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Construction of Lyapunov Certificates for Oscillatory Systems

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Abstract

We develop a method for constructing Lyapunov functions via Semidefinite Programming (SDP) that certifies the stability of oscillatory systems with both Cartesian and angular variables. We utilize the theory of hybrid polynomials (also called mixed trigonometric-polynomials) introduced by Dumitrescu. We use this theory to convert Lyapunov and dual Lyapunov stability conditions for oscillatory systems into SDP problems. Solving these problems using standard convex programming solvers leads to expressions of Lyapunov densities and local Lyapunov functions for these systems, even without apriori knowing the invariant attracting set. To illustrate the applicability of our method, we consider the analysis of Kuramoto models and the state feedback design problem for an inverted pendulum on a cart. Specifically, we establish certificates of almost global synchronization (phase locking) for second-order Kuramoto models. The paper concludes by developing an SDP certificate that enables the design of a swing-up control for an inverted pendulum on a cart. For the analysis, we use our program `vSOS-hybrid`, based on CVX in MATLAB, openly available on GitHub.

Keywords: almost global synchronization; almost global stability; hybrid real-trigonometric polynomials; stability certificates; swing-up control

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1. Introduction

Dynamical systems on different types of manifolds appear as models of electric systems such as power grids [1,2], electronic systems such as Josephson junction arrays [3], mechanical systems such as pendulums on a cart [4], coupled systems of synchronizing units [5–7], and quantum dynamical systems [8]. Such systems usually admit both Cartesian and angular state variables.

Stability [9] and synchronization [10] are critical properties of these systems, as they determine whether the dynamics converges to a coherent operating regime or exhibit undesirable oscillations and loss of coordination. They, thus, determine the reliability, efficiency, and functionality of complex dynamical networks. For the same reason, designing a control to achieve a desired dynamics also becomes an important problem to consider for such systems [11–14].

In this paper, we consider a particular state space structure, a hypercylinder, namely the Cartesian product of an Euclidean space (for Cartesian variables) and a hypertorus (for angular variables), which serves as the state space of most of the dynamical systems mentioned above. In particular, we strive to ensure convergence of trajectories of such systems to some predetermined or unknown attractor by means of Lyapunov certificates in the form of hybrid polynomials [15] defined on the hypercylinder.

Lyapunov certificates ensuring local/global stability [16], or almost globally stability [17] are notoriously difficult to obtain in general. However, the research community has been able to overcome this challenge for certain classes of systems by employing Semidefinite Programming (SDP), which enables the construction of Lyapunov functions with the same "structure" as the class of the system. For

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instance, quadratic Lyapunov functions for linear systems on $\mathbb{R}^n$ [16], polynomial Lyapunov functions for polynomial vector fields on $\mathbb{R}^n$ [18], polynomial Lyapunov functions for recasted non-polynomial vector fields on $\mathbb{R}^n$ [19], and Lyapunov densities as rational functions for polynomial vector fields on $\mathbb{R}^n$ [20]. The construction of Lyapunov certificates (including dual-Lyapunov certificates) for systems on a hypertorus $\mathbb{T}^n$ was recently achieved by the authors in [21], where a stereographic projection is used to map the dynamics on $\mathbb{T}^n$ to a dynamical system on $\mathbb{R}^n$, and in [22], where trigonometric polynomials are used directly as Lyapunov certificates for trigonometric polynomial vector fields on $\mathbb{T}^n$. The latter approach, introduced in [22], can provide less conservative stability certification for systems on a hypertorus than the technique used in [21].

This paper extends the approach in [22] to systems on a hypercylinder using hybrid polynomials as Lyapunov certificates. Hybrid polynomials are proposed by Dumitrescu [15] as tools for the design of adjustable FIR filters and delay-independent absolute stability problems. This type of polynomials has been used to control discrete-time systems in [23] where it goes by the name "mixed trigonometric polynomials". However, to our knowledge, hybrid polynomials have not been leveraged for Lyapunov certification. Here, we obtain sufficient conditions in the form of an SDP problem and illustrate how hybrid certificate construction can be performed for dynamical systems defined on hypercylinders.

The contributions of this paper are as follows:

- We develop a novel method, based on semidefinite programming, for constructing Lyapunov densities (Theorem 1) and local Lyapunov-like functions (Theorem 2) for oscillatory systems.
- The construction of Lyapunov certificates can be achieved through any convex programming solver. We have developed a program `vSOS-hybrid` using CVX [24], which is openly available in GitHub [25].
- The stability results can be utilized to establish various synchronization behaviors of coupled oscillators as illustrated in Section 5.
- The methodology can be used to design control for oscillatory systems as showcased in Section 6.

In the sequel, we will present some preliminaries and the problem statement in Section 2, the theory of hybrid polynomials in Section 3, the main results for almost global stability (Theorem 1) and for local stability (Theorem 2) in Section 4. We apply the methodology to second-order Kuramoto models in Section 5 and to the swing-up control problem of an inverted pendulum system on a cart in Section 6.

2. Preliminaries and Problem Statement

Consider a dynamical system on the product space $\mathbb{R}^c \times \mathbb{T}^d$, where $\mathbb{R}^c$ denotes the c dimensional Euclidean space and $\mathbb{T}^d := [0, 2\pi)^d$ is the d dimensional hypertorus, given by

$$(\dot{\boldsymbol{\omega}}, \dot{\boldsymbol{\theta}}) = \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{\theta}) = \left(f^{(1)}(\boldsymbol{\omega}, \boldsymbol{\theta}), \ldots, f^{(c+d)}(\boldsymbol{\omega}, \boldsymbol{\theta})\right), \tag{1}$$

where the flow is defined for all time. An invariant set I is said to be almost globally stable for the system (1) if $\lim_{t\to\infty} (\boldsymbol{\omega}(t), \boldsymbol{\theta}(t)) \in I$ for almost all initial conditions $(\boldsymbol{\omega}(0), \boldsymbol{\theta}(0)) \in \mathbb{R}^c \times \mathbb{T}^d$. An equilibrium point $\boldsymbol{p} \in \mathbb{R}^c \times \mathbb{T}^d$ for the system (1) is said to be locally stable [16] on an open set U containing $\boldsymbol{p}$ if $\lim_{t\to\infty} (\boldsymbol{\omega}(t), \boldsymbol{\theta}(t)) \to \boldsymbol{p}$ for all $(\boldsymbol{\omega}(0), \boldsymbol{\theta}(0)) \in U$.

A sufficient condition for almost global stability is the existence of a Lyapunov density, or the so-called dual Lyapunov function, first introduced by Rantzer [17]. The theorem has been subject to several generalizations, as mentioned in the introduction. We present a recent form that accommodates almost global stability to invariant sets.

