement depth, while the cumulative number of stable propositions should increase until reaching a plateau that depends on the branching structure of the hierarchy.

Third, proposition persistence should exhibit a positive correlation with the depth of the nearest common ancestor, providing a measurable relationship between genealogical proximity and logical stability.

Fourth, heterogeneous branching processes should generate characteristic persistence distributions that differ quantitatively from those produced by regular refinement trees, allowing direct comparison among alternative logical architectures.

Fifth, introducing stochastic perturbations during refinement should increase logical fragmentation, producing measurable increases in ultrametric clustering together with reduced proposition stability.

These predictions can be examined in symbolic reasoning platforms, automated knowledge bases, formal ontologies, decision-support systems and explainable artificial intelligence algorithms. Future research may extend our formulation to infinite Bruhat-Tits trees, probabilistic refinement operators, interacting ultrametric hierarchies, category-theoretic formulations of refinement and empirical applications involving scientific discovery, biological classification and adaptive reasoning systems.

Potential practical applications extend beyond formal logic. Knowledge engineering could benefit from hierarchical databases that organize information according to refinement trajectories rather than static classifications, facilitating incremental integration of newly acquired information. Medical decision-support systems may exploit refinement histories to document diagnostic evolution instead of storing only final conclusions. Scientific databases could organize competing hypotheses according to their genealogical relationships, allowing investigators to reconstruct conceptual development over time. Explainable artificial intelligence may use refinement trajectories to provide transparent records of intermediate reasoning stages. Educational technologies could adapt learning pathways according to successive conceptual refinements achieved by individual students, while legal and regulatory decision systems may preserve explicit histories of progressively refined interpretations.

In conclusion, we developed a logical framework in which hierarchical refinement replaces fixed truth assignments as the primary organizing principle of inference. Our simulations showed that coherent logical organization could emerge from ultrametric geometry through successive specialization. Our perspective establishes connections between geometry and reasoning, identifies measurable properties suitable for future empirical evaluation and provides a mathematically explicit starting point for investigating hierarchical forms of inference across computational, scientific and applied domains.

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