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*Article*

# Universality in the Exact Renormalization Group: Comparison to Perturbation Theory

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**Abstract:** Various formulations of the exact renormalization group can be compared in the perturbative domain, in which we have reliable expressions for regularization-independent (universal) quantities. We consider the renormalization of the $\lambda\phi^4$ theory in three dimensions and make a comparison between the sharp-cutoff regularization method and other more recent methods. The former gives considerably better results, while the other methods are far from optimal in the perturbative domain. In addition, these methods present theoretical problems that are already manifest in the renormalization of mass.

**Keywords:** renormalization group; universality; nonperturbative field theory

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## 1. Introduction

The exact renormalization group (ERG) [1,2,4–7] is the non-perturbative formulation of the classical renormalization group, which was itself an improvement of perturbative quantum field theory. The ERG has been employed for the calculation of universal quantities for critical phenomena, in particular critical exponents. Unfortunately, the ERG is afflicted by the problem of regularization scheme dependence of the results, which appears more acutely than in perturbation theory. However, considerable studies of regularization scheme dependence have been carried out [8–10], and Litim has proposed an *optimized regulator* [9,10], with which he obtains particularly accurate critical exponents [11]. At any rate, the comparisons made so far, for any given field theory, mostly involve the ERG fixed point and the perturbations about it (which provide the critical exponents). The ERG equations are actually applicable in a much larger parameter region. Thus, our intention is to explore other parts of the parameter space.

Undoubtedly, scale invariance is an important subject and it is natural that many field theorists focus on the renormalization group fixed points. But in many situations one has to deal with field theories that are not scale invariant. Nevertheless, these theories need renormalization, which is normally implemented in perturbation theory [12–14]. The classical renormalization group still is a convenient method of improving the results of perturbation theory, but the ERG opens a new avenue, given its non-perturbative nature. This idea was proposed years ago [15], and it has recently been demonstrated numerically that the ERG equations can outperform perturbative renormalization within a range of coupling constant values [16].

The regularization method employed in Ref. [16] is the simple sharp-cutoff method, which is said to be non optimal [8–11]. However, the sharp-cutoff method is arguably useful within the local potential approximation [17]. At any rate, reviews of the exact renormalization group written after the popularization of Litim's work [18–21] favor his method over the sharp-cutoff method. In a finite (truncated) coupling constant space, both methods lead to ordinary differential equations, whereas general smooth regulators lead to integro-differential equations [8]. Ordinary differential equations are very suitable for numerical calculations and even lend themselves to some analytical investigations.

One class of smooth cutoff functions consists of power-law functions with different exponents [8–10]. In particular cases, these cutoff functions give rise to ordinary differential equations, as do the sharp cutoff or the Litim cutoff functions [8]. Among those cases, we shall pay attention to the 4th power law in three dimensions, which was originally studied by Morris [7,23]. He concluded that it is more convenient than the sharp-cutoff function as a basis for the full derivative expansion of the ERG [23].

Usually, the ERG analysis of regulator performance is concerned with the ERG fixed points. Of course, the choice of regularization scheme affects the renormalization process for other values of the coupling constants [12–14]. Despite the fact that dimensional regularization has become standard in perturbative field theory, there is no fundamental objection to the use of other schemes in scalar field theories. Thus, one is prompted to ask for the effect that the various regulators already tested in the ERG may have in the perturbative domain.

One can consider several aspects of this question. The simplest field theory in three dimensions, the single-field scalar field theory, has only one non-trivial fixed point, namely, the Wilson-Fisher fixed point. This fixed point can be found employing the fixed-dimension perturbative $\lambda\phi^4$-theory (which actually provides very accurate values of the corresponding critical exponents) [13,14]. One important fact to take into account is that this theory is super-renormalizable, that is to say, only a finite (and small) number of Feynman diagrams are *superficially* divergent [13,14]. When these diagrams are regulated, they produce regularization dependent terms, that is to say, non-universal contributions to the renormalized parameters. At any rate, the divergent diagrams only affect the renormalization of the mass $m$, while $\lambda$ is universal, because it does not involve divergences.

The relationship between the bare $\lambda_0$ and the renormalized $\lambda$ is well known in perturbation theory, to a high loop order, and it is employed to calculate the critical exponents of the Wilson-Fisher fixed point [13,14]. This calculation indeed demands a high-order expansion and sophisticated resummation techniques. However, the perturbative series converges very well for small values of $\lambda/m$, so that a few terms of it suffice to obtain very accurate results. Therefore, this fully perturbative region can be the adequate testing ground for a comparison with non-perturbative ERG results, regarding regulator optimality, in particular. Moreover, in addition to testing the relationship between $\lambda_0$ and $\lambda$, we can also test the relationship between $m_0$ and $m$, in spite of its not being universal.

