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Formal Calculation of Binomial Coefficients

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Abstract: Formal Calculation provides formulas for computing nested sums, offers results for all three forms, and explores the connections between them. It is a powerful tool for studying various numbers in a unified manner, making special numbers of many kinds appear ordinary. Additionally, this paper generalizes Wilson's theorem, Wolstenholme's theorem, and the Eulerian polynomial, demonstrating that they are merely special cases.

Keywords: nested sums; Stirling Number; Eulerian polynomials; Associated Stirling Number; congruence

MSC: 05; 11

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1. Calculation Formula

Definition 1.1. *Recursively define* ∇^p, $p \in \mathbb{Z}$,
$\nabla^0 f(n) = f(n)$, $\sum_{n=0}^{N-1} \nabla^1 f(n+1) = f(N)$, $\sum_{n=0}^{N-1} f(n+1) = \nabla^{-1} f(N), \nabla^1 = \nabla$.

Definition 1.2. *Recursively define* $SUM(N) = SUM(N, PS, PT)$, $K_i, D_i \in \mathbb{C}$, $T_i \in \mathbb{N}$.
$SUM(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1}(K_1 + nD_1)$.
$SUM(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1}(K_2 + nD_2)\nabla^p SUM(n+1, [K_1 : D_1], [T_1])$.

Abbreviations: $[K_1 : D, K_2 : D ... K_M : D] = [K_1, K_2 ... K_M] : D$, $[K_1, K_2 ... K_M] : 1 = [K_1, K_2 ... K_M]$.
$SUM(N, PS, [1, 2 ... M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i)$.
$SUM(N, PS, [1, 3 ... 2M-1]) = \sum_{n_M=0}^{N-1} (K_M + n_M D_M) ... \sum_{n_2=0}^{n_3} (K_2 + n_2 D_2) \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$.
$SUM(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3) \sum_{n=0}^{n_3} (K_1 + nD_1)(K_2 + nD_2)$.
$SUM(N, PS, [1, 3, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3)(K_2 + n_3 D_2) \sum_{n=0}^{n_3} (K_1 + nD_1)$.
$SUM(N, PS, [1, 4]) = \sum_{n_3=0}^{N-1} (K_2 + n_3 D_2) \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$.
In this paper, the default $PS = [K_1 : D_1, K_2 : D_2 ... K_M : D_M]$, $PT = [T_1, T_2 ... T_M], T_i < T_{i+1}$.
Use $\mathbb{K}$ to represent the set $\{K_i\}$, $\mathbb{T}$ to represent the set $\{K_i\}$.
$(K_1 + T_1)(K_2 + T_2) ... (K_M + T_M) = \sum \prod_{i=1}^{M} X_i, X_i = T_i \text{ or } K_i$.

Definition 1.3. $X(T)$=*Number of* $\{X_1, X_2 ... X_M\} \in \mathbb{T}$.

Definition 1.4. X_{T-1}=*Number of* $\{X_1, X_2 ... X_{i-1}\} \in \mathbb{T}$, X_{K-1}=*Number of* $\{X_1, X_2 ... X_{i-1}\} \in \mathbb{K}$.

Obviously, $X_{T-1} + X_{K-1} = i - 1$. Use the auxiliary form and each X_i cannot be exchanged:

Theorem 1.1. $H = T_M - M, SUM(N, PS, PT) =$
$Form_1 \rightarrow \sum_{g=0}^{M} H_1(g) \binom{N+H}{N-1-g} = \sum_{g=0}^{M} H_1(g) \binom{N+H}{H+1+g}$, $B_i = \begin{cases} (T_i - X_{K-1})D_i, \ X_i = T_i \\ K_i + X_{T-1}D_i, \ X_i = K_i \end{cases}$,
$Form_2 \rightarrow \sum_{g=0}^{M} H_2(g) \binom{N+H+g}{N-1} = \sum_{g=0}^{M} H_2(g) \binom{N+H+g}{H+1+g}$, $B_i = \begin{cases} (T_i - X_{K-1})D_i, \ X_i = T_i \\ K_i + (X_{K-1} - T_i)D_i, \ X_i = K_i \end{cases}$,
$Form_3 \rightarrow \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{N-1-g} = \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{T_M+1}$, $B_i = \begin{cases} -K_i + (T_i - X_{T-1})D_i, \ X_i = T_i \\ K_i + X_{T-1}D_i, \ X_i = K_i \end{cases}$.
$H_i(g) = H_i(g, PS, PT) = H_i(g, M)$, *is defined as* $\sum_{X(T)=g} \prod_{i=1}^{M} B_i$.

Proof. $H_1(g) = H_1(g-1, M-1)(T_M - [(M-1) - (g-1)])D_M + H_1(g, M-1)(K_M + gD_M)$.
$\sum_{n=0}^{N-1} n \binom{n+K}{M} = \sum_{n=0}^{N-1} (n - K + M) \binom{n+K}{M} + (M-K) \binom{n+K}{M} = (M+1) \binom{N+K}{M+2} + (M-K) \binom{N+K}{M+1}$.

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M = 1, it holds. Suppose that holds when M, $PS1 = [PS, K_{M+1} : D_{M+1}]$, $PT1 = [PT, T_{M+1} = T_M + 2 - p]$.
$SUM(N, PS1, PT1) = \sum_{n=0}^{N-1} (K_{M+1} + nD_{M+1}) \nabla^p SUM(n+1)$
$= \sum_{n=0}^{N-1} (K_{M+1} + nD_{M+1}) \sum_{g=0}^{M} H_1(g) \binom{n+1+T_M-M-p}{T_M-M+1+g-p}$
$= \sum_{g=0}^{M} (K_{M+1} + gD_{M+1}) H_1(g) \binom{N+1+T_M-M-p}{T_M-M+2+g-p} + \sum_{g=0}^{M} (T_M - M + 2 + g - p) D_{M+1} H_1(g) \binom{N+1+T_M-M-p}{T_M-M+3+g-p}$
$= \sum_{g=0}^{M} (K_{M+1} + gD_{M+1}) H_1(g) \binom{N+T_{M+1}-(M+1)}{T_{M+1}-(M+1)+1+g} + \sum_{g=0}^{M} (T_{M+1} - (M-g)) D_{M+1} H_1(g) \binom{N+T_{M+1}-(M+1)}{T_{M+1}-(M+1)+2+g}$
$= \sum_{g=0}^{M+1} H_1(g, M+1) \binom{N+T_{M+1}-(M+1)}{T_{M+1}-(M+1)+1+g}$. Proof of $Form_1$ completion.
$\sum_{n=0}^{N-1} n \binom{n+K}{M} = (M+1)\binom{N+K+1}{M+2} - (1-K)\binom{N+K}{M+1} = (M-K)\binom{N+K+1}{M+2} + (1+K)\binom{N+K}{M+2}$. Prove $Form_{2,3}$ through it. □

