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Gocgen Approach for Bounded Gaps Between Odd Composite Numbers

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Abstract: We developed some questions regarding twin prime conjecture by examining the bounded gaps between composite numbers and explained the relations between the formulas regarding composite numbers. Then we proved the twin prime conjecture to answer the new questions raised by proving the expression:

$$\liminf_{n\to\infty}(c_{n+1} - c_n) \geq 10.$$

In addition, we prove many different approaches have been and it has been proved that there must be a twin prime in the range $(p_{k2} \cdot p_{k1}, p_{k1}^2)$, where $p \neq 2$.

Keywords: number theory; twin prime conjecture; gocgen approach; prime numbers

MSC2020: 11A41

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1. Introduction

Let p_n denote the n-th prime. Twin prime conjecture is conjectured that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) = 2.$$

Can potential twin primes be expressed with the formula composite number after a number forever?

We can add bound to this question by using formulas that generate composite numbers. When the question is simply taken into bound, the targeted range can be examined more easily. The approach in this paper; it involves establishing the bounds mentioned and proving the twin prime conjecture by further examination.

Lemmas

LEMMA 1. According to Aysun and Gocgen [1]:

np+p gives all composite numbers where n is a positive natural numbers and p is a prime number.

Proof. $np + p = p(n+1)$. Then, according to fundamental theorem of arithmetic:

$$(n+1 \in \mathbb{C}) \oplus (n+1 \in \mathbb{P})$$

Let $n+1 \in \mathbb{C}$:

$$n+1 = p_m \times \cdots \times p_{m+k}$$

Then,

$$p(n+1) = p \times (p_m \times \cdots \times p_{m+k})$$

Let $n+1 \in \mathbb{P}$:

$$n+1 = p_m$$

Then,

$$p(n+1) = p \times p_m$$

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LEMMA 2. According to Aysun and Gocgen [1].
2np+p gives all odd composite numbers where n is a positive natural numbers and p is an odd prime numbers.

Proof. $np + p$ gives odd composite numbers where p is a odd number and n is a even number. Then as already proved $np + p$ gives all composite numbers where n is a positive natural number and p is an prime number. Only possibility for odd composite just specified. Therefore, $np + p$ gives all odd composite numbers where p is a odd number and n is a even number. This equal to: $2np + p$ gives all odd composite numbers where n is a positive natural numbers and p is an odd prime numbers.

LEMMA 3. *According to Eratosthenes sieve theory, natural number between 2 and N not divisible by any prime smaller or equal than the square root of N is a prime number.*

Proof. Let $2 \leq n \leq N$. Write $n = p_1 \ldots p_r$ as product of primes, where $r > 0$. If n is not divisible by any prime smaller or equal than $N^{\frac{1}{2}}$, then $p_i > N^{\frac{1}{2}}$, for all i. Hence $n > N^{\frac{r}{2}} \geq n^{\frac{r}{2}}$. Therefore, $1 > \frac{r}{2}$, which implies $r = 1$.

New approach like Lemma 3 by Lemma 1. In the expression $np + p \leq N$, we must give the minimum value of n for the largest value of p. Then: $1 \cdot p + p = 2p \leq N$. Based on this, we can say $p \leq \frac{N}{2}$.

New approach like Lemma 3 by Lemma 2. In the expression $2np + p \leq N$, we must give the minimum value of n for the largest value of p. Then: $2 \cdot 1 \cdot p + p = 3p \leq N$. Based on this, we can say $p \leq \frac{N}{3}$.

LEMMA 4. According to Rhafli [2]:
$2np + p^2$ *gives all odd composite numbers where n is a natural numbers and p is a odd prime numbers.*

Proof. with $n \in \mathbb{N}$ and p are all the primes except 2 which satisfy $p \leq \sqrt{N}$, the equation $2np + p^2$ = all odd composite is true since if we divide it by p we get the trivial equation for odd numbers. For a given interval $I = [a, b]$ one calculates the constant n and iterates to generates the odd composites included in the interval I.