Lemma 1 (Adapted from [26, Theorem 4.2])**.** *Let $I \subseteq \mathbb{R}^c \times \mathbb{T}^d$ be a compact invariant set for the system* (1)*. If there exists a continuously differentiable function $\rho\colon \left(\mathbb{R}^c \times \mathbb{T}^d\right) \setminus I \to \mathbb{R}$ satisfying*

1. $\rho(\boldsymbol{x}) > 0$ *for almost every* $\boldsymbol{x} \in \left(\mathbb{R}^c \times \mathbb{T}^d\right) \setminus I$,

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2. ρ *is integrable away from I, that is, if* $I_\epsilon = \{\boldsymbol{x} \in \mathbb{R}^c \times \mathbb{T}^d : \min_{\boldsymbol{y}\in I} dist(\boldsymbol{x}, \boldsymbol{y}) < \epsilon\}$,

$$\int_{(I_\epsilon)^c} \rho(\boldsymbol{x})\, \mu(\mathrm{dx}) < \infty \text{ for all ffl} > 0,$$

3. $\operatorname{div}(\rho \boldsymbol{f})(\boldsymbol{x}) > 0$ *for almost every* $\boldsymbol{x} \in \left(\mathbb{R}^c \times \mathbb{T}^d\right) \setminus I$,

then almost all solutions of (1) *converge to I as $t \to \infty$.*

The original statement of Lemma 1 in [26] is about systems on $\mathbb{R}^c$, but the proof works for any normal topological space which is equipped with a $\sigma-$finite measure. Note that the condition *C3* implies that the integral of the density, ρ, increases along the flow of positive measure sets ([27], Theorem 3.1). Also, a density satisfying C3 inevitably has a singularity near I ([26], Remark 3).

Let $\mathbf{0}$ denote the vector of zeros of a suitable size. For local stability in an invariant set U, a sufficient condition is the existence of a local Lyapunov-like function on U (or on a larger set), as a consequence of LaSalle's invariance principle ([16], Theorem 4.4).

Lemma 2. *Given a vector field* $(\dot{\boldsymbol{\omega}}, \dot{\boldsymbol{\theta}}) = \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{\theta})$ *with* $\boldsymbol{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. *If* $\mathfrak{K}_{inv} \subseteq \mathbb{R}^c \times \mathbb{T}^d$ *is a compact invariant set containing the origin, and* $v : \mathbb{R}^c \times \mathbb{T}^d \to \mathbb{R}$ *is a continuously differentiable function such that* $-\operatorname{grad} v(\boldsymbol{\omega}, \boldsymbol{\theta}) \cdot \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{\theta})$ *is positive definite in* $\mathfrak{K}_{inv}$ *(meaning it is positive on* $\mathfrak{K}_{inv} \setminus \{\mathbf{0}\}$ *and* 0 *at the origin), then every solution starting in* $\mathfrak{K}_{inv}$ *approaches to the origin.*

Proof. Since $\dot{v}(\boldsymbol{\omega}, \boldsymbol{\theta}) = \operatorname{grad} v(\boldsymbol{\omega}, \boldsymbol{\theta}) \cdot \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{\theta})$ is negative definite in the invariant set $\mathfrak{K}_{inv}$, $\{(\boldsymbol{\omega}, \boldsymbol{\theta}) \in \mathfrak{K}_{inv} : \dot{v}(\boldsymbol{\omega}, \boldsymbol{\theta}) = \mathbf{0}\} = \{\mathbf{0}\}$, hence the result follows by the LaSalle's invariance principle. □

Lyapunov densities and functions are well-established tools for certifying stability properties of dynamical systems. However, as mentioned in the introduction, for vector fields that admit matrix representations (for example, Gram matrix representations in the case of trigonometric polynomial vector fields), the sufficient conditions in Lemmas 1 and 2 reduce to solving linear matrix inequalities (LMIs). Consequently, we can employ SDP solvers to construct both Lyapunov densities and local Lyapunov functions. The main question addressed in this paper is therefore: *what is an adequately broad class of systems on* $\mathbb{R}^c \times \mathbb{T}^d$ *for which each system admits a tractable matrix representation?*

A natural choice is the class of systems representable by hybrid polynomials (Section 3), originally introduced by Dumitrescu [15], as these carry a Gram matrix representation (Section 3.1). As a consequence of Stone-Weierstrass Theorem for product spaces, given any compact set $K \subset \mathbb{R}^c \times \mathbb{T}^d$, the set of restrictions of hybrid polynomials (to K) is dense in the set of smooth functions on K. We will formally define these polynomials in Section 3. Before that, we introduce some notations that will be used throughout the paper: $\mathbf{0}$ (respectively $\mathbf{1}$) for a vector of zeroes (respectively, ones).

3. Hybrid Polynomials

A hybrid polynomial $v(\boldsymbol{\omega}, \boldsymbol{\theta})$ on $\mathbb{R}^c \times \mathbb{T}^d$ is defined as

$$\begin{aligned} v(\boldsymbol{\omega}, \boldsymbol{\theta}) &= \sum_{\boldsymbol{\eta}=\mathbf{0}}^{\boldsymbol{deg}_{v_\omega}} \sum_{\boldsymbol{k}=-\boldsymbol{deg}_{v_\theta}}^{\boldsymbol{deg}_{v_\theta}} v_{\boldsymbol{\eta},\boldsymbol{k}}\, \boldsymbol{\omega}^{\boldsymbol{\eta}}\, \mathrm{e}^{-i\,\boldsymbol{k}\cdot\boldsymbol{\theta}} \\ &:= \sum_{\eta_1=0}^{\boldsymbol{deg}_{v_\omega}(1)} \cdots \sum_{\eta_c=0}^{\boldsymbol{deg}_{v_\omega}(c)} \sum_{k_1=-\boldsymbol{deg}_{v_\theta}(1)}^{\boldsymbol{deg}_{v_\theta}(1)} \cdots \sum_{k_d=-\boldsymbol{deg}_{v_\theta}(d)}^{\boldsymbol{deg}_{v_\theta}(d)} v_{\eta_1,\dots,\eta_c,k_1,\dots,k_d}\, \omega_1^{\eta_1} \dots \omega_c^{\eta_c}\, \mathrm{e}^{-i\,(k_1\theta_1+\dots+k_d\theta_d)}, \end{aligned} \tag{2}$$

where the *hybrid polynomial coefficients* satisfy $v_{\boldsymbol{\eta},-\boldsymbol{k}} = \overline{v_{\boldsymbol{\eta},\boldsymbol{k}}}$ so as to make the polynomial real-valued, [15]. Here, the nonnegative integers $\boldsymbol{deg}_{v_\omega}$ and $\boldsymbol{deg}_{v_\theta}$ are taken as minimal and $\boldsymbol{deg}_v = (\boldsymbol{deg}_{v_\omega}, \boldsymbol{deg}_{v_\omega})$ is called the degree of hybrid polynomial v.

