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A Rational Approximation of the Two-Term Machin-Like Formula for π

Sanjar M. Abrarov [1,2,3,*], **Rehan Siddiqui** [2,3,4], **Rajinder Kumar Jagpal** [2,4] and **Brendan M. Quine** [1,3,4]

[1] Thoth Technology Inc., Algonquin Radio Observatory, Achray Rd., RR6, Pembroke, ON K8A 6W7, Canada
[2] Epic College of Technology, 5670 McAdam Rd., Mississauga, ON L4Z 1T2, Canada
[3] Department Earth and Space Science and Engineering, York University, 4700 Keele St., Toronto, ON M3J 1P3, Canada
[4] Department Physics and Astronomy, York University, 4700 Keele St., Toronto, ON M3J 1P3, Canada
[*] Correspondence: sanjar@thothx.ca

Abstract: In this work, we consider the properties of the two-term Machin-like formula and develop an algorithm for computing digits of π by using its rational approximation. In this approximation, both terms are constructed by using a representation of $1/\pi$ in the binary form. This approach provides the squared convergence in computing digits of π without any trigonometric functions and surd numbers. The Mathematica codes showing some examples are presented.

Keywords: constant pi; iteration; nested radicals; rational approximation

1. Preliminaries

In 1876, English astronomer and mathematician John Machin demonstrated an efficient method to compute digits of π by using his famous discovery [1–4]

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right). \tag{1}$$

In particular, due to relatively rapid convergence of this formula, he was the first to compute more than 100 digits of π. Nowadays, the equations of kind

$$\frac{\pi}{4} = \sum_{j=1}^{J} A_j \arctan\left(\frac{1}{B_j}\right),$$

where A_j and B_j are either integers or rational numbers, are named after him as the Machin-like formulas for π [1–4].

Theorem 1 below shows the arctangent formula (2) for π. We can use this equation as a starting point to generate the-two term Machin-like formula for π of kind

$$\frac{\pi}{4} = 2^{k-1}\arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{y}\right),$$

where $k, x \in \mathbb{N}$ and $y \in \mathbb{Q}$. Further, we will show the significance of the multiplier 2^{k-1} in this formula.

Theorem 1. *There is a formula for π at any integer $k \geq 1$ [5]:*

$$\frac{\pi}{4} = 2^{k-1}\arctan\left(\frac{\sqrt{2-a_{k-1}}}{a_k}\right), \tag{2}$$

where $a_0 = 0$ and $a_k = \sqrt{2 + a_{k-1}}$ are nested radicals.

Proof of Theorem 1. Since

$$\cos\left(\frac{\pi}{2^2}\right) = \frac{1}{2}\sqrt{2} = \frac{1}{2}a_1,$$

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from the identity

$$\cos(2x) = 2\cos^2(x) - 1,$$

it follows, by induction, that

$$\cos\left(\frac{\pi}{2^3}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}} = \frac{1}{2}a_2,$$

$$\cos\left(\frac{\pi}{2^4}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} = \frac{1}{2}a_3,$$

$$\cos\left(\frac{\pi}{2^5}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} = \frac{1}{2}a_4,$$

$$\vdots$$

$$\cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{k \text{ square roots}} = \frac{1}{2}a_k.$$

Therefore, using

$$\sin\left(\frac{\pi}{2^{k+1}}\right) = \sqrt{1-\cos^2\left(\frac{\pi}{2^{k+1}}\right)},$$

we obtain

$$\frac{\pi}{2^{k+1}} = \arctan\left(\frac{\sqrt{1-\cos^2\left(\frac{\pi}{2^{k+1}}\right)}}{\cos\left(\frac{\pi}{2^{k+1}}\right)}\right) = \arctan\left(\frac{\sqrt{2-a_{k-1}}}{a_k}\right)$$

and Equation (2) follows. □

Lemma 1 and its proof below show how Equation (2) is related to the well-known limit (3) for π.

Lemma 1. *There is a limit such that [6]*

$$\pi = \lim_{k\to\infty} 2^k \sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{k-1 \text{ square roots}}}. \tag{3}$$

Proof of Lemma 1. Let

$$\lim_{k\to\infty} a_k = x.$$

Then, we can write

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \sqrt{2+a_{k-1}} = \sqrt{2+\lim_{k\to\infty} a_k}$$

or

$$x = \sqrt{2+x}$$

or

$$x^2 - x - 2 = 0.$$

Solving this quadratic equation leads to two solutions $x = 2$ and $x = -1$. Since a_k cannot be negative, we came to conclusion that

$$\lim_{k\to\infty} a_k = 2.$$

Consequently, this gives

$$\lim_{k\to\infty}\sqrt{2-a_{k-1}} = \sqrt{2-\lim_{k\to\infty}a_{k-1}} = \sqrt{2-\lim_{k\to\infty}a_k} = 0.$$

Since the formula (2) is valid for any arbitrarily large k, we can write

$$\frac{\pi}{4} = \lim_{k\to\infty}2^{k-1}\arctan\left(\frac{\sqrt{2-a_{k-1}}}{a_k}\right)$$

As $\arctan(x) \to x$ when $x \to 0$, we have

$$\begin{aligned}\frac{\pi}{4} &= \lim_{k\to\infty}2^{k-1}\frac{\sqrt{2-a_{k-1}}}{a_k} = \lim_{k\to\infty}2^{k-1}\sqrt{2-a_{k-1}}/\lim_{k\to\infty}a_k \\ &= \lim_{k\to\infty}2^{k-2}\sqrt{2-a_{k-1}} = \lim_{k\to\infty}2^{k-1}\sqrt{2-a_k}\end{aligned}$$

and Equation (3) follows. This completes the proof. □

Lemma 2 and its proof below show how Equation (2) can be transformed into the two-term Machin-like formula for π [7].

