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Article

An Approximation to Riemann Hypothesis

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Abstract: In this version, there is an improvement on Theorem 1.1, the error term of (1.2) is reduced to $O(T^{1/3}(\log T)^{7/3})$. This is achieved mainly by dividing function $\omega(s, T_1, T_2)$ into two parts, the one is dominant, and the other one is minor, and the argument of the dominant one is relatively small (see Corollary 2.1), so the final result is improved.

Keywords:

1. Introduction

Riemann zeta-function $\zeta(s)$ is originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for } \mathrm{Re}\, s > 1.$$

and it also can be expressed as the product form

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s}, \qquad \text{for } \mathrm{Re}\, s > 1.$$

This formula is called Euler's product formula, which indicates the relation between $\zeta(s)$ and prime numbers. About $\zeta(s)$ there is a well-known Riemann hypothesis, states that all the non-trivial zeros of $\zeta(s)$ are on the critical line $\mathrm{Re}\, s = 1/2$. The researches on the conjecture are no doubt a most time-consuming one in mathematics, refer to see the survey paper [3].
The so-called trivial zeros of $\zeta(s)$ are $s = -2, -4, \cdots$, and nontrivial zeros of $\zeta(s)$ are known all in the critical strip $0 \le \mathrm{Re}\, s \le 1$.
Denote by $N(T)$ the number of zeros of $\zeta(\sigma + it)$ in the region $0 \le \sigma \le 1, 0 \le t \le T$, and by $N_0(T)$ the number of zeros on the critical line $\sigma = 1/2, 0 \le t \le T$. Riemann hypothesis is that

$$N_0(T) = N(T).$$

For $N(T)$, it is known that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.1}$$

And for $N_0(T)$, Hardy firstly shown that there are infinity many zeros on the critical line, and then he and Littlewood [5] and Selberg [8] proved that

$$\kappa = \frac{N_0(T)}{N(T)} > 0.$$

Levinson [6] proved

$$\kappa \ge \frac{1}{3}.$$

and then this result has been improved successively. Conrey [2], Feng [4] proved respectively

$$\kappa \ge 0.407, \qquad \kappa \ge 0.412.$$

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In this paper, we will prove that

Theorem 1.1.

$$N(T) = N_0(T) + \mathscr{E}. \tag{1.2}$$

where $\mathscr{E} \ll T^{1/3}(\log T)^{7/3}$.

The main arguments in this paper are based the papers [6],[7] and [1], but instead of using Riemann-Siegel formula, it will be applied an auxiliary function $\omega(s, T_1, T_2)$ defined in Lemma 2.1, which will play a role of mollifier and ferry, it firstly used in [7] but here with a small modification.

2. Some Lemmas

Lemma 2.1. *Suppose that* $T \le T_1 \le T_2 \le 2T$, $\lambda = T^{2/3}(\log T)^{5/3}$, $s_1 = \lambda + c + iu, (c \ge 0)$, $s = v + it$,*define*

$$\omega(s, T_1, T_2) = \frac{e^\lambda}{2\pi i}\int_{T_1}^{T_2} \Gamma(s_1 - s)\lambda^{s - s_1} ds_1. \tag{2.1}$$

Let $\sigma = \lambda + c - v$, $\eta = \min\{|x - t|| x \in [T_1, T_2]\}$, *there is*

$$|\omega(s, T_1, T_2)| \ll \exp(-((c - v)^2 + \eta^2)/2\sigma) \tag{2.2}$$

Let $\Delta = (20\lambda \log T)^{1/2}$, *if* $t \in [T_1 + \Delta, T_2 - \Delta]$, *then*

$$|\omega(c + it, T_1, T_2) - 1| \ll T^{-10}. \tag{2.3}$$

And if $t \le T_1 - \Delta$, *or* $t \ge T_2 + \Delta$, *then*

$$|\omega(c + it, T_1, T_2)| \ll T^{-10}. \tag{2.4}$$

If $|t - u| = o(\sigma^{2/3})$, *then*

$$\arg(\Gamma(s_1 - s)\lambda^{s - s_1}) = \frac{t - u}{2\sigma} - \frac{(t - u)^3}{6\sigma^2} + \epsilon. \tag{2.5}$$

Proof. By Stirling's formula, it has

$$\begin{aligned} \mathrm{Re}(\log\Gamma(\sigma + (u - t)i)) =& \left(\sigma - \frac{1}{2}\right)\frac{\log(\sigma^2 + (u - t)^2)}{2} - \sigma + \frac{1}{2}\log(2\pi) \\ & - (u - t)\arctan\left(\frac{u - t}{\sigma}\right) + O(1/\sigma) \end{aligned}$$

And

$$\begin{aligned} &\mathrm{Re}(\log(\Gamma(\sigma + (u - t)i))) + \mathrm{Re}(\log(\lambda^{-(\sigma + (u - t)i)})) + \lambda \\ &= -\frac{(c - v)^2}{2\sigma} - \frac{(u - t)^2}{4\sigma^2} - \frac{(u - t)^2}{2\sigma} - \frac{1}{2}\log\sigma + \frac{1}{2}\log(2\pi) + O(1/\sigma) \end{aligned}$$

Hence,

$$|\omega(s, T_1, T_2)| \le e^{-(c - v)^2/2\sigma}\frac{\sigma^{-1/2}}{\sqrt{2\pi}}\int_{T_1}^{T_2} e^{-(u - t)^2/2\sigma} du \ll e^{-((c - v)^2 + \eta^2)/2\sigma}.$$

Besides, it is familiar that

$$e^{-\lambda} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s_1 - s)\lambda^{s - s_1} ds.$$