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3.1. Gram Matrix Representations

Hybrid polynomials have a matrix representation with respect to any basis of hybrid monomials [15]. A standard basis for a hybrid polynomial of degree $\boldsymbol{deg}_v = (\boldsymbol{deg}_{v_\omega}, \boldsymbol{deg}_{v_\omega})$ is given as columns of the vector

$$\psi_{\boldsymbol{n}_v}(\boldsymbol{\omega},\boldsymbol{\theta}) \quad = \quad \left(\bigotimes_{j=c}^{1}\left[1\,\omega_j\,\ldots\,\omega_j^{n_{v_\omega}(j)}\right]\right)\otimes\left(\bigotimes_{j=d}^{1}\left[1\,\mathrm{e}^{i\,\theta_j}\,\ldots\,\mathrm{e}^{i\,n_{v_\theta}(j)\,\theta_j}\right]\right), \tag{3}$$

where $\boldsymbol{n}_{v_\theta} = \boldsymbol{deg}_{v_\theta}$ and $\boldsymbol{n}_{v_\omega} = \lceil \boldsymbol{deg}_{v_\omega}/2 \rceil$; and $\boldsymbol{n}_v = (\boldsymbol{n}_{v_\omega}, \boldsymbol{n}_{v_\omega})$ is called the *representation-size vector* of v. The Kronecker product notation is to be read as the lower index (that is c or d, respectively) appearing in the leftmost position. This makes (3) have the vector $\left[1\,\omega_c\,\ldots\,\omega_c^{n_{v_\omega}(c)}\right]$ at the leftmost position and $\left[1\,\mathrm{e}^{i\,\theta_1}\,\ldots\,\mathrm{e}^{i\,n_{v_\theta}(1)\,\theta_1}\right]$ at the rightmost position. A Gram matrix representation V [15] associated with the hybrid polynomial $v(\boldsymbol{\omega},\boldsymbol{\theta})$ is defined as a Hermitian matrix of size $[\![\boldsymbol{n}_v]\!] := \prod_{k=1}^{c+d}(n_v(k)+1) = \left(\prod_{i=1}^{c}(n_{v_\omega}(i)+1)\right)\left(\prod_{j=1}^{d}(n_{v_\theta}(j)+1)\right)$ satisfying

$$v(\boldsymbol{\omega},\boldsymbol{\theta}) = \psi_{\boldsymbol{n}_v}(\boldsymbol{\omega},\boldsymbol{\theta})^{\dagger}\,V\,\psi_{\boldsymbol{n}_v}(\boldsymbol{\omega},\boldsymbol{\theta}). \tag{4}$$

Assumption 1. *Each component $f^{(l)} : \mathbb{R}^c \times \mathbb{T}^d \to \mathbb{R}$ of the vector field $\boldsymbol{f}$ in (1) has a hybrid polynomial expansion*

$$f^{(l)}(\boldsymbol{\omega},\boldsymbol{\theta}) = \sum_{\boldsymbol{\eta}=\boldsymbol{0}}^{\boldsymbol{deg}_{f_\omega}}\ \sum_{\boldsymbol{k}=-\boldsymbol{deg}_{f_\theta}}^{\boldsymbol{deg}_{f_\theta}} f_{\boldsymbol{\eta},\boldsymbol{k}}^{(l)}\,\boldsymbol{\omega}^{\boldsymbol{\eta}}\,\mathrm{e}^{-i\,\boldsymbol{k}\cdot\boldsymbol{\theta}}. \tag{5}$$

This class is polynomial in Euclidean variables and trigonometric in toroidal variables, and it frequently appears in systems with both angular and non-angular variables.

3.2. Trace Parametrization

An important characteristic of hybrid polynomials is their trace parametrization [15]: every hybrid coefficient can be expressed as the trace of any Gram matrix representation via a tensor product of Hankel factors (corresponding to the Euclidean part) and Toeplitz factors (corresponding to the toroidal part). This representation places the polynomial within a framework favorable to semidefinite programming.

The k^{th} elementary Toeplitz matrix of size n, denoted by T_k^n, is a $(0,1)-$matrix with ones on the k^{th} diagonal, meaning $(T_k^n)_{(i,j)} = 1$ if and only if $j - i = k$. The k^{th} elementary Hankel matrix of size n, denoted by B_k^n, is a $(0,1)-$matrix with ones on the k^{th} anti-diagonal, meaning $(B_k^n)_{(i,j)} = 1$ if and only if $i + j - 2 = k$. Define

$$H_{\boldsymbol{\eta},\boldsymbol{k}}^{\boldsymbol{n}_{v_\omega}+\boldsymbol{1},\boldsymbol{n}_{v_\theta}+\boldsymbol{1}} = \left(\bigotimes_{j=c}^{1} B_{\eta_j}^{n_{v_\omega}(j)+1}\right)\otimes\left(\bigotimes_{j=d}^{1} T_{k_j}^{n_{v_\theta}(j)+1}\right). \tag{6}$$

The relation between the coefficients of the hybrid polynomial (2) and the elements of the associated Gram representation (4) is given by trace parametrization [15] as

$$v_{\boldsymbol{\eta},\boldsymbol{k}} = \mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}(V) := \mathrm{tr}\left[H_{\boldsymbol{\eta},\boldsymbol{k}}^{\boldsymbol{n}_{v_\omega}+\boldsymbol{1},\boldsymbol{n}_{v_\theta}+\boldsymbol{1}}\,V\right], \tag{7}$$

where the term $\boldsymbol{n}_{v_\omega}+\boldsymbol{1}, \boldsymbol{n}_{v_\theta}+\boldsymbol{1}$ in the definition of the linear operator $\mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}(\cdot)$ is determined by the representation-size vector $\boldsymbol{n}_v := (\boldsymbol{n}_{v_\omega}, \boldsymbol{n}_{v_\theta})$ of V.