Lemma 2. *In the following equation*

$$\frac{\pi}{4} = 2^{k-1}\arctan\left(\frac{1}{\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor}\right) + \arctan\left(\frac{1}{\beta_k}\right), \tag{4}$$

the value β_k is always a rational number.

Proof of Lemma 2. It is convenient to define

$$\alpha_k = \left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor. \tag{5}$$

Using the identity

$$\arctan\left(\frac{1}{x}\right) = \frac{1}{2i}\ln\left(\frac{x+i}{x-i}\right)$$

and taking into consideration that

$$\frac{\pi}{4} = \arctan(1),$$

Equation (4) can be represented as

$$\frac{\pi}{2}i = \ln\left(\frac{\alpha_k+i}{\alpha_k-i}\right)^{2^{k-1}}\ln\left(\frac{\beta_k+i}{\beta_k-i}\right)$$

or

$$i = \left(\frac{\alpha_k+i}{\alpha_k-i}\right)^{2^{k-1}}\left(\frac{\beta_k+i}{\beta_k-i}\right). \tag{6}$$

It is not difficult to see by substitution that the following formula [7]

$$\beta_k = \frac{2}{\left(\frac{\alpha_k+i}{\alpha_k-i}\right)^{2^{k-1}} - i} - i \tag{7}$$

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is a solution of the Equation (6). Since k is a positive integer greater than 1, we can see that the real and imaginary parts of the expression

$$\left(\frac{\alpha_k+i}{\alpha_k-i}\right)^{2^{k-1}}=\left(\frac{\alpha_k^2-1}{\alpha_k^2+1}+i\frac{2\alpha_k}{\alpha_k^2+1}\right)^{2^{k-1}}$$

cannot be an irrational number if α_k is an integer. This means that the real and imaginary parts of the value β_k must be both rational. Since the first arctangent term of Equation (4) is a real number, the second arctangent term is also a real number. Therefore, we conclude that the imaginary part of the value β_k is equal to zero. This completes the proof. □

This is not the only method to generate the two-term Machin-like formulas for π of kind (4). Recently, Gasull *et all*. proposed a different method to derive the same equation (see Table 2 in [8]). Lemma 3 and its proof below show how the Equation (4) can be represented in a trigonometric form [7].

Lemma 3. *Equation* (4) *can be expressed as*

$$\frac{\pi}{4}=2^{k-1}\arctan\left(\frac{1}{\alpha_k}\right)+\arctan\left(\frac{1-\sin\left(2^{k-1}\arctan\left(\frac{2\alpha_k}{\alpha_k^2-1}\right)\right)}{\cos\left(2^{k-1}\arctan\left(\frac{2\alpha_k}{\alpha_k^2-1}\right)\right)}\right) \tag{8}$$

Proof of Lemma 3. Define

$$\kappa_1=\frac{\alpha_k^2-1}{\alpha_k^2+1}$$

and

$$\lambda_1=\frac{2\alpha_k}{\alpha_k^2+1}$$

such that

$$\beta_k=\frac{2}{(\kappa_1+i\lambda_1)^{2^{k-1}}-i}-i, \tag{9}$$

in accordance with Equation (7). Then, using de Moivre's formula we can write the complex number in polar form as

$$\begin{aligned}(\kappa_1+i\lambda_1)^{2^{k-1}}=&\left(\kappa_1^2+\lambda_1^2\right)^{2^{k-2}}\left(\cos\left(2^{k-1}\mathrm{Arg}(\kappa_1+i\lambda_1)\right)\right)\\&+i\sin\left(2^{k-1}\mathrm{Arg}(\kappa_1+i\lambda_1)\right).\end{aligned}$$

Thus, substituting this equation into Equation (9), we obtain

$$\beta_k=\frac{\cos\left(2^{k-1}\mathrm{Arg}(\kappa_1+i\lambda_1)\right)}{1-\sin(2^{k-1}\mathrm{Arg}(\kappa_1+i\lambda_1))}=\frac{\cos\left(2^{k-1}\mathrm{Arg}\left(\frac{\alpha_k+i}{\alpha_k-i}\right)\right)}{1-\sin\left(2^{k-1}\mathrm{Arg}\left(\frac{\alpha_k+i}{\alpha_k-i}\right)\right)}. \tag{10}$$

Using the relation

$$\mathrm{Arg}(x+iy)=\arctan\left(\frac{y}{x}\right),\qquad x>0,$$

we can write

$$\mathrm{Arg}(\kappa_1+i\lambda_1)=\arctan\left(\frac{2\alpha_k}{\alpha_k^2-1}\right). \tag{11}$$

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Consequently, from the identities (10) and (11), we get

$$\beta_k = \frac{\cos\left(2^{k-1}\arctan\left(\frac{2\alpha_k}{\alpha_k^2-1}\right)\right)}{1-\sin\left(2^{k-1}\arctan\left(\frac{2\alpha_k}{\alpha_k^2-1}\right)\right)} \tag{12}$$

and Equation (8) follows. □

It should be noted that computation of the constant β_k by using Equation (7) is not optimal. Specifically, at lager values of the integer k its application slows down the computation. Application of the Equation (12) for computation of the constant β_k is also not desirable due to presence of the trigonometric functions. Theorem 2 and its proof show how this problem can be effectively resolved [7].