Other aspects of the relation between truncations of the ERG equation and the perturbative renormalization group have been analysed before. Morris and Tighe [24] study the derivative expansion of the $\lambda\phi^4$-theory and compare the ERG beta function with the perturbative beta function to one and two-loop order. However, they focus on the massless case in four dimensions. In three dimensions, the perturbative series is hardly useful for the massless case, unless treated with sophisticated techniques, as noticed above. Kopietz [25] employs Polchinski's ERG and initially keeps the dimension general, but he restricts the study, at some point, to $D \geq 4$. Kopietz's renormalization group equations are further complicated by his keeping the full momentum dependence. Here, we consider the local potential approximation of the ERG, which allows us to carry out simple calculations of the RG flow for the massive $\lambda\phi^4$-theory in three dimensions. Other articles that discuss the connection of the ERG with perturbation theory are Refs. [26,27], but they do not consider the $\lambda\phi^4$-theory in three dimensions.

Of course, the study of the connection between the exact and the perturbative renormalization groups is, generally speaking, as old as the theory of the renormalization group itself [1]. It features in the early articles [4,6] and early modern reviews of the ERG [28–30]. However, these articles and reviews precede (or are simultaneous with) the studies of regularization-scheme optimization and are mainly concerned with the sharp-cutoff scheme, as the only one giving rise to tractable differential equations (it seems that Morris's scheme [7], namely, the 4th power law in three dimensions, was not sufficiently considered, perhaps because it is too specific).

Our study consists of two parts. In the first one, spanning from Sect. 2 to 4, we make a numerical comparison of three well-known ERG regularization methods with the universal perturbative renormalization formulas including up to the three-loop order. The sharp-cutoff regularization method seems to work fine, whereas serious discrepancies appear in the other two methods. In the second part, Sect. 5, we delve deeper into the origin of these discrepancies, by means of an analysis of the relationship between simple forms of the ERG differential equations for $m$ and $\lambda$ in the various schemes,

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on the one hand, and the one-loop gap and bubble equations, on the other. While that relationship appears naturally in the case of the sharp-cutoff regularization method, it seems to be absent in the other cases.

## 2. Sharp-cutoff exact renormalization group and perturbation theory

Here we take the Wegner-Houghton sharp-cutoff ERG equations [2], restricted to the single-field scalar field effective potential [4,22], in three dimensions. We compare to perturbation theory, for small (absolute) values of the bare mass and coupling constant. As before [16], we set a reference UV cutoff $\Lambda_0$ and employ the linearized Wegner-Houghton ERG for a rough approximation. Let us recall two simple conclusions from it: the coupling constant is not renormalized while $m^2$ grows as we lower the running cutoff $\Lambda$, namely,

$$m^2(\Lambda) = m_0^2 + \frac{6\lambda_0}{\pi^2}\,(\Lambda_0 - \Lambda). \tag{1}$$

Hence, a small positive renormalized mass (at $\Lambda = 0$) requires $m_0^2 < 0$. Naturally, a better approximation, for example, a one-loop calculation or the non-perturbative approach of Ref. [16], finds that $\lambda < \lambda_0$, and also finds a (negative) correction to Eq. (1).

Let us make the mass and coupling constant non-dimensional by dividing each by the corresponding power of $\Lambda_0$. This type of non-dimensionalization is not of the usual type, which uses powers of the running cutoff $\Lambda$ [1,2,4,22], but it is more convenient for us to compare with perturbative field theory results. Nevertheless, our redefinition hides the fact that the mass renormalization is non-universal and, in particular, Eq. (1) contains a term proportional to $\Lambda_0$ and divergent for $\Lambda_0 \to \infty$. Let us leave the renormalization of mass for later and consider now the renormalization of the coupling constant in the perturbative domain, namely, for small absolute values of non-dimensional $m_0^2$ and $\lambda_0$.

