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Fractional Equations for the Scaling Limits of Lévy Walks with Position Depending Jump Distributions

Vassili N. Kolokoltsov ⓘ

Faculty of Computation Mathematics and Cybernetics of Moscow State University, 119991 Moscow, Russia;
v.n.kolokoltsov@gmail.com

Abstract: Lévy walks represent important modeling tools for variety of real life processes. Their natural scaling limits are known to be described by the so-called material fractional derivatives. So far these scaling limits were derived for spatially homogeneous walks, where Fourier and Laplace transforms represent natural tools of analysis. Here we derive the corresponding limiting equations in the case of position depending times and velocities of walks, where Fourier transforms cannot be effectively applied. In fact, we derive three different limits (specified by the way the process is stopped at an attempt to cross the boundary), leading to three different multi-dimensional versions of Caputo-Dzherbashian derivatives, which correspond to different boundary conditions for the generators of the related Feller semigroups and processes. Some other extensions and generalisations are analysed.

Keywords: Lévy walks; fractional equations of variable order; Caputo-Dzherbashian and Riemann-Liouville derivatives; material fractional derivatives; scaling limit; continuous time random walks (CTRW); subordinated Markov processes

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1. Introduction

1.1. CTRWs, Lévy walks and Lévy flights

Letters **P** and **E** will be used everywhere to denote probability and expectation. The indicator function of a set M will be denoted $\mathbf{1}(M)$.

The CTRWs (continuous time random walks) found numerous applications in physics. The scaling limits of these CTRW were analysed by many authors, see e.g. [14], [17], [21] and references therein. The crucial points (realised initially by physicists, see [27] or [19]) were as follows: (i) the limits of scaled CTRWs yield Markov processes time-changed by inverse stable subordinators and (ii) these limiting processes solve fractional in time partial differential equations (PDEs). For general properties of fractional PDEs we can refer to [9] and [20].

The simplest CTRWs in $\mathbf{R}^d$ are specified by an i.i.d. sequence $\{W_i\}$ of positive random variables (representing waiting times) with a distribution $Q(dw)$ and an i.i.d. sequence of random variables $\{X_i\}$ in $\mathbf{R}^d$ (representing sizes of jumps) with a distribution $P(dx)$. This pair of random sequences defines the following process in $\mathbf{R}^d$. A particle starting at a point $x \in \mathbf{R}^d$ at a time w waits a random time W_1 in x, then the particle makes a jump to a random point $x + X_1$, then it waits at $x + X_1$ a random time W_2, and then jumps to a random point $x + X_1 + X_2$, etc. The total position and waiting time to the time t are therefore

$$X(t) = x + \sum_{i=1}^{[t]} X_i, \quad W(t) = w + \sum_{i=1}^{[t]} W_i,$$

respectively, where $[t]$ denotes the integer part of t. The pair $(X_x(t), W_w(t))$ is a Markov chain in $\mathbf{R}^d \times \mathbf{R}$ with the transition operator

$$UF(x, w) = \int_{\mathbf{R}^d} \int_0^\infty F(x + y, w + s) Q(ds) P(dy).$$

The CTRW is then defined as the first (spatial) coordinate of this pair subordinated by the inverse process $N_K = \sup\{t : W(t) \le K\}$ to the second coordinate, that is, as the process $x + \sum_{j=1}^{N_K} X_j$. In physics such processes are usually referred to as Lévy flights. Notice that the process N_K can be equivalently defined by the property that $N_K = n \iff W(n) \le K < W_{n+1}$ or yet equivalently by the property that $N_K \ge n \iff W(n) \le K$.

One can naturally extend these simplest CTRWs to the case when the variables X_i, W_i are not independent. For instance, they may have position dependent jumps with dependent X_i, W_i. In such cases the transition operator of the corresponding Markov chain takes the form

$$UF(x,w) = \int_{\mathbf{R}^d} \int_0^\infty (F(x+y, w+s) - F(x,w))Q(x; ds dy),$$

with some transition stochastic kernel Q. Of course, they can also depend on t. The natural general extension leads us to look at the processes of type $X(N_K)$, where $(X, W)(t)$ is some Markov process in $\mathbf{R}^d \times \mathbf{R}$ with an increasing second coordinate $W(t)$ and $N_K = \sup\{t : W(t) \le K\}$ is the inverse process to the second coordinate.

Specific new feature of the limiting equations for dependent (X_i, T_i) is that the corresponding fractional derivatives do not separate in time and spatial additive terms, but appear in a certain mixed form. Moreover, as was found in seminal papers [5], [18], in case of dependent times and sizes of jumps, it is natural to distinguish two versions of subordination defining CTRWs: $X(N_K)$ and $X(N_K + 1)$, referred to as lagging and leading CTRWs, or, in a different terminology, undershooting CTRWs (or just CTRWs) and overshooting CTRWs.

Popular examples of processes arising from dependent X_i, T_i represent the Lévy walks, which are the main objects of the present study. We refer to [3], [4], [7], [22], [28] for recent results and up-to-date reviews of mathematical and physical literature on these processes.

When moving according to a simple Lévy walk, the particle, instead of waiting a random time W_i in a certain location X_{i-1}, is moving during the time W_i with some constant velocity V_i chosen from a given distribution, thus arriving at $X_i = X_{i-1} + V_i W_i$. In the simplest case one assumes the sequences of pairs $\{(V_i, W_i)\}$ to be i.i.d. with also independent V_i and W_i.

Strictly speaking, Lévy walks are quite different from Lévy flights, as the former have continuous trajectories (particles move at constant velocities), while Lévy flights are jump-type processes with discontinuous (piecewise constant) trajectories. In the modern literature, however, it is a standard convention to ignore the behavior of Lévy walks between the switching times, thus (mentally) substituting periods of motion with a constant velocity by the corresponding waiting time and an instantaneous transition (a jump). From this point of view one can look at Lévy walks as examples of general CTRWs with i.i.d. sequence of pairs (X_i, W_i), where X_i and W_i are dependent. Namely, $X_i = V_i W_i$. The transition operator of the corresponding Markov chain gets the following form:

$$UF(x,w) = \int_{\mathbf{R}^d} \int_0^\infty F(x+vs, w+s)Q(ds)P(dv), \tag{1}$$

with a finite measure $Q(ds)$ and a probability measure $P(dv)$.

In case of position depending Lévy walks, the transition operator takes a more general form

$$UF(x,w) = \int_{\mathbf{R}^d} \int_0^\infty F(x+vs, w+s)Q(x; ds)P(x; dv), \tag{2}$$

with some stochastic kernels $Q(x; ds), P(x; dv)$. To simplify exposition we will not consider the extension when these kernels are time-dependent. As for general CTRWs with dependent distribution of jumps and waiting times one distinguishes the lagging Lévy walks $X_x(N_K)$ and the leading (or overshooting) walks $X_x(N_K + 1)$.

As was mentioned, an appropriate scaling of CTRWs leads to a subordinated Markov process with averages evolving according to certain fractional (pseudo)differential equations. The general scheme for

deriving such equations in case of i.i.d. sequences (X_i, T_i) with dependent X_i, T_i was given in [1] based essentially on the method of the Fourier and Laplace transform. The concrete version of this scheme for Lévy walks was performed in [23], [16], [24], which led to the so-called material fractional derivatives.

The general scheme for deriving fractional equations for scaling limits for CTRWs with position dependent jumps was developed in [10] and [11]. For position dependent jumps the methods of Fourier transform cannot be effectively applied, and completely different method had to be used. Modifications and extensions of this method were applied in [15] and [13] for equations with variable orders and kinetic equations. Here we apply this method to derive the equations for the scaling limit of Lévy walks for position dependent distribution of times and velocities (including variable orders of stability for the times of walks) leading to new general equations with fractional material derivatives.

1.2. Scaled Lévy walks with various boundary conditions

In paper [12] it was suggested to look for various scaling limits for general CTRWs specified by the way the particles are considered to cross the boundary at the final jump (including leading and lagging processes as particular cases) and leading to equations with different multi-dimensional extensions of Caputo-Dzherbashian fractional derivatives. Here we argue that, specifically for Lévy walks, the most natural crossing rule is neither lagging nor leading process (where the last jump is included or not, respectively, in the final spatial position). In fact, as was already noted above, CTRWs are only approximations to Lévy walks, where particles are supposed not to jump, but to move with constant velocities between switching times. Thus the natural stopping time should be not before or after the final jump, but in the intermediate time, when the crossing of the boundary really occurs. This version of stopping was considered in some detail in [12] for general CTRWs and the corresponding fractional derivatives were derived under certain technical assumptions. Here we analyse in detail the equations governing the lagging, leading and intermediate limiting processes concretely for Lévy walks and construct the underlying limiting processes and Feller semigroups.