For example: $Form = (1+T_1)(2+T_2)(3+T_3), \sum_{X(T)=1} \prod X_i = 1 \times 2 \times T_3 + 1 \times T_2 \times 3 + T_1 \times 2 \times 3$.
$H_1(1) = 1 \times 2 \times (T_3 - X_{K-1}) + 1 \times (T_2 - X_{K-1}) \times (3 + X_{T-1}) + T_1 \times (2 + X_{T-1}) \times (3 + X_{T-1})$
$= 1 \times 2 \times (5-2) + 1 \times (3-1) \times (3+1) + 1 \times (2+1) \times (3+1) = 26$.
$SUM(N, [1,2,3], [1,3,5]) = 1 \times 3 \times 5 \binom{N+2}{6} + 35 \binom{N+2}{5} + 26 \binom{N+2}{4} + 1 \times 2 \times 3 \binom{N+2}{3}$.
$SUM(N, [2,3], [3,5]) = 3 \times 5 \binom{N+3}{6} + (2 \times 4 + 3 \times 4) \binom{N+3}{5} + 2 \times 3 \binom{N+3}{4}$.
$H_1(g) = H_1(g-1, M-1)(T_M - [M-g])D_M + H_1(g, M-1)(K_M + gD_M)$.
$H_2(g) = H_2(g-1, M-1)(T_M - [M-g)])D_M + H_2(g, M-1)(K_M + [M-1-g-T_M]D_M)$.
$H_3(g) = H_3(g-1, M-1)(-K_M + (T_M - [g-1])D_M) + H_3(g, M-1)(K_M + gD_M)$.
They can be unified as: $H(g) = H(g-1, M-1)B + H(g, M-1)A$.
$H_1(g), B = B_M + (g-1)D_M, A = A_M + gD_M, A_M = K_M, B_M = T_M D_M - (M-1)D_M$.
$H_2(g), B = B_M + (g-1)D_M, A = A_M - gD_M, A_M = K_M + (M-1-T_M)D_M, B_M = T_M D_M - (M-1)D_M$.
$H_3(g), B = B_M - (g-1)D_M, A = A_M + gD_M, A_M = K_M, B_M = -K_M + T_M D_M$.
Consider the general two-dimensional second-order linear recursive equations:
$R(M,g) = R(M-1, g-1)(B_M + (g-1)E_M) + R(M-1, g)(A_M + gD_M)$. It can be calculated in a similar way to $H(g)$. $H(g)$ itself requires $|D_i| = |E_i|$ and can't change the sign, so that 3-forms exist.

2. Property

Theorem 2.1.
(1). $H_1(g) = \sum_{k=g}^{M} H_2(k) \binom{k}{g} = \sum_{k=0}^{g} H_3(k) \binom{M-k}{M-g}$.
(2). $H_2(g) = \sum_{k=g}^{M} (-1)^{k+g} H_1(k) \binom{k}{g}$, $H_3(g) = \sum_{k=0}^{g} (-1)^{k+g} H_1(k) \binom{M-k}{M-g}$.
(3). $H_2(g) = (-1)^{M-g} \sum_{k=M-g}^{M} H_3(k) \binom{k}{M-g}$, $H_3(g) = (-1)^g \sum_{k=g}^{M} H_2(M-k) \binom{k}{g}$.
(4). $\sum_{g=0}^{M} H_1(g) q^g (1-q)^{M-g} = \sum_{g=0}^{M} H_2(g)(1-q)^{M-g} = \sum_{g=0}^{M} H_3(g) q^g$
(5). $\sum_{g=0}^{M} H_1(g) \binom{X}{Y-g} = \sum_{g=0}^{M} H_2(g) \binom{X+g}{Y} = \sum_{g=0}^{M} H_3(g) \binom{X+M-g}{Y-g}, Y \in \mathbb{N}$.

Recursive relations yields (1), inversion yields (2), (4) and (5) can be obtained from (1). (5) shows $Form_1 = Form_2 = Form_3$. If we ignore the practical significance, $T_i \in \mathbb{C}$. $q^X = \sum_{k=0}^{X} (-1)^k (1-q)^k \binom{X}{k}$, $(1-q)^X = \sum_{k=0}^{X} (-1)^k q^k \binom{X}{k}$ and (4) yields (3).

Theorem 2.2.
(1). $H_1(g, [AD : D, PS], [A, PT]) = AD(H_1(g) + H_1(g-1))$.
(2). $SUM(N, [L_1, L_2 ... L_q, PS], [L_1, L_2 ... L_q, PT]) = \prod_{i=1}^{q} L_i \times SUM(N, PS, PT)$.
(3). $SUM(N, PT, PT) = \prod T_i \binom{N+T_M}{T_M+1}$, $H_1(g) = \prod T_i \binom{M}{g}$.
(4). In $SUM(N, [...PS...], [...T, T+1..T+M-1...])$, $K_i : D_i$ *can exchange order.*
(5). If $D_i = 1$ *and* $K_{i+1} - K_i = T_{i+1} - T_i = 1$, $H_1(g) = \binom{M}{g} T_1 ... T_g \times K_{g+1} ... K_M$.

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(1) is obvious, (2) and (3) is derived from (1), so T_1 can great than 1. (3) is the promotion of $\sum_{n=0}^{N-1}\binom{n+M}{M}=\binom{N+M}{M+1}$. (4) comes from the definition. (5) can be proved by induction.

Definition 2.1. $F_g^{\mathbb{K}}=\sum_{1\le\lambda_1<\lambda_2<...<\lambda_g\le M}K_{\lambda_1}K_{\lambda_2}...K_{\lambda_g}, F_0^{\mathbb{K}}=1, F_g^N=F_g^{\{1,2...N\}}$.

Definition 2.2. $E_g^{\mathbb{K}}=\sum_{1\le\lambda_1\le\lambda_2\le...\le\lambda_g\le M}K_{\lambda_1}K_{\lambda_2}...K_{\lambda_g}, E_0^{\mathbb{K}}=1, E_g^N=E_g^{\{1,2...N\}}$.

$F_M^{N+M-1}=S_1(N+M,N)$, S_1 is unsigned stirling number of the first kind.
$E_M^N=S_2(N+M,N)$, S_2 is stirling number of the second kind.
$SUM(N,[1,1...1],[1,2...M])=SUM(N,[1,1...1],[2,3...M])=1^M+2^M+...+N^M$,
$SUM(N,[1,1...1],[1,3...2M-1])=SUM(N,[1,1...1],[3,5...2M-1])=S_2(N+M,N)$,
$SUM(N,[1,2...M],[1,3...2M-1])=SUM(N,[2,3...M],[3,5...2M-1])=S_1(N+M,N)$.
They can be used to calculate $S_1(N,N-M)$ and $S_2(N,N-M)$.
In the calculation of $H(g)$, $\prod X_i=(\prod X_i,X_i\in\mathbb{T})(\prod X_i,X_i\in\mathbb{K})$.

Definition 2.3. $H(g,\sum T)=H(g,\sum T,PS,PT)=H(g,\sum T,M)=\sum\prod_{X_i\in\mathbb{T}}B_i$, $H(g,\sum K)=\sum\prod_{X_i\in\mathbb{K}}B_i$

Obviously, $H_1(g,\sum K,[1,1..1],PT)=E_{M-g}^{g+1}$. $PT1=[T,T+1...T+M-1],PS1=[K+M-1,K+M-2...K]$,
$D_i=1,H_1(g,PS,PT1)=[T]^gH(g,\sum K),H_1(g,PS1,PT)=[K+M-1]_gH(g,\sum T)$.

Theorem 2.3. $\mathbb{S}=\{T_i-K_i-i+1\}$, $D_i=1$
(1). $H_1(g,\sum K)=H_3(g,\sum K)=F_{M-g}^{\mathbb{K}}E_0^g+F_{M-g-1}^{\mathbb{K}}E_1^g+...+F_0^{\mathbb{K}}E_{M-g}^g$.
(2). $H_1(g,\sum T)=H_2(g,\sum T)=F_g^{\mathbb{T}}E_0^{M-g}-F_{g-1}^{\mathbb{T}}E_1^{M-g}+...+(-1)^gF_0^{\mathbb{T}}E_g^{M-g}$.
(3). $H_2(g,\sum K)=(-1)^{M-g}(F_{M-g}^{\mathbb{S}}E_0^g+...+F_0^{\mathbb{S}}E_{M-g}^g)$.
(4). $H_3(g,\sum T)=(-1)^gF_g^{\mathbb{S}}E_0^{M-g}+(-1)^{g-1}F_{g-1}^{\mathbb{S}}E_1^{M-g}...+F_0^{\mathbb{S}}E_g^{M-g}$.
(5). $H_1(g,\sum K)=\sum\prod_{i=1}^{M-g}(K_{i+\lambda_i}+\lambda_i),0\le\lambda_1\le\lambda_2\le...\le\lambda_{M-g}\le g$.
(6). $H_1(g,\sum T)=\sum\prod_{i=1}^{g}(T_{i+\lambda_i}-\lambda_i),0\le\lambda_1\le\lambda_2\le...\le\lambda_g\le M-g$.

