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Dynamic Equilibria with Nonsmooth Utilities and Stocks: An L^∞ Differential GQVI Approach

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Abstract

We develop a comprehensive dynamic Walrasian framework entirely in L^∞ so that prices and allocations are essentially bounded, and market clearing holds *pointwise almost everywhere*. Utilities are allowed to be *locally Lipschitz and quasi-concave*; we employ Clarke subgradients to derive generalized quasi-variational inequalities (GQVIs). We endogenize inventories through a capital-accumulation constraint, leading to a *differential* QVI (dQVI). Existence is proved under either strong monotonicity or pseudo-monotonicity and coercivity. We establish *Walras' law, complementarity, stability and sensitivity* of the equilibrium correspondence in L^2-metrics, incorporate *time-discounting* and *uncertainty* on $\Omega \times [0, T]$, and present convergent *numerical schemes* (Rockafellar–Wets penalties and extragradient). Our results close the "in mean vs pointwise" gap noted in dynamic models and connect to modern decomposition approaches for QVIs.

Keywords: Walrasian dynamic competitive equilibrium; (generalized) quasi-variational inequalities; weak-$*$ compactness; Clarke subgradients; differential QVI; stability; numerical schemes

MSC: 91B50, 90C33, 47J20, 49J52

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1. Introduction

The variational inequality (VI/QVI/GQVI) approach to competitive equilibrium has become a powerful alternative to fixed-point methods for both theory and computation. In dynamic settings, however, classical Lebesgue space formulations with prices and allocations in L^p $(1 \le p < \infty)$ typically guarantee only *clearing in mean*, because feasibility is encoded as integral inequalities. This leaves open whether markets clear almost everywhere in time and whether complementary slackness holds pointwise. The aim of this paper is to close that gap by developing a fully L^∞ framework that ensures *pointwise (a.e.) market clearing and complementarity*, while simultaneously accommodating nonsmooth (locally Lipschitz, quasi-concave) utilities, inventories/stockpiling via dynamic capital laws, discounting, and uncertainty, together with provably convergent algorithms.

The L^∞ setting is natural in continuous time: prices and consumptions are bounded economic processes, budget constraints are well defined as L^∞–L^1 pairings, and the price simplex is weak-$*$ compact by Banach–Alaoglu. This compactness enables existence arguments that are unavailable in L^p for $p < \infty$, where the price set lacks compactness. Moreover, a simple yet robust *testing by simple prices* device converts integral excess-demand inequalities into pointwise complementarity conditions, yielding a.e. clearing.

We organize our main results around seven themes that advance the dynamic Walrasian VI/QVI/GQVI literature:

(C1) *Functional setting in L^∞.* Prices and allocations live in $L^\infty(\mathcal{T})$; the price set is a weak-$*$ compact simplex and budget sets are weak-$*$ compact bands. Existence of equilibrium follows without additional price compactifications, and we obtain *a.e. market clearing and complementarity* by a simple-function testing argument.

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(C2) *From "in mean" to a.e. clearing.* We prove that solutions of the master VI against the convex hull of simple price functions imply $\sum_a(x_a^{(j)} - e_a^{(j)}) \le 0$ and $\bar{p}^{(j)} \cdot \sum_a(x_a^{(j)} - e_a^{(j)}) = 0$ a.e. for each good j. This closes the gap noted in dynamic L^p models where only integral clearing was available.

(C3) *Nonsmooth utilities and GQVI.* Allowing locally Lipschitz, quasi-concave instantaneous utilities, we formulate household optimality via *generalized* VIs using Clarke subgradients, relying on measurable selection theorems to obtain measurable subgradient selections and well-posedness of the generalized VI.

(C4) *Production, stockpiling, and dQVI.* We introduce inventories through linear capital accumulation with depreciation and show that the joint household–firm–inventory system admits an equilibrium characterized as a *differential* QVI (dQVI).

(C5) *Qualitative properties.* Under strong monotonicity (in an L^2 metric) or under pseudo-monotonicity with coercivity, we prove existence; with strong monotonicity we derive *Lipschitz stability* of the equilibrium correspondence and Walras' law. These provide regularity and sensitivity results in the spirit of stability analyses for dynamic price VIs.

(C6) *Numerics.* We give implementable schemes: Rockafellar–Wets penalty methods to enforce budgets and Korpelevich's extragradient for the monotone GVI. Under strong monotonicity, we prove linear convergence; in the merely monotone case we obtain $O(1/k)$ ergodic rates. A detailed dynamic Cobb–Douglas example (single- and multi-good) illustrates closed forms, a price fixed-point map, and discretized extragradient updates.

(C7) *Scalability.* After time discretization the GQVI decomposes by agents and time slabs. We outline a Dantzig–Wolfe–type master/worker decomposition that aligns with contemporary QVI decomposition frameworks, enabling large-scale computations.

Our analysis builds on and sharpens a line of results that positioned dynamic equilibria within VI/QVI theory. Time-dependent Walrasian formulations and computational procedures were developed for continuous time in variational form (e.g., [23–27,42]), and quasi-variational models for quasi-concave preferences were advanced in [3,4]. Conceptual remarks on Lebesgue-space modeling and the limitations of mean-value clearing appear in [15,16]. Regularity and sensitivity for dynamic price VIs (e.g., [51]) highlighted the importance of monotonicity and stability. On the nonsmooth side, generalized subdifferential tools and measurable selection techniques (e.g., [14,43,48]) allow us to treat locally Lipschitz, quasi-concave utilities in a dynamic GQVI form, complementing the static analyses in [3,4]. For the "evolution" and stability viewpoint, we connect to recent work on equilibrium evolution and stability properties [49]. Algorithmically, we lean on classical extragradient methods [39] and penalty approaches [48], and relate our decomposition discussion to recent QVI master/worker architectures (see, e.g., [32]).

With respect to the methodology, three technical devices are central. First, weak-$*$ compactness in L^∞ (Banach–Alaoglu) ensures compact feasible sets for prices and bounded consumption bands. Second, *Mosco convergence* of budget sets with respect to L^1 perturbations of prices yields stability of household solutions and continuity of aggregate excess demand. Third, *testing against simple prices*—indicator functions of measurable sets times simplex vertices—translates the master VI into pointwise complementarity via the Lebesgue differentiation theorem. These yield a.e. market clearing without requiring additional compactifications or restrictive price dynamics. For nonsmooth utilities we adopt Clarke subdifferentials with measurable selections, enabling generalized VI statements while retaining monotonicity (or pseudo-monotonicity) needed for existence and stability.

Conceptually the novelties can be so summarized: (i) our L^∞ formulation delivers a.e. clearing and complementarity in continuous time, closing the "in mean vs a.e." gap for dynamic Walrasian models; (ii) we integrate inventories and stockpiling explicitly via a differential law within a QVI system; (iii) we accommodate locally Lipschitz, quasi-concave utilities and still provide existence and regularity; (iv) we provide constructive, convergent numerical schemes with rates under strong monotonicity; and (v) we connect the theory to scalable decomposition architectures for large agent/time systems.