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Hence,

$$1 - \omega(c + it, T_1, T_2) = R_1 + R_2,$$

where

$$R_1 = \frac{e^{\lambda}}{2\pi} \int_{T_2}^{\infty} \Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} du,$$

$$R_2 = \frac{e^{\lambda}}{2\pi} \int_{-\infty}^{T_1} \Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} du,$$

Hence, if $t \in [T_1 + \Delta, T_2 - \Delta]$, then

$$\begin{aligned} |R_1| &\ll \int_{T_2}^{\infty} \left| e^{\lambda}\Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} \right| du \\ &\ll \lambda^{-1/2} \int_{T_2}^{\infty} \exp(-(u-t)^2/2\lambda) du \\ &\ll T^{-10}. \end{aligned}$$

and similarly

$$|R_2| \ll \lambda^{-1/2} \int_{-\infty}^{T_1} \exp(-(u-t)^2/2\lambda) du \ll T^{-10}.$$

if $t \leq T_1 - \Delta$, or $t \geq T_2 + \Delta$, then

$$|\omega(c + it, T_1, T_2)| \ll \lambda^{-1/2} \int_{T_1}^{T_2} \exp(-(u-t)^2/2\lambda) du \ll T^{-10}.$$

If $|t - u| = o(\sigma^{2/3})$, then

$$\begin{aligned} \mathrm{Im}(\log(\blacksquare(s_1 - s)^{\check{}s - s_1})) &= \frac{t-u}{2\sigma} + \frac{(t-u)^3}{3\sigma^2} - \frac{(t-u)^3}{2\sigma^2} \\ &\quad - \frac{(t-u)^5}{5\sigma^4} + \frac{(t-u)^5}{4\sigma^4} + \epsilon \\ &= \frac{t-u}{2\sigma} - \frac{(t-u)^3}{6\sigma^2} + \frac{(t-u)^5}{20\sigma^4} + \epsilon. \end{aligned}$$

□

Corollary 2.1. *Divide $\omega(s, T_1, T_2)$ into two parts,*

$$\omega(s, T_1, T_2) = \omega_1(s, T_1, T_2) + \omega_2(s, T_1, T_2), \tag{2.6}$$

$$\omega_1(s, T_1, T_2) = \frac{e^{\lambda}}{2\pi i} \int\limits_{|u-t| \leq \Delta} \Gamma(s_1 - s)\lambda^{s - s_1} ds_1,$$

$$\omega_2(s, T_1, T_2) = \frac{e^{\lambda}}{2\pi i} \int\limits_{|u-t| \geq \Delta} \Gamma(s_1 - s)\lambda^{s - s_1} ds_1.$$

Then

$$|\omega_2(s, T_1, T_2)| \leq O(T^{-10}), \tag{2.7}$$

and

$$|\arg(\omega_1(s, T_1, T_2))| \leq O(\Delta^3/\lambda^2). \tag{2.8}$$

This indicates $\omega_1(s, T_1, T_2)$ is the domination of $\omega(s, T_1, T_2)$, and its argument is small.

Lemma 2.2. *Let $\zeta_1(s), \zeta_2(s)$ be $\zeta(s)$ or $\zeta'(s)$ and*

$$\zeta_1(s) = \sum_n a_n n^{-s}, \quad for \ \ \mathrm{Re}\, s > 1,$$

$$\zeta_2(s) = \sum_n b_n n^{-s}, \quad for \ \ \mathrm{Re}\, s > 1.$$

λ same as in Lemma 2.1, $0 \le a \le 1/2$, $s_1 = \lambda + c + iu$, $c \ge 0$, define

$$g(u) = \frac{e^\lambda}{2\pi i} \int_{(a)} \Gamma(s_1 - s)\lambda^{-s_1+s}\zeta_1(s)\overline{\zeta}_2(s)ds \tag{2.9}$$

Then for $T \le u \le 2T$, $1 < \beta < \lambda + c$, there is

$$g(u) = e^\lambda \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{a_k \overline{b}_h}{h^{2\beta}} \left(\frac{h}{k}\right)^{s_1} e^{-\lambda h/k} + O(T^{-10}). \tag{2.10}$$

Proof. We move the integral path from (a) to (β), the residue at the pole $s = 1$ is

$$R \ll e^\lambda \Gamma(\lambda + c - 1 + iu)\lambda^{-(\lambda+c-1+iu)} \ll T^{-10}$$

Hence,

$$\begin{aligned} g(u) &= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{a_k \overline{b}_h}{k^\beta h^\beta} \frac{e^\lambda}{2\pi i} \int_{(\beta)} \Gamma(s_1 - s)\lambda^{-s_1+s}\left(\frac{h}{k}\right)^{it} ds + O(T^{-10}) \\ &= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{a_k \overline{b}_h}{h^{2\beta}} \frac{e^\lambda}{2\pi i} \int_{(\beta)} \Gamma(s_1 - s)\lambda^{-s_1+s}\left(\frac{h}{k}\right)^{s} ds + O(T^{-10}) \\ &= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{a_k \overline{b}_h}{h^{2\beta}} \frac{e^\lambda}{2\pi i} \int_{(\lambda+c-\beta)} \Gamma(w)\lambda^{-w}\left(\frac{h}{k}\right)^{s_1-w} dw + O(T^{-10}) \\ &= e^\lambda \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{a_k \overline{b}_h}{h^{2\beta}} \left(\frac{h}{k}\right)^{s_1} e^{-\lambda h/k} + O(T^{-10}). \end{aligned}$$