The distribution of waiting times T_i in basic CTRWs (or moving times in Lévy walks) are assumed to have heavy tails with power decay at infinity and with unbounded expectation. In order to take into account possible position dependence, our main assumption will be that the time of moving T when started from a position x will have the power law

$$\mathbf{P}(T > t) \sim \frac{1}{\beta(x)} t^{-\beta(x)}, \quad t \to \infty, \tag{3}$$

with some positive function $\beta(x)$ bounded from above and from below. To make formulas more transparent we shall make (3) more precise (though this simplification is not very important). Namely, we assume that these distributions have continuous densities $Q_x(r)$ such that

$$Q_x(r) = r^{-1-\beta(x)} \text{ for } r \geq B, \text{ and } Q_x(r) \leq 1 \text{ for all } r, \tag{4}$$

where $\beta(x)$ is some continuous function such that $\beta_1 \leq \beta(x) \leq \beta_2$, with some constants $B > 0$ and $0 < \beta_1 < \beta_2 < 1$ such that $\beta_1 B^{\beta_1} > 1$ (the latter condition ensures that $\int_B^\infty r^{-1-\beta(x)} dr < 1$ for all x).

In the usual CTRW scaling (see e.g. [17], [16], [11]) one scales the transition times of the Markov chains (1) or (2) by some small parameter τ and the times of walks by τ in the power that equals the inverse value of the stability index of their distributions. Thus the natural scaling of jumps depends on the tails of jump distributions. Here we study Lévy walks without scaling velocities, so that the spatial scaling of jumps arises exclusively from the scaling of moving periods. The corresponding scaled version $(X^\tau, W^\tau)_{x,w}(t)$ of the Markov chain (2) can be defined by its governing transition operator

$$U_\tau F(x, w) = \int_{\mathbf{R}^d} \int_0^\infty F(x + v\tau^{1/\beta(x)} s, w + \tau^{1/\beta(x)} s) Q(x; ds) P(x; dv). \tag{5}$$

Here

$$W^\tau_{x,w}(t) = w + \sum_{i=1}^{[t/\tau]} \tau^{1/\beta(X^\tau(\tau(i-1)))} W_i,$$

and the scaled inverse process is $N^\tau_T = \sup\{t : W^\tau_{x,w}(t) \le T\}$.

The lagging and leading scaled CTRWs are then defined as the processes

$$X^{\tau;T,lag}_{x,w} = X^\tau_{x,w}(N^\tau_T), \quad X^{\tau;T,lead}_{x,w} = X^\tau_{x,w}(N^\tau_T + \tau), \tag{6}$$

respectively.

As was mentioned, both these processes effectively neglect the fact that Lévy walks, strictly speaking, are not jump processes. Away from the boundary, this discrepancy is not essential, but for jumps crossing the boundary it becomes essential. In reality, if at a time $\tau(i-1)$ the process was at $(X^\tau_{x,w}(\tau(i-1)), W^\tau_{x,w}(\tau(i-1)))$ and a new moving period, say W_i, was revealed, which turned out to be final (that is, crossing the boundary $\{w = T\}$, or equivalently, such that $W^\tau_{x,w}(\tau i) > T$), then simultaneously a new velocity, say V_i, was revealed. Then the process starts moving from $X^\tau_{x,w}(\tau(i-1)) = X^{\tau;T,lag}_{x,w}$ with the velocity V_i, until it reaches the spatial position of the intermediate exit point

$$X^{\tau;T,int}_{x,w} = X^\tau_{x,w}(N^{int}_T) = X^\tau_{x,w}(\tau(i-1)) + s\tau^{1/\beta(X^\tau(\tau(i-1)))} W_i V_i, \tag{7}$$

at the time $N^{int}_T = \tau(i-1) + \tau s$, when the process W reaches the boundary $\{w = T\}$, that is, where $s \in (0,1)$ solves the equation

$$W^\tau_{x,w}(N^{int}_T) = W^\tau_{x,w}(\tau(i-1) + \tau s) = W^\tau_{x,w}(\tau(i-1)) + s\tau^{1/\beta(X^\tau(\tau(i-1)))} W_i = T.$$

In other words, the position $(X^\tau_{x,w}(N^{int}_T), W^\tau_{x,w}(N^{int}_T))$ in $\mathbf{R}^d \times \mathbf{R}$ is the point, where the straight line connecting $(X^\tau_{x,w}(\tau(i-1)), W^\tau_{x,w}(\tau(i-1)))$ and $(X^\tau_{x,w}(\tau i), W^\tau_{x,w}(\tau i))$ crosses the hyperplane $\{w = T\}$:

$$X^{\tau;T,int}_{x,w} = X^{\tau;T,int}_{x,w} + \frac{T - W^\tau_{x,w}(\tau(i-1))}{\tau^{1/\beta(X^\tau(\tau(i-1)))} W_i} (X^{\tau;T,lead}_{x,w} - X^{\tau;T,lag}_{x,w})$$

$$= \frac{W^\tau_{x,w}(\tau i) - T}{\tau^{1/\beta(X^\tau(\tau(i-1)))} W_i} X^{\tau;T,lag}_{x,w} + \frac{T - W^\tau_{x,w}(\tau(i-1))}{\tau^{1/\beta(X^\tau(\tau(i-1)))} W_i} X^{\tau;T,lead}_{x,w}. \tag{8}$$

The transition operators for lagging, leading and intermediate stopped processes are the modifications of the transitions (5) for the initial Markov chain that take into account the chosen way of stopping at the boundary. Namely, they are defined for $w \le T$ and take the following forms:

$$U^{T,lag}_\tau F(x,w) = \int_{\mathbf{R}^d} \int_0^\infty$$

$$F(x + v\mathbf{1}(\tau^{1/\beta(x)}s \le T - w)\tau^{1/\beta(x)}s, w + \min\{\tau^{1/\beta(x)}s, T - w\})Q(x;ds)P(x;dv), \tag{9}$$

$$U^{T,lead}_\tau F(x,w) = \int_{\mathbf{R}^d} \int_0^\infty F(x + v\tau^{1/\beta(x)}s, w + \min\{\tau^{1/\beta(x)}s, T - w\})Q(x;ds)P(x;dv), \tag{10}$$

$$U^{T,int}_\tau F(x,w) = \int_{\mathbf{R}^d} \int_0^\infty F(x + v\min\{\tau^{1/\beta(x)}s, T - w\}, w + \min\{\tau^{1/\beta(x)}s, T - w\})Q(x;ds)P(x;dv). \tag{11}$$

We shall use the unified notation $U^{T,*}_\tau$ for these three operators, with $*$ denoting either lag, or lead, or int. It is seen that all $U^{T,*}_\tau$ do not take the process away from any band $\mathbf{R}^d \times [a, T]$ for any $a < T$.

Choosing $a = 0$ without loss of much generality we shall consider, from now on, the corresponding processes as taking values in the band $\mathbf{R}^d \times [0, T]$ with some fixed T. Notice also that

$$U_\tau^{T,lag} F(x, T) = U_\tau^{T,int} F(x, T) = F(x, T),$$

so that the corresponding processes are automatically stopped when reaching the boundary $\{w = T\}$.

1.3. Objectives and content of the paper

The objective of the present paper is to analyse the limits of the discrete Markov chains $(U_\tau^{T,*})^{[t/\tau]}$, as $\tau \to 0$, and the corresponding subordinated processes (first coordinate subordinated by the inverse process to the second coordinate), to derive multidimensional Caputo-Dzherbashian-type and Riemann-Liouville-type fractional equations (with fractional material derivatives of variable order) that govern the evolution of the limiting processes, and to show well-posedness of these equations in natural functional spaces. Our approach is to avoid any difficulties arising from non-Markovian processes by first incorporating stopping rules in the Markov processes (5), identify the corresponding limiting generators and then look at the distributions of the final non-Markovian process as at the stationary distributions of the corresponding stopped Markov process.

We shall also discuss certain modifications of the model, namely when there can be additional waiting times between the walks, and when the walks are performed with parameter depending velocities.

The paper is organised as follows. In Section 2 we obtain preliminary results on the limiting Feller process for the scaled Markov chains (5) governing the sizes and times of jumps. Section 3 is the main one. It is devoted to the presentation of our main results concerning fractional equations that govern the scaling limits of the Markov chains (9) -(11), as well as the related process killed at an attempt to cross the boundary $\{w = T\}$. The latter turns out to be described by a multidimensional version (19) of the Riemann-Liouville fractional operator with material derivative, while the former are given by 3 different multidimensional versions (23) -(26) of the Caputo-Dzherbashian fractional derivative. We derive the corresponding equations and obtain their well-posedness. These results are contained in Theorems 3.4 and 3.5. Last Subsection 3.5 touches briefly on some modifications arising when using constant accelerations (rather than constant velocities) between the switching times, or even more general model with parameter depending velocities. The proofs of the main results are also essentially given in Section 3, up to some technical results, Theorems 3.1 - 3.3. The latter theorems, on the existence of the limiting Feller processes interrupted on an attempt to cross the boundary, are only formulated in Subsection 3.3. In order not to interrupt the main arguments, their proof is postponed and given in special Section 4. Section 5 is devoted to modifications arising from the inclusion of additional waiting times.

1.4. Notations for basic spaces

We conclude the introduction with certain notations that will be used in the paper without further reminder.