Thus, we first set $\lambda_0$ to some small number, say we set $6\lambda_0/\pi^2 = 0.005$ ($\lambda_0 = 0, 008225$). Although Eq. (1) gives only a rough approximation, we can use it to guide ourselves about the choice of $m_0^2$. Thus, let us take $m_0^2 < 0$ but $m_0^2 > -6\lambda_0/\pi^2 = -0.005$, because we want $m$ to be small, but not too small. We do not want to be close to masslessness (criticality) because it is not the truly perturbative domain. In addition, we must not take $m_0^2$ positive, especially, positive and large, because then $\lambda$ is hardly renormalized (as occurs in the linearized ERG). We have tried $m_0^2 = -0.0047 + 0.001k$, $k = 0, \ldots, 5$, and solved numerically the ERG equations, as we now explain.

The Wegner-Houghton ERG equations describe how the couplings in the effective potential flow with $\Lambda$. When truncated to a not-too-small number of coupling constants, the equations are known to be reasonably accurate, at least, for the analysis of critical behavior [22]. We employ them far from the Wilson-Fisher fixed point, namely, for non-vanishing but small initial values of $|m_0^2|$ and $\lambda_0$, and for initially vanishing values of the other couplings.

The numerical integration between $\Lambda = \Lambda_0$ and $\Lambda = 0$ of the set of ordinary differential equations given by the 8th truncation of the Wegner-Houghton equation for the effective potential (up to $\phi^{16}$) yields the following results. For the renormalized mass and quartic coupling constant, we obtain ($k = 0, \ldots, 5$):

$$m = 0.009857,\ 0.03110,\ 0.04343,\ 0.05312,\ 0.06138,\ 0.06869, \tag{2}$$

$$\frac{6\lambda}{\pi^2} = 0.002575,\ 0.003774,\ 0.004051,\ 0.004196,\ 0.004291,\ 0.004359. \tag{3}$$

Hence,

$$\frac{\lambda}{m} = 0.4298,\ 0.1996,\ 0.1534,\ 0.1299,\ 0.115,\ 0.1044. \tag{4}$$

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These values are sufficiently small (except the first one) for us to keep a few terms of a series of powers of $\lambda/m$. Also note the relatively small variation of $6\lambda/\pi^2$ from its initial value $6\lambda_0/\pi^2 = 0.005$ (except in the first case). Actually, Eq. (1) roughly holds, since it gives

$$m = 0.01732,\ 0.03606,\ 0.04796,\ 0.05745,\ 0.06557,\ 0.07280.$$

As to the reliability of the 8th truncation of the ERG equations, we have checked that even truncations of somewhat smaller order yield essentially the same results.

For the comparison with perturbation theory, it is sufficient to keep up to $(\lambda/m)^2$ in the fixed-dimension perturbative series, that is to say, to keep up to the two-loop order in the renormalization of $\lambda$. In addition to this, we can also consider the sextic and octic coupling constants, which were calculated long ago in perturbation theory [31], and which we also obtain in our numerical integration of the ERG equations.

To wit, the expressions that we employ are:

$$\lambda_0 = \lambda\left(1 + \frac{9\lambda}{2\pi m} + \frac{63\,\lambda^2}{4\pi^2 m^2}\right), \tag{5}$$

$$g_6 = \frac{9\lambda^3}{\pi m^3}\left(1 - \frac{3\,\lambda}{\pi m} + 1.389963\,\frac{\lambda^2}{m^2}\right), \tag{6}$$

$$g_8 = -\frac{81\lambda^4}{2\pi m^3}\left(1 - \frac{65\,\lambda}{6\pi\, m} + 7.775001\,\frac{\lambda^2}{m^2}\right), \tag{7}$$

where the non-dimensional sextic and octic coupling constants $g_6$ and $g_8$ refer to the terms next to $\lambda\phi^4$ in the expansion of the effective potential, namely, $g_6\phi^6 + m^{-1}g_8\phi^8$ [31].

The values of $g_6$ and $g_8$ that we obtain with the ERG are:

$$g_6 = 0.1389,\ 0.01728,\ 0.008275,\ 0.005176,\ 0.003658,\ 0.002774,$$

$$-g_8 = 0.09582,\ 0.009248,\ 0.003793,\ 0.002127,\ 0.001380,\ 0.0009759.$$

The preceding perturbative formulas (5,6,7), in combination with Eq. (4), yield

$$\frac{6\lambda_0}{\pi^2} = 0.004919,\ 0.005093,\ 0.005093,\ 0.005090,\ 0.005088,\ 0.005087,$$

$$g_6 = 0.1924,\ 0.01970,\ 0.009166,\ 0.005652,\ 0.003959,\ 0.002983,$$

$$-g_8 = 0.4195,\ 0.01272,\ 0.004669,\ 0.002511,\ 0.001592,\ 0.001109.$$

The comparison is successful, insofar as the ERG integration yields values of $\lambda/m$ and values of $g_6$ and $g_8$ such that the substitution for $\lambda/m$ in the perturbative formulas approximately recovers the value of $\lambda_0$ and obtains values of $g_6$ and $g_8$ similar to the ones of the ERG integration. Naturally, the approximation is better the smaller $\lambda/m$ is, and the last values of $g_6$ and $g_8$ are off by about 10%.