Since the proofs of the following statements are dense and long, only the statements accepted as Lemma are given without citing any evidence, by citing articles directly related to the proof.

LEMMA 5. According to Zhang, Maynard and Polymath project [3–5]:

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 7 \times 10^7,$$

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 4680,$$

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 246.$$

LEMMA 6. According to Goldston and Graham and Pintz and Yildirim [6,7]:
q is exactly the product of two distinct prime numbers:

$$\liminf_{n\to\infty}(q_{n+1} - q_n) \leq 26,$$

$$\liminf_{n\to\infty}(q_{n+1} - q_n) \leq 6.$$

2. Theorems and Proofs

Preliminary Theorem: We can pose a twin prime conjecture like this:
Can at least one of the values formed by the expression $6n + 6 \pm 1$ with a difference of 2 be expressed with the formula $2np + p$ forever after a certain number?

Preliminary Proof: In the $2np + p$ formula, the expression for $p = 3$ is $6n + 3$. This expression produces single composite numbers divisible by 3. Values that cannot be obtained with this expression give odd numbers that cannot be divided by 3. In order to determine the values that cannot be obtained with this expression: it is necessary to find the expressions that create a single value between the expressions $6n + 3$ and $6(n + 1) + 3$. Expressions that create a single value between $6n + 3$ and $6n + 9$: $6n + 5$ and $6n + 7$, i.e. $6n + 6 \pm 1$. Then, the expression $6n + 6 \pm 1$ produces odd numbers that cannot

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be divided by 3, and if at least one of the results produced with a difference of 2 is composite, that is, if it can be expressed with the formula $2np + p$, it is understood that the twin primes are not infinite.

New Question: In order for the desired values to be composites, we can analyze how many gaps should be between the odd composites from a certain value to infinity. 1. group: $6n + 5$, $6n + 7$ 2. group: $6(n + 1) + 5, 6(n + 1) + 7$. Then: 1. group: $6n + 5$, $6n + 7$ 2. group: $6n + 11$, $6n + 13$. We know that at least one value from each group must be composite. In that case:

1) $6n + 5$ and $6n + 11$ can be composite. Bounded of gaps: 6
2) $6n + 5$ and $6n + 13$ can be composite. Bounded of gaps: 8.
3) $6n + 7$ and $6n + 11$ can be composite. Bounded of gaps: 4.
4) $6n + 7$ and $6n + 13$ can be composite. Bounded of gaps: 6.
5) $6n + 5$, $6n + 7$ and $6n + 11$, $6n + 13$ can be composite. Bounded of gaps: 2.

Accordingly, in order for the twin primes not to be infinite, the gap between the composites that must occur forever after a certain number must be 6 and/or 8 and/or 4 and/or 2.

Then we can pose a new question as follows: Can the gap between odd composite numbers that are not divisible by 3 be 6 and/or 8 and/or 4 and/or 2 forever after a certain number?

If the answer to this question is yes, then twin primes are finite. If the answer to this question is no, then twin primes are infinite.

In this logic; 6, 8, 4, 2 can be considered as the key to preventing the infinity of twin primes.

Then the following question can be asked:

c: composite numbers that cannot be divided by 3:

$$\liminf_{n\to\infty}(c_{n+1} - c_n) \geq 10?$$

Inference: The n value in this question only affects the initial value. Therefore $n + a$ can be given by $a \in N^+$. The question will not change:

The expressions $6(n + a) + 3$ and $6(n + a) + 9$ are produces composite divisible by 3.