3.3. Gram Representation of Partial Derivatives of Hybrid Polynomials

For a hybrid polynomial $v(\boldsymbol{\omega},\boldsymbol{\theta})$, the partial derivatives $v_{\omega_l}(\boldsymbol{\omega},\boldsymbol{\theta})$ and $v_{\theta_l}(\boldsymbol{\omega},\boldsymbol{\theta})$ are hybrid polynomial. In this section, we will obtain a Gram representation of derivatives in terms of Gram representations of the original polynomial. Along with the trace parametrization property, these rules will be essential in converting the Lyapunov density condition to matrix inequalities. We define the *derivative matrices* as

$$D_{\theta_l} := \left(\bigotimes_{j=c}^{1} I_{n_{v_\omega}(j)+1}\right) \otimes \left(\bigotimes_{j=d}^{l+1} I_{n_{v_\theta}(j)+1}\right) \otimes \operatorname{diag}(0,i,2i,\dots,n_{v_\theta}(l)\,i) \otimes \left(\bigotimes_{j=l-1}^{1} I_{n_{v_\theta}(j)+1}\right),$$

$$D_{\omega_l} := \left(\bigotimes_{j=c}^{l+1} I_{n_{v_\omega}(j)+1}\right) \otimes \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n_{v_\omega}(l) & 0 \end{pmatrix} \otimes \left(\bigotimes_{j=l-1}^{1} I_{n_{v_\omega}(j)+1}\right) \otimes \left(\bigotimes_{j=d}^{1} I_{n_{v_\theta}(j)+1}\right).$$

The sizes of tensor factors of the derivative matrices coincide with the sizes of the corresponding vectors in (3). Since $(A_1 \otimes \ldots \otimes A_n)(\boldsymbol{w_1} \otimes \ldots \otimes \boldsymbol{w_n}) = A_1\boldsymbol{w_1} \otimes \ldots \otimes A_n\boldsymbol{w_n}$, we can conclude that $\frac{\partial \psi_{\boldsymbol{n_v}}}{\partial \theta_l} = D_{\theta_l}\,\psi_{\boldsymbol{n_v}}$ and $\frac{\partial \psi_{\boldsymbol{n_v}}}{\partial \omega_l} = D_{\omega_l}\,\psi_{\boldsymbol{n_v}}$. Since $v = \psi_{\boldsymbol{n_v}}^\dagger\, V\, \psi_{\boldsymbol{n_v}}$, we have $v_{\theta_l} = \frac{\partial \psi_{\boldsymbol{n_v}}^\dagger}{\partial \theta_l}\, V\, \psi_{\boldsymbol{n_v}} + \psi_{\boldsymbol{n_v}}^\dagger\, V\, \frac{\partial \psi_{\boldsymbol{n_v}}}{\partial \theta_l} = \left(D_{\theta_l}\psi_{\boldsymbol{n_v}}\right)^\dagger V\,\psi_{\boldsymbol{n_v}} + \psi_{\boldsymbol{n_v}}^\dagger\, V\, D_{\theta_j}\psi_{\boldsymbol{n_v}} = \psi_{\boldsymbol{n_v}}^\dagger \left(D_{\theta_l}^\dagger\, V + V D_{\theta_l}\right)\psi_{\boldsymbol{n_v}}$. We, thus, have the result.

Proposition 1. *A Gram representation of $v_{\theta_l}(\boldsymbol{\omega},\boldsymbol{\theta})$ is given by $D_{\theta_l}^\dagger\, V + V D_{\theta_l}$. Moreover, a Gram representation of $v_{\omega_l}(\boldsymbol{\omega},\boldsymbol{\theta})$ is given by $D_{\omega_l}^\dagger\, V + V D_{\omega_l}$.*

3.4. Gram Representation of Product of Hybrid Polynomials

The product of hybrid polynomials is also a hybrid polynomial. In this section, we will obtain a Gram representation of the product in terms of Gram representations of individual polynomials. Let $v(\boldsymbol{\omega},\boldsymbol{\theta})$ and $p(\boldsymbol{\omega},\boldsymbol{\theta})$ be two hybrid polynomials as defined in (2) with representation-size vectors $\boldsymbol{n_v}$ and $\boldsymbol{n_p}$. Define a $(0,1)$–matrix X as

$$X_{i,j} = \begin{cases} 1, & \text{if } (\psi_{\boldsymbol{n_v}} \otimes \psi_{\boldsymbol{n_p}})_i = (\psi_{\boldsymbol{n_v}+\boldsymbol{n_p}})_j \\ 0, & \text{otherwise} \end{cases}. \tag{8}$$

The matrix satisfies $\psi_{\boldsymbol{n_v}} \otimes \psi_{\boldsymbol{n_p}} = X\,\psi_{\boldsymbol{n_v}+\boldsymbol{n_p}}$, hence $v(\boldsymbol{\omega},\boldsymbol{\theta})p(\boldsymbol{\omega},\boldsymbol{\theta}) = (\psi_{\boldsymbol{n_v}}^\dagger\, V\,\psi_{\boldsymbol{n_v}})(\psi_{\boldsymbol{n_p}}^\dagger\, P\,\psi_{\boldsymbol{n_p}}) = (\psi_{\boldsymbol{n_v}} \otimes \psi_{\boldsymbol{n_p}})^\dagger (V \otimes P)(\psi_{\boldsymbol{n_v}} \otimes \psi_{\boldsymbol{n_p}}) = \psi_{\boldsymbol{n_v}+\boldsymbol{n_p}}^\dagger\, X^\dagger (V \otimes P) X\,\psi_{\boldsymbol{n_v}+\boldsymbol{n_p}}$, and we have the following result.

Proposition 2. *A Gram representation of the product of hybrid polynomials $v(\boldsymbol{\omega},\boldsymbol{\theta})$ and $p(\boldsymbol{\omega},\boldsymbol{\theta})$ is given as $X^\dagger(V \otimes P)X$, where X is the matrix defined in (8).*

The trace parametrization property (7), Propositions 1 and 2 will be utilized in the following section to obtain semidefinite programming certificates.

4. Obtaining SDP Certificates Using Hybrid Polynomials

In this section, we obtain semidefinite programming certificates to establish almost global stability and local stability of oscillatory systems.

4.1. Certificates for Almost Global Stability

We now present a sufficient condition for obtaining a Lyapunov density of (1) in terms of a hybrid polynomial. We set the notation $\mathbb{B}_\varepsilon := \{\boldsymbol{\omega} \in \mathbb{R}^c \colon \|\boldsymbol{\omega}\| \le \varepsilon\}$.

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Lemma 3. *Given a vector field (1), if there exist nonzero hybrid polynomial $v(\boldsymbol{\omega},\boldsymbol{\theta}) \geq 0$; nonzero hybrid polynomial $w(\boldsymbol{\omega},\boldsymbol{\theta}) \geq 0$; and $R > 0$ such that*

$$w(\boldsymbol{\omega},\boldsymbol{\theta}) = -\operatorname{grad} v(\boldsymbol{\omega},\boldsymbol{\theta}) \cdot \boldsymbol{f}(\boldsymbol{\omega},\boldsymbol{\theta}) + v(\boldsymbol{\omega},\boldsymbol{\theta}) \operatorname{div} \boldsymbol{f}(\boldsymbol{\omega},\boldsymbol{\theta}), \tag{9}$$

and $v^{-1}(0) \subset \mathbb{B}_R \times \mathbb{T}^d$ is an invariant set under the flow of (1); $1/v(\boldsymbol{\omega},\boldsymbol{\theta})$ is integrable on $\mathbb{B}_r^c \times \mathbb{T}^d$ for some $r > R$, then almost all solutions of (1) converge to $v^{-1}(0)$ as $t \to \infty$.