□

Lemma 2.3. *Let $\theta = (20 \log T/\lambda)^{1/2}$, λ as before, $0 \le c \le \Delta/10$, then*

$$J_1 = \int_{1+\theta}^{\infty} e^\lambda v^{\lambda+c} e^{-\lambda v} dv \ll T^{-8}\lambda^{-1/2}, \tag{2.11}$$

$$J_2 = \int_0^{1-\theta} e^\lambda v^{\lambda+c} e^{-\lambda v} dv \ll T^{-10}. \tag{2.12}$$

$$J_3 = \int_0^{1-\theta} e^\lambda \log(j/v) v^{\lambda+c} e^{-\lambda v} dv \ll (\log j + \theta)T^{-10}, \quad (j \ge 1). \tag{2.13}$$

Proof. For (2.11), by the integration by parts

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$$
\begin{aligned}
J_1 &= -\frac{e^{\lambda} v^{\lambda+c} e^{-\lambda v}}{\lambda}\bigg|_{1+\theta}^{\infty} + \frac{\lambda+c}{\lambda}\int_{1+\theta}^{\infty} e^{\lambda} v^{\lambda+c-1} e^{-\lambda v} dv \\
&\le \frac{\exp(-2\theta^2\lambda/5)}{\lambda} + (1+\theta/10)\frac{J_1}{1+\theta}
\end{aligned}
$$

That is,

$$\frac{9}{10}\frac{\theta\lambda}{1+\theta} J_1 \le \exp(-2\theta^2\lambda/5) \le T^{-8}$$

and

$$J_1 \le T^{-8}\lambda^{-1/2}.$$

For (2.12),

$$J_2 \le e^{\lambda}(1-\theta)^{\lambda} e^{-\lambda(1-\theta)} \le \exp(-\theta^2\lambda/2) \le T^{-10}.$$

For (2.13),

$$J_3 \le e^{\lambda}\log(j/(1-\theta))(1-\theta)^{\lambda+c} e^{-\lambda(1-\theta)} \ll (\log j + \theta) T^{-10}.$$

□

3. The Proof of Theorem 1.1

Proof. Let $h(s) = \pi^{-s/2}\Gamma(s/2)$, then the functional equation of $\zeta(s)$ can be written as

$$h(s)\zeta(s) = h(1-s)\zeta(1-s) \tag{3.1}$$

By Stirling's formula, it has

$$\log h(s) = \frac{1}{2}(s-1)\log\frac{s}{2\pi} - \frac{s}{2} + C_0 + O\left(\frac{1}{s}\right) \tag{3.2}$$

Let $f(s) = \log h(s)$, then

$$f'(s) = \frac{h'(s)}{h(s)} = \frac{1}{2}\log\frac{s}{2\pi} + O\left(\frac{1}{s}\right) \tag{3.3}$$

and for larger t

$$f'(s) + f'(1-s) = \log\frac{t}{2\pi} + O\left(\frac{1}{s}\right) \tag{3.4}$$

Taking logarithm of equation (3.1), and then derivative, it follows

$$h(s)\zeta(s)(f'(s) + f'(1-s)) = -h(s)\zeta'(s) - h(1-s)\zeta'(1-s) \tag{3.5}$$

We note that the right side of (3.5) is a sum of two conjugative complex numbers as $s = 1/2 + it$, so the zeros of the right side of (3.5) occur if and only if

$$\arg(h(s)\zeta'(s)) \equiv \pi/2 \mod \pi \tag{3.6}$$

On the left side of (3.5), clearly, $h(s)$ is never zero, and by (3.4), so these zeros are just the zeros of $\zeta(1/2+it)$.

Moreover, let $\chi(s) = h(1-s)/h(s)$, then $\zeta(s) = \chi(s)\zeta(1-s)$, and

$$\zeta'(s) = -\chi(s)\{(f'(s) + f'(1-s))\zeta(1-s) + \zeta'(1-s)\} \tag{3.7}$$

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By (3.6), the zeros of $\zeta(1/2+it)$ are the ones

$$\arg(h(1-s)\{(f'(s)+f'(1-s))\zeta(1-s)+\zeta'(1-s)\})\equiv \pi/2 \mod \pi$$

on $\sigma=1/2$, equivalently,

$$\arg(h(s)\{(f'(s)+f'(1-s))\zeta(s)+\zeta'(s)\})\equiv \pi/2 \mod \pi \tag{3.8}$$

on $\sigma=1/2$. Write $\mathcal{L}(s)=f'(s)+f'(1-s)$, and denote by

$$G(s)=\zeta(s)+\zeta'(s)/\mathcal{L}(s) \tag{3.9}$$

The investigation above means

$$N_0(T)=\frac{1}{\pi}\Delta_0^T\arg(hG(1/2+it)) \tag{3.10}$$

By(3.2), it can be known that

$$\Delta_0^T\arg(h(1/2+it))=\frac{T}{2}\log\frac{T}{2\pi}-\frac{T}{2}+O(\log T) \tag{3.11}$$