Therefore:

The expressions $6(n + a) + 5$ and $6(n + a) + 7$ produces composite numbers that cannot be divided by 3 primes, that is, cannot be expressed with $6(n + a) + 36$.

$$6(n + a) + 5, 6(n + a) + 7 \ (s.\ \text{group})$$
$$6(n + a + 1) + 5, 6(n + a + 1) + 7 \ (s + 1.\ \text{group})$$

Let's organize the groups:

$$6n + 6a + 5, 6n + 6a + 7 \ (s.\ \text{group})$$
$$6n + 6a + 6 + 5, 6n + 6a + 6 + 7 \ (s + 1.\ \text{group})$$

Let's continue organizing:

$$6n + 6a + 5, 6n + 6a + 7 \ (s.\ \text{group})$$
$$6n + 6a + 11, 6n + 6a13 \ (s + 1.\ \text{group})$$

Accordingly, let's examine the possibilities where at least one value in both groups is a composite, and let's look at the gap that must remain between the composites forever after a certain number so that the twin primes are not infinite:

1) $6n + 6a + 5$ and $6n + 6a + 11$ can be composite. Bounded of gaps: 6.
2) $6n + 6a + 5$ and $6n + 6a + 13$ can be composite. Bounded of gaps: 8.
3) $6n + 6a + 7$ and $6n + 6a + 11$ can be composite. Bounded of gaps: 4.
4) $6n + 6a + 7$ and $6n + 6a + 13$ can be composite. Bounded of gaps: 6.
5) $6n + 6a + 5$, $6n + 6a + 7$ and $6n + 6a + 11$, $6n + 6a + 13$ can be composite. Bounded of gaps: 2.

Accordingly, in order for the twin primes not to be infinite, the gap between the composites that must occur forever after a certain number must be 6 and/or 8 and/or 4 and/or 2.

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Then we can pose a new question as follows: Can the gap between odd composite numbers that are not divisible by $p_1, \dots p_k$ be 6 and/or 8 and/or 4 and/or 2 forever after a certain number?

If the answer to this question is yes, then twin primes are finite. If the answer to this question is no, then twin primes are infinite.

Therefore, the following question arises:

$p \neq 2$ c: composite numbers that cannot be divided by $p_1, \dots, p_k$:

$$\lim_{n\to\infty} \inf(c_{n+1} - c_n) \geq 10?$$

So for the question $n + a$, $a \in N^+$ can be given; it is proven that the problem will not change.

Certain values will repeat in the $2np + p$ formula. For example, since the value 15 is 3×5, the formula $2np + p$ returns 15 with different n values both when $p = 3$ and when $p = 5$. p values that do not create this situation are composite numbers greater than p^2 and p^2. Since values smaller than p^2 can already occur with other p values in the formula, the constant in the formula is designated as p^2 and naturally n as $\mathbb{N}$ rather than $\mathbb{N}^+$. replaceable. This ensures there are fewer repeating values in the formula.

Thus, the natural relationship between two formulas produced independently of each other is understood.

Considering $2np + p^2$ on the question, the question will not change:

1. group odd values between $6n + 9$, $6(n + 1) + 9$.

2. group: odd values between $6(n + 1) + 9$, $6(n + 2) + 9$.

If we arrange the expressions in the groups:

1. group: odd values between $6n + 9$, $6n + 15$.

2. group: odd values between $6n + 15$, $6n + 21$.

As a result:

1. group: $6n + 11, 6n + 13$.

2. group: $6n + 17, 6n + 19$.

In order for twin primes not to be infinite, at least one of the two values in each group must be composite forever after a certain number. Accordingly, let's examine the possibilities in which at least one value in both groups is a composite, and let's look at the gap that must remain between the composites forever after a certain number so that the twin primes are not infinite:

1) $6n + 11$ and $6n + 17$ can be composite. Bounded of gaps: 6.

2) $6n + 11$ and $6n + 19$ can be composite. Bounded of gaps: 8.

3) $6n + 13$ and $6n + 17$ can be composite. Bounded of gaps: 4.

4) $6n + 13$ and $6n + 19$ can be composite. Bounded of gaps: 6.

