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Short Note

Functional van Lint-Seidel Relative and Gerzon Universal Bounds

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Abstract

We introduce the notion of functional equiangular lines in finite rank modules over subrings of $\mathbb{R}$. We show that van Lint-Seidel relative and Gerzon universal bounds hold for this most general lines.

Keywords: equiangular lines

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1. Introduction

Let $d \in \mathbb{N}$ and $\gamma \in [0, 1]$. Recall that a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ is said to be **γ -equiangular lines** [1,2] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j, k \leq n, j \neq k.$$

A fundamental problem associated with equiangular lines is the following.

Problem 1. *Given $d \in \mathbb{N}$ and $\gamma \in [0, 1]$, what is the upper bound on n such that there exists a collection $\{\tau_j\}_{j=1}^n$ of γ -equiangular lines in $\mathbb{R}^d$?*

Two answers to Problem (1) which are driving forces in the study of equiangular lines is the following relative bound of van Lint and Seidel [2,3] and universal bound of Gerzon [4].

Theorem 1. [2,3] (*van Lint-Seidel Relative Bound*) *Let $\{\tau_j\}_{j=1}^n$ be γ -equiangular lines in $\mathbb{R}^d$. Then*

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}.$$

Theorem 2. [4] (*Gerzon Universal Bound*) *Let $\{\tau_j\}_{j=1}^n$ be γ -equiangular lines in $\mathbb{R}^d$. Then*

$$n \leq \frac{d(d + 1)}{2}.$$

The notion of functional equiangular lines is hinted in [5]. In this paper, we define it in most general form and derive functional forms of Theorems 1 and 2.

2. Functional Equiangular Lines

In the paper, $\mathcal{R}$ denotes a subring of $\mathbb{R}$ and $\mathcal{M}$ is a rank d module over $\mathcal{R}$. By $\mathcal{M}^*$ we mean the module of all homomorphisms from $\mathcal{M}$ to $\mathcal{R}$. Module of all homomorphisms from $\mathcal{M}$ to $\mathcal{M}$ is denoted by $\text{Mor}(\mathcal{M})$.

Definition 1. Let $\{\tau_j\}_{j=1}^n$ be a collection in a module $\mathcal{M}$ of rank d over $\mathcal{R}$ and $\{f_j\}_{j=1}^n$ be a collection in $\mathcal{M}^*$. Let $\gamma \geq 0$. The pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be **functional γ -equiangular** if following conditions hold.

- (i) $f_j(\tau_j) = 1$ for all $1 \leq j \leq n$.
- (ii) $|f_j(\tau_k)f_k(\tau_j)| = \gamma^2$ for all $1 \leq j, k \leq n, j \neq k$.
- (iii) The operator

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto \sum_{j=1}^n f_j(x)\tau_j \in \mathcal{M}$$

is similar (through invertible operator) to a diagonal operator.

Theorem 3. (Functional van Lint-Seidel Relative Bound) If $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is functional γ -equiangular for rank d module $\mathcal{M}$, then

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}.$$

Proof. Define

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto S_{f,\tau}x := \sum_{j=1}^n f_j(x)\tau_j \in \mathcal{M}.$$

Let $\lambda_1, \dots, \lambda_d$ be eigenvalues of $S_{f,\tau}$. Since $S_{f,\tau}$ is diagonalizable,

$$\begin{aligned} n^2 &= \left(\sum_{j=1}^n f_j(\tau_j) \right)^2 = (\text{Tra}(S_{f,\tau}))^2 = \left(\sum_{k=1}^d \lambda_k \right)^2 \\ &\leq d \sum_{k=1}^d \lambda_k^2 = d \text{Tra}(S_{f,\tau}^2) = d \sum_{j=1}^n \sum_{k=1}^n f_j(\tau_k)f_k(\tau_j) \\ &= d \sum_{j=1}^n f_j(\tau_j)^2 + d \sum_{j,k=1, j \neq k}^n f_j(\tau_k)f_k(\tau_j) = dn + d \sum_{j,k=1, j \neq k}^n f_j(\tau_k)f_k(\tau_j) \\ &\leq dn + d \sum_{j,k=1, j \neq k}^n |f_j(\tau_k)f_k(\tau_j)| = dn + d(n^2 - n)\gamma^2. \end{aligned}$$

Therefore

$$n \leq d + d(n-1)\gamma^2 \implies n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

□

We are unable to derive Gerzon universal bound for functional equiangular lines. However, we derive following Gerzon bound for functionals.

Theorem 4. (Functional Gerzon Universal Bound) Let $\{\tau_j\}_{j=1}^n$ be a collection in a module $\mathcal{M}$ of rank d over $\mathcal{R}$ and $\{f_j\}_{j=1}^n$ be a collection in $\mathcal{M}^*$. Assume the following.

- (i) $f_j(\tau_j) = 1$ for all $1 \leq j \leq n$.
- (ii) There is a $\gamma \neq 1$ such that $f_j(\tau_k)f_k(\tau_j) = \gamma^2$ for all $1 \leq j, k \leq n, j \neq k$.

Then

$$n \leq d^2.$$

Proof. For $1 \leq j \leq n$, define

$$\tau_j \otimes f_j : \mathcal{M} \ni x \mapsto (\tau_j \otimes f_j)(x) := f_j(x)\tau_j \in \mathcal{M}$$

We show that the collection $\{\tau_j \otimes f_j\}_{j=1}^n$ in $\text{Mor}(\mathcal{M})$ is linearly independent over $\mathcal{R}$. Let $c_1, \dots, c_n \in \mathcal{R}$ be such that

$$\sum_{j=1}^n c_j(\tau_j \otimes f_j) = 0.$$

Let $1 \leq k \leq n$ be fixed. Then previous equation gives

$$\sum_{j=1}^n c_j(\tau_j \otimes f_j)(\tau_k \otimes f_k) = 0.$$

By taking trace we get

$$\begin{aligned} 0 &= \sum_{j=1}^n c_j \text{Tra}((\tau_j \otimes f_j)(\tau_k \otimes f_k)) = \sum_{j=1}^n c_j f_j(\tau_k) f_k(\tau_j) \\ &= \sum_{j=1, j \neq k}^n c_j f_j(\tau_k) f_k(\tau_j) + c_k f_k(\tau_k)^2 = \sum_{j=1, j \neq k}^n c_j \gamma^2 + c_k \\ &= \gamma^2 \left(\sum_{j=1}^n c_j - c_k \right) + c_k = \gamma^2 \left(\sum_{j=1}^n c_j \right) + (1 - \gamma^2) c_k. \end{aligned}$$

Therefore

$$c_k = \frac{\gamma^2}{\gamma^2 - 1} \sum_{j=1}^n c_j =: c, \quad \forall 1 \leq k \leq n.$$

Now

$$\begin{aligned} 0 &= \text{Tra} \left(\sum_{j=1}^n c_j(\tau_j \otimes f_j) \right) = \text{Tra} \left(\sum_{j=1}^n c(\tau_j \otimes f_j) \right) \\ &= \sum_{j=1}^n c \text{Tra}(\tau_j \otimes f_j) = \sum_{j=1}^n c f_j(\tau_j) = cn. \end{aligned}$$

Hence $c = 0$. Therefore $\{\tau_j \otimes f_j\}_{j=1}^n$ is linearly independent. Since the rank of $\text{Mor}(\mathcal{M})$ is d^2 we must have $n \leq d^2$. $\square$

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