For a set Ω, which is either $\mathbf{R}^d$ or the band $\mathbf{R}^d \times [0, T]$ in $\mathbf{R}^{d+1}$ (with some $T > 0$) let $C(\Omega)$ denote the spaces of bounded continuous functions on Ω, equipped with the standard sup-norm $\|.\|$. For $k \in \mathbf{N}$, let $C^k(\Omega)$ denote the spaces of k times continuously differentiable functions on Ω (in the case of a band, the corresponding one-sided derivatives are meant on the boundary) with bounded derivatives equipped with the standard norm

$$\|f\|_{C^k} = \max\{\|f\|, \max_{m=1}^{k} \|f^{(m)}\|\},$$

where $\|f^{(m)}\|$ denotes the maximum of sup-norms of all partial derivatives of f of order m. Let $C_\infty(\Omega)$ denote the closed subspaces of $C(\Omega)$ consisting of functions vanishing at infinity, $C_\infty^k(\Omega)$ the closed subspace of $C^k(\Omega) \cap C_\infty(\Omega)$ consisting of functions such that all its derivatives up to order k belong to $C_\infty(\Omega)$.

2. Preliminary results

Here we provide some auxiliary results on the limiting Feller process for the scaled Markov chains governing the sizes and times of jumps. These results are obtained by combining the method of proving well-posedness of processes generated by operators of order at most one (from [11]) with general convergence results from [5].

Our main assumptions are as follows:

Condition (A) on the distribution of times: the family $Q_x(r)$ is given by (4) with $\beta_1 \leq \beta(x) \leq \beta_2$ with some constants $0 < \beta_1 < \beta_2 < 1$, and $\beta(x) \in C^1(\mathbf{R}^d)$;

Condition (B) on the distribution of velocities: $P(x; dv)$ is a family of probability laws on $\mathbf{R}^d$ such that the family of measures $|v|P(x; dv)$ is tight (in particular, uniformly bounded);

Condition (C) on the first order regularity of spatial distributions: the derivative of $P(x; dv)$ with respect to x, $\nabla_x P(x; dv)$ exists as a family of signed vector-valued uniformly bounded measures such that the family $|v|\nabla_x P(x; dv)$ is tight;

Condition (D) on the second order regularity: the second order derivatives of $P(x; dv)$ with respect to x exist as uniformly bounded and tight families of signed measures, and $\beta(x) \in C^2(\mathbf{R}^d)$.

It is known (see e.g. Theorem 19.28 of [8] or Theorem 8.1.1 of [11]) that if the chains with transitions $U_\tau^{k\tau}$ (with a family of transitions given by (5)) converges to a Feller process $(X, W)_{x,w}(t)$, as $k\tau \to t$, then the generator Λ of the corresponding limiting semigroup can be obtained as the limit

$$\Lambda F = \lim_{\tau \to 0} \frac{1}{\tau}(U_\tau F - F). \tag{12}$$

By (5),

$$\frac{1}{\tau}(U_\tau F - F)(x, w) = \frac{1}{\tau} \int_{\mathbf{R}^d} \int_0^\infty [F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s) - F(x, w)]Q(x; ds)P(x; dv)$$

We shall need the following simple result (a proof can be found e.g. in [13] or [15]):

Lemma 2.1. *Let $p(y)$ be a probability density on $\mathbf{R}_+$ such that $p(y) = y^{-1-\beta}$ for $y \geq B$ with some $\beta \in (0, 1)$ and $B > 0$ such that $\beta B^\beta > 1$ (the latter condition comes from the requirement that $\int_B^\infty p(y)dy \leq 1$). Then for any Lipschitz continuous $f \in C(\mathbf{R})$ vanishing at zero, it follows that*

$$\left| h^{-1} \int_0^\infty f(h^{1/\beta}y)p(y)dy - \int_0^\infty \frac{f(y)dy}{y^{1+\beta}} \right| \leq \frac{B}{1-\beta}h^{-1+1/\beta}L, \tag{13}$$

where L is the Lipschitz constant of f.

Applying this result, yields

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\infty [F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s)) - F(x, w)]Q(x; ds)$$

$$= \int_0^\infty [F(x + vs, w + s)) - F(x, w)] \frac{ds}{s^{1+\beta(x)}}.$$

Consequently,

$$\lim_{\tau \to 0} \frac{1}{\tau}(U_\tau F - F) = LF$$

with

$$LF(x, w) = \int_{\mathbf{R}^d} \int_0^\infty [F(x + vs, w + s) - F(x, w)] \frac{ds}{s^{1+\beta(x)}} P(x; dv). \tag{14}$$

The next simple proposition is our first (preliminary) result.

Proposition 2.1. *Assume conditions (A) - (C) hold. Then operator (14) generates a Feller process $(X, W)_{x,w}(t)$ in $\mathbf{R}^{d+1}$ and a corresponding Feller semigroup in $C_\infty(\mathbf{R}^{d+1})$, which has $C_\infty^1(\mathbf{R}^{d+1})$ as an invariant core.*

The conditions are just slightly different from that of Theorem 5.1.1 from [11] (or Theorem 5.14.1 from [13]). The proof is exactly the same. We omit it, but will show the arguments below in a more involved case of processes with a boundary.

Thus the limit (12) exists for functions F from the core of the limiting process. Hence the standard results (Theorem 19.28 of [8]) implies the following direct consequence of Proposition 2.1:

Proposition 2.2. *Assume conditions (A) - (C) hold. Then the chains with transitions $U_\tau^{[t/\tau]}$ arising from (5) converge in distribution to the Feller process $(X, W)_{x,w}(t)$, as $\tau \to 0$.*

Let us define the right continuous inverse process to $W_{x,w}(t)$ by the formula

$$E_T = \sup\{t : W_{x,w}(t) \le T\} = \inf\{t : W_{x,w}(t) > T\}. \tag{15}$$

As usual, by E_T- we denote the left continuous modification of E_T.

Once Propositions 2.1 and 2.2 are obtained, the fundamental Theorem 3.6 from [5] can be applied to conclude the following:

Corollary 1. *The leading (or overshooting) and lagging CTRWs, $X_x^\tau(N_T^\tau + \tau)$ and $X_x^\tau(N_T^\tau)$, from (6) converge in distribution to the process $X_{x,w}^{T,lead} = X_{x,w}(E_T)$ and, respectively, to the process $X_{x,w}^{T,lag}$, which is the right continuous version of the process $X_{x,w}(E_T-)$.*

By (8), this implies the following:

Corollary 2. *The intermediate CTRWs $X_{x,w}^{\tau;T,int}$ converge to $X_{x,w}^{T,int}$, which is the spatial coordinate of the point of intersection of the line, joining $(X_{x,w}^{T,lag}, E_T-)$ and $(X_{x,w}^{T,lead}, E_T)$, with the boundary hyperplane $\{w = T\}$.*

Our aim is to introduce and to analyse the fractional pseudo-differential equations that govern the evolution of the processes $X^{T,lag}$, $X^{T,lead}$, $X^{T,int}$.

3. Main results

3.1. Material derivatives

Integrating by parts, (14) can be rewritten as

$$LF(x,w) = \int_{\mathbf{R}^d} \int_0^\infty \frac{ds}{\beta(x)s^{\beta(x)}} \left(v\frac{\partial F}{\partial x} + \frac{\partial F}{\partial w} \right)(x + vs, w + s)P(x; dv).$$

Recalling the expression for the standard (right) fractional derivative of an order $\beta \in (0,1)$:

$$D_-^\beta f(w) = \frac{d^\beta}{d(-w)^\beta}f(w) = \frac{1}{\Gamma(-\beta)} \int_0^\infty [f(w+s) - f(w)]\frac{ds}{s^{1+\beta}}$$

$$= \frac{1}{\beta\Gamma(-\beta)} \int_0^\infty f'(w+s)\frac{ds}{\beta s^\beta},$$

one can say that the integral

$$\mathbf{D}_v^{\beta(x)}F(x,w) = -\int_0^\infty \frac{ds}{\beta(x)s^{\beta(x)}} \left(v\frac{\partial F}{\partial x} + \frac{\partial F}{\partial w} \right)(x + sv, w + s), \tag{16}$$

represents (up to a positive multiplier that we shall neglect) the fractional material derivative, where the material derivative (in the direction v) is defined as

$$\mathbf{D}_v F(x,w) = \left(v\frac{\partial F}{\partial x} + \frac{\partial F}{\partial w}\right)(x,w).$$

Thus we can write down the generator L in the form

$$LF(x,w) = -\int_{\mathbf{R}^d} \mathbf{D}_v^{\beta(x)} F(x,w) P(x;dv), \qquad (17)$$

where the r.h.s. is the averaged (over v) fractional material derivative.