So, the main task to determine $N_0(T)$ is to calculate $\Delta_0^T\arg(G(1/2+it))$.
Let $L=\log(T/2\pi)$, $U\le T$, and let D be the rectangle with the vertices $1/2+iT$, $c+iT$, $c+i(T+U)$, $1/2+i(T+U)$,$(c\ge 3)$. First of all, we might as well assume there are no zeros of $G(s)$ on the boundary of D, then by the principle of argument, the change of $\arg G(s)$ around D is equal to 2π times $N_G(D)$, the number of zeros of $G(s)$ in D.
On the right side of D

$$|G(c+it)-1|\le\sum_{n\ge 2}n^{-c}+O(1/L)\le 1/3$$

so, $\arg G(s)$ change less than π. On the lower side and the upper side of D, by a known result [9, §9.4], a extension of Jessen's theorem, taking account on the order of $G(s)$, we can know that $\arg G(s)=O(L)$ as $0<\sigma\le 3$, and $\arg G(\sigma+it)=O(2^{-\sigma})$ as $\sigma\ge 3$, hence, for any $0\le b\le c$, it has

$$\int_b^c\arg G(\sigma+iT)d\sigma,\ \int_b^c\arg G(\sigma+i(T+U))d\sigma\ll O(L) \tag{3.12}$$

So,

$$\Delta_T^{T+U}\arg(G(1/2+it))=-2\pi N_G(D)+O(\log T) \tag{3.13}$$

Now the work is turned into to evaluate $N_G(D)$.
Take $a=1/2-1/L$, and let $\mathscr{D}$ be the rectangle with vertices $a+iT$, $c+iT$, $c+i(T+U)$, $a+i(T+U)$.
Taking the integral $\int_{\mathscr{D}}\log G(s)ds$, by the Littlewood's Lemma [9, §9.9], it has

$$\int_T^{T+U}\log|G(a+it)|dt-\int_T^{T+U}\log|G(c+it)|dt+\int_a^c\arg G(\sigma+i(T+U))d\sigma \\ -\int_a^c\arg G(\sigma+iT)d\sigma=2\pi\sum\text{dist} \tag{3.14}$$

where $\sum$dist is the sum of the distances of the zeros of $G(s)$ from the left.
By (3.9), it is easy to know

$$\int_T^{T+U}\log G(c+it)dt=\int_T^{T+U}\log\zeta(c+it)dt+O(1/L)$$

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and it is familiar that

$$\log \zeta(s) = \sum_n \frac{-\Lambda(n)}{n^s \log n}$$

So

$$\int_T^{T+U} \log |G(c+it)| dt \ll 1.$$

With (3.12), the rest is to calculate the first integral of (3.14).
By the concavity of logarithm, it has

$$\begin{aligned}\int_T^{T+U} \log |G(a+it)| dt &= \frac{1}{2} \int_T^{T+U} \log |G(a+it)|^2 dt \\ &\leq \frac{1}{2} U \log \left(\frac{1}{U} \int_T^{T+U} |G(a+it)|^2 dt \right) \end{aligned} \tag{3.15}$$

At first, we simplify $G(s)$ as

$$G_0(s) = \zeta(s) + \frac{\zeta'(s)}{L}. \tag{3.16}$$

Then

$$G(s) = G_0(s) + E(s).$$

$$E(s) = \left(\frac{1}{\mathcal{L}(s)} - \frac{1}{L} \right) \zeta'(s) \ll \frac{1}{L^3} \zeta'(s).$$

And

$$\begin{aligned}\int_{T_1}^{T_2} |G(a+it)|^2 dt = \int_{T_1}^{T_2} |G_0(a+it)|^2 dt + 2\mathrm{Re} \int_{T_1}^{T_2} G_0(a+it) E(a-it) dt \\ + \int_{T_1}^{T_2} |E(a+it)|^2 dt \end{aligned}$$

By Cauchy inequality

$$\int_{T_1}^{T_2} G_0(a+it)E(a-it)dt \leq \left(\int_{T_1}^{T_2} |G_0(a+it)|^2 dt \int_{T_1}^{T_2} |E(a+it)|^2 dt \right)^{1/2}$$

The third integral in the right side of (3.16) is much smaller than the first one, which will be actually calculated later, hence

$$\int_{T_1}^{T_2} |G(a+it)|^2 dt = (1+\epsilon) \int_{T_1}^{T_2} |G_0(a+it)|^2 dt.$$

Moreover, let

$$\phi(s, T_1 - \Delta, T_2 + \Delta) = \omega_1^{1/2}(s, T_1 - \Delta, T_2 + \Delta). \tag{3.17}$$

By (2.8), we can know that on the upper side and the lower side of $\mathscr{D}$, there is

$$\arg(\phi(s, T_1 - \Delta, T_2 + \Delta)) \leq O(\Delta^3/\lambda^2) \tag{3.18}$$

and

$$\int_a^c \arg(\phi(v + T_j i, T_1 - \Delta, T_2 + \Delta)) dv \leq O(c\Delta^3/\lambda^2), \quad (j = 1, 2). \tag{3.19}$$

It is assumed that $c \leq \Delta/10$.
Moreover, by Lemma 2.1, there is

$$\int_{T_1}^{T_2} \log|\phi(c+it, T_1-\Delta, T_2+\Delta)|dt \ll T^{-9}. \tag{3.20}$$

(3.19) and (3.20) indicate that function $\phi(s, T_1-\Delta, T_2+\Delta)$ may be used as a mollifier. Let

$$\mathcal{G}(s) = G_0(s)\phi(s, T_1-\Delta, T_2+\Delta). \tag{3.21}$$

By (2.8), for $\operatorname{Re} s \leq c$, it has

$$\begin{aligned}
&\int_{T_1}^{T_2} |\mathcal{G}(s)|^2 dt = \int_{T_1}^{T_2} |G_0(s)|^2 |\phi(s, T_1-\Delta, T_2+\Delta)|^2 dt \\
&= \int_{T_1}^{T_2} |G_0(s)|^2 |\omega_1(s, T_1-\Delta, T_2+\Delta)| dt \\
&\leq (1+O(L^3/\lambda)) \int_{T_1}^{T_2} |G_0(s)|^2 \omega_1(s, T_1-\Delta, T_2+\Delta) dt \\
&\leq (1+O(L^3/\lambda)) \int_{T_1}^{T_2} |G_0(s)|^2 \omega(s, T_1-\Delta, T_2+\Delta) dt + O(T^{-8})
\end{aligned} \tag{3.22}$$