## 3. Results for Litim's optimized exact renormalization group

Here we carry out the analogous calculations for Litim's optimally regulated ERG flow, employing his flow equation [11, eq. 2.13]. That flow equation applies to the $O(N)$ scalar field theory in $d$ dimensions, so we take the particular case $d = 3$ and $N = 1$. We employ again the 8th truncation of the flow equations. Litim studies the reliability of truncations (for fixed-point calculations) and finds that even lower order truncations are reliable [11, sect. 3]. We have also checked the reliability of the 8th truncation for our calculations.

For the calculation of renormalized mass and coupling constants through Litim's flow equation, we need to set initial ("bare") values. We can choose again $6\lambda_0/\pi^2 = 0.005$ ($\lambda_0 = 0{,}008225$) and $m_0^2 = -0.0047 + 0.001k,\ k = 0, \ldots, 5$. However, we should not expect to recover the same values of renormalized mass, namely, the values in Eq. (2): The mass renormalization is not universal and we have changed the regularization scheme. Nevertheless, our aim is to compare the results of the ERG integration with the results of the perturbative formulas (5,6,7), in which the bare mass $m_0$ does not feature. Indeed, these formulas are regularization-scheme independent. Hence, we can choose again the same values of $m_0^2$, despite the fact that we obtain, for each value of $m_0^2$, a value of $m$ that is different from the one in the preceding section. We only need $m$ and not $m_0$ in the perturbative formulas, which are regularization-scheme independent, and we only need to assess to what extent the formulas are fulfilled.

The numerical integration is again straight-forward and yields the following results:

$$m = 0.05975,\ 0.06670,\ 0.07305,\ 0.07895,\ 0.08448,\ 0.08970, \tag{8}$$

$$\frac{6\lambda}{\pi^2} = 0.003398,\ 0.003521,\ 0.003620,\ 0.003702,\ 0.003771,\ 0.003832, \tag{9}$$

$$g_6 = 0.005347,\ 0.004343,\ 0.003637,\ 0.003115,\ 0.002713,\ 0.002396, \tag{10}$$

$$-g_8 = 0.001123,\ 0.0008897,\ 0.0007273,\ 0.0006088,\ 0.0005189,\ 0.0004489. \tag{11}$$

Hence,

$$\frac{\lambda}{m} = 0.09355,\ 0.08683,\ 0.08150,\ 0.07712,\ 0.07343,\ 0.07027. \tag{12}$$

In the present case, the relative variation of $6\lambda/\pi^2$ from its initial value $6\lambda_0/\pi^2 = 0.005$ is not as small as before. Nevertheless, the values of $\lambda/m$ are small (smaller than before) and warrant the comparison with the low-order perturbative formulas.

Perturbative formulas (5,6,7), in combination with Eq. (12), yield

$$\frac{6\lambda_0}{\pi^2} = 0.003901,\ 0.004001,\ 0.004080,\ 0.004146,\ 0.004200,\ 0.004248,$$
$$g_6 = 0.002165,\ 0.001740,\ 0.001444,\ 0.001228,\ 0.001063,\ 0.0009340,$$
$$-g_8 = 0.0007361,\ 0.0005564,\ 0.0004383,\ 0.0003558,\ 0.0002956,\ 0.0002502.$$

These numbers show that there are serious discrepancies between Litim's flow equation results and perturbation theory. The values of $6\lambda_0/\pi^2$ are not close to the value $6\lambda_0/\pi^2 = 0.005$. Moreover, the respective values of $g_6$ and $g_8$ are considerably different form the ones in Eqs. (10,11), even for $k = 5$, such that $\lambda/m = 0.07027$ and $\lambda^2/m^2 = 0,004938$. For this small value, the perturbative formulas should be accurate.

## 4. Results for Morris's power-law cutoff function

Here we carry out the calculations for Morris's power-law cutoff function, employing his differential equation in $D = 3$ [7, eq. 12], truncated at $\phi^{16}$.