Continuing the analogy, let us note that if $f \in C^1(\mathbf{R})$ vanishes at $w = T$, then the right Riemann-Liouiville derivative can be defined as the restriction of D_-^β to the space of functions vanishing for $w \geq T$, that is, as the operator

$$D_{T-}^\beta f(w) = \frac{1}{\Gamma(-\beta)}\left[\int_0^{T-w} (f(w+s) - f(w))\frac{ds}{s^{1+\beta}} - \frac{f(w)}{\beta(T-w)^\beta}\right]$$

$$= \frac{1}{\Gamma(-\beta)}\frac{1}{\beta}\int_0^{T-w} f'(w+s)s^{-\beta}\,ds.$$

Similarly, operators (16) and (17), reduced to the space of smooth functions vanishing for $w \geq T$, take the forms

$$\mathbf{D}_{v,T-}^{\beta(x)} F(x,w) = -\int_0^{T-w}[F(x+vs,w+s) - F(x,w)]\frac{ds}{s^{1+\beta(x)}} + \frac{F(x,w)}{\beta(x)(T-w)^{\beta(x)}}$$

$$= -\int_0^{T-w}\frac{ds}{\beta(x)s^{\beta(x)}}\left(v\frac{\partial F}{\partial x} + \frac{\partial F}{\partial w}\right)(x+sv, w+s), \qquad (18)$$

and

$$L_{T-}F(x,w) = -\int_{\mathbf{R}^d} \mathbf{D}_{v,T-}^{\beta(x)} F(x,w) P(x;dv). \qquad (19)$$

Thus L_{T-} represents a multidimensional analog of the standard Riemann-Liouville fractional derivative (of variable order in our case), so that the inverse of this operator (if well defined) will represent a multidimensional analog of the standard Riemann-Liouville fractional integral.

When looking for a probabilistic interpretation of fractional derivatives in [12], it was noted that the Riemann-Liouville derivative is obtained from the free (without boundary) fractional operator by restricting it to the space of functions vanishing identically beyond the boundary, which in terms of the underlying stochastic process means its killing on the attempt to cross the boundary. In its turn, the Caputo-Dzherbashian derivative is obtained from the free fractional operator by restricting it to the space of functions that are constant beyond the boundary, which in terms of the underlying stochastic process means its stopping on the attempt to cross the boundary. This interpretation of fractional derivatives led to the natural extension of the fractional derivatives not only to a two-sided case, but to a variety of multidimensional settings. However, while killing on the boundary has always clear meaning, stopping for a multidimensional jump-type process really depends on the way one projects the result of the final jump (that crosses the boundary) to the boundary, which leads to several different version of the Caputo-Dzherbashian fractional operators. Three of them are analysed in this paper.

3.2. Stopped and killed limiting generators

Using Lemma 2.1 allows us to conclude that the limits of

$$L_\tau^{T,*} F(x,w) = \mathbf{1}(w < T)\frac{1}{\tau}(U_\tau^{T,*} F - F)(x,w),$$

as $\tau \to 0$, exist for $F \in C^1(\mathbf{R}^d \times [0,T])$ and equal, respectively,

$$L^{T,lag}F(x,w) = \int_{\mathbf{R}^d} \int_0^\infty$$

$$[F(x + vs\mathbf{1}(s \leq T-w), w + \min\{s, T-w\}) - F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv), \tag{20}$$

$$L^{T,lead}F(x,w) = \int_{\mathbf{R}^d} \int_0^\infty [F(x+vs, w+\min\{s,T-w\}) - F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv), \tag{21}$$

$$L^{T,int}F(x,w) = \int_{\mathbf{R}^d} \int_0^\infty$$

$$[F(x + v\min\{s,T-w\}, w+\min\{s,T-w\}) - F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv). \tag{22}$$

Remark 1. *Let us comment for clarity that the processes described by these three generators differ only by the last jump, that is, by the jump that is meant to cross the boundary. Saying more precisely, the last jump means really the last period of motion with a constant velocity. In $L^{T,lag}$ the last jump (when $s > T-w$) just does not occur at all (hence the shift vs is multiplied by the indicator $\mathbf{1}(s \leq T-w)$). In $L^{T,lead}$ the last jump occurs in full (thus confirming the term overshooting). In $L^{T,int}$ the last period of motion with constant velocity is interrupted exactly on crossing the boundary $\{w = T\}$, which makes this process the most natural one from author's point of view.*

One can rewrite these expressions in the following equivalent forms:

$$L^{T,lag}F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [F(x+sv, w+s) - F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv)$$

$$+ \mathbf{1}(w < T)[F(x,T) - F(x,w)]\frac{1}{\beta(x)(T-w)^{\beta(x)}}, \tag{23}$$

$$L^{T,lead}F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [F(x+sv, w+s) - F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv)$$

$$+ \int_{\mathbf{R}^d} \int_{T-w}^\infty [F(x+vs, T) - F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv), \tag{24}$$

$$L^{T,int}F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [F(x+sv, w+s) - F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv)$$

$$+ \mathbf{1}(w < T) \int_{\mathbf{R}^d} [F(x+v(T-w), T) - F(x,w)]\frac{1}{\beta(x)(T-w)^{\beta(x)}}P(x;dv). \tag{25}$$

Integrating by parts yields

$$L^{T,int}F(x,w) = \int_0^{T-w} \int_{\mathbf{R}^d} \left[v\frac{\partial F}{\partial x} + \frac{\partial F}{\partial w}\right](x+sv, w+s)\frac{ds}{\beta(x)s^{\beta(x)}}P(x;dv), \tag{26}$$

which is exactly the averaged fractional material derivative (18).

We shall denote by $L^{T,*}$ the general operators with $*$ denoting either lag, or lead, or int. Our main technical results, given below, concern the existence of well defined Feller processes in $\mathbf{R}^d \times [0,T]$ generated by $L^{T,*}$.

To work with these operators let us denote by $C_{\infty,0}(\mathbf{R}^d \times [0,T])$ (or sometimes shorter by $C_{\infty,0}$) the subspace of $C_\infty(\mathbf{R}^d \times [0,T])$ of functions vanishing at the boundary $\{t = T\}$, and by $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$ the subspace of $C^1_\infty(\mathbf{R}^d \times [0,T]) \cap C_{\infty,0}(\mathbf{R}^d \times [0,T])$ with all partial derivatives belonging to $C_{\infty,0}(\mathbf{R}^d \times [0,T])$.

The elementary properties of $L^{T,*}$ are collected in the following proposition, its proof being obtained by a direct inspection that we omit.

Proposition 3.1. *Assume conditions (A)-(B). Then all three $L^{T,*}$ are bounded operators from $C^1_\infty(\mathbf{R}^d \times [0,T])$ to $C_\infty(\mathbf{R}^d \times [0,T])$. Moreover, the image of $L^{T,lag}$ and $L^{T,int}$ belong to $C_{\infty,0}(\mathbf{R}^d \times [0,T])$.*

We are also interested in the versions of these processes killed on the boundary $\{w = T\}$. The semigroups arising from killed processes act on the space $C_{\infty,0}(\mathbf{R}^d \times [0,T])$. It is seen that in this space all three operators $L^{T,*}$ coincide. Let us denote them by $L^{T,kill}$:

$$L^{T,kill}F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [F(x+sv, w+s) - F(x,w)] \frac{ds}{s^{1+\beta(x)}} P(x; dv)$$

$$- \mathbf{1}(w < T)F(x,w)\frac{1}{\beta(x)(T-w)^{\beta(x)}}. \tag{27}$$

As expected, this is nothing else, but the operator (19), which represents (up to a sign) a multi-dimensional analog of the Riemann-Liouville fractional operator.

3.3. Formulation of the technical results: stopped and killed limiting processes

Theorem 3.1. *Under conditions(A)-(C) the operator $L^{T,kill}$ generates a Feller semigroup in the space $C_{\infty,0}(\mathbf{R}^d \times [0,T])$ with $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$ being its invariant core. Moreover, this semigroup is also strongly continuous in $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$. Finally, the potential operator $(L^{T,kill})^{-1}$ is well defined as a bounded operator both in $C_{\infty,0}(\mathbf{R}^d \times [0,T])$ and $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$.*

Theorem 3.2. *Under conditions(A)-(D) the operators $L^{T,lag}$ and $L^{T,int}$ generate Feller semigroups $S^{T,lag}$ and $S^{T,int}$, respectively, in the space $C_\infty(\mathbf{R}^d \times [0,T])$ such that all points of the boundary $\{w = T\}$ are rest points for the corresponding Feller processes. For $S^{T,lag}$ an invariant core can be taken as the subspace $C^1_{\infty,w0}(\mathbf{R}^d \times [0,T])$ of $C^1_\infty(\mathbf{R}^d \times [0,T])$ consisting of functions with the derivative with respect to w vanishing at the boundary $\{w = T\}$. For $S^{T,int}$ an invariant core can be taken as the subspace $C^1_{\infty,m0}(\mathbf{R}^d \times [0,T])$ of $C^1_\infty(\mathbf{R}^d \times [0,T])$ consisting of functions with the averaged material derivative*

$$\int \left(\frac{\partial F}{\partial w} + v \frac{\partial F}{\partial x} \right)(x,w)P(x;dv)$$

vanishing at the boundary $\{w = T\}$. Moreover, these semigroups are also strongly continuous in these cores considered as Banach subspaces of $C^1_\infty(\mathbf{R}^d \times [0,T])$.

Remark 2. *In the case of symmetric distribution of velocities, e.g. if $\int vP(x;dv) = 0$, the spaces $C^1_{\infty,w0}(\mathbf{R}^d \times [0,T])$ and $C^1_{\infty,m0}(\mathbf{R}^d \times [0,T])$ coincide.*

Theorem 3.3. *Under conditions(A)-(D) the operator $L^{T,lead}$ generates a Feller semigroup $S^{T,lead}$ in the space $C_\infty(\mathbf{R}^d \times [0,T])$ with an invariant core $C^1_{\infty,w0}(\mathbf{R}^d \times [0,T])$. This semigroup is strongly continuous in this core considered as a Banach subspace of $C_\infty(\mathbf{R}^d \times [0,T])$.*

The proof of all these results follow the same line of arguments. We shall give details for Theorem 3.1 in Section 4 and briefly comment on modifications arising in other cases.