Proof. We will use Lemma 1 to prove the result. We first prove that $\mu(w^{-1}(0)) = 0$ using a proof inspired by [28]. Since w is a continuous function, $w^{-1}(0)$ is Lebesgue measurable. Thus, its characteristic function $\chi_{w^{-1}(0)}$ is nonnegative and measurable. Note that $\mathbb{R}^c \times \mathbb{T}^d$ is a $\sigma-$finite space. Without loss of generality, fix $\omega_1, \ldots, \omega_c \in \mathbb{R}$ and $\theta_1, \ldots, \theta_{d-1} \in \mathbb{T}$, then $w(\omega_1, \ldots, \omega_c, \theta_1, \ldots, \theta_{d-1}, x)$ is a nonzero trigonometric polynomial of degree $n_{v_\theta}(d)$ and has at most $2n_{v_\theta}(d) + 1$ zeroes. Hence, the set $\{x \in \mathbb{T}\colon\ (\omega_1, \ldots, \omega_c, \theta_1, \ldots, \theta_{d-1}, x) \in w^{-1}(0)\}$ has measure 0. Thus by Tonelli's theorem, $\mu(w^{-1}(0)) = 0$. Let $I = v^{-1}(0)$. Taking $\rho(\boldsymbol{\omega},\boldsymbol{\theta}) = 1/v(\boldsymbol{\omega},\boldsymbol{\theta})$, we have

$$\operatorname{div}(\rho \boldsymbol{f}) = \frac{w(\boldsymbol{\omega},\boldsymbol{\theta})}{v(\boldsymbol{\omega},\boldsymbol{\theta})^2} > 0 \ \text{ for a.e. } (\boldsymbol{\omega},\boldsymbol{\theta}) \in I^c$$

where the equality follows due to equation (9) and positivity follows since w is positive almost everywhere, hence C3 is satisfied. Also $\rho(\boldsymbol{\omega},\boldsymbol{\theta}) > 0$ for every $(\boldsymbol{\omega},\boldsymbol{\theta}) \in I^c$ since v is nonnegative, hence C1 is satisfied. For any $\epsilon > 0$, we have

$$\iint_{\mathbb{R}^c \times \mathbb{T}^d \setminus I_\epsilon} \rho(\boldsymbol{\omega},\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\omega}\, \mathrm{d}\boldsymbol{\theta} = \iint_{(\mathbb{T}^d \times \mathbb{B}_r^c) \setminus I_\epsilon} \left| \frac{1}{v(\boldsymbol{\omega},\boldsymbol{\theta})} \right| \mathrm{d}\boldsymbol{\omega}\, \mathrm{d}\boldsymbol{\theta} + \iint_{(\mathbb{T}^d \times \mathbb{B}_r) \setminus I_\epsilon} \left| \frac{1}{v(\boldsymbol{\omega},\boldsymbol{\theta})} \right| \mathrm{d}\boldsymbol{\omega}\, \mathrm{d}\boldsymbol{\theta} < \infty,$$

due to the integrability hypothesis and the integrability of a positive function on a compact set. Hence, C2 is satisfied. Since all hypotheses of Lemma 1 are satisfied, the result follows. □

Using Gram matrix representations of hybrid polynomials, Lemma 3 can be used to obtain sufficient conditions for the existence of a hybrid polynomial Lyapunov density as an SDP problem. We now present the main theorem of this section.

Theorem 1. *Let $\boldsymbol{n}_{v_\omega} \in \mathbb{Z}^c_{\geq 0}$; $\boldsymbol{n}_{v_\theta} \in \mathbb{Z}^d_{\geq 0}$; $\psi_{\boldsymbol{n}_v}$ as in (3); $\mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}(\cdot)$ as defined in (7); D_{θ_l}, D_{ω_l} and X as defined in Section 3; and a vector field (1) satisfying Assumption 1 with degree not more than $(2\boldsymbol{n}_{f_\omega}, \boldsymbol{n}_{f_\theta})$. If there exist*

1. *a nonzero Hermitian matrix $V \geq 0$ of size $[\![\boldsymbol{n}_v]\!]$,*
2. *a nonzero Hermitian matrix $W \geq 0$ of size $[\![\boldsymbol{n}_v + \boldsymbol{n}_f]\!]$,*

such that the zero set of $v(\boldsymbol{\omega},\boldsymbol{\theta}) = \psi_{\boldsymbol{n}_v}(\boldsymbol{\omega},\boldsymbol{\theta})^\dagger\, V\, \psi_{\boldsymbol{n}_v}(\boldsymbol{\omega},\boldsymbol{\theta})$ is forward-invariant under the flow of (1), $v^{-1}(0) \subset \mathbb{B}_R \times \mathbb{T}^d$, $1/v$ is integrable on $\mathbb{B}_r^c \times \mathbb{T}^d$ for some $r > R$, and

$$\mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}(W) \;=\; \mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}\left(X^\dagger \left(\sum_{l=1}^{c+d} V \otimes \left(D_{z_l}^\dagger F^{(l)} + F^{(l)} D_{z_l} \right) - \sum_{l=1}^{c+d} \left(D_{z_l}^\dagger V + V D_{z_l} \right) \otimes F^{(l)} \right) X \right), \tag{10}$$

for all $\boldsymbol{0} \leq \boldsymbol{\eta} \leq 2(\boldsymbol{n}_{f_\omega} + \boldsymbol{n}_{v_\omega})$ and $-(\boldsymbol{n}_{f_\theta} + \boldsymbol{n}_{v_\theta}) \leq \boldsymbol{k} \leq \boldsymbol{n}_{f_\theta} + \boldsymbol{n}_{v_\theta}$, where $z_i = \omega_i$, when $1 \leq i \leq c$ and $z_i = \theta_{i-c}$, when $c < i < c + d$, then almost all solutions of (1) converge to $v^{-1}(0)$ as $t \to \infty$.