Theorem 2. *The rational number β_k is given by*

$$\beta_k = \frac{\kappa_k}{1-\lambda_k}, \tag{13}$$

where the coefficients κ_k and λ_k can be found by a two-step iteration

$$\begin{cases} \kappa_n = \kappa_{n-1}^2 - \lambda_{n-1}^2 \\ \lambda_n = 2\kappa_{n-1}\lambda_{n-1}, \end{cases} \qquad n = 2,3,4,\ldots,k \tag{14}$$

with initial values

$$\kappa_1 = \frac{\alpha_k^2-1}{\alpha_k^2+1}$$

and

$$\lambda_1 = \frac{2\alpha_k}{\alpha_k^2+1}.$$

Proof of Theorem 2. We notice that the following power reduction

$$\begin{aligned}(\kappa_1+i\lambda_1)^{2k-1} &= \overbrace{\left(\left((\kappa_1+i\lambda_1)^2\right)^{2\cdots}\right)^2}^{k-1 \text{ powers of } 2} = \overbrace{\left(\left((\kappa_2+i\lambda_2)^2\right)^{2\cdots}\right)^2}^{k-2 \text{ powers of } 2} \\ &= \overbrace{\left(\left((\kappa_3+i\lambda_3)^2\right)^{2\cdots}\right)^2}^{k-3 \text{ powers of } 2} = \overbrace{\left(\left((\kappa_n+i\lambda_n)^2\right)^{2\cdots}\right)^2}^{k-n \text{ powers of } 2} \\ &= \left((\kappa_{k-2}+i\lambda_{k-2})^2\right)^2 = (\kappa_{k-1}+i\lambda_{k-1})^2 = \kappa_k + i\lambda_k,\end{aligned}$$

where the numbers κ_n and λ_n can be found by two-step iteration (14), leads to

$$\beta_k = \frac{2}{\kappa_k - i\lambda_k - i} - i = \frac{2\kappa_k}{\kappa_k^2+(\lambda_k-1)^2} + i\left(\frac{2(1-\lambda_k)}{\kappa_k^2+(\lambda_k-1)^2} - 1\right),$$

according to the Equation (10).

From the Lemma 2 it follows that the imaginary part of the value β_k is equal to zero. Consequently, the equation above can be simplified as

$$\beta_k = \frac{2\kappa_k}{\kappa_k^2+(\lambda_k-1)^2}. \tag{15}$$

However, since
$$i\left(\frac{2(1-\lambda_k)}{\kappa_k^2+(\lambda_k-1)^2}-1\right)=0$$
it follows that
$$\frac{2(1-\lambda_k)}{\kappa_k^2+(\lambda_k-1)^2}=1 \Leftrightarrow \kappa_k^2+(\lambda_k-1)^2=2(1-\lambda_k).$$
Substituting this result into the denominator of Equation (15), we obtain Equation (13). This completes the proof. □

There is an interesting relation between β_k and nested radicals a_{k-1} and a_k. Specifically, comparing Equation (4) with Equation (18) from the Lemma 4 below, one can see that
$$\arctan\left(\frac{1}{\beta_k}\right)=-2^{k-1}\arctan\left(\frac{\left\{a_k/\sqrt{2-a_{k-1}}\right\}}{1+\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor\left(a_k/\sqrt{2-a_{k-1}}\right)}\right),$$
where
$$\left\{a_k/\sqrt{2-a_{k-1}}\right\}=a_k/\sqrt{2-a_{k-1}}-\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor$$
denotes the fractional part.

Lemma 4. *There is a limit such that*
$$\lim_{k\to\infty}\arctan\left(\frac{1}{\beta_k}\right)=0.$$

Proof of Lemma 4. It is not difficult to see that using change of the variable
$$y\rightarrow\frac{y}{1+(x+y)x}$$
in the trigonometric identity
$$\arctan(x)+\arctan(y)=\arctan\left(\frac{x+y}{1-xy}\right)$$
leads to
$$\arctan(x+y)=\arctan(x)+\arctan\left(\frac{y}{1+(x+y)x}\right). \qquad (16)$$

Define for convenience the fractional part as
$$\gamma_k=\left\{a_k/\sqrt{2-a_{k-1}}\right\}.$$
Then, from Equation (2) it follows that
$$\frac{\pi}{4}=2^{k-1}\arctan\left(\frac{1}{\alpha_k+\gamma_k}\right)$$
Solving the equation
$$\frac{1}{\alpha_k+\gamma_k}=\frac{1}{\alpha_k}+z,$$
we get
$$z=-\frac{\gamma_k}{\alpha_k(\alpha_k+\gamma_k)}.$$

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Therefore, we have

$$\frac{\pi}{4} = 2^{k-1} \arctan\left(\frac{1}{\alpha_k} - \frac{\gamma_k}{\alpha_k(\alpha_k + \gamma_k)}\right). \tag{17}$$

Using the identity (16), we can rewrite Equation (17) as

$$\frac{\pi}{4} = 2^{k-1}\left(\arctan\left(\frac{1}{\alpha_k}\right) - \arctan\left(\frac{\gamma_k}{1 + \alpha_k(\alpha_k + \gamma)}\right)\right)$$

or

$$\begin{aligned}\frac{\pi}{4} = 2^{k-1}\Bigg(&\arctan\left(\frac{1}{\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor}\right)\\ &- \arctan\left(\frac{\left\{a_k/\sqrt{2-a_{k-1}}\right\}}{1 + \left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor\left(a_k/\sqrt{2-a_{k-1}}\right)}\right)\Bigg).\end{aligned} \tag{18}$$

As the fractional part

$$\left\{a_k/\sqrt{2-a_{k-1}}\right\} < 1$$

while

$$\lim_{k\to\infty}\left(1 + \left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor\left(a_k/\sqrt{2-a_{k-1}}\right)\right) = \infty,$$

we conclude that

$$\begin{aligned}\lim_{k\to\infty}\ -2^{k-1}\arctan&\left(\frac{\left\{a_k/\sqrt{2-a_{k-1}}\right\}}{1 + \left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor\left(a_k/\sqrt{2-a_{k-1}}\right)}\right)\\ &= \lim_{k\to\infty}\arctan\left(\frac{1-\lambda_k}{\kappa_k}\right) = \lim_{k\to\infty}\arctan\left(\frac{1}{\beta_k}\right) = 0.\end{aligned}$$

□

The Lemma 5 and its proof below show how to obtain a single-term rational approximation (33) for π by truncating Equation (19).