The following result is a straightforward but important corollary of the theorems given above.

Proposition 3.2. *The semigroups $S^{T,lag}$, $S^{T,int}$, $S^{T,lead}$ represent different extensions of the semigroup $S^{T,kill}$ from the space $C_{\infty,0}(\mathbf{R}^d \times [0,T])$ to the space $C_\infty(\mathbf{R}^d \times [0,T])$. The domain of the operator $L^{T,kill}$ lies in the intersection of the domains of the operators $L^{T,lag}$, $L^{T,int}$, $L^{T,lead}$.*

Finally, when working with $L^{T,lead}$, we shall need to use functions from the domain that are not differentiable up to the boundary. Let us introduce the following functional space $H_\infty(\mathbf{R}^d \times [0,T])$, which is the subspace of $C_\infty(\mathbf{R}^d \times [0,T])$ consisting of functions $F(x,w)$ such that the spatial gradient $\nabla_x F$ exists

and belongs to $C_\infty(\mathbf{R}^d \times [0,T])$ and, with respect to the second variable, F is locally Hölder in the sense that the function

$$H_F(x) = \sup_{0 < w_1 - w_2 \le 1} \frac{|F(x, w_1) - F(x, w_2)|}{|w_1 - w_2|^{\beta(x)}}$$

is well defined and belongs to $C_\infty(\mathbf{R}^d)$. It is seen that for any $F \in H(\mathbf{R}^d \times [0,T])$ formulas (20), (21), and (22) yield well defined functions from $C_\infty(\mathbf{R}^d \times [0,T])$. Consequently, using the fact that the generator of any Feller semigroup is a closed operator, and approximating the functions from $H_\infty(\mathbf{R}^d \times [0,T])$ by the functions from the corresponding invariant cores of $L^{T,*}$ (given by the above Theorems) we obtain the following fact.

Proposition 3.3. *The space $H_\infty(\mathbf{R}^d \times [0,T])$ belongs to the domain of the generators of all semigroups $S^{T,lag}$, $S^{T,int}$, $S^{T,lead}$, and the space $H_\infty(\mathbf{R}^d \times [0,T]) \cap C_{\infty,0}(\mathbf{R}^d \times [0,T])$ belongs to the domain of the generator of the semigroup $S^{T,kill}$.*

3.4. Main results on the limiting fractional equations

Let us start with the analogs of the Riemann-Liouville fractional operators.

Theorem 3.4. *(i) For any $F \in C_{\infty,0}(\mathbf{R}^d \times [0,T])$ there exists a unique classical solution $G = (L^{T,kill})^{-1} \in C_{\infty,0}(\mathbf{R}^d \times [0,T])$ (classical in the sense that G lies in the domain of $L^{T,kill}$) to the equation*

$$L^{T,kill} G(x, w) = -\int_{\mathbf{R}^d} \mathbf{D}^{\beta(x)}_{v,T-} G(x, w) P(x; dv) = F(x, w).$$

(ii) The solution G has the following path integral (probabilistic) interpretation:

$$G(x, w) = \mathbf{E} \int_0^{E_T} F[(X, W)^{T,kill}_{x,w}(t)] \, dt, \tag{28}$$

where E_T is given by (15).

   (iii) If $F \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$, then $G \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$ as well.

Proof. Statements (i) and (iii) are direct consequences of Theorem 3.1. Representation (28) is the standard probabilistic representation for the potential operator that is routinely derived from the Dynkin martingale (see detail of a similar derivation in the proof of the next Theorem). □

Theorem 3.5. *(i) For any $\phi \in C^1_\infty(\mathbf{R}^d)$ there exists a unique classical (in the sense that it belongs to the domain of $L^{T,*}$) solution of the multi-dimensional fractional Cauchy problem (with material fractional derivatives)*

$$L^{T,*} F = 0, \quad F(x, T) = \phi(x), \tag{29}$$

where $*$ in $L^{T,*}$ denotes either lag, or int, or lead;
   (ii) This solution has the following probabilistic representation:

$$F(x, w) = \mathbf{E}\phi(X^{T,*}_{x,w}(E_T)), \tag{30}$$

and where E_T is given by (15).
   (iii) If $\phi \in C^2_\infty(\mathbf{R}^d)$, then, in case of either lag or int, this solution F belongs to the space $F \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$.

Proof. (i) We claim that there exists a function ϕ^* from the domain of the generator $L^{T,*}$ such that $\phi^*(x, T) = \phi(x)$ and $L^{T,*}\phi^*(x, T) \in C_{\infty,0}(\mathbf{R}^d \times [0,T])$.

   For the case of either lag or int such a function can be easily chosen from the space $C^1_\infty(\mathbf{R}^d \times [0,T])$ (implying, by Proposition 3.1, that $L^{T,*}\phi^* \in C_{\infty,0}(\mathbf{R}^d \times [0,T])$). In fact, one can take $\phi^{lag}(x, w) = \phi(x)$,

and ϕ^{int} must be chosen from the requirement that its material derivative vanishes on the boundary $\{w = T\}$.

The case of $L^{T,lead}$ is a bit more subtle, as ϕ^{lead} cannot be chosen from $C^1_\infty(\mathbf{R}^d \times [0,T])$. By Proposition 3.3, we can search for an appropriate ϕ^{lead} in the space $H_\infty(\mathbf{R}^d \times [0,T])$. And this is possible, because, as follows from (24), if $F \in H_\infty(\mathbf{R}^d \times [0,T])$, then

$$L^{T,lead}(x,T) = \int_{\mathbf{R}^d} \int_0^\infty [F(x+vs,T) - F(x,T)] \frac{ds}{s^{1+\beta(x)}} P(x;dv) + \lim_{w \to T} \frac{F(x,T) - F(x,w)}{\beta(x)(T-w)^{\beta(x)+1}}.$$

Consequently, for a given smooth $F(x,T) = \phi(x)$, one can choose $\phi^{lead}(x,w) = F(x,w)$ such that the last two terms in the last expression cancel.

With ϕ^* chosen in the way, required above, we see that the function $F - \phi^*$ belongs to $C_{\infty,0}(\mathbf{R}^d \times [0,T])$ and satisfies the equation

$$L^{T,kill}(F - \phi^*) = L^{T,*}(F - \phi^*) = -L^{T,*}\phi^*. \tag{31}$$

Since $L^{T,*}\phi^* \in C_{\infty,0}(\mathbf{R}^d \times [0,T])$, we can conclude by Theorem 3.4, that there exists a unique classical solution

$$(L^{T,kill})^{-1}(-L^{T,*}\phi^*) \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$$

of problem (31). Therefore, by Proposition 3.2, the function

$$F = (L^{T,kill})^{-1}(-L^{T,*}\phi^*) + \phi^* \tag{32}$$

belongs to the domain of $L^{T,*}$ and represents the unique solution of the original problem.

(ii) Representation (30) is obtained by the straightforward application of the Dynkin martingale. Namely, since $X^{T,*}$ is a Feller process, it follows that the process

$$M(t) = F((X,W)^{T,*}_{x,w}(t)) - \int_0^t L^{T,*}F((X,W)^{T,*}_{x,w}(s))\, ds$$

is a martingale for any $F \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$. By (29), $M(t) = F((X,W)^{T,*}_{x,w}(t))$. Then (30) follows from Doob's optional sampling theorem and an evident observation (see the end of the proof of Theorem 3.1, if necessary) that the stopping time E_T has a finite expectation.

(iii) This follows from Theorem 3.4 (iii) and the observation that $L^{T,*}\phi \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$ whenever $\phi \in C^2_\infty(\mathbf{R}^d)$. □

Remark 3. *Of course, once Theorem 3.5 or 3.4 are proved, one can use formulas (30) or (28) to define generalised solutions for the corresponding problems for an arbitrary continuous function ϕ.*

3.5. Modification: motions with a fixed random acceleration or parameter depending velocity

For a particle in random media a reasonable model is represented by a process that moves with a constant acceleration between random stops, see e.g. [6]. This suggests to look at a modification of Lévy walks, that can be called Lévy runs, where, after each switching, the particle starts moving with some constant acceleration (rather than velocity, as in Lévy walks) drawn randomly from some distribution. Fractional equations arising in the natural scaling limit of such processes are straightforward modifications of the above case with constant velocity.

Namely, the corresponding operator (14) of the limiting Markov process without a boundary changes to the operator

$$LF(x,w) = \int_{\mathbf{R}^d} \int_0^\infty [F(x+as^2/2, w+s) - F(x,w)] \frac{ds}{s^{1+\beta(x)}} P(x;da), \tag{33}$$

where $P(x; da)$ is the distribution of accelerations chosen at the position x. The Riemann-Liouville-type operator (27) of the killed process changes to the operator

$$L^{T,kill} F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [F(x + as^2/2, w + s) - F(x,w)] \frac{ds}{s^{1+\beta(x)}} P(x; dv)$$

$$- \mathbf{1}(w < T) F(x,w) \frac{1}{\beta(x)(T - w)^{\beta(x)}}. \tag{34}$$

Similar modifications can be written for the three versions of Caputo-Dzherbashian fractional derivatives. All results above have straightforward extension to this new model with constant accelerations between switching times.