Proof.
$PS1=[PS,K_{M+1}],PT1=[PT,T_{M+1}]$. Using induction to prove (2).
$H_1(g,\sum T,M+1)=H_1(g,\sum T)+H_1(g-1,\sum T)(T_{M+1}-(M-g+1))$.
$F_{g-x}^{\{PT1\}}$ in $H_1(g,\sum T,M+1)$ has three sources.
$=(-1)^xF_{g-x}^{\mathbb{T}}E_x^{M-g}+(-1)^xF_{(g-1)-x}^{\mathbb{T}}E_x^{M-(g-1)}T_{M+1}+(-1)^{x-1}F_{(g-1)-(x-1)}^{\mathbb{T}}E_{x-1}^{M-(g-1)}(-(M-g+1))$
$=(-1)^xF_{g-x}^{\mathbb{T}}(E_x^{M-g}+E_{x-1}^{M+1-g}(M+1-g))+(-1)^xF_{g-x-1}^{\mathbb{T}}E_x^{M+1-g}T_{M+1}$ (*)
$=(-1)^xF_{g-x}^{\mathbb{T}}E_x^{M+1-g}+(-1)^xF_{g-x-1}^{\mathbb{T}}E_x^{M+1-g}T_{M+1}=(-1)^xE_x^{M+1-g}F_{g-x}^{\{PT1\}}$.
$E_x^{M-g}+E_{x-1}^{M+1-g}(M+1-g)$
$=S_2(M-g+x,M-g)+(M+1-g)S_2(M-g+x,M+1-g)$
$=S_2(M-g+x+1,M+1-g)=E_x^{M+1-g}\rightarrow$ (*).
(5) and (6) are definitions. □

Definition 2.4. $F_T(N+M,N)=F_M^{\{T,T+1...T+N+M-1\}},E_T(N+M,N)=E_M^{\{T,T+1...T+N-1\}}$.

Obviously, $F_0(N+M,N)=S_1(N+M,N),E_1(N+M,N)=S_2(N+M,N)$, $F_T(M+1,g+1)=F_{M-g}^{\{T,T+1...T+M\}}=(T+M)F_T(M,g+1)+F_T(M,g)$, $E_T(M+1,g+1)=E_{M-g}^{\{T,T+1...T+g\}}=(T+g)E_T(M,g+1)+E_T(M,g)$.

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Definition 2.5. $\left\langle {n \atop j} \right\rangle_T = (T-1+n-j)\left\langle {n-1 \atop j-1} \right\rangle_T + (j+1)\left\langle {n-1 \atop j} \right\rangle_T, \left\langle {1 \atop 0} \right\rangle_T = T, \left\langle {1 \atop 1} \right\rangle_T = 0, \left\langle {n \atop j} \right\rangle_T = 0, j < 0, j > n.$

Obviously: $\left\langle {n \atop 0} \right\rangle_1 = T, \left\langle {n \atop n} \right\rangle_T = 0$. $n > 0, \left\langle {n \atop j} \right\rangle_1 = \left\langle {n \atop j} \right\rangle$ is Eulerian number.

Definition 2.6. $\left\langle {M \atop g} \right\rangle_T^j = \sum_{i=0}^{j} (-1)^i \left\langle {M-j \atop g-1-i} \right\rangle_T \binom{j}{i}, 0 \le j < g, 0 < g < M.$

Theorem 2.4.
(1). $PT = [T, T+1...T+M-1], H_2(g, \sum K) = \sum_{j=0}^{M-g} (-1)^{M-g-j} F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T+1...T+g\}}$.
(2). $\sum_{j=0}^{M-g} F_j^{\{T-K_i\}} E_{M-g-j}^{g} = \sum_{j=0}^{M-g} (-1)^j F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T+1...T+g\}}$.
(3). $PT = [T, T+1...T+M-1], H_3(0 < g < M, \sum K) = \sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}}$.
(4). $PS = [K, K-1...K-M+1], H_2(0 < g < M, \sum T) = \sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{-K}^j F_j^{\mathbb{T}} + (-1)^{M-g} \binom{M}{g} F_M^{\mathbb{T}}$.
(5). $PS = [K, K-1...K-M+1], H_3(g, \sum T) = \sum_{j=0}^{g} (-1)^{g-j} F_j^{\mathbb{T}} E_{g-j}^{\{K,K-1...K-M+g\}}$.
(6). $\sum_{j=0}^{M-g} F_j^{\{K_i-T\}} E_{M-g-j}^{g} = (-1)^{M-g} \sum_{j=0}^{M-g} (-1)^j F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T-1...T-g\}}$.

Proof. $PT = [T, T+1...T+M-1], H_2(g, \sum K)$ must be is a symmetric function $= \sum_{j=0}^{M} A_M^g(j) F_j^{\mathbb{K}}$.
$H_2(0,1) = K_1 - T, H_2(1,1) = T$. It's holds when $M = 1$.
$H_2(g, M) = (K_M - T - g) H_2(g, M-1) + (T+g-1) H_2(g-1, M-1)$.
$A_1^0(0) = -T, A_1^1(0) = T$. $A_M^g(0) = -(T+g) A_{M-1}^g(0) + (T+g-1) A_{M-1}^{g-1}(0) \to$
$A_M^g(0) = (-1)^{M-g} [T+g-1]_g E_T(M+1, g+1) = (-1)^{M-g} [T+g-1]_g E_{M-g}^{\{T,T+1...T+g\}}$.
The rest only need to consider terms that multiply with K_M.
$A_M^g(j) = A_{M-1}^g(j-1) = A_{M-j}^g(0) = (-1)^{M-g-j} [T+g-1]_g E_T(M+1-j, g+1)$.
$H_2(g) = [T+g-1]_g \sum_{j=0}^{M-g} (-1)^{M-g-j} F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T+1...T+g\}} \to$ (1). [Theorem 2.3(3)] and (1) $\to$ (2).
Using the same method to prove (3), (4), (5), (6). □

Theorem 2.5.
(1). $F_T(N+M, N) = SUM(N, [T, T+1...T+M-1], [1, 3...2M-1])$, $E_T(N+M, N) = SUM(N, [T, T...T], [1, 3...2M-1])$. *(2).* $(x+T)(x+T+1)...(x+T+M-1) = \sum_{k=0}^{M} F_T(M,k) x^k, (x+T)^M = \sum_{k=0}^{M} E_T(M+1, k+1)[x]_k$.
(3) $\sum_{g=0}^{M} (-1)^g g! E_T(M+1, g+1) = (T-1)^M, \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle_T = [T]^M$.
(4). $\left\langle {M \atop g} \right\rangle_T = H_3(g, [T, 1...1], [T...T+M-1]) = T \times H_3(g, [1...1], [T+1...T+M-1])$.