The rest of manuscript is organized as follows. Section 2 surveys variational (VI/QVI/GQVI) approaches to (general) competitive equilibrium and clarifies where our model fits and extends this line. Section 3 reviews the L^∞ functional setting, compactness and stability tools for our variational formulation, and records measurable-selection facts used throughout. Section 4 presents the L^∞ model, feasible sets, budget constraints, and the price simplex, and introduces the Clarke-based GVI form of household problems with discounting. Section 5 states and proves the main existence theorem with a.e. clearing and complementarity, Walras' law, stability bounds, and the dQVI extension with inventories. Section 6 builds on these bounds to prove convergence of a projected extragradient scheme on the price simplex and to quantify first–order time–discretization errors for the L^∞ model. Section 7 develops a dynamic Cobb–Douglas economy in L^∞, deriving closed-form demands when band constraints are slack, constructing a price fixed-point with a.e. clearing, and illustrating computation via a discretized extragradient scheme. Finally, Section 8 concludes and outlines directions on nonconvex technologies, robust preferences, and accelerated/primal–dual algorithms. An expanded technical appendix collects weak-$*$ compactness, Mosco convergence, and measurable selection lemmas used in the proofs.

2. Literature Overview

To situate our contribution within the VI/QVI/GQVI literature on competitive equilibrium, we group works by methodological theme and indicate how our L^∞ framework advances each line.

(L1) *Variational formulations of equilibrium.* Casting equilibria as VIs/QVIs has roots in classical monotone operator theory and variational analysis; see, among others, [33,36,48]. In the exchange setting, [33] formulated price determination as a VI on a convex feasible set, opening a path beyond fixed-point methods. Our analysis follows this route but works entirely in L^∞ and moves from integral feasibility to *pointwise (a.e.)* complementarity.

(L2) *Time-dependent/dynamic Walrasian models.* For continuous-time markets, [42] and the series [23–27] developed evolutionary VI/QVI formulations with prices and allocations in Lebesgue spaces, typically obtaining *clearing in mean*. These papers established existence and computational procedures under various monotonicity assumptions. Our contribution complements this line by proving existence *in* L^∞ and converting the master VI into a.e. complementarity through a simple-function testing device, thereby closing the "in mean vs a.e." gap.

(L3) *Quasi-concavity, quasi-variational inequalities, and nonsmooth utilities.* Allowing quasi-concave utility weakens convexity and naturally leads to QVIs; see [3,4]. Dynamic settings with locally Lipschitz utilities require generalized subdifferentials; we rely on Clarke calculus and measurable selections (cf. [14,43,48]) to formulate household optimality as a *generalized* VI (GVI) in continuous time. This bridges static QVI treatments with dynamic, nonsmooth preferences (see also [46]).

(L4) *Lebesgue-space modeling and the role of* L^∞*.* Conceptual remarks on choosing Lebesgue spaces for dynamic equilibrium and consequences for feasibility appear in [15,16]. When $p \in L^p$ for $p < \infty$, price sets are not compact and clearing emerges only in integral form. By placing prices and consumptions in L^∞ we exploit weak-$*$ compactness (Banach–Alaoglu) and obtain a.e. clearing via testing on indicator-price simple functions. This shift is the cornerstone of our existence and complementarity results.

(L5) *Production, stockpiling, and dynamic constraints.* Production and inventories can be incorporated into dynamic Walrasian models through time-dependent constraints and state equations. Variational formulations for time-dependent equilibria are surveyed in [42]. We formalize inventories through a linear capital-accumulation law with depreciation and derive a *differential* QVI (dQVI), extending the exchange-only formulations in [25–27].

(L6) *Stability, sensitivity, and evolution.* Regularity and sensitivity for price-based dynamic VIs are treated in [51]. The broader stability/evolution viewpoint for equilibria has been advanced in [49], which studies how equilibria vary under perturbations and in time. In our framework, strong monotonicity (in an L^2 metric) yields *Lipschitz* dependence of optimal allocations on prices and leads to stability of the aggregate excess map. We also provide a pointwise Walras' law in the L^∞ setting.

(L7) *Stochasticity, discounting, and measurability.* Discounting is standard in intertemporal utility and integrates seamlessly in VI formulations (e.g., [25]). For uncertainty on $\Omega \times \mathcal{T}$, measurability issues are handled through Komlós-type subsequences and measurable selections [14,37]. We extend existence and a.e. clearing to the product space, obtaining pointwise complementarity in (ω, t).

(L8) *Computational methods.* Early computational procedures for time-dependent Walrasian VIs are discussed in [23]. For monotone operators on convex sets, the extragradient method [39] is a robust baseline, while penalty methods provide a principled way to enforce budget and complementarity constraints [48]. We analyze both in our L^∞ model and establish linear convergence under strong monotonicity and $O(1/k)$ ergodic rates in the merely monotone case. Our dynamic Cobb–Douglas example offers closed forms and a price fixed-point iteration that is readily discretized.

(L9) *Decomposition and scalability.* Network and decomposition ideas for equilibrium computation have a long pedigree (e.g., [22,44,45]). The rise of QVI decomposition is pushing scalability to large agent/time systems; see [32] for a recent Dantzig–Wolfe style architecture for QVIs. After time discretization, our GVI/GQVI separates by agents and time slabs, enabling master–worker price updates and parallel household subproblems, consistent with these decomposition paradigms.

Relative to the above lines of investigation, our contribution is to: (i) formulate the dynamic competitive equilibrium purely in L^∞ and prove existence with *a.e. clearing and complementarity*; (ii) accommodate locally Lipschitz, quasi-concave preferences via Clarke GVI and establish stability; (iii) embed inventories through a dQVI; (iv) extend to discounted and stochastic environments with measurable selections; and (v) provide convergent algorithms with practical discretizations and decomposition pathways. Together these results resolve the mean-clearing limitation and align dynamic Walrasian theory with modern monotone operator methods and large-scale computation.

3. Preliminaries

In this section we collect basic facts on weak-* compactness in L^∞, Mosco convergence of budget sets, measurable Clarke subgradients, and a simple testing principle for a.e. inequalities.

Notation

For $f \in L^\infty(\mathcal{T}, \mathbb{R}^l)$ and $g \in L^1(\mathcal{T}, \mathbb{R}^l)$ the duality pairing is $\langle f, g \rangle = \int_0^T f(t) \cdot g(t)\, dt$. For $x = (x_a)_{a=1}^n$ we write $\|x\|_2^2 = \sum_{a=1}^n \int_0^T |x_a(t)|^2\, dt$ (an L^2 metric used for stability). For a measurable multifunction $M : \mathcal{T} \rightrightarrows \mathbb{R}^l$, $\operatorname{gph} M = \{(t, y) : y \in M(t)\}$.

Weak-∗ compactness of bands

Lemma 3.1 (Bands are weak-∗ compact (Banach–Alaoglu [1]; see also [2, Thm. 5.116])). *Let $\overline{x} \in L^\infty_+(\mathcal{T}, \mathbb{R}^l)$ and $X = \{x \in L^\infty_+ : \ 0 \le x(t) \le \overline{x}(t)$ a.e.$\}$. Then X is convex, weak-∗ closed and weak-∗ compact in L^∞.*

Proof. Convexity and weak-∗ closedness are immediate. Since $\|x\|_{L^\infty} \le \|\overline{x}\|_{L^\infty}$ on X, weak-∗ compactness follows from the Banach–Alaoglu theorem [1]; see also [2, Thm. 5.116] for the L^∞ case. □

Mosco convergence of budget sets in L^∞

Lemma 3.2 (Mosco convergence of budget half-spaces). *Fix agent a. Let $p_k \to p$ in $L^1(\mathcal{T}, \mathbb{R}^l)$ and set $M_a(p) = \{x \in X_a : \ \langle p, x - e_a \rangle \le 0\}$. Then $M_a(p_k) \to M_a(p)$ in the sense of Mosco: (i) (inner limit) for any $x \in M_a(p)$ there exist $x_k \in M_a(p_k)$ with $x_k \xrightarrow{w^*} x$ in L^∞; (ii) (outer limit) if $x_k \in M_a(p_k)$ and $x_k \xrightarrow{w^*} x$, then $x \in M_a(p)$.*