We now set the following initial ("bare") values. We choose again $6\lambda_0/\pi^2 = 0.005$ ($\lambda_0 = 0,008225$) but we now choose $m_0^2 = -0.0016 + 0.001k$, $k = 0, \ldots, 5$, because it is more convenient for obtaining suitable renormalized values (as $\Lambda \to 0$). Our purpose is always to assess to what extent the perturbative formulas are fulfilled.

The numerical integration is again straight-forward and yields the following results:

$$m = 0.01122,\ 0.03282,\ 0.04526,\ 0.05501,\ 0.06330,\ 0.07065, \tag{13}$$

$$\frac{6\lambda}{\pi^2} = 0.003834,\ 0.004519,\ 0.004643,\ 0.004705,\ 0.004743,\ 0.004770, \tag{14}$$

$$g_6 = 0.1312,\ 0.009637,\ 0.004078,\ 0.002389,\ 0.001618,\ 0.001190, \tag{15}$$

$$-g_8 = 0.2101,\ 0.008086,\ 0.002674,\ 0.001337,\ 0.0008052,\ 0.0005393. \tag{16}$$

Hence,

$$\frac{\lambda}{m} = 0.5621,\, 0.2265,\, 0.1687,\, 0.1407,\, 0.1232,\, 0.1111. \tag{17}$$

In the present case, the relative variation of $6\lambda/\pi^2$ from its initial value $6\lambda_0/\pi^2 = 0.005$ is smaller than in the two preceding cases, but this is not really significant and we must assess the concordance with the perturbative formulas.

Perturbative formulas (5,6,7), in combination with Eq. (17), yield

$$\begin{aligned}
\frac{6\lambda_0}{\pi^2} &= 0.008853,\, 0.006354,\, 0.005977,\, 0.005801,\, 0.005695,\, 0.005623,\\
g_6 &= 0.4590,\, 0.02845,\, 0.01209,\, 0.007123,\, 0.004845,\, 0.003576,\\
-g_8 &= 1.953,\, 0.02095,\, 0.006685,\, 0.003376,\, 0.002061,\, 0.001398.
\end{aligned}$$

We obtain values of $6\lambda_0/\pi^2$ that are not quite close to the value $6\lambda_0/\pi^2 = 0.005$. Moreover, the values of $g_6$ and $g_8$ are too large (in absolute value) in comparison with the respective ones obtained in the numerical integration. This also occurs in the last case ($k = 5$), in spite of having $\lambda/m = 0.1111$, which is a reasonably small value.

## 5. Regularization and renormalization in the exact renormalization group

So far, we have tested the renormalization of coupling constants, which is regularization-scheme independent. This is not the case of the relationship between $m_0$ and $m$, but this non-universal relationship is also worth being considered.

We have already derived a first approximation to sharp-cutoff mass renormalization in Eq. (1), by employing the linearized Wegner-Houghton ERG. It is also straight-forward to linearize Litim's flow equation. Naturally, this linearized flow is such that the coupling constant $\lambda$ is not renormalized, while $m^2$ grows, ruled by

$$m^2(\Lambda) = m_0^2 + \frac{12\lambda_0}{\pi^2}\,(\Lambda_0 - \Lambda). \tag{18}$$

When we replace $6\lambda_0/\pi^2 = 0.005$ in Eq. (18), at $\Lambda = 0$, we obtain

$$m = 0.07280,\, 0.07937,\, 0.08544,\, 0.09110,\, 0.09644,\, 0.1015, \tag{19}$$

which roughly match the result of the numerical integration of Litim's flow equation, shown in Eq. (8).

However, we should not expect great accuracy from linearized ERG equations, which do not even renormalize the coupling constant. Fortunately, this method can be considerably improved by means of a simple non-perturbative formula, without considering coupling constant renormalization, namely, the "gap equation" [13]. Assuming a sharp cutoff $\Lambda_0$, the gap equation reads

$$m^2 = m_0^2 + \int_0^{\Lambda_0} \frac{d^3k}{(2\pi)^3}\, \frac{12\lambda_0}{k^2+m^2}\,. \tag{20}$$

This equation can actually be connected with the ERG [6,15,16]. In addition, it can be easily integrated to give (suppressing inverse powers of $\Lambda_0$):

$$m^2 = m_0^2 + \frac{6\Lambda_0}{\pi^2}\,\lambda_0 - \frac{3}{\pi}\,m\lambda_0\,. \tag{21}$$

This mass renormalization equation improves on Eq. (1) for $\Lambda = 0$. For example, when solved for our values of $m_0$ and $\lambda_0$, it yields

$$m = 0.01383,\, 0.03234,\, 0.04419,\, 0.05365,\, 0.06176,\, 0.06898.$$

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The agreement with the result of the numerical integration of the Wegner-Houghton ERG equation in Eq. (2) is quite good. Further improvements can be achieved with the method employed in Ref. [16].