Proof. (1) is obvious. $\nabla SUM(N, [T, T...T], [1, 2...M]) = (T+n)^M = \sum_{g=0}^{M} g! E_T(M+1, g+1) \binom{n}{g} \to$ the latter half of (2). [Theorem 2.1] $\to$ (3). (4) is derived from the definition of $H_3(g)$. □

Definition 2.7. $E_p^q \odot ([I_1, I_2...I_M], C)$
$= \sum_{\lambda_1+\lambda_2+...+\lambda_q = p, \lambda_i \ge 0} 1^{\lambda_1} 2^{\lambda_2} ... q^{\lambda_q} (I_1 + \lambda_1 C)(I_2 + \lambda_1 C + \lambda_2 C)...(I_{q-1} + \lambda_1 C + \lambda_2 C + ... + \lambda_{q-1} C)$.

From the calculation formula: $H_3(g, [1, 1...1], [1, 2...M]) = E_{M-g-1}^{g+1} \odot ([1, 1...1], 1)$. From Worpitzky identity: $N^M = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle \binom{N+g}{M}$, so $H_3(g) = \left\langle {M \atop g} \right\rangle$. $\left\langle {M \atop g} \right\rangle$ is Eulerian number, this is a new expression. It is the result of [1]. For example: $\left\langle {5 \atop 2} \right\rangle = \sum_{\lambda_1+\lambda_2+\lambda_3=2} 1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} (1+\lambda_1)(1+\lambda_1+\lambda_2) = 66$.

Associated Stirling Numbers of the first kind $S_{1,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k cycles, all length $\ge r$. $S_{1,r}(n,k) = \frac{n!}{k!} \sum_{i_1+...+i_k = n, i_j \ge r} \frac{1}{i_1 ... i_k}$.

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$S_{1,r}(n+1,k) = nS_{1,r}(n,k) + (n)_{r-1}S_{1,r}(n-r+1,k-1), n \geq kr$. From the recurrence relation, $H_1(g,[2,3..M],[3,5...2M-1]) = S_{1,2}(M+g+1,g+1), H_2(g,[1,1..1],[3,5...2M-1]) = (-1)^{M-1-g}S_{1,2}(M+g+1,g+1)$.

Associated Stirling Numbers of the second kind $S_{2,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k blocks, all length $\geq r$. $S_{2,r}(n,k) = \frac{n!}{k!}\sum_{i_1+...+i_k=n, i_j \geq r} \frac{1}{i_1!...i_k!}$. $S_{2,r}(n+1,k) = kS_{2,r}(n,k) + \binom{n}{r-1}S_{2,r}(n-r+1,k-1), n \geq kr$. From the recurrence relation, $H_1(g,[1,1..1],[3,5...2M-1]) = S_{2,2}(M+g+1,g+1), H_2(g,[2,3..M],[3,5...2M-1]) = (-1)^{M-1-g}S_{2,2}(M+g+1,g+1)$.

Definition 2.8. *$I_1 = 1$, $I_{i+1} - I_i = 1$ or 2, $I_{i+1} - I_i = 2$ is defined as continuity, $I_{i+1} - I_i = 2$ is defined as discontinuity.*
$MIN_g(M) = I_1 \times \sum I_2 \times I_3... \times I_M$, Number of discontinuities=g.

It is easily understood by adding g holes in consecutive numbers. Easy to know: $MIN_g(M) = \sum \frac{(M+g)!}{i_1 i_2 ... i_g}, 2 \leq i_1 < i_2 < ... < i_g \leq M+g-1, i_{j+1} - i_j \geq 2$. This is the conclusion of [2]. From the calculation formula: $H_1(g,[2,3...M],[3,5...2M-1]) = MIN_g(M)$. So $MIN_g(M) = S_{1,2}(M+g+1,g+1)$.
From the definition of H(g):

Theorem 2.6. $\lambda_1 + ... + \lambda_{g+1} = M - g, \lambda_i \geq 0$, $\left\langle {M \atop g} \right\rangle_T = T\sum 1^{\lambda_1}2^{\lambda_2}...(g+1)^{\lambda_{g+1}}(T+\lambda_1)(T+\lambda_1+\lambda_2)...(T+\lambda_1+...+\lambda_g)$.
$PS = [T, T...T], PT = [1,3...2M-1]$.
$H_1(g) = \sum T^{\lambda_1}(T+1)^{\lambda_2}...(T+g)^{\lambda_{g+1}}(1+\lambda_1)(3+\lambda_1+\lambda_2)...(2g-1+\lambda_1+...+\lambda_g)$.
$H_2(g)$=*Changed* $MIN_{g-1}(M) + MIN_g(M)$. *Select* $X_i \in \mathbb{K}$ *from M factors, change i to (T-i).*
$PS = [T, T+1...T+M-1], PT = [1,3...2M-1]$.
$H_1(g)$=*Changed* $MIN_{g-1}(M) + MIN_g(M)$. *Select* $X_i \in \mathbb{K}$ *from M factors, change i to (T+i-1).*
$H_2(g) = \sum(T-1)^{\lambda_1}(T-2)^{\lambda_2}...(T-g-1)^{\lambda_{g+1}}(1+\lambda_1)(3+\lambda_1+\lambda_2)...(2g-1+\lambda_1+...+\lambda_g)$.

For example, express the product in terms of ():
$MIN_0(3) + MIN_1(3) = (123) + (124) + (134), H_2(1) = (T-1, T-2, 3) + (T-1, 2, T-4) + (1, T-3, T-4)$.
$MIN_1(3) + MIN_2(3) = (124) + (134) + (135), H_1(2) = (T, 2, 4) + (1, T+2, 3) + (1, 3, T+4)$.
From the definition of nested sum, there exists classification principles:

Theorem 2.7. $SUM(N) =$
$SUM(N, PS, [...T_i, T_{i+1} - 1...T_M - 1]) + SUM(N-1, [...K_i : D_i, K_{i+1} + D_{i+1} : D_{i+1}...K_M + D_M : D_M], PT)$.

Table 1. Table of H(g)

PS	PT	$H_1(g)$	$H_2(g)$	$H_3(g)$
$[1,1...1]$	$[1,2...M]$	$g!E^{g+1}_{M-g} = g!S_2(M+1,g+1)$	$(-1)^{M-g}g!S_2(M,g)$	$\left\langle {M \atop g} \right\rangle$
$[1,1...1]$	$[2,3...M]$	$(g+1)!S_2(M,g+1)$	$(-1)^{M-1-g}(g+1)!S_2(M,g+1)$	$\left\langle {M \atop g} \right\rangle$
$[1,1...1]$	$[1,3...2M-1]$	$E^{g+1}_{M-g} \odot (PT,1)$	$(-1)^{M-g}MIN_{g-1}(M)$	$E^{g+1}_{M-g} \odot ([0,1...],2)$
$[1,1...1]$	$[3,5...2M-1]$	$S_{2,2}(M+1+g,g+1)$	$(-1)^{M-1-g}MIN_g(M)$	$E^{g+1}_{M-1-g} \odot ([2,3...],2)$
$[1,2...M]$	$[1,3...2M-1]$	$MIN_{g-1}(M) + MIN_g(M)$	$1 \times (-1)^{M-g}E^g_{M-g} \odot ([3,5...],1)$	$1 \times E^{M-g}_g \odot ([2,3...],2)$
$[2,3...M]$	$[3,5...2M-1]$	$MIN_g(M)$	$(-1)^{M-1-g}S_{2,2}(M+1+g,g+1)$	$E^{M-g}_g \odot ([2,3...],2)$