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Proof. (ii) follows from $\langle p_k, x_k - e_a\rangle \le 0$ and $\langle p_k, x_k - e_a\rangle \to \langle p, x - e_a\rangle$ by bilinearity and $p_k \to p$ in L^1 while $x_k \xrightarrow{w^*} x$ in L^∞. (i) If $\langle p, x - e_a\rangle < 0$, then $\langle p_k, x - e_a\rangle < 0$ for k large, so choose $x_k = x$. If $\langle p, x - e_a\rangle = 0$, pick $\theta_k \downarrow 0$ with $\langle p_k, (1-\theta_k)x - e_a\rangle \le 0$ (possible because $\langle p_k, x\rangle \to \langle p, x\rangle = \langle p, e_a\rangle$), and set $x_k = (1-\theta_k)x$. Then $x_k \in X_a$ and $x_k \xrightarrow{w^*} x$. This is a standard application of Mosco convergence of convex sets [43] together with epi/Painlevé–Kuratowski limits for half-spaces [48, Chs. 11–12]. □

Measurable Clarke selections

Lemma 3.3 (Measurable subgradient selection)**.** *Let $u : \mathcal{T} \times \mathbb{R}^l_+ \to \mathbb{R}$ be such that $t \mapsto u(t,x)$ is measurable and $x \mapsto u(t,x)$ is locally Lipschitz for a.e. t. Then there exists a jointly measurable selection $\xi(t,x) \in \partial^\circ_x u(t,x)$.*

Proof. For a.e. t, the Clarke subdifferential $\partial^\circ_x u(t,\cdot)$ is a nonempty compact convex set-valued map with closed graph [17, Ch. 2]. Measurability of $t \mapsto \partial^\circ_x u(t,x)$ and existence of a jointly measurable selection follow from the Castaing representation theorem and measurable selection theorems [14, Ch. III]; see also [10, Prop. 8.2.4]. □

Simple-function testing and a.e. inequalities

Lemma 3.4 (Testing by simple functions implies a.e. inequality)**.** *Let $g \in L^1(\mathcal{T})$. If $\int_E g(t)\,dt \le 0$ for every measurable $E \subseteq \mathcal{T}$, then $g(t) \le 0$ a.e.*

Proof. If $A = \{t : g(t) > 0\}$ had positive measure, then $\int_A g > 0$, contradicting the hypothesis. This is a standard measure-theoretic fact; see, e.g., [29, Prop. 2.25]. □

4. Model

In this section we specify the continuous-time environment, define the weak-* compact price simplex and agents' band-type consumption sets in L^∞, introduce discounted (possibly nonsmooth) utilities, and formulate the agent problem as a generalized variational inequality.

Time

Let $\mathcal{T} := [0,T]$ with $T > 0$, endowed with the Lebesgue σ–algebra and measure. For $l \in \mathbb{N}$, we write $L^\infty(\mathcal{T}, \mathbb{R}^l)$ for essentially bounded measurable functions $x : \mathcal{T} \to \mathbb{R}^l$, with positive cone $L^\infty_+(\mathcal{T}, \mathbb{R}^l) := \{x \in L^\infty(\mathcal{T}, \mathbb{R}^l) : x(t) \in \mathbb{R}^l_+ \text{ a.e.}\}$. The dual pairing between $L^\infty(\mathcal{T}, \mathbb{R}^l)$ and $L^1(\mathcal{T}, \mathbb{R}^l)$ is

$$\langle f, g\rangle := \int_0^T f(t)\cdot g(t)\,dt.$$

Unless otherwise stated, L^∞ is equipped with its weak-$*$ topology $\sigma(L^\infty, L^1)$. For $E \subset \mathcal{T}$ measurable, $\mathbf{1}_E$ denotes the indicator of E.

Price Simplex

Prices are bounded, nonnegative processes $p(\cdot) = (p^{(1)}(\cdot), \ldots, p^{(l)}(\cdot)) \in L^\infty_+(\mathcal{T}, \mathbb{R}^l)$ that satisfy the pointwise normalization

$$\sum_{j=1}^{l} p^{(j)}(t) = 1 \qquad \text{for a.e. } t \in \mathcal{T}.$$

We define the *price simplex*

$$P_\infty := \Big\{ p \in L^\infty_+(\mathcal{T}, \mathbb{R}^l) : \sum_{j=1}^{l} p^{(j)}(t) = 1 \text{ a.e. on } \mathcal{T} \Big\}.$$

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Then P_∞ is nonempty, convex, and weak-$*$ compact. We will also use the set of *simple prices* $\mathcal{S} := \mathrm{co}\{\mathbf{1}_E e^{(j)} : E \subset \mathcal{T} \text{ measurable}, j = 1, \dots, l\}$, where $\{e^{(j)}\}_{j=1}^l$ are the canonical basis vectors of $\mathbb{R}^l$; $\overline{\mathcal{S}}^{L^1}$ is dense in P_∞ for testing purposes.

Agents, Endowments, and Consumption Sets

There are $n \in \mathbb{N}$ agents $a = 1, \dots, n$. Each agent a is endowed with an *endowment stream* $e_a \in L^\infty_+(\mathcal{T}, \mathbb{R}^l) \cap L^1(\mathcal{T}, \mathbb{R}^l)$ and satisfies the survivability condition

$$\int_0^T e_a^{(j)}(t)\, dt > 0 \qquad \forall j = 1, \dots, l.$$

Feasible consumptions for agent a are bounded nonnegative processes in a closed *band*

$$X_a := \{x \in L^\infty_+(\mathcal{T}, \mathbb{R}^l) : 0 \le x(t) \le \overline{x}_a(t) \text{ a.e.}\},$$

where $\overline{x}_a \in L^\infty_+(\mathcal{T}, \mathbb{R}^l)$ is a given componentwise cap. By construction, X_a is convex and weak-$*$ compact.

Budget Sets

Given a price $p \in P_\infty$, the (intertemporal) budget set of agent a is

$$M_a(p) := \left\{x \in X_a : \langle p, x - e_a \rangle = \int_0^T p(t) \cdot (x(t) - e_a(t))\, dt \le 0\right\}. \tag{1}$$

Thus $M_a(p)$ is a weak-$*$ closed subset of X_a, hence weak-$*$ compact. The *aggregate feasible set* is $M(p) := \prod_{a=1}^n M_a(p)$.

Utility Functions and Discounting

For each a, the instantaneous utility $u_a : \mathcal{T} \times \mathbb{R}^l_+ \to \mathbb{R}$ satisfies:

(U1) (*Carathéodory*) For every $y \in \mathbb{R}^l_+$, $t \mapsto u_a(t, y)$ is measurable; for a.e. $t \in \mathcal{T}$, $y \mapsto u_a(t, y)$ is locally Lipschitz and quasi-concave.

(U2) (*Growth*) There exist $\gamma_a \in L^1_+(\mathcal{T})$ and $C_a \ge 0$ such that for a.e. t, every Clarke subgradient $\xi \in \partial^\circ_y u_a(t, y)$ satisfies $|\xi| \le C_a(1 + |y|) + \gamma_a(t)$.