We can also find an improved version of Eq. (18) for Litim's regulator, and reproduce the success above, in a certain sense. Nevertheless, we encounter insurmountable problems to connect the result with standard perturbative field theory.

Let us firstly expound the connection between Eq. (20) and the Wegner-Houghton ERG equation. This equation admits an integral formulation, whose second derivative with respect to $\phi$ at $\phi = 0$ yields [15,16]:

$$m^2(\Lambda) = m^2(\Lambda_0) + 12 \int_{\Lambda}^{\Lambda_0} \frac{d^3k}{(2\pi)^3} \frac{\lambda(k)}{k^2 + m^2(k)} \,. \tag{22}$$

Taking $\Lambda = 0$ and assuming that we can set $\lambda(k) = \lambda_0$ and $m^2(k) = m^2(0) = m^2$, we obtain Eq. (20). Equation (22) is equivalent to the differential equation

$$\frac{dm^2}{d\Lambda} = -\frac{6\,\lambda_0\,\Lambda^2}{\pi^2\,(\Lambda^2 + m^2)} \,. \tag{23}$$

This differential equation cannot be solved analytically (to our knowledge). However, when $|m_0^2|/\Lambda_0^2 \ll 1$, and as far as $|m^2|/\Lambda^2 \ll 1$, we can neglect the $m^2$ in the denominator, so we have a trivial differential equation, whose solution is Eq. (1).

Actually, equation (23) derives from a truncation of the Wegner-Houghton ERG equation in which we assume $\lambda$ to be constant and, consistently, the higher-order coupling constants to vanish. Thus, we are left with only the first equation of the hierarchy of ordinary differential equations. Of course, taking Litim's flow equation [11, eq. 2.13], and under the same assumptions, we can also restrict ourselves to the first differential equation, which can be written as

$$\frac{dm^2}{d\Lambda} = -\frac{12\,\lambda_0\,\Lambda^4}{\pi^2\,(\Lambda^2 + m^2)^2} \,. \tag{24}$$

In analogy with Eq. (23), the solution of this equation, when $|m_0^2|/\Lambda_0^2 \ll 1$, is Eq. (18). However, the integral equation that is equivalent to Eq. (24) is now

$$m^2(\Lambda) = m_0^2 + \int_{\Lambda}^{\Lambda_0} \frac{d^3k}{(2\pi)^3} \frac{24\lambda(k)\,k^2}{[k^2 + m^2(k)]^2} \,. \tag{25}$$

Taking $\Lambda = 0$, $m^2(k) = m^2(0)$, and $\lambda(k) = \lambda_0$, as we did in Eq. (22), we now obtain, after integrating over $k$:

$$m^2 = m_0^2 + \frac{6\lambda_0}{\pi^2}\left(3\Lambda_0 - \frac{\Lambda_0^3}{\Lambda_0^2 + m^2} - 3m \arctan\frac{\Lambda_0}{m}\right) \approx m_0^2 + \frac{12\lambda_0\Lambda_0}{\pi^2} - \frac{9\lambda_0\,m}{\pi} \,, \tag{26}$$

where we have suppressed inverse powers of $\Lambda_0$ in the last expression.

The solution of Eq. (26) with $6\lambda_0/\pi^2 = 0.005$ and $m_0^2 = -0.0047 + 0.001k$, $k = 0, \ldots, 5$, is (we also set $\Lambda_0 = 1$):

$$m = 0.06197,\ 0.06846,\ 0.07447,\ 0.08008,\ 0.08537,\ 0.09039,$$

Let us recall how the gap equation result (21) improved on Eq. (1) for $\Lambda = 0$. Likewise, Eq. (26) improves on Eq. (18) for $\Lambda = 0$, and on the corresponding numerical values in (19) (compare again with Eq. 8). The improved numerical accuracy was to be expected, because we are considering a better approximation of Litim's flow equation and we are in the perturbative domain. However, Eq. (26) is questionable, as will be shown shortly.