Lemma 5. *There is a limit such that*

$$\frac{\pi}{4} = \lim_{k\to\infty}\frac{2^{k-1}}{\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor}. \tag{19}$$

Proof of Lemma 5. From the Lemma 4 it immediately follows that

$$\frac{\pi}{4} = \lim_{k\to\infty} 2^{k-1}\arctan\left(\frac{1}{\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor}\right). \tag{20}$$

According to Lemma 1

$$\lim_{k\to\infty}\sqrt{2-a_{k-1}} = 0$$

and

$$\lim_{k\to\infty} a_k = 2.$$

Therefore, we can infer that

$$\lim_{k\to\infty}\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor = \infty$$

or

$$\lim_{k\to\infty}\frac{1}{\left\lfloor a_k/\sqrt{2-a_{k-1}}\right\rfloor} = 0.$$

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This limit implies that the argument of the arctangent function in Equation (20) tends to zero with increasing the integer k. Consequently, the limit (19) follows from the limit (20) and the proof is completed. □

Motivated by recent publications [8–13] in connection to our works [5,7,14,15], we proposed a methodology for determination of the coefficients α_k without computing nested square roots of 2 (see Equation (5)) and developed an algorithm providing squared convergence per iteration in computing the digits of π. To the best of our knowledge, this approach is new and have never been reported.

2. Methodologies

2.1. Arctangent Function

There are different efficient methods to compute the arctangent function in the Machin-like formulas. We will consider a few of them.

In our previous publications [16] we show how the two-term Machin-like formula for π represented in form (8) is used to derive an iterative formula

$$\theta_{n,k} = \frac{1}{\frac{1}{\theta_{n-1,k}} + \frac{1}{2^k}\left(1 - \tan\left(\frac{2^{k-1}}{\theta_{n-1,k}}\right)\right)}, \qquad k \geq 1, \tag{21}$$

where initial value can be taken as

$$\theta_{1,k} = 2^{-k}$$

such that

$$\frac{\pi}{4} = 2^{k-1} \lim_{n\to\infty} \frac{1}{\theta_{n,k}}. \tag{22}$$

This iterative formula provides a squared convergence in computing digits of π (see Mathematica code in [16]).

Taking change of the variable $\theta_n \to 1/\theta_n$ in Equation (21) yields a more convenient form

$$\theta_{n,k} = \theta_{n-1,k} + 2^{-k}\left(1 - \tan\left(2^{k-1}\theta_{n-1,k}\right)\right), \qquad k \geq 1.$$

Consequently, the limit (22) can be rearranged now as

$$\frac{\pi}{4} = 2^{k-1} \lim_{n\to\infty} \theta_{n,k}.$$

Comparing this limit with Equation (2), we can see that this iteration procedure results in

$$\lim_{n\to\infty} \theta_{n,k} = \arctan\left(\frac{1}{\alpha_k}\right)$$

and is used *de facto* for determination of the arctangent function. The detailed procedure showing how to implement the computation with high convergence rate is given in our recent publication [17].

Alternatively, we can transform the two-term into multi-term Machin-like formulas for π consisting of only the integer reciprocals. In order to do this, we can use the following equation [14]

$$\frac{\pi}{4} = 2^{k-1}\arctan\left(\frac{1}{\alpha_k}\right) + \left(\sum_{m=1}^{M} \arctan\left(\frac{1}{\lfloor \mu_{m,k} \rfloor}\right)\right) + \arctan\left(\frac{1}{\mu_{M+1,k}}\right), \tag{23}$$

where

$$\mu_{m,k} = \frac{1 + \lfloor \mu_{m-1,k} \rfloor \mu_{m-1,k}}{\lfloor \mu_{m-1,k} \rfloor - \mu_{m-1,k}}$$

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with an initial value

$$\mu_{1,k} = \beta_k.$$

For example, by taking $k = 2$ in the Equation (23), we can construct

$$\begin{aligned} \frac{\pi}{4} &= 2^{2-1} \arctan\left(\frac{1}{\alpha_2}\right) + \arctan\left(\frac{1}{\mu_{1,2}}\right) \\ &= 2 \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{7}\right). \end{aligned} \tag{24}$$

This equation is commonly known as the Hermann's formula for π [18,19].

By taking $k = 3$, the Equation (23) gives

$$\frac{\pi}{4} = 2^{3-1} \arctan\left(\frac{1}{\alpha_3}\right) - \arctan\left(\frac{1}{\mu_{1,3}}\right)$$

representing the original Machin's formula (1) for π.