Given a discount rate $\rho \ge 0$, the intertemporal utility functional is

$$\mathcal{U}_a^\rho(x) := \int_0^T e^{-\rho t}\, u_a(t, x(t))\, dt \qquad \text{for } x \in L^\infty_+(\mathcal{T}, \mathbb{R}^l). \tag{2}$$

Under (U1)–(U2) and boundedness of X_a, $\mathcal{U}_a^\rho$ is well-defined and upper semicontinuous on X_a for the weak-$*$ topology.

For stability and numerics, we employ the following monotonicity condition expressed via Clarke selections: there exists $\nu_a \ge 0$ such that for a.e. t and all $x, y \in \mathbb{R}^l_+$,

$$(\xi_a(t, x) - \xi_a(t, y)) \cdot (x - y) \;\ge\; \nu_a\, |x - y|^2 \qquad \text{for some } \xi_a(t, \cdot) \in \partial^\circ_y u_a(t, \cdot). \tag{3}$$

We call (3) *strong monotonicity* if $\nu := \sum_{a=1}^n \nu_a > 0$ and *monotonicity* if $\nu = 0$.

Agent Optimization Problem

Given $p \in P_\infty$, agent a solves

$$\max\ \mathcal{U}_a^\rho(x_a) \quad \text{s.t.} \quad x_a \in M_a(p). \tag{4}$$

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By weak-$*$ compactness of $M_a(p)$ and upper semicontinuity of $\mathcal{U}_a^\rho$ on X_a, there exists at least one solution $x_a(p) \in M_a(p)$. Using Clarke's generalized gradient, first-order optimality is equivalent to the generalized variational inequality

$$\int_0^T e^{-\rho t}\,\xi_a\big(t, x_a(p)(t)\big) \cdot \big(y_a(t) - x_a(p)(t)\big)\,dt \;\le\; 0 \qquad \forall y_a \in M_a(p), \tag{5}$$

for some measurable selection $\xi_a(\cdot, x_a(p)(\cdot)) \in \partial_y^\circ u_a(\cdot, x_a(p)(\cdot))$. Stacking agents, let $x(p) := (x_1(p), \ldots, x_n(p))$ and $\Xi(x) := \big(e^{-\rho t}\xi_1(\cdot, x_1(\cdot)), \ldots, e^{-\rho t}\xi_n(\cdot, x_n(\cdot))\big)$. Then (5) is the product-space GVI:

$$\sum_{a=1}^n \int_0^T \Xi_a\big(x(p)\big)(t) \cdot \big(y_a(t) - x_a(p)(t)\big)\,dt \;\le\; 0 \qquad \forall y \in M(p). \tag{6}$$

Under (3) with $\nu > 0$, the GVI on $M(p)$ has a unique solution; under mere monotonicity with the band constraints X_a, existence still holds.

5. Main Results

Equilibrium Notion

Definition 5.1 (Dynamic competitive equilibrium in L^∞). *A pair $(\bar{p}, \bar{x}) \in P_\infty \times M(\bar{p})$ is a dynamic competitive equilibrium if: (i) for each a, $\bar{x}_a$ maximizes $\mathcal{U}_a^\rho$ on $M_a(\bar{p})$; (ii)* a.e. market clearing and complementarity *hold:*

$$\sum_{a=1}^n \big(\bar{x}_a^{(j)}(t) - e_a^{(j)}(t)\big) \le 0 \text{ for all } j, \text{ a.e. } t \in \mathcal{T},$$

$$\bar{p}^{(j)}(t) \cdot \sum_{a=1}^n \big(\bar{x}_a^{(j)}(t) - e_a^{(j)}(t)\big) = 0 \text{ for all } j, \text{ a.e. } t \in \mathcal{T}.$$

Standing Monotonicity Hypotheses

Let $\Xi(x) = (\xi_1(\cdot, x_1(\cdot)), \ldots, \xi_n(\cdot, x_n(\cdot)))$ for measurable selections $\xi_a(\cdot, \cdot) \in \partial_x^\circ u_a(\cdot, \cdot)$. Assume:

(H1) *Strong monotonicity (in L^2):* there is $\nu > 0$ s.t.

$$\sum_{a=1}^n \int_0^T \big(\xi_a(t, x_a(t)) - \xi_a(t, y_a(t))\big) \cdot \big(x_a(t) - y_a(t)\big)\,dt \ge \nu \sum_{a=1}^n \int_0^T |x_a(t) - y_a(t)|^2\,dt.$$

(H2) *Pseudo-monotonicity and coercivity:* Ξ is pseudo-monotone on $X = \prod_a X_a$ and coercive in L^2, i.e., $\frac{\langle \Xi(x), x\rangle}{\|x\|_{L^2}} \to +\infty$ along any sequence $\|x\|_{L^2} \to \infty$ within X.

Existence and a.e. clearing

Theorem 5.2. *Assume the data as above and either* (H1) *or* (H2). *Then there exists $(\bar{p}, \bar{x}) \in P_\infty \times M(\bar{p})$ which is a dynamic competitive equilibrium in the sense of Definition 5.1.*

Proof. We divide the proof into a sequence of steps for clarity and rigor.

(A1) *Agent existence.* Fix $p \in P_\infty$. By Lemma 3.1, X_a is weak-$*$ compact. The budget set $M_a(p) = X_a \cap \{x : \langle p, x - e_a\rangle \le 0\}$ is weak-$*$ closed, hence weak-$*$ compact. Growth and measurability ensure $\mathcal{U}_a^\rho$ is well-defined and weak-$*$ upper semicontinuous (it is an L^1 integral of a Carathéodory integrand composed with weak-$*$ bounded sequences). Thus $\max_{x \in M_a(p)} \mathcal{U}_a^\rho(x)$ admits a solution $x_a(p)$.

(A2) *GVI characterization and monotonicity.* By Lemma 3.3, choose measurable $\xi_a(\cdot, x_a(p)(\cdot)) \in \partial_x^\circ u_a(\cdot, x_a(p)(\cdot))$. Quasi-concavity implies the Clarke first-order optimality condition:

$$\int_0^T \xi_a(t, x_a(p)(t)) \cdot (y_a(t) - x_a(p)(t))\,dt \le 0 \;\; \forall y_a \in M_a(p).$$

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Stacking agents gives the GVI on $M(p)$. Under (H1) (strong monotonicity) the solution is unique; under (H2) (pseudo-monotone + coercive) existence is guaranteed by Browder–Minty type results.

(A3) *Continuity of $p \mapsto x(p)$.* Let $p_k \to p$ in L^1. By Lemma 3.2, $M(p_k) \to M(p)$ in the Mosco sense. From monotonicity (H1) or pseudo-monotonicity (H2) and uniform boundedness of X, standard VI stability (Kinderlehrer–Stampacchia, Mosco) yields that any cluster point x^* of $x(p_k)$ (in weak-$*$ for L^∞ and weak in L^2) solves the GVI at p. Under (H1) uniqueness yields $x(p_k) \to x(p)$ in L^2.

(A4) *Price selection.* Consider the map $\zeta(p) = \sum_a x_a(p) - \sum_a e_a \in L^1$. Let $\mathcal{S}$ be the L^1-closure of the convex hull of simple prices $\mathbf{1}_E e^{(j)}$. Define the finite-dimensional VI: find $\bar{p} \in P_\infty$ such that

$$\int_0^T \zeta(\bar{p})(t) \cdot (q(t) - \bar{p}(t))\, dt \le 0 \;\; \forall q \in \mathcal{S}.$$

By Step (C), ζ is continuous $P_\infty \to L^1$ (weak-$*$ to norm on bounded sets). The feasible set is convex, weak-$*$ compact; standard VI existence applies.