Before, let us turn to Morris's power-law cutoff function. From his differential equation in $D = 3$ [7], we can derive, instead of (23) or (24):

$$\frac{dm^2}{d\Lambda} = -\frac{6\,\lambda_0\,\Lambda^3}{\pi^2\,(2\Lambda^2 + m^2)^{3/2}}\,. \tag{27}$$

Within the same approximation as above, we obtain instead of (21) or (26):

$$m^2 \approx m_0^2 + \frac{6\lambda_0\Lambda_0}{2^{3/2}\pi^2} - \frac{3\lambda_0\,m}{\pi^2}\,. \tag{28}$$

Equations (26) and (28) are both questionable from a theoretical standpoint. Indeed, unlike Eqs. (20,21), equations (26) or (28) do not agree with standard renormalized one-loop perturbation theory, as they should. Let us see why.

Let us recall that the gap equation, as the "cactus approximation" to the Dyson-Schwinger equation for the two-point function, is just an elaboration of the one-loop perturbation theory [13]. The one-loop mass renormalization is given by

$$m^2 = m_0^2 + \int \frac{d^3k}{(2\pi)^3}\,\frac{12\lambda_0}{k^2 + m_0^2}\,. \tag{29}$$

This integral is ultraviolet divergent, of course, and needs to be regularized. There are several methods of regularization in field theory, namely, modifications of the kinetic term in the action (or Hamiltonian), proper-time regularization, lattice regularization, etc [12–14]. Usually, every method introduces a new parameter and gives a form of the integral that, in the divergent limit, can be split into a divergent term and a parameter-independent term. The latter term can also be calculated with methods that do not introduce a new parameter, such as subtraction methods or the method of differentiation. In fact, when we take the derivative with respect to $m_0^2$ of the integrand in Eq. (29), we obtain a convergent integral, proportional to $(m_0^2)^{-1/2}$. The indefinite integral over $m_0^2$ recovers the divergent part as the arbitrary constant of integration and obtains the finite term $-3m_0\lambda_0/\pi$. This term is the one obtained in Eq. (21) with the simple sharp-cutoff regularization. It must be reproduced by every method of regularization. In particular, it is obtained with the economical methods of dimensional or analytic regularization.

Given that the term proportional to $m\lambda_0$ in the expression of the renormalized mass is universal and is the one in Eq. (21), we deduce that the terms proportional to $m\lambda_0$ in Eqs. (26) or (28) cannot arise in any field-theory regularization. To be precise, these regularization methods depart from what is allowed in renormalized perturbation theory. Let us notice that the form of the integrands in Eq. (25) or in the analogous integral corresponding to Eq. (27), with what look like anomalous powers of the propagator in the integrands, already makes one suspect that they are unrelated to the one-loop mass renormalization Eq. (29). The forms of the integrands are due to the flow equations (24) or (27) having anomalous powers of $(\Lambda^2 + m^2)^2$ in the denominators instead of simply $\Lambda^2 + m^2$, as appears in the sharp-cutoff differential equation (23).

In this regard, we may recall the "mean approach" to the sharp-cutoff limit of the ERG equation (for the local potential) that is defined by Liao, Polonyi and Strickland [8]. They obtain, within that approach, a differential equation for $m^2$ with the squared denominator and compare it to the simple denominator of the corresponding Wegner-Houghton equation [8, eqs. 26–27]. Although they admit that the Wegner-Houghton equation is the correct equation, they claim that "the difference between the two equations does not affect the critical properties significantly" and are satisfied with it. Of course, they do not consider the perturbative domain.

Let us recall a method of regularization that consists in modifying the propagator as

$$G_\Lambda(k) = \frac{1}{k^2 + m^2 + k^4/\Lambda^2}\,.$$

This form arises from adding $(\Delta\phi)^2/(2\Lambda^2)$ to the field-theory kinetic term, to suppress very rough field configurations [13]. A related method is Pauli-Villars's regularization, which can actually give rise to denominators with powers of $k^2 + M_i^2$, but with different masses $M_i$ [12,14]. Naturally, all these methods produce the same universal term $-3m\lambda_0/\pi$.

Finally, let us consider the one-loop perturbative renormalization of $\lambda$, in connection with the second differential equation of the Wegner-Houghton equation hierarchy. This equation can be written as

$$\frac{d\lambda}{d\Lambda} = -\frac{18\,\lambda^2\,\Lambda^2}{\pi^2\,(\Lambda^2+m^2)^2}\,, \tag{30}$$

where we have neglected the sextic coupling constant. To integrate this equation, let us assume that $m$ is constant (with $\Lambda$) and takes its renormalized value at $\Lambda = 0$ (like we did to integrate Eq. 22). This approximation is equivalent to the so-called *bubble approximation* of the Schwinger-Dyson equation and obtains a simple expression of $\lambda_0$ as a function of $\lambda$ and $m$ [15, eq. 16]; namely,

$$\lambda_0 = \frac{\lambda}{1 - 9\lambda/(2\pi m)} = \lambda\left(1 + \frac{9\lambda}{2\pi m} + \frac{81\,\lambda^2}{4\pi^2 m^2} + \cdots\right). \tag{31}$$

Of course, this function matches Eq. (5) to one-loop order.