5) $6n + 11$, $6n + 13$ and $6n + 17$, $6n + 19$ can be composite. Bounded of gaps: 2.

Accordingly, in order for the twin primes not to be infinite, the gap between the composites that must occur forever after a certain number must be 6 and/or 8 and/or 4 and/or 2.

Then we can pose a new question as follows: Can the gap between odd composite numbers that are not divisible by 3 be 6 and/or 8 and/or 4 and/or 2 forever after a certain number?

If the answer to this question is yes, then twin primes are finite. If the answer to this question is no, then twin primes are infinite.

In this way, it is proven that there will be no change when a question is generated from the formula $2np + p^2$.

Cumulative Question: The expressions $6n + p_1 \times \cdots \times p_k - 6$ and $6n + p_1 \times \cdots \times p_k$ produce composite numbers divisible by $p_1 \times \cdots \times p_k$.

Therefore:

The expressions $6n + p_1 \times \cdots \times p_k - 4$ and $6n + p_1 \times \cdots \times p_k - 2$ produce composite numbers that cannot be divided by $p_1 \dots p_k$ primes, that is, cannot be expressed with $6n + p_1 \times \cdots \times p_k - 6$.

$$6n + p_1 \times \cdots \times p_k - 4, 6n + p_1 \times \cdots \times p_k - 2 \text{ (}s\text{. group)}$$

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$$6(n+1)n + p_1 \times \cdots \times p_k - 4, 6(n+1)n + p_1 \times \cdots \times p_k - 2 \ (s+1.\ \text{group})$$

Let's organize the groups:

$$6n + p_1 \times \cdots \times p_k - 4, 6n + p_1 \times \cdots \times p_k - 2 \ (s.\ \text{group})$$
$$6n + p_1 \times \cdots \times p_k + 2, 6n + p_1 \times \cdots \times p_k + 4 \ (s+1.\ \text{group})$$

Accordingly, let's examine the possibilities where at least one value in both groups is a composite, and let's look at the gap that must remain between the composites forever after a certain number so that the twin primes are not infinite:

1) $6n + p_1 \times \cdots \times p_k - 4$ and $6n + p_1 \times \cdots \times p_k + 2$ can be composite. Bounded of gaps: 6.
2) $6n + p_1 \times \cdots \times p_k - 4$ and $6n + p_1 \times \cdots \times p_k + 4$ can be composite. Bounded of gaps: 8.
3) $6n + p_1 \times \cdots \times p_k - 2$ and $6n + p_1 \times \cdots \times p_k + 2$ can be composite. Bounded of gaps: 4.
4) $6n + p_1 \times \cdots \times p_k - 2$ and $6n + p_1 \times \cdots \times p_k + 4$ can be composite. Bounded of gaps: 6.
5) $6n + p_1 \times \cdots \times p_k - 4, 6n + p_1 \times \cdots \times p_k - 2$ and $6n + p_1 \times \cdots \times p_k + 2, 6n + p_1 \times \cdots \times p_k + 4$ can be composite. Bounded of gaps: 2.

Accordingly, in order for the twin primes not to be infinite, the gap between the composites that must occur forever after a certain number must be 6 and/or 8 and/or 4 and/or 2.

Then we can pose a new question as follows: Can the gap between odd composite numbers that are not divisible by $p_1, \ldots p_k$ be 6 and/or 8 and/or 4 and/or 2 forever after a certain number?

If the answer to this question is yes, then twin primes are finite. If the answer to this question is no, then twin primes are infinite.

Therefore, the following question arises:

$p \neq 2$

c: composite numbers that cannot be divided by $p_1, \ldots, p_k$:

$$\liminf_{n\to\infty}(c_{n+1} - c_n) \geq 10?$$

Basis 1: At this point, no matter what the difference between the first composites is, there must be a twin prime in that range.