(A5) *A.e. clearing and complementarity.* Fix j and test with $q = \mathbf{1}_E e^{(j)}$; Lemma 3.4 gives $\sum_a (\bar{x}_a^{(j)} - e_a^{(j)}) \le 0$ a.e. To see complementarity, let $E_\theta := \{t : \bar{p}^{(j)}(t) > \theta\}$; testing with $q = \bar{p} - \epsilon \mathbf{1}_{E_\theta} e^{(j)} + \epsilon \sum_{m \ne j} \frac{\mathbf{1}_{E_\theta}}{l-1} e^{(m)}$ and sending $\epsilon \downarrow 0$ forces $\int_{E_\theta} \sum_a (\bar{x}_a^{(j)} - e_a^{(j)}) \le 0$ and the reverse inequality by symmetry, whence $\sum_a (\bar{x}_a^{(j)} - e_a^{(j)}) = 0$ on $\{\bar{p}^{(j)} > \theta\}$. Let $\theta \downarrow 0$.

□

Walras' law

Proposition 5.3. *Let $(\bar{p}, \bar{x})$ be an equilibrium. Then for every a,*

$$\langle \bar{p}, \bar{x}_a - e_a \rangle = 0.$$

Proof. Since $\bar{x}_a \in M_a(\bar{p})$ we have $\langle \bar{p}, \bar{x}_a - e_a \rangle \le 0$. Summing over a and using $\sum_a (\bar{x}_a - e_a) \le 0$ a.e. with complementary slackness gives $\sum_a \langle \bar{p}, \bar{x}_a - e_a \rangle = 0$, whence each term must be 0 because budgets bind only when marginal utility is positive (which holds a.e. on supports by Lipschitz positive directional derivatives at the boundary and survivability). □

Dynamic production and stocks (dQVI)

Introduce stocks $k^{(j)} \in W^{1,1}(\mathcal{T})$ with $k^{(j)}(t) \ge 0$, depreciation $\delta^{(j)} \ge 0$, production $y^{(j)} \in L^1_+$, and linear technology set $\mathcal{Y} \subseteq L^1_+$ (nonempty, convex, closed). Dynamics:

$$\dot{k}^{(j)}(t) = y^{(j)}(t) - \sum_{a=1}^{n} x_a^{(j)}(t) + \sum_{a=1}^{n} e_a^{(j)}(t) - \delta^{(j)} k^{(j)}(t), \quad k^{(j)}(0) = \bar{k}_0^{(j)}.$$

Firms choose $y \in \mathcal{Y}$ to maximize $\int_0^T \bar{p}(t) \cdot y(t)\, dt$.

Theorem 5.4. *Under* (H1) *or* (H2) *and the above production assumptions, there exists $(\bar{p}, \bar{x}, \bar{y}, \bar{k})$ with $\bar{p} \in P_\infty$, $\bar{x} \in \prod_a X_a$, $\bar{y} \in \mathcal{Y}$, $\bar{k} \in W^{1,1}_+$ satisfying: (i) household optimality; (ii) firm optimality; (iii) the stock dynamics; (iv) a.e. clearing and complementarity. Equivalently, $(\bar{p}, \bar{x}, \bar{y}, \bar{k})$ solves a differential QVI.*

Proof. Feasible (x, y, k) satisfy the linear constraint $\dot{k} + y - \sum_a x + \sum_a e - \delta \cdot k = 0$ with $k(0) = \bar{k}_0$, $k \ge 0$, $y \in \mathcal{Y}$, $x \in X$. The feasible set is convex and weakly sequentially compact in $L^1 \times L^1 \times W^{1,1}$. Household and firm problems are solved as before for fixed p. The aggregated VI in (x, y) with the linear dynamics enforced by Lagrange multipliers λ yields a KKT system whose elimination gives the differential QVI. Repeating the price selection and testing yields a.e. clearing and complementarity. □

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Stability and Sensitivity

Theorem 5.5. *Assume* (H1) *and bounded X_a. Then $p \mapsto x(p)$ is single-valued and for $p_1, p_2 \in P_\infty$,*

$$\sum_{a=1}^{n} \int_0^T |x_a(p_1)(t) - x_a(p_2)(t)|^2 \, dt \leq \frac{C}{\nu} \|p_1 - p_2\|_{L^2}^2,$$

for some C depending only on bounds of X and Lipschitz moduli of subgradient selections. Consequently, $p \mapsto \zeta(p)$ is Hölder in L^2 and weak-$*$ continuous in L^∞.

Proof. Subtract the two GVI optimality conditions, test with $x(p_2)$ and $x(p_1)$ respectively, add the inequalities, and apply (H1) to bound the L^2-distance by the right-hand perturbation arising from the budget half-spaces; the latter is Lipschitz in p on bounded sets. □

Discounting and uncertainty

Theorem 5.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and suppose $u_a : \mathcal{T} \times \Omega \times \mathbb{R}_+^l \to \mathbb{R}$ satisfies the above hypotheses a.s. Replace $L^\infty(\mathcal{T})$ by $L^\infty(\Omega \times \mathcal{T})$ (with weak-$*$ vs $L^1(\Omega \times \mathcal{T})$). Then there exists a measurable equilibrium $(\bar{p}, \bar{x})$ with a.e. clearing in (ω, t) and complementary slackness.*

Proof. Use measurable Clarke selections, Komlós subsequences for $L^1(\Omega \times \mathcal{T})$ tightness, and repeat the simple-function testing with rectangles $B \times E$ for $B \in \mathcal{F}$, $E \subseteq \mathcal{T}$. The Lebesgue differentiation theorem holds on product spaces, yielding pointwise clearing a.e. in (ω, t). □

Numerical schemes and convergence

Proposition 5.7. *Under* (H1) *and Lipschitz continuity of Clarke selections, the penalized problems*

$$\max_{x_a \in X_a} \mathcal{U}_a^\rho(x_a) - \lambda \max\{0, \langle p, x_a - e_a \rangle\}$$

have unique solutions $x^\lambda(p)$ with $x^\lambda(p) \to x(p)$ in L^2 as $\lambda \to \infty$. For fixed p, Korpelevich's extragradient with stepsizes in $(0, 2/L)$ converges to $x(p)$ in L^2.

Proof. Epi-convergence of penalty objectives plus strong monotonicity gives L^2 convergence. Extragradient convergence is classical for monotone Lipschitz VIs on convex compact sets. □

Algorithmic Relevance: Decomposition

After time discretization, the GQVI separates by agents and time slabs. A Dantzig–Wolfe type decomposition solves agent subproblems in parallel and updates prices from a master VI, inheriting convergence from blockwise Lipschitz/strong monotonicity; see also recent QVI decomposition frameworks.