In the next is mainly to calculate the last integral.
By Lemma 2.1, it has

$$\begin{aligned}
&\int_{T_1}^{T_2} \omega(a+it, T_1-\Delta, T_2+\Delta)|G_0(a+it)|^2 dt \\
&= \frac{e^\lambda}{2\pi} \int_{T_1}^{T_2} \int_{T_1-\Delta}^{T_2+\Delta} \Gamma(s_1-(a+it))\lambda^{-(s_1-(a+it))} du |G_0(a+it)|^2 dt \\
&= \frac{e^\lambda}{2\pi} \int_{T_1-\Delta}^{T_2+\Delta} \int_{T_1}^{T_2} \Gamma(s_1-(a+it))\lambda^{-(s_1-(a+it))} |G_0(a+it)|^2 dt du \\
&\leq \frac{e^\lambda}{2\pi} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1-(a+it))\lambda^{-(s_1-(a+it))} |G_0(a+it)|^2 dt du + \epsilon \\
&= I_{11} + I_{12} + I_{21} + I_{22} + \epsilon.
\end{aligned}$$

where

$$\begin{aligned}
I_{11} &= \frac{e^\lambda}{2\pi} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1-(a+it))\lambda^{-(s_1-(a+it))} |\zeta(a+it)|^2 dt du \\
I_{12} &= \frac{e^\lambda}{2\pi L} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1-(a+it))\lambda^{-(s_1-(a+it))} \zeta(a+it)\zeta'(a-it) dt du \\
I_{21} &= \frac{e^\lambda}{2\pi L} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1-(a+it))\lambda^{-(s_1-(a+it))} \zeta(a-it)\zeta'(a+it) dt du \\
I_{22} &= \frac{e^\lambda}{2\pi L^2} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1-(a+it))\lambda^{-(s_1-(a+it))} |\zeta'(a+it)|^2 dt du.
\end{aligned}$$

In the following specify $T_1 = T, T_2 = T+U$.
We first calculate I_{11}, by Lemma 2.2,

$$\begin{aligned}
I_{11} &= e^\lambda \int_{T_1-\Delta}^{T_2+\Delta} \sum_{j_1, j_2} \frac{1}{j_2^{2\beta}} \rho^{s_1} \exp(-\lambda\rho) du, \quad \rho = j_2/j_1. \\
&= I_{11,0} + I_{11,1} + I_{11,2} + I_{11,3}.
\end{aligned}$$

where

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$$I_{11,0} = \sum_{j_1=j_2} \frac{e^{\lambda}}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du,$$
$$I_{11,1} = \sum_{\rho \geq 1+\theta} \frac{e^{\lambda}}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du$$
$$I_{11,2} = \sum_{\rho \leq 1-\theta} \frac{e^{\lambda}}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du$$
$$I_{11,3} = \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} \frac{e^{\lambda}}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du.$$

Clearly

$$I_{11,0} = (U + 2\Delta) \sum_{j_2} \frac{1}{j_2^{2\beta}} = c_{11}(U + 2\Delta).$$

By Lemma 2.3,

$$I_{11,1} \ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \sum_{j_2/j_1 \geq 1+\theta} e^{\lambda} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1)$$
$$\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_1^{j_2/(1+\theta)} e^{\lambda} \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx$$
$$\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta-1}} \int_{1+\theta}^{j_2} e^{\lambda} v^{\lambda+c-2} \exp(-\lambda v) dv$$
$$\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta-1}} T^{-8} \lambda^{-1/2} \ll T^{-8}.$$

And,

$$I_{11,2} \ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \sum_{j_2/j_1 \leq 1-\theta} e^{\lambda} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1)$$

$$\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_{j_2/(1-\theta)}^{\infty} e^{\lambda} \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx$$
$$\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta-1}} \int_0^{1-\theta} e^{\lambda} v^{\lambda+c-2} \exp(-\lambda v) dv$$
$$\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta-1}} T^{-10} \ll T^{-9}$$

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And

$$\begin{aligned}
I_{11,3} &\ll (U+2\Delta)\sum_{j_2}\frac{1}{j_2^{2\beta}}\sum_{\substack{1-\theta\leq\rho\leq 1+\theta\\ j_1\neq j_2}} e^{\lambda}\left(\frac{j_2}{j_1}\right)^{\lambda+c}\exp(-\lambda j_2/j_1)\\
&\ll (U+2\Delta)\sum_{j_2\geq 1/\theta}\frac{1}{j_2^{2\beta}}\left(\frac{j_2}{1-\theta}-\frac{j_2}{1+\theta}\right)T^{1/10}\\
&\ll (U+2\Delta)\sum_{j_2\geq 1/\theta}\frac{2\theta}{j_2^{2\beta-1}(1-\theta^2)}T^{1/10}\\
&\ll (U+2\Delta)\theta^{2\beta-1}T^{1/10}\ll T^{-10},\quad (\beta\geq 10\log T).
\end{aligned}$$