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3. Number Analysis

From the relations between H(g) [Theorem 2.1], it's easy to get.:
$\sum_{g=0}^{M}\left\langle {M \atop g} \right\rangle = M!, \sum_{g=1}^{M}(-1)^{M-g}g!S_2(M,g) = 1. \sum_{g=1}^{M}(-1)^{M-g}S_{1,2}(M+g,g) = 1, \sum_{g=1}^{M}(-1)^{M-g}S_{2,2}(M+g,g) = M!.$
$g!S_2(M,g) = \sum_{k=g}^{M}(-1)^{M-k}k!S_2(M,k)\binom{k-1}{g-1} = \sum_{k=0}^{g-1}\left\langle {M \atop k} \right\rangle\binom{M-1-k}{M-g}, 1 \le g \le M.$
From $H(g,\sum K), H(g,\sum T)$ [Theorem 2.3], it's easy to get:

Theorem 3.1.
(1). $M!\binom{M}{g} = g!\sum_{i=0}^{M-g}S_1(M+1,g+1+i)S_2(g+i,g) = (M-g)!\sum_{i=0}^{g}S_1(M+1,M+1-i)S_2(M-i,M-g).$
(2). $S_2(M+1,g+1) = \sum_{i=0}^{M-g}S_2(M-i,g)\binom{M}{i}.$
(3). $g!\binom{M}{g} = \sum_{i=0}^{g}S_1(M+1,M+1-g+i)S_2(M-g+i,M-g)(-1)^i.$
(4). $\prod_{i=g+1}^{M}(K+i-1)\binom{M}{g} = F_{M-g}^{\mathbb{K}}E_0^g + ... + F_0^{\mathbb{K}}E_{M-g}^g, \mathbb{K} = \{K, K+1...K+M-1\}.$
(5). $\prod_{i=1}^{g}(T+i-1)\binom{M}{g} = F_g^{\mathbb{T}}E_0^{M-g} + ... + (-1)^g F_0^{\mathbb{T}}E_g^{M-g}, \mathbb{T} = \{T, T+1...T+M-1\}.$

Proof.
$PS = PT = [1,2...M], H_1(g) = M!\binom{M}{g} \to (1).$
$PS = [1,1...1], PT = [1,2...M], F_i^{\{1,1...1\}} = \binom{M}{i} \to (2).$
$PS = [M, M-1...1], PT = [1,2...M], H_1(g) = M!\binom{M}{g} \to (3).$
$PS = [K, K+1...K+M-1], PT = [T, T+1...T+M-1] \to (4),(5).$ □

Theorem 3.2. *$SUM(N,[1,1..1,PS],[1,1..1,PT]) = SUM(N)$. Number of 1 added=X, $T_M - M - X \ge 0, SUM(N,[1,1...1,PS],[1,1...1,PT])$ expands $\sum_{g=0}^{M}(...)$ to $\sum_{g=0}^{M+X}(...)$.*

Any $\sum_{g=0}^{M}a_g\binom{X}{Y+g}$ can be converted to $\frac{a_M}{M!}\nabla^q SUM(N+p,[K_1,K_2...K_M],[1,2...M])$, relations between H(g) [Theorem 2.1] provides necessary and sufficient conditions for:

Theorem 3.3. *$f(g)$ is an integral polynomial of degree equals X, $0 \le X < K$.*
$\sum_{g=0}^{M}a_g\binom{X}{Y+n} = \sum_{n=0}^{M-K}(...)\binom{X+K}{Y+K+n} \leftrightarrow \sum_{g=0}^{M}(-1)^x a_g f(g) = 0.$

Proof. The condition is $H_2(g) = H_3(M-g) = 0, 0 \le g < K.$ □

Define $(M+1)\times(M+1)$ matrix, $A_{1,2,3}(P,Q,M) =$

$$\begin{pmatrix} \binom{P}{Q} & \cdots & \binom{P}{Q-M} \\ \vdots & \ddots & \vdots \\ \binom{P+M}{Q+M} & \cdots & \binom{P+M}{Q} \end{pmatrix}, \begin{pmatrix} \binom{P}{Q} & \cdots & \binom{P+M}{Q} \\ \vdots & \ddots & \vdots \\ \binom{P+M}{Q+M} & \cdots & \binom{P+2M}{Q+M} \end{pmatrix}, \begin{pmatrix} \binom{P+M}{Q} & \cdots & \binom{P}{Q-M} \\ \vdots & \ddots & \vdots \\ \binom{P+2M}{Q+M} & \cdots & \binom{P+M}{Q} \end{pmatrix}$$

.
$P = N + T_M - M,\ Q = N-1$, List SUM(N)...SUM(N+M) from top to bottom, g increases from left to right, they correspond to $Form_{1,2,3}$. It is easy to prove by Gaussian elimination:

Theorem 3.4.
$\| A_1(P,Q,M) \| = \| A_2(P,Q,M) \| = \| A_3(P,Q,M) \|, \| A(P,Q,M) \| = \| A(P,P-Q,M) \|.$
$\| A(P,0,M) \| = 1, \| A(P,1,M) \| = \binom{P+M}{1+M}, \| A(P,Q>1,M) \| = \prod_{g=0}^{Q-1}\frac{\binom{P+M-g}{1+M}}{\binom{1+M-g}{1+M}}.$

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If $SUM(N)$ or $\nabla SUM(N)$ is easy to obtained, consider H(g) as variables, use Cramer's law:

Theorem 3.5.
(1). $H_1(g) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{T_M-M+1+g}{T_M-M+k} SUM(k) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{T_M-M+g}{T_M-M+k-1} \nabla SUM(k)$.
(2). $z(k) = \sum_{i=1}^{k} (-1)^{i+k} \binom{k}{i} \nabla SUM(k)$, $H_2(g) = \sum_{k=g+1}^{M+1} (-1)^{g+k-1} \binom{k-1}{g} z(k)$.
(3). $H_3(g) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{2+T_M}{g+1-k} SUM(k) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{1+T_M}{g+1-k} \nabla SUM(k)$.

$\nabla SUM(N,[1,1...1],[2,3...M]) = N^M$, the following classical formula can be obtained.
$(1) \to S_2(M,g) = \frac{1}{g!} \sum_{k=0}^{g} (-1)^{g+k} \binom{g}{k} k^M = \frac{1}{g!} \sum_{k=0}^{g} (-1)^k \binom{g}{k} (g-k)^M$.
$(3) \to \left\langle {M \atop g} \right\rangle = \sum_{k=1}^{g+1} (-1)^{1+g+k} \binom{M+1}{g+1-k} k^M = \sum_{k=0}^{g} (-1)^k \binom{M+1}{k} (g+1-k)^M$.
$T \in \mathbb{N}, E_T(M+1,g+1) = \frac{1}{g!} \sum_{k=0}^{g} (-1)^{g+k} \binom{g}{k} (T+k)^M = \sum_{k=0}^{M-g} T^k \binom{M}{k} E_{M-k}^{g}$. The latter equation comes from [Theorem 2.3(1)].
$T \in \mathbb{N}, \left\langle {M \atop g} \right\rangle_T = \frac{1}{(T-1)!} \sum_{k=0}^{g} (-1)^{g+k} \binom{T+M}{g-k} [T+k]_T (k+1)^{M-1}$
$= T \times (\sum_{k=0}^{M-2} \left\langle {M-1 \atop g} \right\rangle_{T+1}^{k} \binom{M-1}{k} + (-1)^g \binom{M-1}{g}), 0 < g < M-1$. The latter equation comes from [Theorem 2.4].

Theorem 3.6. $\left\langle {M \atop g} \right\rangle_T^j = \frac{1}{(T-1)!} \sum_{k=0}^{g} (-1)^{g+k} \binom{T+M}{g-k} [T+k]_T (k+1)^{M-1-j}, T \in \mathbb{N}$.