This model is related to the model with parameter dependent velocity suggested in [26]. To combine these models we can suggest to substitute vs in (2) by a more general smooth function $\phi(v,s)$ such that $\partial\phi/\partial s(v,0) = 0$ for all v. The theory above can be carried out for this situation with more or less obvious corrections. Namely, possible growth of ϕ in v should be compensated by the assumption of the existence of appropriate moments of $P(x; dv)$.

4. Proofs of Theorems 3.1–3.3

4.1. Approximations

To build the processes generated by $L^{T,*}$ (including $L^{T,kill}$) we shall use appropriate approximations. For an $\epsilon > 0$ let $\chi_\epsilon(r)$ be a smooth function $\mathbf{R} \to [0,1]$ such that $\chi_\epsilon(r) = 0$ for $r \leq \epsilon$ and $\chi_\epsilon(r) = 1$ for $r \geq 2\epsilon$. Let $L^{T,*}_{\chi,\epsilon}$ denote the operator obtained by changing ds to $\chi_\epsilon(s)ds$ in the formula for $L^{T,*}_{\chi,\epsilon}$. One sees that all $L^{T,*}_{\chi,\epsilon}$ are bounded operators in the space $C_\infty(\mathbf{R}^d \times [0,T])$ such that the images of $L^{T,lag}_{\chi,\epsilon}$ and $L^{T,int}_{\chi,\epsilon}$ belong to $C_{\infty,0}(\mathbf{R}^d \times [0,T])$. Consequently, all $L^{T,*}_{\chi,\epsilon}$ generate Feller semigroups $S^{T,*}_{\chi,\epsilon}$ in $C_\infty(\mathbf{R}^d \times [0,T])$ and hence the corresponding Feller processes in $\mathbf{R}^d \times [0,T]$. For the cases of $L^{T,int}_{\chi,\epsilon}$ and $L^{T,lag}_{\chi,\epsilon}$ all points of the boundary $\{w = T\}$ are rest points for these processes.

We are going to construct the processes generated by $L^{T,*}$ as the limits of the corresponding processes generated by $L^{T,*}_{\chi,\epsilon}$. To perform these limits we are going to show that the semigroups $S^{T,*}_{\chi,\epsilon}$ are uniformly (in ϵ) bounded as semigroups in certain subspaces of $C^1_\infty(\mathbf{R}^d \times [0,T])$.

4.2. Proof of Theorem 3.1

Recall that we consider the operator $L^{T,kill}_{\chi,\epsilon}$ as a bounded operator in $C_{\infty,0}(\mathbf{R}^d \times [0,T])$. Denoting

$$\Sigma_{\epsilon,x}(r) = \int_r^\infty \frac{\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}},$$

we obtain

$$L^{T,kill}_{\chi,\epsilon} F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [F(x + sv, w + s) - F(x,w)] \frac{\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} P(x; dv)$$

$$- F(x,w) \Sigma_{\epsilon,x}(T - w). \tag{35}$$

Differentiating with respect to w (taking into account that $\int P(x, dv) = 1$ and that F vanishes on the boundary $\{w = T\}$) yields

$$\frac{\partial}{\partial w} L^{T,kill}_{\chi,\epsilon} F(x,w) = L^{T,kill}_{\chi,\epsilon} \frac{\partial F}{\partial w}(x,w)$$

$$+ F(x,w) \int_{\mathbf{R}^d} \chi_\epsilon(T - w)(T - w)^{-(1+\beta(x))} P(x; dv) + F(x,w) \Sigma'_{\epsilon,x}(T - w)).$$

The last two terms cancel yielding

$$\frac{\partial}{\partial w} L^{T,kill}_{\chi,\epsilon} F(x,w) = L^{T,kill}_{\chi,\epsilon} \frac{\partial F}{\partial w}(x,w). \tag{36}$$

Differentiating with respect to x yields

$$\frac{\partial}{\partial x} L^{T,kill}_{\chi,\epsilon} F(x,w) = L^{T,kill}_{\chi,\epsilon} \frac{\partial F}{\partial x}(x,w)$$

$$+ \int_{\mathbf{R}^d} \int_0^{T-w} [F(x+sv,w+s) - F(x,w)] \frac{\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} \nabla_x P(x;dv)$$

$$- \int_{\mathbf{R}^d} \int_0^{T-w} [F(x+sv,w+s) - F(x,w)] \frac{\beta'(x)\ln s\,\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} P(x;dv)$$

$$- F(x,w) \int_{T-w}^\infty \frac{\beta'(x)\ln s\,\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}}.$$

Since $|F(x,w)|$ is bounded by $(T-w)$ times the C^1-norm of F, it follows that all terms in this expression apart from the first one (that generates a contraction semigroup) are uniformly bounded in ϵ. Moreover, since $\partial F/\partial x$ vanishes at the boundary $\{w = T\}$, it follows that $\partial L^{T,kill}_{\chi,\epsilon} F/\partial x$ also vanishes at this boundary. Therefore, due to the perturbation theory, the operators $L^{T,kill}_{\chi,\epsilon}$ generate strongly continuous semigroups in $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$, which are uniformly bounded in ϵ.

Consequently, we may conclude that the derivatives of $S^{T,kill}_{\chi,\epsilon}(t)F$ are uniformly bounded functions (at least for t from any compact interval, which is sufficient for our purposes) for any initial $F \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$. Therefore, writing

$$S^{T,kill}_{\chi,\epsilon_1}(t)F - S^{T,kill}_{\chi,\epsilon_2}(t)F = \int_0^t S^{T,kill}_{\chi,\epsilon_1}(t-s)(L^{T,kill}_{\chi,\epsilon_1} - L^{T,kill}_{\chi,\epsilon_2})S^{T,kill}_{\chi,\epsilon_2}(s)F\,ds,$$

we conclude that, for $\epsilon_1 < \epsilon_2$,

$$\sup_{x,w} |[S^{T,kill}_{\chi,\epsilon_1}(t)F - S^{T,kill}_{\chi,\epsilon_2}(t)F](x,w)|$$

$$= o(1) \sup_{s \in [0,t]} \|S^{T,kill}_{\chi,\epsilon_2}(s)F\|_{C^1_{\infty,0}(\mathbf{R}^d \times [0,T])} = o(1)\|F\|_{C^1_{\infty,0}(\mathbf{R}^d \times [0,T])}, \qquad (37)$$

as $\epsilon_2 \to 0$. Hence the functions $S^{T,kill}_{\chi,\epsilon_1}(t)F$ converge to a function $S^{T,kill}(t)F$.

Convergence for $F \in C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$ extends to the convergence for $F \in C_{\infty,0}(\mathbf{R}^d \times [0,T])$ by the standard density argument. Therefore the family of contraction operators $S^{T,kill}_{\chi,\epsilon_1}(t)$ converges to a family $S^{T,kill}(t)F$, as $\epsilon \to 0$. Clearly the limiting family $S^{T,kill}(t)F$ is also a strongly continuous semigroup of contractions in $C_{\infty,0}(\mathbf{R}^d \times [0,T])$.

Writing

$$\frac{S^{T,kill}(t)F - F}{t} = \frac{S^{T,kill}(t)F - S^{T,kill}_{\chi,\epsilon_1}(t)F}{t} + \frac{S^{T,kill}_{\chi,\epsilon_1}(t)F - F}{t}$$

and noting that by (37) the first term is of order $o(1)\|f\|_{C^1_{\infty,0}}$, as $\epsilon \to 0$, allows one to conclude that $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$ belongs to the domain of the generator of the semigroup $S^{T,kill}(t)$ in $C_{\infty,0}(\mathbf{R}^d \times [0,T])$ and that it is given there by (27).

To show that $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$ is an invariant core, we can apply to $S^{T,kill}(t)$ the procedure applied above to $S^{T,kill}_{\chi,\epsilon_1}(t)$. Namely, differentiating $L^{T,kill}F(x,w)$ we find that, on the partial derivatives of F, the operator $L^{T,kill}$ acts as the diagonal operator (with $L^{T,kill}$ on the diagonal) plus a uniformly bounded operator. Thus, again referring to the standard perturbation theory, we conclude that the operators $S^{T,kill}(t)$ act as a uniformly bounded strongly continuous semigroup in $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$.