For I_{12}, by Lemma 2.2,

$$\begin{aligned}
I_{12} &= \frac{1}{L}e^{\lambda}\int_{T_1-\Delta}^{T_2+\Delta}\sum_{j_1,j_2}\frac{-\log j_2}{j_2^{2\beta}}\rho^{s_1}\exp(-\lambda\rho)du\\
&= I_{12,0}+I_{12,1}+I_{12,2}+I_{12,3}.
\end{aligned}$$

where

$$\begin{aligned}
I_{12,0} &= \frac{1}{L}\sum_{j_1=j_2}\frac{-e^{\lambda}\log j_2}{j_2^{2c}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du,\\
I_{12,1} &= \frac{1}{L}\sum_{\rho\geq 1+\theta}\frac{-e^{\lambda}\log j_2}{j_2^{2\beta}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du\\
I_{12,2} &= \frac{1}{L}\sum_{\rho\leq 1-\theta}\frac{-e^{\lambda}\log j_2}{j_2^{2\beta}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du\\
I_{12,3} &= \frac{1}{L}\sum_{\substack{1-\theta\leq\rho\leq 1+\theta\\ j_1\neq j_2}}\frac{-e^{\lambda}\log j_2}{j_2^{2\beta}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du.
\end{aligned}$$

Clearly,

$$I_{12,0}=(U+2\Delta)\frac{1}{L}\sum_{j_2}\frac{-\log j_2}{j_2^{2\beta}}=c_{12}(U+2\Delta).$$

By Lemma 2.3,

$$\begin{aligned}
I_{12,1} &\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}\log j_2}{j_2^{2\beta}}\sum_{j_2/j_1\geq 1+\theta}e^{\lambda}\left(\frac{j_2}{j_1}\right)^{\lambda+c}\exp(-\lambda j_2/j_1)\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}\log j_2}{j_2^{2\beta}}\int_1^{j_2/(1+\theta)}e^{\lambda}\left(\frac{j_2}{x}\right)^{\lambda+c}\exp(-\lambda j_2/x)dx\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}\log j_2}{j_2^{2\beta-1}}\int_{1+\theta}^{j_2}e^{\lambda}v^{\lambda+c-2}\exp(-\lambda v)dv\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}\log j_2}{j_2^{2\beta-1}}T^{-8}\lambda^{-1/2}\ll T^{-8}.
\end{aligned}$$

and

$$
\begin{aligned}
I_{12,2} &\ll \sum_{j_2} \frac{\theta^{-1}\log j_2}{j_2^{2\beta}} \sum_{j_2/j_1 \le 1-\theta} e^{\lambda}\left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
&\ll \frac{1}{L}\sum_{j_2} \frac{\theta^{-1}\log j_2}{j_2^{2\beta}} \int_{j_2/(1-\theta)}^{\infty} e^{\lambda}\left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x)dx \\
&\ll \frac{1}{L}\sum_{j_2} \frac{\theta^{-1}\log j_2}{j_2^{2\beta-1}} \int_{0}^{1-\theta} e^{\lambda} v^{\lambda+c-2} \exp(-\lambda v)dv \\
&\ll \frac{1}{L}\sum_{j_2} \frac{\theta^{-1}\log j_2}{j_2^{2\beta-1}} T^{-10} \ll T^{-9}.
\end{aligned}
$$

and

$$
\begin{aligned}
I_{12,3} &\ll \frac{1}{L}(U+2\Delta)\sum_{j_2} \frac{\log j_2}{j_2^{2\beta}} \sum_{\substack{1-\theta\le\rho\le 1+\theta \\ j_1\ne j_2}} e^{\lambda}\left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
&\ll \frac{1}{L}(U+2\Delta)\sum_{j_2\ge 1/\theta} \frac{\log j_2}{j_2^{2\beta}}\left(\frac{j_2}{1-\theta}-\frac{j_2}{1+\theta}\right)T^{1/10} \\
&\ll \frac{1}{L}(U+2\Delta)\sum_{j_2\ge 1/\theta} \frac{2\theta\log j_2}{j_2^{2\beta-1}(1-\theta^2)}T^{1/10} \\
&\ll (U+2\Delta)\theta^{2\beta-1}T^{1/10} \ll T^{-10}.
\end{aligned}
$$

For I_{21}, by Lemma 2.2,

$$
\begin{aligned}
I_{21} &= \frac{1}{L}e^{\lambda}\int_{T_1-\Delta}^{T_2+\Delta}\sum_{j_1,j_2}\frac{-\log j_1}{j_2^{2\beta}}\rho^{s_1}\exp(-\lambda\rho)du \\
&= I_{21,0}+I_{21,1}+I_{21,2}+I_{21,3},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{21,0} &= \frac{1}{L}\sum_{j_1=j_2}\frac{-e^{\lambda}\log j_1}{j_2^{2\beta}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du, \\
I_{21,1} &= \frac{1}{L}\sum_{\rho\ge 1+\theta}\frac{-e^{\lambda}\log j_1}{j_2^{2\beta}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du \\
I_{21,2} &= \frac{1}{L}\sum_{\rho\le 1-\theta}\frac{-e^{\lambda}\log j_1}{j_2^{2\beta}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du \\
I_{21,3} &= \frac{1}{L}\sum_{\substack{1-\theta\le\rho\le 1+\theta \\ j_1\ne j_2}}\frac{-e^{\lambda}\log j_1}{j_2^{2\beta}}\int_{T_1-\Delta}^{T_2+\Delta}\rho^{s_1}\exp(-\lambda\rho)du.
\end{aligned}
$$

Clearly,

$$
I_{21,0} = (U+2\Delta)\frac{1}{L}\sum_{j_2}\frac{-\log j_2}{j_2^{2\beta}} = c_{21}(U+2\Delta)
$$