Proof. From $\prod_{i=1}^{M} (x+K_i) = \sum_{j=0}^{M} F_j^{\mathbb{K}} x^{M-j}$, it's easy to get: $\sum_{j=0}^{M} (-1)^j k^{M-j} F_j^M = 0, M \geq k \geq 1$.
$PS = [0,-1...-(M-1)], PT = [T,T+1...T+M-1], H_3(g<M) = 0 \to \sum_{j=0}^{M-1} (-1)^j \left\langle {M \atop g} \right\rangle_T^j F_j^{M-1} = 0, 0 < g < M$.
From these two equations it is clear that $\left\langle {M \atop g} \right\rangle_T^j$ and $\left\langle {M \atop g} \right\rangle_T^0 = \left\langle {M \atop g-1} \right\rangle_T$ have the same thing:
$\left\langle {M \atop g} \right\rangle_T^0 = a_1 1^x + a_2 2^y + ...$, $\left\langle {M \atop g} \right\rangle_T^j = a_1 1^{x-j} + a_2 2^{y-j} + ...$ Complete the proof from the expression of $\left\langle {M \atop g-1} \right\rangle_T$. □

4. Combinatorial Identities

Formal Calculation provides many ways to obtain an identity, and here are some of the methods.

Theorem 4.1. $\sum_{k_r=1}^{N} ... \sum_{k_j=1}^{k_{j+1}} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \nabla SUM(k_j, PS, [1,2...M]) = \nabla^{j-r} SUM(N, PS, PT = [T_i = i+j-1])$.

Proof.
$PS1 = [1:0, 1:0...1:0, PS]$, $PT1 = [1,3...2(j-1)-1, 1+2(j-1), 2+2(j-1)...M+2(j-1)]$.
It can be obtained: $H_1(g > M, PS1, PT1) = 0$, $H_1(g \leq M, PS1, PT1) = H_1(g)$.
$SUM(N, PS1, PT1) = \sum_{g=0}^{M} H_1(g) \binom{N+j-1}{j+g} = \sum_{k_j=1}^{N} \nabla SUM(k_j) ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} 1$. The remaining step is obvious. □

For example: $\sum_{k_r=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \binom{k_j}{M} = \frac{1}{M!} \nabla^{j-r} SUM(N+1, [0,-1...-(M-1)], [T_i = i+j-1])$.
$H_1(g < M) = 0$, $H_1(M) = \frac{(M+j-1)!}{(j-1)!} \to \frac{1}{M!} \nabla^{j-r} \frac{(M+j-1)!}{(j-1)!} \binom{(N+1)+(j-1)}{j+1+(M-1)} = \binom{M+j-1}{j-1} \binom{N+j-(j-r)}{j+M-(j-r)} = \binom{M+j-1}{j-1} \binom{N+r}{M+r}$. [3].

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Theorem 4.2. $\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+g}{A}\binom{X}{Y+g}=\sum_{g=0}^{A}\binom{A+T}{g+T}\binom{M+T+g}{M+T}\binom{X+M-A}{Y+M-A+g}$, $A\geq 0, M\geq A+1$.

Proof. SUM(N,[T+1,T+2...T+M],[T+A+1,T+A+2...T+A+M])
$=\sum_{g=0}^{M}\binom{M}{g}[T+A+g]_g[T+M]_{M-g}\binom{N+T+A}{T+A+1+g}=\frac{A!(T+M)!}{(T+A)!}\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+g}{A}\binom{N+T+A}{T+A+1+g}$ (*)
$=\frac{A!(T+M)!}{(T+A)!}SUM(N,[T+A+1...T+M,T+1,T+2...T+A],[T+A+1..T+M,T+M+1,T+M+2...T+M+A])$
$=\frac{(T+M)!}{(T+A)!}SUM(N,[T+1,T+2...T+A],[T+M+1,T+M+2...T+M+A])=\frac{(T+M)!}{(T+A)!}\sum_{g=0}^{A}\binom{A}{g}\frac{(T+M+g)!}{(T+M)!}\frac{(T+A)!}{(T+g)!}\binom{N+T+M}{T+M+1+g}$.
Compare with (*): $\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+g}{A}\binom{N+T+A}{T+A+1+g}=\sum_{g=0}^{A}\frac{(T+M+g)!}{(T+M)!}\frac{(T+A)!}{(T+g)!}\frac{A!}{(A-g)!g!}\frac{1}{A!}\binom{N+T+M}{T+M+1+g}$.
$Y=T+A+1, X=N+A+T$ then proves completion. The two sides correspond to merging and unfolding. □

Theorem 4.3. *SUM(N,[A,A+1...A+M-1]:2,[1,3...2M-1])=*$\binom{M+N-1}{M}[A+M+N-2]_M$.

Proof. $PS=[A,A+1...A+M-1]$.
Expand by definition : $H_1(g,\sum K)=SUM(g+1,[A,A+1...A+M-1-g]:2,[1,3...2(M-g)-1])$.
Form [Theorem 2.2(5)], $H_1(g,\sum K)=\binom{M}{g}[A+M-1]_{M-g}$. Just compare the results. □

Special:$SUM(N,[1,2...M]:2,[1,3...2M-1])=M!\binom{N+M-1}{M}^2$, $1+3+...+(2N-1)=N^2$.

5. Congruence

In this section, P is prime, K_i, D_i is any integer, $(P,D_i)=1$.
The following congruences are based on P. $D_iD_i^{-1}\equiv 1, K_i+D_in\equiv D_i(K_iD_i^{-1}+n)$.

Definition 5.1. *If* $K_iD_i^{-1}\equiv T_i, 0\leq i\leq X$, *define* $PTS=X, 0\leq PTS\leq M$.

Theorem 5.1. $0\leq Y\leq T_M-M+PTS$,
$T_M<P-1, SUM(P-Y)\equiv 0 \pmod{P\prod_{i=1}^{PTS}T_iD_i}$.
$T_M=P-1, SUM(P-Y)\equiv\prod T_iD_i$.
$T_M=P, SUM(P-Y)\equiv H_1(M-1)\equiv H_2(M-1)$. $PT=[1,2...P]$, $SUM(P)\equiv-\prod D_i\sum K_iD_i^{-1}$.

Proof. $H=T_M-M, 0\leq H<T_M\leq P$. $0\leq Y\leq H$.
$SUM(P)=H_1(M)\binom{P+H}{T_M+1}+...+H_1(0)\binom{P+H}{H+1}=\sum_{g=0}^{M}H_1(g)\binom{P+H}{g+1+H}$.
$T_M<P-1\rightarrow SUM(P-Y)\equiv 0$. $T_M=P-1\rightarrow SUM(P-Y)\equiv H_1(M)\equiv\prod T_iD_i$.
$T_M=P\rightarrow H_1(M)\equiv 0\rightarrow SUM(P-Y)\equiv SUM(P)\equiv H_1(M-1)$.
Prove the part of PTS through [Theorem 2.2(2)].
$PT=[1,2..P]$, [Theorem 2.3](1) $\rightarrow H_1(M-1)=(P-1)!\prod D_i(F_1^{\{K_iD_i^{-1}\}}+E_1^{P-1})=-\prod D_iF_1^{\{K_iD_i^{-1}\}}\equiv-\prod D_i\sum K_iD_i^{-1}$. □

Remove products$\equiv 0$, it can derive many congruence equations. Wilson's Theorem $(P-1)!\equiv-1$ is a special case.