Finally, the potential operator $(L^{T,kill})^{-1}$ is known to be expressed via the semigroup by the following formula:

$$(L^{T,kill})^{-1} F(x,w) = \int_0^\infty S^{T,kill}(t)F(x,w)\,dt = \int_0^\infty \mathbf{E}F(X,W)^{kill}_{x,w}(t)\,dt.$$

Since the coordinate $W_{x,w}^{kill}(t)$ increases faster than certain Poisson process $W_w^P(t)$ with the generator $L_P f(w) = \Omega(f(w+1) - f(w))$ with some $\Omega > 0$, one can very roughly (but sufficiently for us) estimate the probability that $W_{x,w}^{kill}(t) < T$ by the probability

$$\mathbf{P}(W_0^P(t) < T) \leq \sum_{k=0}^{[T]} \frac{(\Omega t)^k}{k!} e^{-\Omega t} \leq T \max\{1, (\Omega t)^T\} e^{-\Omega t}.$$

Consequently,

$$|(L^{T,kill})^{-1} F(x,w)| \leq \|F\| \int_0^\infty T \max\{1, (\Omega t)^T\} e^{-\Omega t} \, dt \leq \frac{T}{\Omega}[1 + \Gamma(T+1)].$$

Thus the potential operator $(L^{T,kill})^{-1}$ is a bounded operator in $C_{\infty,0}(\mathbf{R}^d \times [0,T])$, as was claimed. Quite similarly, one shows that this operator is bounded in the space $C_{\infty,0}^1(\mathbf{R}^d \times [0,T])$.

4.3. Proof of Theorem 3.2

Differentiating $L_{\chi,\epsilon}^{T,lag} F$ and $L_{\chi,\epsilon}^{T,int} F$ with respect to x shows again (as for the case of $L_{\chi,\epsilon}^{T,kill}$) that the action of these operators on the spatial derivatives is the same as that of $L_{\chi,\epsilon}^{T,lag} F$ and $L_{\chi,\epsilon}^{T,int} F$, respectively, up to some uniformly (in ϵ) bounded operators. Moreover, $\partial L_{\chi,\epsilon}^{T,lag} F / \partial x$ and $\partial L_{\chi,\epsilon}^{T,int} F / \partial x$ vanish at the boundary for any $F \in C_{\infty,0}^1(\mathbf{R}^d \times [0,T])$. New features arise when differentiating with w. After some cancellations, similar to the case of $L_{\chi,\epsilon}^{T,kill} F$ we find that, for $F \in C_{\infty,0}^1(\mathbf{R}^d \times [0,T])$,

$$\frac{\partial}{\partial w} L_{\chi,\epsilon}^{T,lag} F(x,w) = L_{\chi,\epsilon}^{T,lag} \frac{\partial F}{\partial w}(x,w)$$

$$- \Sigma_{\epsilon,x}(T-w) \frac{\partial F}{\partial w}(x,T) - \int_{\mathbf{R}^d} [F(x + v(T-w), T) - F(x,T)] \frac{\chi_\epsilon(T-w)}{(T-w)^{1+\beta(x)}} P(x; dv), \quad (38)$$

and

$$\frac{\partial}{\partial w} L_{\chi,\epsilon}^{T,int} F(x,w) = L_{\chi,\epsilon}^{T,int} \frac{\partial F}{\partial w}(x,w) - \Sigma_{\epsilon,x}(T-w) \int_{\mathbf{R}^d} \left(\frac{\partial F}{\partial w} + v \frac{\partial F}{\partial x} \right)(x + v(T-w), T) P(x; dv). \quad (39)$$

It follows that if $(\partial F/\partial w)(x,T) = 0$, then $L_{\chi,\epsilon}^{T,lag} F \in C_{\infty,0}^1(\mathbf{R}^d \times [0,T])$, and thus the subspace $C_{\infty,w0}^1(\mathbf{R}^d \times [0,T])$ is invariant under the action of the semigroup $S_{\chi,\epsilon 1}^{T,lag}(t)$. Similarly, if the averaged material derivative vanishes at the boundary $\{w = T\}$, then $L_{\chi,\epsilon}^{T,int} F \in C_{\infty,0}^1(\mathbf{R}^d \times [0,T])$, and thus the subspace $C_{\infty,m0}^1(\mathbf{R}^d \times [0,T])$ is invariant under the action of the semigroup $S_{\chi,\epsilon}^{T,int}(t)$.

Arguing now as for case of the killed process we find that, for any $F \in C_{\infty,w0}^1(\mathbf{R}^d \times [0,T])$ the functions $S_{\chi,\epsilon}^{T,lag}(t) F$ converge, as $\epsilon \to 0$, in the space $C_\infty(\mathbf{R}^d \times [0,T])$ to some functions $S_\chi^{T,lag}(t) F$. Extending this convergence by the density argument we conclude that the contraction operators $S_{\chi,\epsilon}^{T,lag}(t)$ converge strongly in the space $C_\infty(\mathbf{R}^d \times [0,T])$ to some contraction operators $S^{T,lag}(t)$ that form a strongly continuous semigroup in the space $C_\infty(\mathbf{R}^d \times [0,T])$ such that the space $F \in C_{\infty,w0}^1(\mathbf{R}^d \times [0,T])$ belongs to the core of its generator.

Similarly we find that, for any $F \in C_{\infty,m0}^1(\mathbf{R}^d \times [0,T])$ the functions $S_{\chi,\epsilon}^{T,int}(t) F$ converge in the space $C_\infty(\mathbf{R}^d \times [0,T])$ to some functions $S_{\chi,\epsilon}^{T,int}(t) F$. Extending this convergence by the density argument we conclude that the contraction operators $S_{\chi,\epsilon}^{T,int}(t)$ converge strongly in the space $C_\infty(\mathbf{R}^d \times [0,T])$ to some contraction operators $S^{T,int}(t)$ that form a strongly continuous semigroup in the space $C_\infty(\mathbf{R}^d \times [0,T])$ such that the space $F \in C_{\infty,m0}^1(\mathbf{R}^d \times [0,T])$ belongs to the core of its generator.

However, we cannot complete the proof as for the killed process, because it is not obvious that the derivatives of $L^{T,lag} F$ or $L^{T,int} F$ with respect to w remain bounded under the action of the corresponding semigroups. Therefore, in this case, we have to use the second order regularity condition (D), to work

with the second order derivatives and then show, in the same way as for the first order derivatives, that the semigroups $S^{T,lag}(t)$ and $S^{T,int}(t)$ are strongly continuous in the space $C^1_\infty(\mathbf{R}^d \times [0,T])$. For instance, in the case of $L^{T,lag}$, we first check that the subspace of $C^2_\infty(\mathbf{R}^d \times [0,T])$ consisting of functions with the first and the second derivatives in w vanishing at the boundary $\{w = T\}$ is invariant under $S^{T,int}_{\chi,\epsilon}(t)$, and then show the convergence, as $\epsilon \to 0$, of the functions $S^{T,int}_{\chi,\epsilon}(t)F$ for F in this subspace, the convergence being in the space $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$. Then we extend this convergence by the density argument to all F from $C^1_{\infty,0}(\mathbf{R}^d \times [0,T])$, and thus complete the proof.

4.4. Proof of Theorem 3.3

We have

$$
L^{T,lead}_{\chi,\epsilon} F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [F(x+sv, w+s) - F(x,w)] \frac{\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} P(x;dv)
$$

$$
+ \int_{\mathbf{R}^d} \int_{T-w}^\infty [F(x+vs, T) - F(x,w)] \frac{\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} P(x;dv), \tag{40}
$$

Differentiating with respect to x yields

$$
\frac{\partial}{\partial x} L^{T,lead}_{\chi,\epsilon} F(x,w) = L^{T,lead}_{\chi,\epsilon} \frac{\partial F}{\partial x}(x,w)
$$

$$
+ \int_{\mathbf{R}^d} \int_0^\infty [F(x+sv, w+\min\{s, T-w\}) - F(x,w)] \frac{\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} \nabla_x P(x;dv)
$$

$$
- \int_{\mathbf{R}^d} \int_0^\infty [F(x+sv, w+\min\{s, T-w\}) - F(x,w)] \frac{\beta'(x)\ln s\,\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} P(x;dv).
$$

Further on,

$$
\frac{\partial}{\partial w} L^{T,lead}_{\chi,\epsilon} F(x,w) = \int_{\mathbf{R}^d} \int_0^{T-w} [\frac{\partial}{\partial w} F(x+sv, w+s) - \frac{\partial}{\partial w} F(x,w)] \frac{\chi_\epsilon(s)\,ds}{s^{1+\beta(x)}} P(x;dv)
$$

$$
- \int_{\mathbf{R}^d} [F(x+(T-w)v, T) - F(x,w)] \frac{\chi_\epsilon(T-w)}{(T-w)^{1+\beta(x)}} P(x;dv)
$$

$$
+ \frac{\chi_\epsilon(T-w)}{(T-w)^{1+\beta(x)}} \int_{\mathbf{R}^d} [F(x+v(T-w), T) - F(x,w)] P(x,dv) - \Sigma_{\epsilon,x}(T-w) \frac{\partial}{\partial w} F(x,w)
$$

$$
= \int_{\mathbf{R}^d} \int_0^{T-w} [\frac{\partial}{\partial w} F(x+sv, w+s) - \frac{\partial}{\partial w} F(x,w)] \frac{\chi_\epsilon(s)}{s^{1+\beta(x)}} ds\, P(x;dv) - \Sigma_{\epsilon,x}(T-w) \frac{\partial}{\partial w} F(x,w)
$$

$$
= L^{T,lead} \frac{\partial F}{\partial w} - \int_{\mathbf{R}^d} \int_{T-w}^\infty \frac{\partial F}{\partial w}(x+vs, T) \frac{\chi_\epsilon(s)}{s^{1+\beta(x)}} ds\, P(x;dv).
$$

It follows that the space $C^1_{\infty,w0}(\mathbf{R}^d \times [0,T])$ is invariant under the action of the semigroup $S^{T,lead}_{\chi,\epsilon}(t)$, as in the case of the semigroup $S^{T,int}_{\chi,\epsilon}(t)$. However, unlike the latter, the generator $L^{T,lead}_{\chi,\epsilon}$ does not vanish on the boundary $\{t = T\}$. The rest of the proof is the same as for Theorem 3.2.