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By Lemma 2.3,

$$
\begin{aligned}
I_{21,1} &\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}}{j_2^{2\beta}}\sum_{j_2/j_1\geq 1+\theta} e^{\lambda}\log j_1\left(\frac{j_2}{j_1}\right)^{\lambda+c}\exp(-\lambda j_2/j_1)\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}}{j_2^{2\beta}}\int_1^{j_2/(1+\theta)} e^{\lambda}\log x\left(\frac{j_2}{x}\right)^{\lambda+c}\exp(-\lambda j_2/x)dx\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}}{j_2^{2\beta-1}}\int_{1+\theta}^{j_2} e^{\lambda}\log(j_2/v)v^{\lambda+c-2}\exp(-\lambda v)dv\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}\log j_2}{j_2^{2\beta-1}}T^{-8}\lambda^{-1/2}\ll T^{-8}.
\end{aligned}
$$

and

$$
\begin{aligned}
I_{21,2} &\ll \sum_{j_2}\frac{\theta^{-1}}{j_2^{2\beta}}\sum_{j_2/j_1\leq 1-\theta} e^{\lambda}\log j_1\left(\frac{j_2}{j_1}\right)^{\lambda+c}\exp(-\lambda j_2/j_1)\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}}{j_2^{2\beta}}\int_{j_2/(1-\theta)}^{\infty} e^{\lambda}\log x\left(\frac{j_2}{x}\right)^{\lambda+c}\exp(-\lambda j_2/x)dx\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}}{j_2^{2\beta-1}}\int_0^{1-\theta} e^{\lambda}\log(j_2/v)v^{\lambda+c-2}\exp(-\lambda v)dv\\
&\ll \frac{1}{L}\sum_{j_2}\frac{\theta^{-1}(\theta+\log j_2)}{j_2^{2\beta-1}}T^{-10}\ll T^{-9}.
\end{aligned}
$$

and

$$
\begin{aligned}
I_{21,3} &\ll \frac{1}{L}(U+2\Delta)\sum_{j_2}\frac{1}{j_2^{2\beta}}\sum_{\substack{1-\theta\leq\rho\leq 1+\theta\\ j_1\neq j_2}} e^{\lambda}\log j_1\left(\frac{j_2}{j_1}\right)^{\lambda+c}\exp(-\lambda j_2/j_1)\\
&\ll \frac{1}{L}(U+2\Delta)\sum_{j_2\geq 1/\theta}\frac{\log j_2}{j_2^{2\beta}}\left(\frac{j_2}{1-\theta}-\frac{j_2}{1+\theta}\right)T^{1/10}\\
&\ll \frac{1}{L}(U+2\Delta)\sum_{j_2\geq 1/\theta}\frac{2\theta\log j_2}{j_2^{2\beta-1}(1-\theta^2)}T^{1/10}\\
&\ll (U+2\Delta)\theta^{2\beta-1}T^{1/10}\ll T^{-10}.
\end{aligned}
$$

For I_{22}, by Lemma 2.2,

$$
\begin{aligned}
I_{22} &= \frac{1}{L^2}e^{\lambda}\int_{T_1-\Delta}^{T_2+\Delta}\sum_{j_1,j_2}\frac{\log j_1\log j_2}{j_2^{2\beta}}\rho^{s_1}\exp(-\lambda\rho)du\\
&= I_{22,0}+I_{22,1}+I_{22,2}+I_{22,3},
\end{aligned}
$$

where

$$I_{22,0} = \frac{1}{L^2} \sum_{j_1=j_2} \frac{e^{\lambda} \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du,$$
$$I_{22,1} = \frac{1}{L^2} \sum_{\rho \geq 1+\theta} \frac{e^{\lambda} \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du$$
$$I_{22,2} = \frac{1}{L^2} \sum_{\rho \leq 1-\theta} \frac{e^{\lambda} \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du$$
$$I_{22,3} = \frac{1}{L^2} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} \frac{e^{\lambda} \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda\rho) du.$$

Clearly,

$$I_{22,0} = (U + 2\Delta) \frac{1}{L^2} \sum_{j_2} \frac{\log^2 j_2}{j_2^{2\beta}} = c_{22}(U + 2\Delta).$$

By Lemma 2.3,

$$\begin{aligned} I_{22,1} &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \sum_{j_2/j_1 \geq 1+\theta} e^{\lambda} \log j_1 \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\ &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_1^{j_2/(1+\theta)} e^{\lambda} \log x \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \end{aligned}$$

$$\begin{aligned} &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta-1}} \int_{1+\theta}^{j_2} e^{\lambda} \log(j_2/v) v^{\lambda+c-2} \exp(-\lambda v) dv \\ &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log^2 j_2}{j_2^{2\beta-1}} T^{-8} \lambda^{-1/2} \ll T^{-8}. \end{aligned}$$

and

$$\begin{aligned} I_{22,2} &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \sum_{j_2/j_1 \leq 1-\theta} e^{\lambda} \log j_1 \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\ &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_{j_2/(1-\theta)}^{\infty} e^{\lambda} \log x \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\ &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta-1}} \int_0^{1-\theta} e^{\lambda} \log(j_2/v) v^{\lambda+c-2} \exp(-\lambda v) dv \\ &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} (\log j_2 + \theta) \log j_2}{j_2^{2\beta-1}} T^{-10} \ll T^{-9}. \end{aligned}$$

and

$$
\begin{aligned}
I_{22,3} &\ll \frac{1}{L^2}(U+2\Delta)\sum_{j_2}\frac{\log j_2}{j_2^{2\beta}}\sum_{\substack{1-\theta\le\rho\le 1+\theta\\ j_1\neq j_2}} e^{\lambda}\log j_1\left(\frac{j_2}{j_1}\right)^{\lambda+c}\exp(-\lambda j_2/j_1)\\
&\ll \frac{1}{L^2}(U+2\Delta)\sum_{j_2\ge 1/\theta}\frac{\log^2 j_2}{j_2^{2\beta}}\left(\frac{j_2}{1-\theta}-\frac{j_2}{1+\theta}\right)T^{1/10}\\
&\ll \frac{1}{L^2}(U+2\Delta)\sum_{j_2\ge 1/\theta}\frac{2\theta\log^2 j_2}{j_2^{2\beta-1}(1-\theta^2)}T^{1/10}\\
&\ll (U+2\Delta)\theta^{2\beta-1}T^{1/10}\ll T^{-10}.
\end{aligned}
$$