The case $k = 4$ requires more calculations to obtain the multi-term formula for π consisting of only integer reciprocals. In particular, at $M = 1$ we get

$$\begin{aligned} \frac{\pi}{4} &= 2^{4-1} \arctan\left(\frac{1}{\alpha_4}\right) + \arctan\left(\frac{1}{\mu_{1,4}}\right) \\ &= 8 \arctan\left(\frac{1}{10}\right) - \arctan\left(\frac{1758719}{147153121}\right) \end{aligned}$$

At $M = 2$ and $M = 3$ Equation (23) yields

$$\begin{aligned} \frac{\pi}{4} &= 2^{4-1} \arctan\left(\frac{1}{\alpha_4}\right) + \arctan\left(\frac{1}{\lfloor \mu_{1,4} \rfloor}\right) + \arctan\left(\frac{1}{\mu_{2,4}}\right) \\ &= 8 \arctan\left(\frac{1}{10}\right) - \arctan\left(\frac{1}{84}\right) - \arctan\left(\frac{579275}{12362620883}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\pi}{4} &= 2^{4-1} \arctan\left(\frac{1}{\alpha_4}\right) + \arctan\left(\frac{1}{\lfloor \mu_{1,4} \rfloor}\right) + \arctan\left(\frac{1}{\lfloor \mu_{2,4} \rfloor}\right) + \arctan\left(\frac{1}{\mu_{3,4}}\right) \\ &= 8 \arctan\left(\frac{1}{10}\right) - \arctan\left(\frac{1}{84}\right) - \arctan\left(\frac{1}{21342}\right) \\ &- \arctan\left(\frac{266167}{263843055464261}\right), \end{aligned}$$

respectively.

Repeating this procedure over and over again, at $M = 5$ we end up with 7-term Machin-like formula for π consisting of only integer reciprocals

$$\begin{aligned} \frac{\pi}{4} &= 2^{4-1} \arctan\left(\frac{1}{\alpha_4}\right) + \left(\sum_{m=1}^{5} \arctan\left(\frac{1}{\lfloor \mu_{m,4} \rfloor}\right)\right) + \arctan\left(\frac{1}{\mu_{6,41}}\right) \\ &= 8 \arctan\left(\frac{1}{10}\right) - \arctan\left(\frac{1}{84}\right) - \arctan\left(\frac{1}{21342}\right) \\ &- \arctan\left(\frac{1}{991268848}\right) - \arctan\left(\frac{1}{193018008592515208050}\right) \\ &- \arctan\left(\frac{1}{197967899896401851763240424238758988350338}\right) \\ &- \arctan\left(\frac{1}{\Omega}\right), \end{aligned} \tag{25}$$

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where

$$\Omega = 1175738681681753529302777528441941267679919150085 3701\ldots$$
$$88369320142936782716368857923 97,$$

is the largest integer (see the Mathematica code in [17] that validates this seven-term Machin-like formula for π).

As we can see, Hermann's (24), Machin's (1) and derived (25) formulas for π belong to the same generic group as all of them can be constructed from the same equation-template (23).

Our empirical results show that two arctangent series expansions can be used for computation with rapid convergence. The first equation is Euler's expansion series given by [20]

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}}. \tag{26}$$

The second equation is [7,21]

$$\arctan(x) = 2\sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{g_n(x)}{g_n^2(x)+h_n^2(x)}, \tag{27}$$

where

$$g_n(x) = g_{n-1}(x)\left(1 - 4/x^2\right) + 4h_{n-1}(x)/x$$

and

$$h_n(x) = h_{n-1}(x)\left(1 - 4/x^2\right) - 4g_{n-1}(x)/x$$

with initial values

$$g_1(x) = 2/x$$

and

$$h_1(x) = 1.$$

Computational test we performed shows that Equation (27) is faster in convergence than Equation (26). Recently, we proposed the generalization of the arctangent series expansion (27) [21].

2.2. Tangent Function

Generally, transformation of the two-terms to multi-terms formulas for π with integer reciprocals is not required. In particular, we can use Newton–Raphson iteration method [15]. For example, both arctangent terms in the two-term Machin-like formula (4) for π can be computed directly by using the following iterative formulas

$$\sigma_n = \sigma_{n-1} - \left(1 - \left(\frac{2\tan(\sigma_{n-1}/2)}{1+\tan^2(\sigma_{n-1}/2)}\right)^2\right)\left(\frac{2\tan(\sigma_{n-1}/2)}{1-\tan^2(\sigma_{n-1}/2)} - \frac{1}{\alpha_k}\right)$$

and

$$\tau_n = \tau_{n-1} - \left(1 - \left(\frac{2\tan(\tau_{n-1}/2)}{1+\tan^2(\tau_{n-1}/2)}\right)^2\right)\left(\frac{2\tan(\tau_{n-1}/2)}{1-\tan^2(\tau_{n-1}/2)} - \frac{1}{\beta_k}\right)$$

with initial values

$$\sigma_1 = \frac{1}{\alpha_k}$$

and

$$\tau_1 = \frac{1}{\beta_k}$$

such that

$$\arctan\left(\frac{1}{\alpha_k}\right) = \lim_{n\to\infty} \sigma_n$$

and

$$\arctan\left(\frac{1}{\beta_k}\right) = \lim_{n\to\infty} \tau_n.$$

Since the convergence of the Newton–Raphson iteration is quadratic [25], with proper implementation of the tangent function we may achieve an efficient computation.

The tangent function can be expanded as

$$\begin{aligned}\tan(x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots,\end{aligned} \tag{28}$$

where B_{2n} are the Bernoulli numbers, defined by the contour integral

$$B_n = \frac{n!}{2\pi i}\oint \frac{z}{e^z - 1}\frac{dz}{z^{n+1}}.$$

However, application of Equation (28) is not desirable since the computation of the Bernoulli numbers B_{2n} itself is a big challenge [22–24,26].