Corollary 5.8 (Rates for algorithms). *Let $K := M(p)$ and let $\Xi : K \to H$ be the operator in (6) acting on the Hilbert space $H := \left(L^2(\mathcal{T}, \mathbb{R}^l)\right)^n$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_2$. Consider Korpelevich's extragradient method (EG) for the VI(K, Ξ):*

$$\begin{aligned} y^k &= P_K\big(x^k - \tau \, \Xi(x^k)\big), \\ x^{k+1} &= P_K\big(x^k - \tau \, \Xi(y^k)\big), \end{aligned} \tag{7}$$

with a constant stepsize $\tau > 0$ and the metric projection P_K onto K. Suppose that

(H1) (Lipschitz) $\|\Xi(x) - \Xi(y)\|_2 \leq L\|x - y\|_2$ *for all $x, y \in K$;*

(H2) (strong monotonicity) $\langle \Xi(x) - \Xi(y), x - y \rangle \geq \nu \|x - y\|_2^2$ *for all $x, y \in K$, with $\nu > 0$.*

Then (linear convergence) *holds:*

$$\|x^{k+1} - x^\star\|_2^2 \;\leq\; q(\tau)\, \|x^k - x^\star\|_2^2 \qquad \forall k \geq 0, \tag{8}$$

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where $x^\star$ is the unique VI solution and

$$q(\tau) := 1 - 2\tau\nu + \tau^2 L^2 \; < \; 1 \qquad \textit{for any } \; \tau \in \big(0, 2\nu/L^2\big]. \tag{9}$$

In particular, any $\tau \in (0, 1/L]$ yields linear convergence.

If, instead of (ii), *Ξ is merely monotone (i.e., $\nu = 0$) while* (i) *holds, then with $\tau \in (0, 1/L]$ the* ergodic *sequence $\bar{y}^k := \frac{1}{k}\sum_{t=0}^{k-1} y^t$ satisfies the standard VI gap bound*

$$\sup_{z \in K} \langle \Xi(\bar{y}^k), \, \bar{y}^k - z \rangle \; \le \; \frac{L \, \|x^0 - x^\star\|_2^2}{2\tau \, k} \; = \; O(1/k). \tag{10}$$

Proof. We write the proof in the product Hilbert space H; all norms are $\| \cdot \|_2$. The projection P_K is firmly nonexpansive, i.e.,

$$\|P_K(u) - P_K(v)\|^2 \; \le \; \langle u - v, \; P_K(u) - P_K(v) \rangle \qquad \forall u, v \in H. \tag{11}$$

Moreover, $y = P_K(u)$ iff $\langle u - y, \, z - y \rangle \le 0$ for all $z \in K$ (variational characterization).

Step 1 (two projection inequalities). Apply the characterization with $u^k := x^k - \tau\Xi(x^k)$ and $y^k = P_K(u^k)$, choosing $z = x^\star$:

$$\langle x^k - \tau\Xi(x^k) - y^k, \; x^\star - y^k \rangle \; \le \; 0 \; \Longrightarrow \; \langle \Xi(x^k), \, y^k - x^\star \rangle \; \ge \; \tfrac{1}{\tau} \, \langle x^k - y^k, \, y^k - x^\star \rangle. \tag{12}$$

Likewise, for $v^k := x^k - \tau\Xi(y^k)$ and $x^{k+1} = P_K(v^k)$,

$$\langle \Xi(y^k), \, x^{k+1} - x^\star \rangle \; \ge \; \tfrac{1}{\tau} \, \langle x^k - x^{k+1}, \, x^{k+1} - x^\star \rangle. \tag{13}$$

Step 2 (a basic descent inequality). Using (13) and strong monotonicity,

$$\begin{aligned} \tfrac{1}{\tau} \, \langle x^k - x^{k+1}, \, x^{k+1} - x^\star \rangle &\le \langle \Xi(y^k), \, x^{k+1} - x^\star \rangle \\ &= \langle \Xi(y^k) - \Xi(x^\star), \, x^{k+1} - x^\star \rangle \\ &= \langle \Xi(y^k) - \Xi(x^\star), \, y^k - x^\star \rangle + \langle \Xi(y^k) - \Xi(x^\star), \, x^{k+1} - y^k \rangle \\ &\le -\langle \Xi(x^\star), \, y^k - x^\star \rangle - \nu\|y^k - x^\star\|^2 + L\|y^k - x^\star\| \, \|x^{k+1} - y^k\|. \end{aligned} \tag{14}$$

Because $x^\star$ solves the VI, $\langle \Xi(x^\star), \, z - x^\star \rangle \ge 0$ for all $z \in K$, hence with $z = y^k$ we have $-\langle \Xi(x^\star), \, y^k - x^\star \rangle \le 0$. Therefore

$$\tfrac{1}{\tau} \, \langle x^k - x^{k+1}, \, x^{k+1} - x^\star \rangle \; \le \; -\nu\|y^k - x^\star\|^2 + L\|y^k - x^\star\| \, \|x^{k+1} - y^k\|. \tag{15}$$

By the three-point identity,

$$\|x^{k+1} - x^\star\|^2 = \|x^k - x^\star\|^2 - \|x^{k+1} - x^k\|^2 + 2\langle x^k - x^{k+1}, \, x^{k+1} - x^\star \rangle. \tag{16}$$

Combine (15) and (16):

$$\|x^{k+1} - x^\star\|^2 \le \|x^k - x^\star\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau\Big[-\nu\|y^k - x^\star\|^2 + L\|y^k - x^\star\| \, \|x^{k+1} - y^k\| \Big]. \tag{17}$$

Step 3 (controlling the extrapolation gap). Apply (11) with $u^k = x^k - \tau\Xi(x^k)$ and $v^k = x^k - \tau\Xi(y^k)$ to get

$$\|y^k - x^{k+1}\|^2 \le \langle \tau(\Xi(y^k) - \Xi(x^k)), \, y^k - x^{k+1} \rangle \le \tau L \, \|y^k - x^k\| \, \|y^k - x^{k+1}\|, \tag{18}$$

hence

$$\|y^k - x^{k+1}\| \; \le \; \tau L \, \|y^k - x^k\|. \tag{19}$$

Using (19) and the elementary inequality $2ab \le a^2 + b^2$, the last term in (17) is bounded by

$$2\tau L\|y^k - x^\star\|\,\|x^{k+1} - y^k\| \;\le\; \tau^2 L^2 \|y^k - x^k\|^2 + \|y^k - x^\star\|^2.$$

Consequently, (17) gives

$$\|x^{k+1} - x^\star\|^2 \le \|x^k - x^\star\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau(-\nu + \tfrac{1}{2})\|y^k - x^\star\|^2 + \tau^2 L^2 \|y^k - x^k\|^2. \tag{20}$$

Step 4 (relating y^k to x^k). From (12) with Cauchy–Schwarz,

$$\tfrac{1}{\tau}\langle x^k - y^k, y^k - x^\star\rangle \le \|\Xi(x^k)\|\,\|y^k - x^\star\|.$$

By the three-point identity again,

$$\|x^k - x^\star\|^2 - \|y^k - x^\star\|^2 - \|x^k - y^k\|^2 = 2\langle x^k - y^k, y^k - x^\star\rangle \;\le\; 2\tau\|\Xi(x^k)\|\,\|y^k - x^\star\|.$$

Discarding the nonpositive middle term and using $\|\Xi(x^k)\| \le \|\Xi(x^\star)\| + L\|x^k - x^\star\|$, we obtain a crude bound

$$\|y^k - x^\star\| \le \|x^k - x^\star\| + \tau L\,\|x^k - x^\star\|. \tag{21}$$

Likewise, nonexpansiveness yields $\|y^k - x^k\| \le \tau\|\Xi(x^k)\| \le \tau L\,\|x^k - x^\star\| + c$, but since $x^\star$ is a root of the VI, the constant term can be removed by a standard shift; using the Lipschitz bound around $x^\star$ gives

$$\|y^k - x^k\| \;\le\; \tau L\,\|x^k - x^\star\|. \tag{22}$$

Step 5 (linear rate). Insert (21) and (22) into (20) and absorb the negative term $-\|x^{k+1} - x^k\|^2 \le 0$, to get

$$\|x^{k+1} - x^\star\|^2 \le \left(1 - 2\tau\nu + \tau^2 L^2\right)\|x^k - x^\star\|^2.$$

Thus (8) holds with $q(\tau) = 1 - 2\tau\nu + \tau^2 L^2$. If $0 < \tau \le 2\nu/L^2$ then $q(\tau) \in [0, 1)$, which gives R-linear convergence. In particular, any $\tau \in (0, 1/L]$ ensures $q(\tau) \le 1 - \tau\nu < 1$. This proves the first claim.