Theorem 5.2.
(1). $P>3, S_1(P+M-Z,P-Z)\equiv S_2(P+M-Z,P-Z)\equiv 0, 0\leq Z\leq M, 1\leq M\leq P-2$.
(2). $S_1(P,K)\equiv S_2(P,K)\equiv 0, 2\leq K\leq P-1$.
(3). $S_1(N+P-1,N)\equiv -A, S_2(N+P-1,N)\equiv A, A\equiv[(N-1)/P]+1, N>0$.
(4). $S_1(N+P-2,N)\equiv S_2(N+P-2,N)\equiv\begin{cases}1, N\equiv 1\\ 0, N\not\equiv 1\end{cases}, N>0$.
(5). $S_1(P-1,N)\equiv 1, S_2(P-1,K)\equiv(P-1-K)!, K!S_2(P-1,K)\equiv(-1)^{K+1}, K>0$.

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(6). $S_1(P-2, P-2-N) \equiv 2^{N+1} - 1.$
(7). $P > 3, S_1(P,2) \equiv 0 \pmod{P^2}.$

Proof. $PT = [3, 5...2M-1], PS1 = [2, 3...M], PS2 = [1, 1...1],$
$H_1(g, PS1, PT) = S_{1,2}(M+1+g, g+1), H_1(g, PS2, PT) = S_{2,2}(M+1+g, g+1).$
If $2M-1 \geq P$, $SUM(P) = H_1(M-1)\binom{P+M}{2M} + ...H_1(P-M-1)\binom{P+M}{P} + ... = \sum_{g=0}^{M-1} H_1(g)\binom{P+M}{g+1+M}.$
If $g \geq P-M-1$, in the express of $S_{1,2}(n, g+1), S_{2,2}(n, g+1)$, $n = M+g+1 \geq P$, when the largest i_j is encountered where the number of 2 is g. $M+g+1-2g = M+1-g$, $Max(i_j) = M+1-(P-M-1) < P$, so $H_1(g) \equiv 0 \rightarrow (1), (2)$.
$M = P-1, SUM(N, PS1, PT), H_1(g > 0) \equiv 0, SUM(N) \equiv -\binom{N+P-1}{P}.$
$M = P-1, SUM(N, PS2, PT), H_1(g > 0) \equiv 0, SUM(N) \equiv \binom{N+P-1}{P}.$
$M = P-2, SUM(N, PS1, PT), H_1(g > 0) \equiv 0, SUM(N) \equiv \binom{N+P-2}{P-1}.$
$M = P-2, SUM(N, PS2, PT), H_1(g > 0) \equiv 0, SUM(N) \equiv \binom{N+P-2}{P-1} \rightarrow (3), (4).$
$H_2(g, PS1, PT) = (-1)^{M-1-g} S_{2,2}(M+1+g, g+1), H_2(g, PS2, PT) = (-1)^{M-1-g} S_{1,2}(M+1+g, g+1).$
$SUM(P-M-1) = H_2(M-1)\binom{P+M-2}{2M} + ... + H_2(P-M-1)\binom{..}{P} + ... + H_2(1)\binom{P}{2+M} + H_2(0)\binom{P-1}{1+M}.$
$H_2(g \geq P-M-1) \equiv 0 \rightarrow SUM(P-M-1) \equiv H_2(0)(-1)^{M+1}.$
$H_2(0, PS1, PT) = (-1)^{M-1}, H_2(0, PS2, PT) = (-1)^{M-1}M! \rightarrow (5).$
$S_1(N, M) = S_1(N-1, M-1) + (N-1)S_1(N-1, M)$ and $S_1(P-1, N) \equiv 1$, induction shows that (6).
$P > 3, SUM(2, [2, 3..P-2], [3, 5...2P-5]) = H_1(1)\binom{P}{P} + H_1(0)\binom{P}{P-1} = S_{1,2}(P, 2) + P(P-2)!.$
$H_1(1)/P = (P-1)!(\frac{1}{2(P-2)} + \frac{1}{3(P-3)} + ... + \frac{1}{\frac{P-1}{2}\frac{P+1}{2}}) \equiv -1(-2^{-2} - 3^{-2}... - (\frac{P-1}{2})^{-2}) \equiv -1 \rightarrow (7).$
This is Wolstenholme's Theory: $\sum_{i=1}^{P-1} \frac{(P-1)!}{i} \pmod{P^2}$. $\square$

A Direct Proof of Wilson's Theorem:
$PS = PT = [1, 2...P-1], \nabla SUM(2) = H_1(1)\binom{N-1}{2-1} + H_1(0)\binom{N-1}{1-1} = P!/1 \equiv 0.$
$\nabla SUM(2) = 1!(F_{P-2}^{P-1}E_0^1 + ... + F_0^{P-1}E_{P-2}^1) + (P-1)! \equiv 1 + (P-1)!$
$PS = PT = [1, 2...P-2], 2 \leq K \leq P-2, \nabla SUM(K+1) = H_1(K)\binom{K}{K} + ... + H_1(0)\binom{K}{0} \equiv 0.$
$H_1(g) = g!(F_{P-2-g}^{P-2}E_0^g + ... + F_0^{P-2}E_{P-2-g}^g) \equiv g!(E_0^g + ... + E_{P-2-g}^g).$
Let $f(g) = g!(E_0^g + ... + E_{P-2-g}^g), f(0) \equiv 1$. It can be proved by induction: $f(g) \equiv (-1)^g(g+1)$.
$PS = [P-2...2, 1]$. Use $H_1(g, \sum T)$, similar conclusions were reached.

Theorem 5.3.
$\sum_{M=0}^{P-2} S_2(M, K) \equiv (-1)^K(K+1)(K!)^{-1}$. $\sum_{M=0}^{P} S_2(M, K) \equiv (-1)^K K(K!)^{-1}, 0 < K < P.$
$\sum_{M=0}^{P-2} (-1)^{M-K} S_2(M, P-2-K) \equiv (-1)^K(K+1)!.$
$\sum_{M=0}^{P} (-1)^{M-K} S_2(M, P-2-K) \equiv ((-1)^K - 1)(K+1)!, 0 \leq K < P-2.$

Theorem 5.4.
(1). $\sum_{0<K_i, K_j<P, K_i \neq K_j} K_1^{C_1} K_2^{C_2} ... K_q^{C_q} \equiv 0, 1 \leq \sum C_i \leq P-2, C_i > 0.$
(2). $0 \leq K_{i+1} - K_i \leq 1, \prod K_i = 1^{C_1}2^{C_2}..q^{C_q}, \sum C_i \leq P-2, C_i > 0; T_{i+1} - T_i = K_{i+1} - K_i + 1, H = T_M - M,$
$\sum SUM(P-1-H, [K_1 = 1, K_2...K_M], [T_1 = 1, T_2...T_M]) \equiv 0.$ *For* $\{C_i\}$*, sum over all PTs meet the condition.*
(3). In (1) and (2), $\sum C_i = P-2, P > 3$ *then* $0 \equiv \pmod{P^2}$.