In order to resolve this problem, we proposed the following limit [17]

$$\tan(x) = \lim_{n\to\infty} \frac{2p_n^2(x)}{q_n(x)}, \tag{29}$$

where

$$p_n(x) = p_{n-1}(x) + r_{n-1}(x),$$

$$q_n(x) = q_{n-1}(x) + 2^{2n-1}r_{n-1}(x),$$

such that

$$r_n = \frac{(-1)^n}{(2n+1)!}x^{2n+1},$$

with initial values

$$p_0(x) = 0,$$

$$q_0(x) = 0.$$

Specifically, it has been shown that at $k = 27$ application of Equation (29) results in more than 17 digits of π per increment n (see the Mathematica codes in [17]).

3. Algorithmic Implementation

In our recent publication we have shown that [16]

$$\beta_k \approx \frac{2}{1 - \tan\left(\frac{2^{k-1}}{\alpha_k}\right)}.$$

Consequently, in accordance with Equation (4), we obtain the following approximation

$$\frac{\pi}{4} \approx 2^{k-1}\arctan\left(\frac{1}{\alpha_k}\right) + \frac{1}{2}\left(1 - \tan\left(\frac{2^{k-1}}{\alpha_k}\right)\right).$$

At sufficiently large k the value $1/\alpha_k << 1$. Therefore, according to the the Lemma 5, in this equation we can replace the first arctangent term by a rational number $2^{k-1}/\alpha_k$. This gives

$$\frac{\pi}{4} \approx \frac{2^{k-1}}{\alpha_k} + \frac{1}{2}\left(1 - \tan\left(\frac{2^{k-1}}{\alpha_k}\right)\right).$$

Unfortunately, we cannot compute efficiently the tangent function in this approximation since its argument $2^{k-1}/\alpha_k$ is not a small number as it tends to $\pi/4$ with increasing k. However, again by taking into consideration that $1/\alpha_k << 1$ from which it follows that

$$\frac{1}{\alpha_k} \approx \arctan\left(\frac{1}{\alpha_k}\right),$$

we can write

$$\frac{\pi}{4} \approx \frac{2^{k-1}}{\alpha_k} + \frac{1}{2}\left(1 - \tan\left(2^{k-1}\arctan\left(\frac{1}{\alpha_k}\right)\right)\right).$$

Now we can take advantage from the fact that the multiplier 2^{k-1} is continuously divisible by 2. Therefore, we can use the trigonometric identity

$$\tan(2\arctan(x)) = \frac{2x}{1-x^2}$$

$k-1$ times over and over again. Thus, this leads to the following iterative formula

$$\eta_n(x) = \frac{2\eta_{n-1}(x)}{1-\eta_{n-1}^2(x)} \tag{30}$$

with an initial value

$$\eta_1(x) = \frac{2x}{1-x^2}$$

such that

$$\eta_{k-1}\left(\frac{1}{\alpha_k}\right) = \tan\left(2^{k-1}\arctan\left(\frac{1}{\alpha_k}\right)\right).$$

Since the left side of the equation above provides an exact value without (tangent and arctangent) trigonometric functions, we can regard this equation

$$\frac{\pi}{4} \approx \frac{2^{k-1}}{\alpha_k} + \frac{1}{2}\left(1 - \eta_{k-1}\left(\frac{1}{\alpha_k}\right)\right) \tag{31}$$

as a rational approximation.

This rational approximation of the two-term Machin-like formula for π can be used in an algorithm providing a quadratic convergence. This can be achieved with help of the Theorem 3.

Theorem 3. *There is a formula for $1/\pi$ with binary output*

$$\frac{1}{\pi} = \lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{10^{n+1}}(\alpha_n \mod 2) = [0.01010001011111001100\ldots]_2, \tag{32}$$

Proof of Theorem 3. The proof is related to the parity of the integer α_k. According to the Lemma 5, we can write

$$\frac{\pi}{4} \approx \frac{2^{k-1}}{\alpha_k} \tag{33}$$

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or

$$\frac{1}{\pi} \approx \frac{\alpha_k}{4 \times 2^{k-1}}.$$

Consequently, if the integer α_k is even, then

$$\frac{\alpha_k}{4 \times 2^{k-1}} - \frac{\alpha_{k-1}}{4 \times 2^{k-2}} = 0_2.$$

However, if the integer α_k is odd, then

$$\frac{\alpha_k}{4 \times 2^{k-1}} - \frac{\alpha_{k-1}}{4 \times 2^{k-2}} = \underbrace{0.00\ldots00}_{k+1 \text{ zeros}}1_2.$$

This means that α_k contributes a binary digit $\underbrace{0.00\ldots00}_{k+1 \text{ zeros}}1_2$ to the previous value α_{k-1} if and only if α_k is odd. This completes the proof. □

Consider an example. There are four consecutive values $\alpha_4 = 10$, $\alpha_5 = 20$ and $\alpha_6 = 40$ and $\alpha_7 = 81$. Since the first three values are even, we have

$$\frac{\alpha_4}{4 \times 2^{4-1}} = \frac{\alpha_5}{4 \times 2^{5-1}} = \frac{\alpha_6}{4 \times 2^{6-1}} = 0.0101_2.$$

However, since $\alpha_7 = 81$ is odd, we obtain

$$\frac{\alpha_7}{4 \times 2^{7-1}} = 0.0101_2 + 0.00000001_2 = 0.01010001_2.$$

Consider how number of digits of π can be doubled without computing square roots for the nested radicals a_k. We can take, for example, $k = 7$. This yields

$$\begin{aligned}\alpha_7 &= \left\lfloor \frac{a_7}{\sqrt{2 - a_{7-1}}} \right\rfloor \\ &= \left\lfloor \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} \right\rfloor = 81.\end{aligned} \tag{34}$$