5. Extension: including waiting times

In the literature on Lévy walks one often assumes additionally that a particle waits some random time after a move, before starting a new one.

Allowing for additional waiting time means that the transitions (2) are modified and turn to the transitions

$$
UF(x,w) = \int_{\mathbf{R}^d} \int_0^\infty F(x+vs, w+s+r) Q(x;ds) P(x;dv) R(x;dr), \tag{41}
$$

with some family of probabilities $R(x; dr)$ with the tails given by some $\alpha(x) \in (0, 1)$ with $0 < \alpha_1 \leq \alpha(x) \leq \alpha_2 < 1$. To be more concrete, we assume, analogously to (4) that $R(x, dr)$ has density $R_x(r)$ such that

$$R_x(r) = r^{-1-\alpha(x)} \text{ for } r \geq A; \ R_x(r) \leq 1 \text{ for all } r; \ \alpha_1 \leq \alpha(x) \leq \alpha_2, \tag{42}$$

with some constants $0 < \alpha_1 < \alpha_2 < 1$ and $A > 0$.

Then the scaled version (5) extends to the transitions

$$U_\tau F(x, w) = \int_{\mathbf{R}^d} \int_0^\infty F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s + \tau^{1/\alpha(x)}r)Q(x; ds)P(x; dv)R(x; dr). \tag{43}$$

The corresponding prelimiting operator (12) converges on the set of smooth functions to the operator

$$LF(x, w) = \int_{\mathbf{R}^d} \int_0^\infty [F(x + vs, w + s) - F(x, w)]\frac{ds}{s^{1+\beta(x)}}P(x; dv)$$

$$+ \int_0^\infty [F(x, w + r) - F(x, w)]\frac{dr}{r^{1+\alpha(x)}}. \tag{44}$$

To get (44) one just writes down

$$F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s + \tau^{1/\alpha(x)}r)$$

$$= [F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s + \tau^{1/\alpha(x)}r) - F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s)]$$

$$+ [F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s) - F(x, w)], \tag{45}$$

and apply Lemma 2.1 to both terms.

Thus, the sequential shift of the second (time) coordinate in (43) turns to the sum of independent shifts, when passing to the limit.

Straightforward extension of Propositions 2.1, 2.2 yields the following:

Proposition 5.1. *Assume that conditions (A) - (C) and (42) hold and $\alpha(x)$ is continuously differentiable. Then operator (44) generates a Feller process $(X, W)_{x,w}(t)$ in $\mathbf{R}^{d+1}$ and a corresponding Feller semigroup in $C_\infty(\mathbf{R}^{d+1})$, which has $C_\infty^1(\mathbf{R}^{d+1})$ as an invariant core. The chains with transitions $U_\tau^{[t/\tau]}$ arising from (43) converge in distribution to the Feller process $(X, W)_{x,w}(t)$, as $\tau \to 0$.*

Let us now write down the corresponding extensions of stopped processes. Since we first wait and then jump, we will be stopped if either the waiting time is crossing the boundary $\{w = T\}$ or, otherwise, if we cross the boundary when moving. Thus the lagging stopped version of (43) will be

$$U_{\tau,wait}^{T,lag}F(x, w) = F(x, T)\left[\int_{(T-w)\tau^{-1/\alpha(x)}}^\infty R_x(dr) + \int_0^{(T-w)\tau^{-1/\alpha(x)}} R_x(dr)\int_{(T-w-r\tau^{1/\alpha(x)})\tau^{-1/\beta(x)}}^\infty Q_x(ds)\right]$$

$$+ \int_{\mathbf{R}^d} P(x, dv)\int_0^{(T-w)\tau^{-1/\alpha(x)}} R_x(dr)\int_0^{(T-w-r\tau^{1/\alpha(x)})\tau^{-1/\beta(x)}} Q_x(ds)F(x + v\tau^{1/\beta(x)}s, w + \tau^{1/\beta(x)}s + \tau^{1/\alpha(x)}r). \tag{46}$$

Similarly other transitions $U_{\tau,wait}^{T,*}$ are defined by adding additional waiting times to the transitions of $U_\tau^{T,*}$.

To find the limiting generator, we are looking for the limit of $(U_{\tau,wait}^{T,*} - 1)/\tau$. By (42), as $\tau \to 0$,

$$\frac{1}{\tau}\int_{(T-w)\tau^{-1/\alpha(x)}}^\infty R_x(dr) \to \frac{1}{\alpha(x)}(T - w)^{-\alpha(x)},$$

$$\frac{1}{\tau} \int_0^{(T-w)\tau^{-1/al(x)}} R_x(dr) \int_{(T-w-r\tau^{1/\alpha(x)})\tau^{-1/\beta(x)}}^\infty Q_x(ds)$$

$$\sim \int_0^{(T-w)\tau^{-1/\alpha(x)}} R_x(dr)(T-w-r\tau^{1/\alpha(x)})^{-\beta(x)}\frac{1}{\beta(x)} \to \frac{1}{\beta(x)}(T-w)^{-\beta(x)}.$$

Thus

$$L_{wait}^{T,lag}F(x,w) = \lim_{\tau\to0}(U^\tau F - F)(x,w)/\tau = \frac{1}{\alpha(x)}(T-w)^{-\alpha(x)} + \frac{1}{\beta(x)}(T-w)^{-\beta(x)} + \lim_{\tau\to0} I(x,w),$$

where

$$I(x,w) = \frac{1}{\tau}\int_{\mathbf{R}^d} P(x,dv)\int_0^{(T-w)\tau^{-1/\alpha(x)}} R_x(dr)\int_0^{(T-w-r\tau^{1/\alpha(x)})\tau^{-1/\beta(x)}} Q_x(ds)$$

$$\times[F(x+v\tau^{1/\beta(x)}s, w+\tau^{1/\beta(x)}s+\tau^{1/\alpha(x)}r)-F(x,w)].$$

To deal with this expression we again use (45) and Lemma 2.1 yielding

$$L_{wait}^{T,lag}F(x,w) = \int_{\mathbf{R}^d}\int_0^{T-w}[F(x+sv,w+s)-F(x,w)]\frac{ds}{s^{1+\beta(x)}}P(x;dv)$$

$$+\int_0^{T-w}[F(x,w+r)-F(x,w)]\frac{dr}{r^{1+\alpha(x)}}$$

$$+\mathbf{1}(w<T)[F(x,T)-F(x,w)]\left[\frac{1}{\beta(x)(T-w)^{\beta(x)}}+\frac{1}{\alpha(x)(T-w)^{\alpha(x)}}\right]. \quad (47)$$

Similar calculations work for other $L_{wait}^{T,*}$ leading to the following formulas:

$$L_{wait}^{T,*}F(x,w) = L^{T,*}F(x,w)+\int_0^{T-w}[F(x,w+r)-F(x,w)]\frac{dr}{r^{1+\alpha(x)}}$$

$$+\mathbf{1}(w<T)[F(x,T)-F(x,w)]\frac{1}{\alpha(x)(T-w)^{\alpha(x)}}. \quad (48)$$

The results for $L^{T,*}$ and the corresponding processes $S^{T,*}$ extend to the version with additional waiting times. However, to avoid technical complications, we make additional simplifying assumption:

Condition (E): for the results below concerning $L_{wait}^{T,int}$ we assume that the distribution of velocities is symmetric: $\int vP(x;dv) = 0$ for all x; for the results concerning $L_{wait}^{T,lead}$ we assume that either $\beta(x) > \alpha(x)$ for all x, or $\alpha(x) > \beta(x)$ for all x; nothing additional for $L_{wait}^{T,lag}$ and $L_{wait}^{T,kill}$.

Theorem 5.1. *Under conditions of Proposition 5.1 supplemented by Condition (E), the results of Theorems 3.1, 3.2, 3.3, as well as Theorems 3.4, 3.5, extend literally to the operator $L_{wait}^{T,*}$.*

Proof. The extension of all proofs is straightforward. Let us note only that condition (E) for $L_{wait}^{T,int}$ is needed, while otherwise the boundary conditions of spaces $C_{\infty,w0}^1(\mathbf{R}^d\times[0,T])$ and $C_{\infty,m0}^1(\mathbf{R}^d\times[0,T])$ do not coincide and therefore neither can be chosen as an invariant subspace for $L_{wait}^{T,int}$ such that the application of $L_{wait}^{T,int}$ to this subspace belongs to $C_{\infty,0}^1(\mathbf{R}^d\times[0,T])$. The condition (E) for $L_{wait}^{T,lead}$ is needed for choosing ϕ^{lead} in the extension of the proof of Theorem 3.5. $\square$

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