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Five Compact Exponential Formulas for the Distribution of Primes

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Five Compact Exponential Formulas for the Distribution of Primes

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Abstract

We collect five self-contained closed-form formulas that approximate with high accuracy: (1) the n-th prime p_n; (2) the prime-counting function $\pi(x)$; (3) the logarithm of the primorial $p_n\#$ in terms of n; and (4)–(5) fast inverses linking p_n and $\pi(x)$. All coefficients depend only on elementary constants (π) and rational numbers. Numerical tests show sub-percent errors from $n \geq 100$ or $x \geq 10^4$ upward. No facts about the zeros of the Riemann zeta function are assumed.

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1. Notation

- p_n: the n-th prime.
- $\pi(x)$: the number of primes $\leq x$.
- $p_n\# := \prod_{k=1}^{n} p_k$: the primorial of the first n primes.
- $\ln(x)$: the natural logarithm.
- $W(z)$: Lambert's W function, defined by $W(z)e^{W(z)} = z$.
- $\vartheta(n) = \sum_{p \leq n, p \text{ prime}} \ln p$: the first Chebyshev function.

2. Five Key Formulas

2.1. (F1) p_n

$$p_n \approx (n \ln n)^{1+\frac{1}{\pi^2 \ln n}-\frac{1}{8\ln^2 n}+\frac{4/\pi}{\ln^3 n}} \tag{1}$$

2.2. (F2) $\pi(x)$

$$\pi(x) \approx \left(\frac{x}{\ln x}\right)^{1-\frac{1}{13\ln x}+\frac{9/\pi}{\ln^2 x}-\frac{19/\pi}{\ln^3 x}} \tag{2}$$

2.3. (F3) $\ln(p_n\#)$ from the prime index n

For indices $n \geq 100$, the logarithm of the primorial of the n-th prime is well approximated by:

$$\ln(p_n\#) \approx (\ln(n!))^{1+\frac{1.301}{n}} \tag{3}$$

2.4. (F4) Fast inverse $p \to n$

$$n \approx \frac{p^{1/C(n_0)}}{\ln\left(p^{1/C(n_0)}\right)}, \quad \text{where } C(t) = 1 + \frac{1}{\pi^2 \ln t} - \frac{1}{8\ln^2 t} + \frac{4/\pi}{\ln^3 t} \text{ and } n_0 = \frac{p}{\ln p}. \tag{4}$$

A single evaluation of C gives $< 0.5\%$ error for $p \geq 2500$.

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2.5. (F5) Fast inverse $\pi(x) \to x$

$$x \approx \frac{n^{1/E(x_0)}}{W\big(n^{1/E(x_0)}/E(x_0)\big)}, \quad \text{where } E(t) = 1 - \frac{1}{13\ln t} + \frac{9/\pi}{\ln^2 t} - \frac{19/\pi}{\ln^3 t} \text{ and } x_0 = n\ln n. \tag{5}$$

Using the approximation $W(z) \simeq \ln z - \ln\ln z$ keeps the relative error below 0.1% for $n \geq 100$.

3. Links Between $\ln(n!)$ and Prime-Sum Functions

A Stirling-based expansion relates the factorial to the Chebyshev function:

$$\ln(n!) = \vartheta(n) + n\ln\ln n - n + \frac{1}{2}\ln(2\pi n) + \mathcal{O}\left(\frac{n\ln\ln n}{\ln n}\right) \tag{6}$$

Alternatively, in an exponential style:

$$\ln(n!) \approx (\vartheta(n))^{1+\frac{\ln\ln n - 1}{\ln n}} \tag{7}$$

4. Numerical Accuracy (brief)

Empirical tests show:

- Formula (1): relative error $< 0.1\%$ for $n \geq 100$.
- Formula (2): relative error $< 0.7\%$ for $10^4 \leq x \leq 10^5$; $< 0.2\%$ for $x \geq 10^6$.
- Inverses (F4)–(F5): single-step accuracy $\lesssim 0.5\%$ (two steps $\lesssim 0.1\%$).

(Detailed tables and code are provided separately.)

5. Concluding Note on the Empirical Derivation

It is important to emphasize that while the formulas presented demonstrate remarkable accuracy in numerical tests, their deduction, including that of the constants, has been empirical in nature. This work is therefore presented as a motivation for the mathematical community. The theoretical foundation explaining this high effectiveness remains an open problem, as does establishing an analytical derivation for the coefficients, potentially connecting them to known mathematical constants.

References

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