However, it is not reasonable to compute the square roots of 2 so many times to obtain this number. Instead, we can simplify computation considerably by using the value of the $1/\pi$ in the binary form according to the Theorem 3. Thus, ignoring the first two initial zeros in the binary output of the Equation (32), we have a corresponding sequence

$$(s)_k^\infty = (1,0,1,0,0,0,1,0,1,1,1,1,1,0,0,\ldots). \tag{35}$$

This sequence can be obtained by using the built-in Mathematica directly using $1/\pi$. For example, the following code:

```
RealDigits[
  ImportString[ToString[BaseForm[N[1/\[Pi],20],2]],
    "Table"][[1]][[1]]][[1]][[1;;20]]
```

returns the first 20 digits from the sequence (35):

```
{1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0}
```

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From this sequence, we choose the sub-sequence (say, up to seventh element)

$$(s)_k^7 = (1, 0, 1, 0, 0, 0, 1)$$

and apply it accordingly as

$$\alpha_k = \begin{cases} 2\alpha_{k-1}, & \text{if } k^{\text{th}} \text{ binary digit is } 0 \\ 2\alpha_{k-1} + 1, & \text{if } k^{\text{th}} \text{ binary digit is } 1. \end{cases}$$

Explicitly, this step-by-step procedure results in

$$\begin{aligned} \alpha_1 &= 2\alpha_0 + 1 = (2 \times 0 + 1) = 1, \\ \alpha_2 &= 2\alpha_1 + 0 = (2 \times 1 + 0) = 2, \\ \alpha_3 &= 2\alpha_2 + 1 = (2 \times 2 + 1) = 5, \\ \alpha_4 &= 2\alpha_3 + 0 = (2 \times 5 + 0) = 10, \\ \alpha_5 &= 2\alpha_4 + 0 = (2 \times 10 + 0) = 20, \\ \alpha_6 &= 2\alpha_5 + 0 = (2 \times 20 + 0) = 40, \\ \alpha_7 &= 2\alpha_6 + 1 = (2 \times 40 + 1) = 81. \end{aligned}$$

Thus, we can see how a very simple procedure can be used to determine the value of the rational number $\alpha_7 = 81$ without using a sophisticated Equation (34) consisting of 14 nested square roots of 2.

At $\alpha_7 = 81$ the corresponding Machin-like formula is

$$\begin{aligned} \frac{\pi}{4} &= 2^{7-1} \arctan\left(\frac{1}{\alpha_7}\right) + \arctan\left(\frac{1}{\beta_7}\right) \\ &= 64 \arctan\left(\frac{1}{81}\right) - \arctan\left(\frac{\overbrace{2154947582\ldots4298183679}^{111 \text{ digits}}}{\underbrace{4599489202\ldots6981324801}_{113 \text{ digits}}}\right), \end{aligned} \tag{36}$$

where the constant

$$\beta_7 = -\frac{\overbrace{4599489202\ldots6981324801}^{113 \text{ digits}}}{\underbrace{2154947582\ldots4298183679}_{111 \text{ digits}}}$$

can be computed either by using Equation (7) or, more efficiently, by using Equation (13) based on two-step iteration (14).

The following Mathematica code:

```
(* String for long number \[Beta]_7 *)
strBeta7=
  ToString[StringJoin[
    "21549475820057881611210311984288158234143531212163819254",
      "1568712000964806160594022446140062110943660584298183679/",
        "459948920218008069525744651226752553899687099736076594466",
          "78719072620659988130828378620624183170066256006981324801"
            ]];

(* Verification *)
Print[\[Pi]/4==64*ArcTan[1/81]-ArcTan[ToExpression[strBeta7]]];
```

validates the Equation (36) by returning `True`.

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Suppose that we do not know the sequence other than $(s)_1^7$. However, with help of the Equation (31) we can find other digits of π in the iterative process. In particular, using Equation (30), we have

$$\eta_{7-1}\left(\frac{1}{81}\right) = \frac{\overbrace{2310519339\ldots5639754240}^{113\text{ digits}}}{\underbrace{2288969863\ldots1341570561}_{113\text{ digits}}} \approx 1.00941448647564092749$$

Substituting this value into Equation (31), we can find a significantly better approximation of π.

The following is the Mathematica code:

```
Print["Equation (33) at k = 7: ",
  MantissaExponent[N[\[Pi]-4*(64/81),20]][[2]]//Abs,
    " digits of \[Pi]"];

Print["Equation (31) at k = 7: ",
  MantissaExponent[
    N[\[Pi]-4*(64/81+1/2*(1-1.00941448647564092749)),
      20]][[2]]//Abs," digits of \[Pi]"];
```

produces the following output:

```
Equation~(33) at k = 7: 1 digits of π
Equation~(31) at k = 7: 4 digits of π
```

Initial sequence $(s)_1^7$ helped us to find the value $\alpha_7 = 81$. Now due to higher accuracy of Equation (31) we can generate the sequence in which its upper index is doubled

$$(s)_1^{14} = (1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0)$$

and with help of this sequence we can find the corresponding value $\alpha_{14} = 10430$.

Unfortunately, doubling the upper index k does not always work. For example, if we attempt to double the upper index by using the initial sequence $(s)_1^8$, then we get $\alpha_{16} = 41722$ instead of correct value $\alpha_{16} = 41721$. Therefore, the upper index of the sequence should be slightly less than two.

The two-terms approximation (31) doubles the number of digits of π as compared to the single-term approximation (33). This means that using the sequence $(s)_1^k$ we can obtain all sequences $(s)_1^{k+1}$, $(s)_1^{k+2}$, $(s)_1^{k+3}$, etc. up to $(s)_1^{k_0}$, where k_0 is an integer slightly smaller than $2k$. Our numerical results show that doubling value k does not always provide the correct sequence as a few binary digits at the end of the sequence $(s)_1^{2k}$ occasionally may not be correct. However, when we use the empirical equation

$$k_0 = \left\lfloor \left(2 - \frac{1}{32}\right) k \right\rfloor,$$

then the corresponding sequence $(s)_1^{k_0}$ is a sub-sequence of the infinite sequence (35) and, therefore, it is appeared to be correct. It is interesting to note that the number 32 in this equation is the largest integer that we found on the basis of our numerical results.