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Proof. Using trace parametrization (7) followed by Propositions 1 and 2, we have

$$\begin{aligned} w(\boldsymbol{\omega},\boldsymbol{\theta}) &= \psi^{\dagger}_{\boldsymbol{n}_v+\boldsymbol{n}_f} W \psi_{\boldsymbol{n}_v+\boldsymbol{n}_f} \\ &= \psi^{\dagger}_{\boldsymbol{n}_v+\boldsymbol{n}_f} \left(X^{\dagger} \left(\sum_{l=1}^{c+d} V \otimes \left(D^{\dagger}_{z_l} F^{(l)} + F^{(l)} D_{z_l} \right) - \sum_{l=1}^{c+d} \left(D^{\dagger}_{z_l} V + V D_{z_l} \right) \otimes F^{(l)} \right) X \right) \psi_{\boldsymbol{n}_v+\boldsymbol{n}_f} \\ &= \sum_{l=1}^{c+d} \left(v(\boldsymbol{\omega},\boldsymbol{\theta})\, f^{(l)}_{z_l}(\boldsymbol{\omega},\boldsymbol{\theta}) - v_{z_l}(\boldsymbol{\omega},\boldsymbol{\theta})\, f^{(l)}(\boldsymbol{\omega},\boldsymbol{\theta}) \right) \\ &= -\operatorname{grad} v(\boldsymbol{\omega},\boldsymbol{\theta}) \cdot \boldsymbol{f}(\boldsymbol{\omega},\boldsymbol{\theta}) + v(\boldsymbol{\omega},\boldsymbol{\theta}) \operatorname{div} \boldsymbol{f}(\boldsymbol{\omega},\boldsymbol{\theta}). \end{aligned}$$

Hence, (9) is satisfied for almost every $(\boldsymbol{\omega},\boldsymbol{\theta}) \in \left(\mathbb{R}^c \times \mathbb{T}^d\right) \setminus v^{-1}(0)$. Since $v^{-1}(0) \subset \mathbb{B}_R \times \mathbb{T}^d$ and $1/v$ is integrable on $\mathbb{B}^c_r \times \mathbb{T}^d$ for some $r > R$, all hypotheses of Lemma 3 are satisfied and the result folows. □

Theorem 1 is a novel method to construct Lyapunov density for oscillatory systems. The integrability of $\rho(\boldsymbol{\omega},\boldsymbol{\theta}) = 1/v(\boldsymbol{\omega},\boldsymbol{\theta})$, on $\mathbb{B}^c_r \times \mathbb{T}^d$ for some $r > R$, cannot be imposed as an SDP and has to be manually checked for. The condition, however, can be simplified when there is exactly one real variable, as given below.

Remark 1 (Integrability of ρ for $c = 1$)**.** *For systems on $\mathbb{R} \times \mathbb{T}^d$, the hybrid polynomial $v(\omega,\boldsymbol{\theta})$ obtained via Theorem 1 has the form*

$$v(\omega,\boldsymbol{\theta}) \quad = \quad \omega^L \left(v_L(\boldsymbol{\theta}) + \sum_{l=1}^{L} \frac{1}{\omega^l} v_{L-l}(\boldsymbol{\theta}) \right) \tag{11}$$

where $L = deg_{v_\omega}$. Since $v(\omega,\boldsymbol{\theta}) \geq 0$ and $v^{-1}(0) \subset B_R \times \mathbb{T}^d$, it follows that $v_L(\boldsymbol{\theta}) > 0$ for all $\boldsymbol{\theta} \in \mathbb{T}^d$. Choose $r > R$ such that

$$\frac{1}{2}|v_L(\boldsymbol{\theta})| \geq \left| \sum_{l=1}^{L} \frac{1}{\omega^l} v_{L-l}(\boldsymbol{\theta}) \right|, \quad \forall\, (\omega,\boldsymbol{\theta}) \in [-r,r]^c \times \mathbb{T}^d.$$

Using $\left|v(\boldsymbol{\omega},\boldsymbol{\theta})/\omega^L\right| \geq |v_L(\boldsymbol{\theta})| - \left|\sum_{l=1}^{L} \frac{1}{\omega^l} v_{L-l}(\boldsymbol{\theta})\right| \geq |v_L(\boldsymbol{\theta})|/2$, we have

$$\int_{\mathbb{T}^d} \int_{\mathbb{B}^c_r} \left| \frac{1}{v(\omega,\boldsymbol{\theta})} \right| \mathrm{d}\omega\, \mathrm{d}\boldsymbol{\theta} \quad = \quad \int_{\mathbb{T}^d} \int_{\mathbb{B}^c_r} \left| \frac{2}{\omega^L v_L(\boldsymbol{\theta})} \right| \mathrm{d}\omega\, \mathrm{d}\boldsymbol{\theta} = K \int_{\mathbb{B}^c_r} \left| \frac{1}{\omega^L} \right| \mathrm{d}\omega,$$

which is integrable if and only if $L = deg_{v_\omega} > 1$. Thus, for systems on $\mathbb{R} \times \mathbb{T}^d$, the integrability of $\rho = 1/v$ sufficiently far away from the zero set is automatically established if $deg_{v_\omega} > 1$.

In the following section, we obtain a result to construct local/global LaSalle's function for a given oscillatory system.

4.2. Certificates for Local Stability

The local stability of (1) can be established using LaSalle's Invariance Principle (Lemma 2). We now discuss certain domains $\mathcal{D}$ that can be expressed as the common region of positivity of a fixed set of hybrid polynomials. Hybrid polynomials that are positive in such domains have a characterization in terms of the fixed set of hybrid polynomials. Furthermore, this characterization can be used to find a continuously differentiable function v such that $\dot{v}$ is negative definite in this domain by using semidefinite programming.

Lemma 4 (adapted from Appendix C of [15]). *Let q_ℓ be hybrid polynomials for $\ell = 1,\ldots,\mathscr{L}-1$ and $q_{\mathscr{L}}(\boldsymbol{\omega},\boldsymbol{\theta}) = \rho^2 - \omega_1^2 - \ldots - \omega_c^2$. Let $\mathcal{D}$ be the bounded domain*

$$\mathcal{D} = \{(\boldsymbol{\omega},\boldsymbol{\theta}) \in \mathbb{R}^c \times \mathbb{T}^d : \; q_\ell(\boldsymbol{\omega},\boldsymbol{\theta}) \geq 0, \; \ell = 1,\ldots,\mathscr{L}\}. \tag{12}$$

If a hybrid polynomial $r(\boldsymbol{\omega},\boldsymbol{\theta})$ is positive on $\mathcal{D}$ (that is, $r(\boldsymbol{\omega},\boldsymbol{\theta}) > 0$ for all $(\boldsymbol{\omega},\boldsymbol{\theta}) \in \mathcal{D}$), then there exist sum-of-squares polynomials $s_\ell(\boldsymbol{\omega},\boldsymbol{\theta})$ for $\ell = 0,\ldots,\mathscr{L}$ such that

$$r(\boldsymbol{\omega},\boldsymbol{\theta}) = s_0(\boldsymbol{\omega},\boldsymbol{\theta}) + \sum_{\ell=1}^{\mathscr{L}} s_\ell(\boldsymbol{\omega},\boldsymbol{\theta}) q_\ell(\boldsymbol{\omega},\boldsymbol{\theta}). \tag{13}$$

If the polynomials $r(\boldsymbol{\omega},\boldsymbol{\theta})$ and $q_\ell(\boldsymbol{\omega},\boldsymbol{\theta})$ have real coefficients, then the sum-of-squares polynomials $s_\ell(\boldsymbol{\omega},\boldsymbol{\theta})$ also have real coefficients.

We now provide a sufficient SDP condition for local stability of a vector field in the set $\mathcal{D}$ defined in (12).