Answer: For the difference between composite numbers that are not divisible by $p_1, \ldots, p_k$:

$$\lim_{n\to 1}(c_{n+1} - c_n) = ((p_{k+2} \cdot p_{k+1}) - (p_{k+1} \cdot p_{k+1})$$

Let's edit:

$$\lim_{n\to 1}(c_{n+1} - c_n) = ((p_{k+2} \cdot p_{k+1}) - p_{k+1}^2)$$

According to Lemma 5: Since p is a prime number, and

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 7 \times 10^7,$$

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 4680,$$

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 246$$

are proved, the number of cases where 7×10^7 differs between p_{k+1} and p_{k+2} is infinite. Therefore ($p_{k+1} = x$, $p_{k+2} = x + 7 \times 10^7$),

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = ((x + 7 \times 10^7 \cdot x) - x^2)$$

Let's edit this expression:

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = (x^2 + 7 \times 10^7 x - x^2)$$

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = (7 \times 10^7 x)$$

The number of cases where there is a 4680 difference between the expressions p_{k+1} and p_{k+2} is infinite. Therefore $(p_{k+1} = x, p_{k+2} = x + 4680)$,

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = ((x + 4680 \cdot x) - x^2)$$

Let's edit this expression:

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = (x^2 + 4680x - x^2)$$

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = (4680x)$$

The number of cases where there is a 246 difference between the expressions p_{k+1} and p_{k+2} is infinite. Therefore $(p_{k+1} = x, p_{k+2} = x + 246)$,

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = ((x + 246 \cdot x) - x^2)$$

Let's edit this expression:

$$\liminf_{n\to\infty}(c_{n+1} - c_n) = (x^2 + 246x - x^2)$$

$$\liminf_{n\to\infty}(c_{n+1}1 - c_n) = (246x)$$

When it is not forgotten that x is prime:

$$\liminf_{n\to\infty}(c_{n+1} - c_n) \geq 10.$$

Accordingly, the gap between odd composite numbers that cannot be divided by $p_1, \ldots p_k$ cannot be 6 and/or 8 and/or 4 and/or 2 forever after a certain number.

Therefore twin primes are infinite.

New Result: This outcome affords an opportunity to refine the outcomes derived from Lemma 6 (named as Gift for Aygul shortly GFA):

Since the twin primes are infinite, there are an infinite number of cases that are divisible by p_k and where there is a $p \times 2$ difference between the composites consisting of the product of only two primes.

That is, $q_{n+1} = p_k \times p_{n+1}$ and $q_n = p_k \times p_n$:

$$\liminf_{n\to\infty}(p_k \times p_{n+1} - p_k \times p_n) = p \times 2.$$

New Approach: Using the proof of infinity of twin primes, the following conclusion can be made for the detection of twin primes; p being prime: As long as $p_{k+2} \times p_{k+1} - p_{k+1}^2 \geq 10$, there must be twin primes in the range $(p_{k+2} \times p_{K+1}, p_{K+1}^2)$. We can create a new approach by making an observation about this:

The expression $(p_{k+2} \times p_{K+1} - p_{k+1}^2)$ can be expressed as $(p_{k+2} - p_{k+1}) \cdot p_{k+1}$. Then, as long as $(p_{k+2} - p_{k+1}) \cdot p_{k+2} \geq 10$, a twin prime occurs. The only case where $(p_{k+2} - p_{k+1}) \cdot p_{k+1} < 10$ is when $p_{k+1} = 3, p_{k+2} = 5$. For these values, $(5 - 3) \cdot 3 = 6 < 10$ occurs. When looked at, these values will exceptionally provide the desired situation due to *Basis 1*. Since $(p_{k+2} - p_{k+1}) \cdot p_{k+1} \geq 10$ at all other values; there must be twin primes in the range $(p_{k+2} \times p_{k+1}, p_{k+1}^2)$ for $p \in \mathbb{P}$ and $p \neq 2$.

Q. E. D.

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