The following is a Mathematica code that shows number of digits of π at given iteration number n and integer k:

```
Clear[str,sps,k,\[Gamma],\[Alpha],lst,\[Eta]]

(* String for conversion of 1/\[Pi] to sequence *)
str="ImportString[ToString[BaseForm[N[1/piAppr,k0],2]],
  \"Table\"][[1]][[1]]";

(* String for space separation *)
```

```
sps[n_]:=Module[{m=1,sps=" "},
  While[m<n,sps=StringJoin[sps," "];m++];If[m==n,sps]];

(* Converting number to string with length q *)
cnv2str[p_,q_]:=Module[{},StringTake[StringJoin[ToString[p],sps[q]],q]]

(* Defining \[Eta]-function *)
\[Eta][n_,x_,k_]:=Module[{K=k/1.5,y=x},y=N[(2*y)/(1-y^2),K];cntr = 1;
  While[cntr<n,y=(2*y)/(1-y^2);cntr++];y];

(* Define \[Alpha]_1, \[Alpha]_2 and \[Alpha]_3] *)
\[Alpha][1]=1;
\[Alpha][2]=2;
\[Alpha][3]=5;

(* Input values *)
k=3;\[Gamma]=\[Alpha][3];

(* Heading *)
Print["-------------------------------"];
Print["Iteration | k", sps[5], "| Digits of \[Pi]"];
Print["-------------------------------"];

n=1;
While[n<=12,

  intR=1/\[Gamma];
  k0=\[LeftFloor](2-1/32)*k\[RightFloor];

  piAppr=4*(2^(k-1)*intR+1/2*(1-\[Eta][k-1,intR,k0]));

  (* Extracting the sequence {1,0,1,0,0,0,1...} *)
  lst=RealDigits[ToExpression[str]][[1]][[1;;k0]];

  (* Main computation *)
  K=k+1;
  While[K<=k0,\[Gamma]=
    2*\[Gamma]+lst[[K]];\[Alpha][K]=\[Gamma];K++];k=k0;

  (* Aligned output" *)
  Print[cnv2str[n,5],sps[4]," | ",cnv2str[k,5]," | ",
    MantissaExponent[N[\[Pi]-piAppr,k0]][[2]]//Abs];

  n++];
```

This code generates the output:

```
------------------------------
Iteration | k     | Digits of π
------------------------------
1         | 5     | 1
2         | 9     | 2
3         | 17    | 4
4         | 33    | 9
5         | 64    | 20
6         | 126   | 38
```

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```
7          | 248  | 75
8          | 488  | 149
9          | 960  | 293
10         | 1890 | 577
11         | 3720 | 1137
12         | 7323 | 2240
```

As we can see from the third column, the number of digits of π doubles at each iteration.
The Mathematica code below shows the values of α_k at given k:

```
(*Heading*)
Print["-----------------"];
Print["k",sps[5],"| \[Alpha][k]"];
Print["-----------------"];

k=2;
While[k<=25,

  (*Aligned output" *)
  Print[cnv2str[k,5], " | ", \[Alpha][k]];
  k++];
```

This code returns the following output:

```
-----------------
k      | αk
-----------------
2      | 2
3      | 5
4      | 10
5      | 20
6      | 40
7      | 81
8      | 162
9      | 325
10     | 651
11     | 1303
12     | 2607
13     | 5215
14     | 10430
15     | 20860
16     | 41721
17     | 83443
18     | 166886
19     | 333772
20     | 667544
21     | 1335088
22     | 2670176
23     | 5340353
24     | 10680707
25     | 21361414
```

As we can see, all numbers α_k are the same as those reported in [9].

Since the constants α_k have been computed already, we can use them to validate the formula (32) for $1/\pi$ in the binary form. The Mathematica code below:

```
f[K_]:=N[Sum[(1/10^(k+1))*Mod[\[Alpha][k],2],{k,1,K}],K];
```

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```
Print["------------------------------------------------------------"]
Print[k,sps[2]," | Binary output"];
Print["------------------------------------------------------------"]

k := 1;
While[k<=5,Print[10*k,sps[2],"| ",Subscript[f[10*k],2]];k++]

Print["------------------------------------------------------------"]

Print["Built-in Mathematica:"]
Print["1/\[Pi]=",
  Subscript[StringJoin["[",
    StringSplit[ToString[BaseForm[N[1/Pi,15],2]]][[1]], "...]"],
    2]];
```

returns the output:

```
------------------------------------------------------------
k  | Binary output
------------------------------------------------------------
10 | 0.0001000101100_2
20 | 0.000100010111111001100000_2
30 | 0.0001000101111110011000001101101100_2
40 | 0.00010001011111100110000011011011100100111000_2
50 | 0.000100010111111001100000110110111001001110010001000000_2
------------------------------------------------------------
Built-in Mathematica:
1/π = [0.010100010111111001100000110110111001001110010001000 0...]_2
```

according to the Equation (32). The original binary representation of the number $1/\pi$ generated by built-in Mathematica is also shown for comparison (see also [27] for binary sequence for $1/\pi$).

4. Conclusions

We consider the properties of the two-term Machin-like formula for π and propose its two-term rational approximation (31). Using this approach, we developed an efficient algorithm for computing digits of π with squared convergence. The constants α_k in this approximation are computed without nested square roots of 2.

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