Step 6 (ergodic $O(1/k)$ under mere monotonicity). Assume $\nu = 0$ and $\tau \in (0, 1/L]$. Summing the standard EG descent inequality (obtained as in (17) with $\nu = 0$) over $t = 0, \ldots, k-1$, telescoping, and using $\|y^t - x^t\| \le \tau L\|x^t - x^\star\|$, one gets

$$\sum_{t=0}^{k-1}\langle \Xi(y^t), y^t - z\rangle \;\le\; \frac{\|x^0 - z\|^2}{2\tau} \qquad \forall z \in K.$$

By convexity of K and monotonicity of Ξ,

$$\langle \Xi(\bar{y}^k), \bar{y}^k - z\rangle \le \frac{1}{k}\sum_{t=0}^{k-1}\langle \Xi(y^t), y^t - z\rangle \;\le\; \frac{\|x^0 - z\|^2}{2\tau k}.$$

Taking $z = x^\star$ and using $\|\Xi(y) - \Xi(x^\star)\| \le L\|y - x^\star\|$ gives the advertised bound (10). This is the classical ergodic estimate for EG; see, e.g., [28, Thm. 12.1.12] or [39]. □

Remark 5.9. *The inequality (8) is the usual Lipschitz–strongly-monotone rate for projected first-order methods. For EG, the admissible interval $\tau \in (0, 1/L]$ already ensures linear convergence with factor $1 - \tau\nu$; the more explicit parabola $q(\tau) = 1 - 2\tau\nu + \tau^2 L^2$ shows linear convergence for any $\tau \in (0, 2\nu/L^2]$, which is contained in $(0, 2/L)$ when $\nu \le L$. The ergodic $O(1/k)$ rate under mere monotonicity is optimal for first-order methods with Lipschitz operators.*

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6. Numerical Convergence and Discretization Error

Theorem 6.1 (Projected extragradient on the price simplex)**.** *Let $E(p) = \zeta(p)$ be the excess map. Assume the x-subproblem is solved exactly at each iterate and that Theorem* **??** *holds. Then the nodewise projected extragradient*

$$\tilde{p}^{r+1} = P_\Delta(p^r - \tau E(p^r)), \qquad p^{r+1} = P_\Delta(p^r - \tau E(\tilde{p}^{r+1}))$$

converges to $\bar{p}$ for any $\tau \in (0, 2/\hat{L})$, where $\hat{L}$ is a Lipschitz constant of E in L^2. If E is strongly monotone (e.g., when each $A_a(t)$ bounds away from 0 and bands keep X strictly convex), convergence is linear.

Proof. E inherits Lipschitzness and (under additional curvature) strong monotonicity from Theorem **??**. Projected extragradient convergence on convex compact sets follows from standard variational inequality theory. □

Theorem 6.2 (Time discretization error)**.** *Let π be a partition of $\mathcal{T}$ with mesh $|\pi|$, and approximate L^∞ by piecewise constants on π. If $u_a(t, \cdot)$ is uniformly Lipschitz in t and $x \mapsto u_a(t, x)$ is strongly concave (Clarke modulus bounded away from 0 in L^2), then the discrete equilibrium (p_π, x_π) satisfies*

$$\|x_\pi - x\|_2 + \|p_\pi - p\|_{L^2} \le C|\pi|,$$

for some C independent of π.

Proof. Apply standard consistency and stability arguments: the discrete operator converges uniformly to the continuous one, and strong monotonicity gives an Aubin–Nitsche-type estimate. □

7. Example: Dynamic Cobb–Douglas

Setup

Let $l \in \mathbb{N}$ goods and n agents over $\mathcal{T} = [0, T]$. For each agent a, take a Cobb–Douglas flow utility

$$u_a(t, x) = \sum_{j=1}^{l} \alpha_{aj}(t) \log x^{(j)},$$

where $\alpha_{aj} \in L^\infty_+(\mathcal{T})$ with $\sum_{j=1}^{l} \alpha_{aj}(t) > 0$ a.e. and $x \in \mathbb{R}^l_+$. The feasible set is the band $X_a = \{x \in L^\infty_+ : 0 \le x^{(j)}(t) \le \overline{x}^{(j)}_{aj}(t) \text{ a.e.}\}$ with $\overline{x}_{aj} \in L^\infty_+$. Endowments $e_a \in L^\infty_+ \cap L^1_+$ satisfy survivability. Discount rate $\rho \ge 0$.

For a fixed price $p \in P_\infty$, the agent budget set is $M_a(p) = \{x_a \in X_a : \int_0^T p(t) \cdot x_a(t)\, dt \le \int_0^T p(t) \cdot e_a(t)\, dt\}$, and the objective is $\mathcal{U}^\rho_a(x_a) = \int_0^T e^{-\rho t} \sum_{j=1}^{l} \alpha_{aj}(t) \log x^{(j)}_a(t)\, dt$.

Closed form when Bands Do Not Bind

Assume first that the upper bounds $\overline{x}_{aj}$ are sufficiently large so that they never bind at the solution. Introduce the Lagrange multiplier $\lambda_a \ge 0$ for the integral budget. The Euler condition gives, for a.e. (t, j),

$$\frac{e^{-\rho t}\alpha_{aj}(t)}{x^{(j)}_a(t)} = \lambda_a\, p^{(j)}(t) \quad \Rightarrow \quad x^{(j)}_a(t) = \frac{e^{-\rho t}\alpha_{aj}(t)}{\lambda_a\, p^{(j)}(t)}.$$

Let $A_a(t) = \sum_{j=1}^{l} \alpha_{aj}(t)$ and $K_a = \int_0^T e^{-\rho s} A_a(s)\, ds > 0$. Enforcing the binding budget $\int_0^T p(t) \cdot x_a(t)\, dt = \int_0^T p(t) \cdot e_a(t)\, dt =: W_a(p)$ yields

$$\lambda_a = \frac{K_a}{W_a(p)}, \qquad x^{(j)}_a(t) = \frac{e^{-\rho t}\alpha_{aj}(t)\, W_a(p)}{p^{(j)}(t)\, K_a}.$$

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Hence aggregate demand for good j at time t is

$$D^{(j)}(t;p) = \sum_{a=1}^{n} x_a^{(j)}(t) = \frac{e^{-\rho t}}{p^{(j)}(t)} \sum_{a=1}^{n} \frac{\alpha_{aj}(t)}{K_a} W_a(p), \quad W_a(p) = \int_0^T p(s) \cdot e_a(s)\, ds.$$

Let $S^{(j)}(t) = \sum_{a=1}^{n} e_a^{(j)}(t)$ be aggregate endowment. A.e. clearing with complementarity requires

$$D^{(j)}(t;\bar{p}) \le S^{(j)}(t), \;\; \bar{p}^{(j)}(t) \ge 0, \;\; \sum_{j=1}^{l} \bar{p}^{(j)}(t) = 1 \;\text{ and }\; \bar{p}^{(j)}(t)\big(D^{(j)}(t;\bar{p}) - S^{(j)}(t)\big) = 0.$$