Theorem 2. *Given a dynamical system $(\dot{\boldsymbol{\omega}},\dot{\boldsymbol{\theta}}) = \boldsymbol{f}(\boldsymbol{\omega},\boldsymbol{\theta})$ satisfying $\boldsymbol{f}(\mathbf{0},\mathbf{0}) = \mathbf{0}$, where each component function $f^{(l)} : \mathbb{R}^c \times \mathbb{T}^d \to \mathbb{R}$ is a hybrid polynomial (5); q_ℓ are hybrid polynomials of degree at most $\boldsymbol{n_q}$ with Gram representations Q_ℓ for $\ell = 1,\ldots,\mathscr{L}$ with $q_{\mathscr{L}}(\boldsymbol{\omega},\boldsymbol{\theta}) = \rho^2 - \omega_1^2 - \ldots - \omega_c^2$; $\mathcal{D}$ is the bounded domain (12); $\boldsymbol{n_v} \geq \boldsymbol{n_q}$; $\mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}(\cdot)$ as defined in (7); D_{θ_l}, D_{ω_l} and X as defined in Section 3. If there exist*

1. *a Hermitian matrix V of size $[\![\boldsymbol{n_v}]\!]$;*
2. *a Hermitian matrix $S_0 \geq 0$ of size $[\![\boldsymbol{n_v}+\boldsymbol{n_f}]\!]$; and Hermitian matrices $S_\ell \geq 0$ of sizes $[\![\boldsymbol{n_v}+\boldsymbol{n_f}-\boldsymbol{n_q}]\!]$ for each $\ell \in \{1,\ldots,\mathscr{L}\}$ such that*
   - (a) *$\psi_{\boldsymbol{n_v}+\boldsymbol{n_f}}(\mathbf{0},\mathbf{0})$ is an eigenvector corresponding to 0 eigenvalue for S_0^w,*
   - (b) *$\psi_{\boldsymbol{n_v}+\boldsymbol{n_f}-\boldsymbol{n_q}}(\mathbf{0},\mathbf{0})$ is an eigenvector corresponding to 0 eigenvalue for each S_ℓ^w,*
   - (c) *at least one of the S_ℓ^w has nullity 1; and*

$$\mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}\left(S_0^w + \sum_{\ell=1}^{L} X^\dagger (S_\ell^w \otimes Q_\ell) X\right) = -\mathscr{H}_{\boldsymbol{\eta},\boldsymbol{k}}\left(\sum_{l=1}^{c+d} X^\dagger \left(\left(D_{z_l}^\dagger V + V D_{z_l}\right) \otimes F^{(l)}\right) X\right) \tag{14}$$

for each $-(\boldsymbol{n_{v_\theta}}+\boldsymbol{n_{f_\theta}}) \leq \boldsymbol{k} \leq \boldsymbol{n_{v_\theta}}+\boldsymbol{n_{f_\theta}}$ and $\mathbf{0} \leq \boldsymbol{\eta} \leq 2\,(\boldsymbol{n_{v_\omega}}+\boldsymbol{n_{f_\omega}})$, and where $z_i = \omega_i$, when $1 \leq i \leq c$ and $z_i = \theta_{i-c}$, when $c < i < c+d$; then all solutions of system (1) originating in any compact invariant subset $\mathfrak{K}_{inv} \subseteq \mathcal{D}$ converge to the origin.

Proof. Using equation (14), trace parametrization (7) along with Propositions 1 and 2, we have

$$\begin{aligned}
w(\boldsymbol{\omega},\boldsymbol{\theta}) \;&:=\; s_0^w(\boldsymbol{\omega},\boldsymbol{\theta}) + \sum_{\ell=1}^{\mathscr{L}} s_\ell^w(\boldsymbol{\omega},\boldsymbol{\theta}) q_\ell(\boldsymbol{\omega},\boldsymbol{\theta}) \\
&=\; \psi_{\boldsymbol{n_v}+\boldsymbol{n_f}}^\dagger \left(S_0^w + \sum_{\ell=1}^{L} X^\dagger (S_\ell^w \otimes Q_\ell) X\right) \psi_{\boldsymbol{n_v}+\boldsymbol{n_f}} \\
&=\; \psi_{\boldsymbol{n_v}+\boldsymbol{n_f}}^\dagger \left(-\sum_{l=1}^{c+d} X^\dagger \left(\left(D_{z_l}^\dagger V + V D_{z_l}\right) \otimes F^{(l)}\right) X\right) \psi_{\boldsymbol{n_v}+\boldsymbol{n_f}} \\
&=\; -\sum_{l=1}^{c+d} v_{z_l}(\boldsymbol{\omega},\boldsymbol{\theta}) f^{(l)}(\boldsymbol{\omega},\boldsymbol{\theta}) \\
&=\; -\operatorname{grad} v(\boldsymbol{\omega},\boldsymbol{\theta}) \cdot \boldsymbol{f}(\boldsymbol{\omega},\boldsymbol{\theta})
\end{aligned}$$

Note that $w(\boldsymbol{\omega},\boldsymbol{\theta}) \geq 0$ on $\mathcal{D}$ by construction; $s_\ell(0,0) = 0$ by the eigenvalue condition; and at least one $s_\ell(\boldsymbol{\omega},\boldsymbol{\theta}) > 0$ elsewhere due to the nullity condition elsewhere. Hence, $w(\boldsymbol{\omega},\boldsymbol{\theta})$ is positive definite. The result follows from LaSalle's invariance principle (Lemma 2). □

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In the following sections, we exhibit the utility of the semidefinite programming certificates obtained in this section via some examples of second-order coupled oscillators (in Section 5) and via a feedback design problem for an inverted pendulum on a cart (in Section 6).

5. Application to Kuramoto Models

In this section, we will apply the results obtained in Section 4 via our `vSOS-hybrid` program, [25], which was built using CVX in MATLAB R2025b, to study synchronization properties of first and second-order Kuramoto models. A system (1) is said to exhibit

- *almost global synchronization* if for almost all $(\boldsymbol{\omega}(0), \boldsymbol{\theta}(0)) \in \mathbb{R}^c \times \mathbb{T}^d$, we have $\lim_{t\to\infty}(\theta_i(t) - \theta_j(t)) = 0$, for all $i, j \in \{1, \ldots, d\}$, and $\lim_{t\to\infty}(\omega_i(t) - \omega_j(t)) = 0$, for all $i, j \in \{1, \ldots, c\}$.
- *local synchronization* if there exists a neighborhood $U \subset \mathbb{R}^c \times \mathbb{T}^d$ containing the origin such that $\lim_{t\to\infty}(\theta_i(t) - \theta_j(t)) = 0$, for all $i, j \in \{1, \ldots, d\}$, and $\lim_{t\to\infty}(\omega_i(t) - \omega_j(t)) = 0$, for all $i, j \in \{1, \ldots, c\}$ and for all $(\boldsymbol{\omega}(0), \boldsymbol{\theta}(0)) \in U$.