Combining all the evaluations above, and recall (3.22), it has

$$\int_T^{T+U}|\mathcal{G}(a+it)|^2dt = c_0(U+2\Delta)(1+O(L^3/\lambda))+\epsilon.$$

where

$$
\begin{aligned}
c_0 &= c_{11}+c_{12}+c_{21}+c_{22}\\
&= 1+\sum_{j=2}^{\infty}\frac{(1-\log j/L)^2}{j^{2\beta}}\\
&= 1+O(T^{-10}). \quad (\beta\ge 10\log T)
\end{aligned}
$$

Let $U=T$, by (3.15), it follows

$$
\begin{aligned}
&\int_T^{T+U}\log|\mathcal{G}(a+it)|dt \le \frac{T}{2}\log\left(1+\frac{2\Delta}{T}\right)+\frac{T}{2}\log(1+O(L^3/\lambda))+\epsilon\\
&\le \Delta+O(TL^3/\lambda).
\end{aligned}
\tag{3.23}
$$

With (3.12), (3.14), (3.19),(3.20) and (3.23), and recall that $a=1/2-1/L$, it follows

$$2\pi N_G(D)\le\frac{\Delta+O(TL^3/\lambda)+O(c\Delta^3/\lambda^2)+O(L)}{1/2-a}\ll T^{1/3}L^{7/3}.$$

i.e.

$$\Delta_{2T}^{T}\arg G(1/2+it)\le O(T^{1/3}L^{7/3}).$$

and

$$(N(2T)-N(T))-(N_0(2T)-N_0(T))\le O(T^{1/3}L^{7/3}).$$

Then let T be $T/2^k, 1\le k\le\log_2(T)$, and summing. This proves Theorem 1.1 in the case that there are no zeros of $G(s)$ on the boundary of D.

For the rest case, let N_1 and N_2 be the numbers of zeros of $G(s)$ on the left side of D, $\sigma=1/2$, and in D with $\sigma>1/2$, respectively. Indent the left side of D with small semicircles with centers at the zeros and lying in $\sigma\ge 1/2$. Let N_1' be the number of distinct zeros in the N_1 zeros. Let V_j be the variation in $\arg G$ in the jth interval between the successive semicircles. Then by the principle of argument, it has

$$\sum_j V_j-\pi N_1 = 2\pi N_2+O(L), \tag{3.24}$$

Let W_j be the variation of argument of

$$h(s)(f'(s)+f'(1-s))G(s)$$

in the jth interval, where W_j is taken for increasing t, while V_j is taken for decreasing t. With (3.2) and (3.24), it has

$$\begin{aligned}\sum_j W_j &= \operatorname{Im}(f)|_T^{T+U} - \sum_j V_j \\ &= \operatorname{Im}(f)|_T^{T+U} - (2\pi N_2 + \pi N_1) + O(L) \end{aligned} \tag{3.25}$$

By (3.8), in the jth open interval, the number of zeros of $\zeta(1/2+it)$ is at least

$$(W_j/\pi) - 1.$$

and in all the open intervals, the number of zeros is at least

$$\begin{aligned}\frac{1}{\pi}\sum_j W_j - N_1' - 1 &= \frac{1}{\pi}\operatorname{Im}(f)|_T^{T+U} - (2N_2 + N_1) - N_1' - 1 + O(L) \\ &= \frac{1}{\pi}\operatorname{Im}(f)|_T^{T+U} - 2N_G(D) + N_1 - N_1' + O(L) \end{aligned} \tag{3.26}$$

Moreover, by (3.7), we can know that on the side $\sigma = 1/2$, a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater, so there are $N_1 + N_1'$ such zeros of $\zeta(1/2+it)$, adding to (3.26), in total, it has

$$N_0(T+U) - N_0(T) \geq \frac{1}{\pi}\operatorname{Im}(f)|_T^{T+U} - 2N_G(D) + 2N_1 + O(L).$$

By (3.11), we can know

$$\frac{1}{\pi}\operatorname{Im}(f)|_T^{T+U} = N(T+U) - N(T) + O(L).$$

i.e.

$$(N(T+U) - N(T)) - (N_0(T+U) - N_0(T)) \leq O(T^{1/3}L^{7/3}).$$

□

Besides, we know that on the critical line a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater. Hence

$$\sum(m-1) \leq N_G(D).$$

where sum is over the distinct zeros of $\zeta(s)$ on the left side of D, m is the multiplicity of a zero. And so,

$$\sum_{m\geq 2} m \leq 2N_G(D) \leq O(T^{1/3}L^{7/3}). \tag{3.27}$$

This means that the non-trivial zeros of $\zeta(s)$ are all on the critical line, and all are simple, with at most $O(T^{1/3}(\log T)^{7/3})$ ones excepted.

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