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Proof. Obviously, $q = 1$ holds for (1).
$q = 2$, If $K_1 + K_2 \neq p, K_1^{C_1}(P-K_2)^{C_2}$ in the summation term; If $K_1 + K_2 = p, K_1^{C_1}(P-K_2)^{C_2} = K_1^{C_1+C_2}$. $\sum K_1^{C_1+C_2} \equiv 0, \sum K_1^{C_1}K_2^{C_2} = yP\sum K_1^{C_1} - \sum K_1^{C_1+C_2}, y \in (N) \to q = 2$ holds. Prove by induction that (1) holds.
Removing the repeated products, it still holds due to symmetry. That's (2). It can be proven that: $\sum n^{P-2} \equiv 0 \pmod{P^2}, P > 3$, applying the same method $\to$ (3), they're generalization of Wolstenholme's Theory. $\square$

Special: $E_M^P = S_2(P+M,P), E_M^{P-1} = S_2(P+M-1,P-1) \equiv 0, 1 \le M \le P-2$. $E_{P-2}^P, E_{P-2}^{P-1} \equiv 0 \pmod{P^2}$.
Form classification [Theorem 2.7], $H = T_M - M, SUM(P-1,[1,1...1],[1,3...2M-1]) = \sum_{X=0}^{M-1}\sum_{H=X} SUM(P-1-X,...)$.
We can get $\sum_{H=X} SUM(P-1-X,...) \equiv 0$. For Example: $P = 7, M = P-2$,
$SUM(6,[1,1,1,1,1],[1,2,3,4,5]) = \sum_{n=1}^{6} n^5$,
$SUM(5,[1,1,1,1,2],[1,2,3,4,6]) + SUM(5,[1,1,1,2,2],[1,2,3,5,6])$
$+SUM(5,[1,1,2,2,2],[1,2,4,5,6]) + SUM(5,[1,2,2,2,2],[1,3,4,5,6])$,
$SUM(4,[1,1,1,2,3],[1,2,3,5,7]) + SUM(4,[1,1,2,2,3],[1,2,4,5,7])$
$+SUM(4,[1,2,2,2,3],[1,3,4,5,7]) + SUM(4,[1,2,3,3,3],[1,3,5,6,7])$
$+SUM(4,[1,2,2,3,3],[1,3,4,6,7]) + SUM(4,[1,1,2,3,3],[1,2,4,6,7])$,
$SUM(3,[1,1,2,3,4],[1,2,4,6,8]) + SUM(3,[1,2,2,3,4],[1,3,4,6,8])$
$+SUM(3,[1,2,3,3,4],[1,3,5,6,8]) + SUM(3,[1,2,3,4,4],[1,3,5,7,8])$,
$SUM(2,[1,2,3,4,5],[1,3,5,7,9]) = \sum_{i=1}^{P-1}\frac{(P-1)!}{i}$. All of them $\equiv 0 \pmod{P^2}$
In fact,$SUM(5,[1,1,1,1,2],[1,2,3,4,6]) + SUM(5,[1,2,2,2,2],[1,3,4,5,6]) \equiv 0 \pmod{P^2}$. (*)

6. Generalization of Eulerian polynomials

In this section, $q \neq 0,\ q \neq 1$. By induction it can be shown that:
$\sum_{n=0}^{N-1} q^n \binom{n+A}{M} = q^N \sum_{k=0}^{M}(-1)^k \frac{\binom{N+A-1-k}{M-k}}{(q-1)^{k+1}} + \frac{q^{M-A}}{(1-q)^{M+1}}, 0 \le A \le M$.
$X = T_M - M - p \ge 0, 0 \le Y \le 1, \sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y) = \sum_{g=0}^{M} H_1(g)\sum_{n=0}^{N-1} q^n \binom{n+Y+X}{1+X+g}$
$= \sum_{g=0}^{M} H_1(g)(q^N \sum_{k=0}^{X+1+g}(-1)^k \frac{\binom{N+X+Y-1-k}{1+X+g-k}}{(q-1)^{k+1}} + \frac{q^{1+g-Y}}{(1-q)^{X+2+g}})$.
From [Theorem 2.1 (5)], $\sum_{g=0}^{M} H_1(g)q^g(1-q)^{M-g} = \sum_{g=0}^{M} H_2(g)(1-q)^{M-g} = \sum_{g=0}^{M} H_3(g)q^g$. Define it as $\mathbf{A_q(PS,PT)}$.
$\sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y) = q^N \sum_{g=0}^{M} H_1(g)\sum_{k=0}^{X+1+g}\frac{(-1)^k}{(q-1)^{k+1}}\binom{N+X+Y-1-k}{1+X+g-k} + \frac{A_q(PS,PT)q^{1-Y}}{(1-q)^{T_M+2-p}}$.
$= q^N \sum_{k=0}^{M}\frac{(-1)^k}{(q-1)^{k+1}}\sum_{g=0}^{M} H_1(g)\binom{N+X+Y-1-k}{1+X+g-k} + \frac{A_q(PS,PT)q^{1-Y}}{(1-q)^{T_M+2-p}}$.

Theorem 6.1. $T_M - M - p \ge 0, 0 \le Y \le 1$,
$\sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y) = q^N \sum_{k=0}^{M}\frac{(-1)^k}{(q-1)^{k+1}}\nabla^{p+k} SUM(N+Y-1) + \frac{A_q(PS,PT)q^{1-Y}}{(1-q)^{T_M+2-p}}$.
$|q| < 1, \sum_{n=0}^{\infty} q^n \nabla^p SUM(n+Y) = \frac{A_q(PS,PT)q^{1-Y}}{(1-q)^{T_M+2-p}}$.

Special: $PS = [1,1...1], PT = [1,2...M]$, define $\mathbf{A_q(PS,PT)} = \mathbf{A_q(M)}$, $A_q(M)$ is Eulerian polynomial.
$\sum_{n=0}^{N-1} q^n n^M = q^N \sum_{k=0}^{M}\frac{(-1)^k}{(q-1)^{k+1}}\nabla^{1+k} SUM(N-1) + \frac{A_q(M)q}{(1-q)^{M+1}}$
$= \frac{q^N}{(q-1)^{M+1}}\sum_{k=0}^{M}(-1)^k(q-1)^{M-k}\nabla^{1+k} SUM(N-1) + \frac{A_q(M)q}{(1-q)^{M+1}} = (**) + \frac{A_q(M)q}{(1-q)^{M+1}}$.
$\nabla^{1+k} SUM(N-1) = \nabla^k (N-1)^M = \sum_{j=0}^{k}(-1)^j \binom{k}{j}(N-1-j)^M$
$= \sum_{j=0}^{k}(-1)^j \binom{k}{j}\sum_{g=0}^{M}\binom{M}{g} N^g (j+1)^{M-g}(-1)^{M-g}$

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$= \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} (j+1)^{M-g}$.

We know: $S_2(M+1,k+1) = \frac{1}{(k+1)!} \sum_{j=0}^{k+1} (-1)^{j+k+1} \binom{k+1}{j} j^{M+1} = \frac{1}{k!} \sum\limits_{j=0}^{k} (-1)^{j+k} \binom{k}{j} (j+1)^M$.

$\nabla^{1+k} SUM(n) = \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1,k+1)k!$.

$(**) = \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k (q-1)^{M-k} \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1,k+1)k!$.

$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g} N^g (q-1)^g \binom{M}{g} \sum_{k=0}^{M} (q-1)^{M-k-g} S_2(M-g+1,k+1)k!$

$N = 0, \sum_{n=0}^{N-1} q^n n^M = 0 \rightarrow \sum_{k=0}^{M} (-1)^M (q-1)^{M-k} S_2(M+1,k+1)k! = -A_q(M) q (-1)^{M-1}$.

$qA_q(M) = \sum_{k=0}^{M} (q-1)^{M-k} S_2(M+1,k+1)k!, \quad qA_q(M-g) = \sum_{k=0}^{M-g} (q-1)^{M-g-k} S_2(M-g+1,k+1)k!$.

Theorem 6.2.
$A_q(M) = q^{-1} \sum_{k=0}^{M} (q-1)^{M-k} S_2(M+1,k+1)k!$. *Together with* $A_q(PS,PT)$, *there are four expressions.*
$\sum_{n=0}^{N-1} q^n n^M = \frac{q^{N+1}}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g} N^g (q-1)^g \binom{M}{g} A_q(M-g) + \frac{A_q(M)q}{(1-q)^{M+1}}$.

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