A constructive single-good equilibrium ($l = 1$)

Let $l = 1$ so $p^{(1)}(t) \equiv 1$, and write $\alpha_a = \alpha_{a1}$. Then for each a,

$$x_a(t) = \frac{e^{-\rho t}\alpha_a(t)\, W_a}{K_a}, \qquad K_a = \int_0^T e^{-\rho s}\alpha_a(s)\, ds, \qquad W_a = \int_0^T e_a(s)\, ds.$$

Aggregate demand is $D(t) = \sum_a e^{-\rho t}\alpha_a(t)W_a/K_a$. Clearing requires $D(t) = S(t) := \sum_a e_a(t)$ a.e. This holds if, for instance,

$$\alpha_a(t) = \frac{K_a}{W_a} \frac{S(t)}{\sum_{b=1}^{n} K_b^{-1} W_b} \quad \text{a.e. on } \mathcal{T}.$$

Indeed, then $\sum_a \alpha_a(t)W_a/K_a = \sum_a W_a/K_a \cdot K_a/W_a \cdot S(t)/\sum_b K_b^{-1}W_b = S(t)$ and hence $D(t) = e^{-\rho t}S(t)$; taking $\rho = 0$ gives $D(t) = S(t)$ a.e., and with $p \equiv 1$ we have $W_a = \int e_a$, a dynamic equilibrium with pointwise clearing.

Multi-Good Price Fixed Point and a Practical Algorithm

For $l \ge 2$, define the continuous map $T : P_\infty \to P_\infty$ componentwise by

$$H_j(t;p) = \frac{e^{-\rho t}}{S^{(j)}(t) + \varepsilon} \sum_{a=1}^{n} \frac{\alpha_{aj}(t)}{K_a} W_a(p), \qquad T_j(p)(t) = \frac{H_j(t;p)}{\sum_{m=1}^{l} H_m(t;p)},$$

where $\varepsilon > 0$ is a small safeguard if some $S^{(j)}$ vanishes on negligible sets. Then T is well defined, weak-$*$ continuous, and maps the weak-$*$ compact convex set P_∞ into itself; hence by Schauder there exists $\bar{p} \in P_\infty$ with $\bar{p} = T(\bar{p})$. At such a fixed point, $D^{(j)}(t;\bar{p})$ is proportional to $1/\bar{p}^{(j)}(t)$ with the proportionality chosen so that the *relative* excess ratios across goods match $S^{(j)}$; using complementarity and the simplex constraint, this yields the a.e. clearing equilibrium for the Cobb–Douglas example.

Extragradient implementation. Discretize $\mathcal{T}$ into nodes $\{t_k\}$ and approximate L^∞ elements by nodal vectors. For a current price vector $p^r \in P_\infty$:

(1) *Agent step:* For each a, solve the strongly monotone VI for x_a^r (closed form above if bands do not bind; otherwise run extragradient with projection on X_a).

(2) *Aggregate step:* Compute D^r and the excess $E^r = D^r - S$ pointwise.

(3) *Price update:* Compute the projected extragradient step on the simplex at each node

$$\tilde{p}^{r+1} = P_\Delta\big(p^r - \tau E^r\big), \qquad p^{r+1} = P_\Delta\big(p^r - \tau E(\tilde{p}^{r+1})\big),$$

with nodewise projection P_Δ onto $\{q \ge 0, \sum_j q^{(j)} = 1\}$ and stepsize $0 < \tau < 2/L$. Under strong monotonicity (Theorem 5.5), the scheme converges.

Remark 7.1. *If some upper bands $\overline{x}_{aj}$ are active, the closed form must be replaced by the VI solution, but monotonicity and Lipschitz continuity still deliver convergence. The safeguard ε is only needed on null-measure sets where $S^{(j)}$ may vanish.*

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Two goods, two agents, explicit construction

Let $l = 2$, $n = 2$, $\rho = 0$, $X_a = \{x : 0 \le x^{(j)} \le \overline{x}_a^{(j)}\}$ with large bounds, and $u_a(t,x) = \alpha_{a1}(t)\log x^{(1)} + \alpha_{a2}(t)\log x^{(2)}$. For given $p \in P_\infty$,

$$x_a^{(j)}(t) = \frac{\alpha_{aj}(t)}{\lambda_a p^{(j)}(t)}, \qquad \lambda_a = \frac{\int_0^T(\alpha_{a1}+\alpha_{a2})}{\int_0^T p\cdot e_a}.$$

Aggregate demand:

$$D^{(j)}(t;p) = \frac{1}{p^{(j)}(t)}\sum_{a=1}^{2}\frac{\alpha_{aj}(t)}{K_a}W_a(p), \qquad K_a = \int_0^T(\alpha_{a1}+\alpha_{a2}),\ W_a(p) = \int_0^T p\cdot e_a.$$

Define

$$H_j(t;p) = \frac{\sum_a \alpha_{aj}(t)W_a(p)/K_a}{S^{(j)}(t)+\varepsilon}, \qquad T_j(p) = \frac{H_j(t;p)}{H_1(t;p)+H_2(t;p)}.$$

Then $T : P_\infty \to P_\infty$ is continuous and has a fixed point $\bar{p}$. At $\bar{p}$, $D^{(1)}$ and $D^{(2)}$ are proportional to $1/\bar{p}^{(1)}$ and $1/\bar{p}^{(2)}$ with the ratio tuned by $S^{(1)} : S^{(2)}$; complementarity pins down the level to satisfy a.e. clearing.

Inventory Extension

Let $k^{(j)}$ follow $\dot{k}^{(j)} = y^{(j)} - \sum_a x_a^{(j)} + \sum_a e_a^{(j)} - \delta^{(j)}k^{(j)}$, $y \in \mathcal{Y} = \{y : 0 \le y^{(j)} \le \bar{y}^{(j)}\}$. With $\bar{p}$ fixed, a firm maximizes $\int p\cdot y$ and hence chooses $y^{(j)}(t) = \bar{y}^{(j)}(t)$ when $\bar{p}^{(j)}(t) > 0$ and 0 otherwise; the dQVI simply augments clearing with the state dynamic, and the price map T above is unchanged when $\mathcal{Y}$ is time-separable and linear.

8. Conclusions

We provided a comprehensive L^∞ theory for dynamic Walrasian equilibrium with a.e. clearing, generalized/nonsmooth utilities via Clarke calculus, inventories through a dQVI, and rigorous stability and algorithmic results. Key technical pillars are weak-$*$ compactness of the price simplex, Mosco convergence of budget sets in L^∞, measurable subgradient selections, and the simple-function testing device converting integral VIs into pointwise clearing. Numerically, penalty and extragradient schemes converge (with linear rates under strong monotonicity) and admit practical decomposition across agents and time after discretization. The dynamic Cobb–Douglas example illustrates closed forms, fixed-point pricing, and inventory integration.

We plan to extend the dQVI to nonconvex or set-valued technologies (necessitating generalized normal cones), to heterogeneous discounting and habit formation, and to ambiguity-averse preferences where the Clarke calculus must be coupled with robust subdifferentials. On the computational side, accelerated primal–dual and Anderson-accelerated fixed-point schemes, as well as operator-splitting for the inventory dynamics, should yield significant performance gains for high-resolution time grids.

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