An expression equivalent to Eq. (31) results from the classic renormalization-group-improved perturbation theory to one-loop order, with the beta-function [13,14]

$$\left(m\frac{\partial}{\partial m}\frac{\lambda}{m}\right)_{\lambda_0} = \frac{\lambda}{m}\left(-1 + \frac{9\lambda}{2\pi m}\right). \tag{32}$$

This beta-function does not refer to a flow with the cutoff $\Lambda$ but to the effect that a change of $m$ has on $\lambda$, once renormalization has been carried out, for a given value of $\lambda_0$. The integration of Eq. (32) between $m_1$ and $m_2$ yields:

$$\lambda_1 = \frac{\lambda_2}{1 - (1/m_2 - 1/m_1)9\lambda_2/(2\pi)}\,. \tag{33}$$

We have seen that, for some $m_1$ large ($m_1 \gg \Lambda_0$), $\lambda_1(\Lambda)$ hardly changes with $\Lambda$; hence, $\lambda_1 \approx \lambda_0$ (the latter being its value at $\Lambda_0$). Therefore, for $m_1 \gg \Lambda_0 > m_2$, we neglect $1/m_1$ in Eq. (33) and it becomes equivalent to Eq. (31).

In contrast to the above, from Litim's equation hierarchy, in place of Eq. (30), we have

$$\frac{d\lambda}{d\Lambda} = -\frac{72\,\lambda^2\,\Lambda^4}{\pi^2\,(\Lambda^2+m^2)^3}\,, \tag{34}$$

whereas, from Morris's equation hierarchy, we have

$$\frac{d\lambda}{d\Lambda} = -\frac{27\,\lambda^2\,\Lambda^3}{\pi^2\,(2\Lambda^2+m^2)^{5/2}}\,. \tag{35}$$

These two differential equations can be integrated with the same approximation made above. However, the results do not match either Eq. (5) or the classic renormalization-group-improved perturbation theory. The problem is again that abnormal denominators replace the now right denominator $(k^2 + m^2)^2$ (as it appears in the "bubble" Feynman diagram).

The additional ERG differential equation in Ref. [8, Eq. 43], which has not been employed here (or elsewhere, to our knowedlege) is also afflicted by this problem.

## 6. Conclusion

Our analysis of the Wegner-Houghton sharp-cutoff exact renormalization group equation demonstrates that it is a useful tool in the perturbative domain of $\lambda\phi^4$ theory in three dimensions.

Unlike Morris [17], who cautions that, with the sharp-cutoff method in the local potential approximation, "truncations of the field dependence have limited accuracy and reliability," we do find sufficient accuracy and reliability with a moderate truncation. This conclusion can be expected to hold for more general field theories.

We observe, in the perturbative domain, a good numerical concordance of the Wegner-Houghton sharp-cutoff ERG flow results with standard perturbative formulas, whereas this concordance is lacking in other forms of the ERG flow, e.g., in Litim's or Morris's forms. For Litim's flow, the values of $\lambda/m$ that we use are smaller than for the Wegner-Houghton form and, nonetheless, the concordance with perturbation theory is worse.

Moreover, a theoretical study of the effect of changes of the regularization scheme on universal magnitudes in standard renormalized perturbation theory leads us to unveil that Litim's or Morris's flow equations modify regularization-independent terms in the mass and coupling-constant renormalization formulas. This problem cannot be solved by numerical manipulations and actually resides in the nature of of the regularization methods themselves; namely, those methods cannot be consistent with standard renormalized perturbation theory.

Of course, such a strong statement holds if we only deal with the renormalization of the effective potential, as performed in the local potential approximation of the ERG. The renormalization of the complete effective action, including the derivative terms in it, could perhaps compensate for the terms in the effective potential. However, the complete effective action is quite a complex entity. In three dimensions, in which the field anomalous dimension is quite small, we expect the effective potential to be sufficiently accurate. Indeed, it appears to be so with the sharp-cutoff method.

Hopefully, the results presented here will contribute to make the exact renormalization group a more useful tool in quantum field theory.

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