We first showcase a brief comparison of the density obtained via Theorem 1, for a system without Cartesian coordinates, with an SDP-based method for obtaining densities exclusive to systems on a hypertorus [22].

Example 1 (Comparison with earlier results)**.** *Consider a system of three first-order oscillators coupled using a sinusoidal function, given by*

$$\dot{\theta}_j = \sum_{k=1}^{3} \sin 2(\theta_k - \theta_j), \quad \theta_i \in \mathbb{T}, \quad i = 1, \ldots, 3. \tag{15}$$

Note that this is a system on $\mathbb{T}^3$. *Taking phase-difference variables* $\varphi_1 := \theta_1 - \theta_3$, $\varphi_2 := \theta_2 - \theta_3$, *we obtain the reduced phase-difference system on* $\mathbb{T}^2$ *as*

$$\begin{aligned} \dot{\varphi}_1 &= \sin 2(\varphi_2 - \varphi_1) - 2\sin 2\varphi_1 - \sin 2\varphi_2, \\ \dot{\varphi}_2 &= \sin 2(\varphi_1 - \varphi_2) - 2\sin 2\varphi_2 - \sin 2\varphi_1. \end{aligned} \tag{16}$$

The almost global stability of (16) *is equivalent to almost global synchronization of* (15). *The system* (16) *was analyzed in* [22] *to obtain a Lyapunov density. As discussed by the authors, the density obtained via* [22] *blows up at an invariant set (which attracts almost all trajectories) and is not minimal. However, using Theorem 1 with* $c = 0$ *and* $\boldsymbol{n}_{v_\theta} = (2, 2)$, *we obtained a feasible result via* `vSOS-hybrid` *program* [25]. *We obtained* $v(\varphi_1, \varphi_2) = -0.7969 \sin 2\varphi_1 \sin 2\varphi_2 + 0.0714 \cos 2\varphi_1 \cos 2\varphi_2 - 2.9016 \cos 2\varphi_1 - 2.9016 \cos 2\varphi_2 + 5.7318$. *The plot of the density* $\rho(\boldsymbol{\varphi}) = 1/v(\boldsymbol{\varphi})$ *is given along with the dynamics of the phase-difference system in Figure 1. It is evident that the set where* ρ *is unbounded is the set of stable equilibrium points. This shows that Theorem 1 may give a more precise solution than* [22], *possibly due to the constraints being more similar than* [22] *to a standard LMI.*

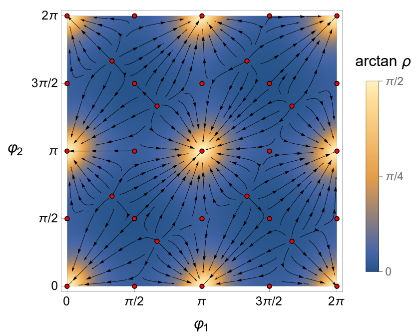

Figure 1. Phase portrait of the phase-difference system corresponding to the multistable system on the hypertorus in Example 1. The density ρ obtained via `vSOS-hybrid` [25] blows up precisely at the locally stable equilibrium points.

The second-order Kuramoto model [29] is widely used to understand the behavior of coupled phase oscillators [1,2]. It is given by the set of d coupled second-order differential equations

$$\ddot{\theta}_i(t) = -\alpha\dot{\theta}_i(t) + \omega_i^0 + K\sum_{j=1}^{d} A_{ji}\sin(\theta_j(t) - \theta_i(t)), \tag{17}$$

$i = 1, 2, \ldots, d$, where $\alpha > 0$ is the dissipation factor, ω_i^0 is the natural frequency of the i^{th} oscillator, K is the coupling strength, and (A_{ji}) accounts for the underlying topology. Here, $A_{ji} = 1$ if j^{th} oscillator influences the i^{th} oscillator; and $A_{ji} = 0$ otherwise. A second-order phase-difference system of (17) can be defined by removing the phase-shift symmetry $(\theta_1, \ldots, \theta_d) \mapsto (\theta_1 + \epsilon, \ldots, \theta_d + \epsilon)$ and the angular velocity-shift symmetry $(\theta_1, \ldots, \theta_d) \mapsto (\theta_1 + \beta t, \ldots, \theta_d + \beta t)$ from the system (17). One way of doing this is to introduce phase-difference variables $\varphi_i = \theta_i - \theta_d$ for $i = 1, \ldots, d-1$. Then the difference system can be rewritten as a system on $\mathbb{R}^{d-1} \times \mathbb{T}^{d-1}$

$$\ddot{\varphi}_i(t) = -\alpha\dot{\varphi}_i(t) + \varphi_i^0 + K\left(\sum_{j=1}^{d-1} A_{ji}\sin(\varphi_j(t) - \varphi_i(t)) - \sum_{j=1}^{d-1} A_{jd}\sin(\varphi_j(t)) - A_{di}\sin(\varphi_i)\right). \tag{18}$$

Note that (17) is almost global phase synchronized if and only if the phase-difference system is almost globally stable to the origin. We now establish almost global phase synchrony for one such system using Theorem 1. The densities obtained can be verified using `vSOS-hybrid`, [25], on GitHub. In the sequel, we will write their expressions by neglecting coefficients smaller than 0.1, but use the original expression for plotting purposes.

Example 2 (Almost global phase synchronization)**.** *Consider the second-order Kuramoto model with uniform natural frequency $\omega_1^0 = \omega_2^0 = \omega_0$, that is*

$$\begin{aligned} \ddot{\theta}_1(t) &= -0.8\,\dot{\theta}_1(t) + \omega_0 + 5\,\sin(\theta_2(t) - \theta_1(t)) \\ \ddot{\theta}_2(t) &= -0.8\,\dot{\theta}_2(t) + \omega_0 + 5\,\sin(\theta_1(t) - \theta_2(t)). \end{aligned}$$

Its phase-difference system (18) takes the form

$$\begin{bmatrix} \dot{\Omega} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -0.8\,\Omega + 5\,i\,(\mathrm{e}^{i\,\varphi} - \mathrm{e}^{-i\,\varphi}) \\ \Omega \end{bmatrix}. \tag{19}$$

The `vSOS-hybrid` program [25] establishes the feasibility of Theorem 1 with $\boldsymbol{n}_v = (5, 3)$. The program returns a 24×24 matrix V, which corresponds to the hybrid polynomial $v(\varphi, \Omega)$ given by

$$\begin{aligned} v(\Omega, \varphi) = &\,\Omega^3(0.1784\,\sin\varphi - 0.1576\,\sin 2\varphi) + \Omega^2(-0.8612\,\cos\varphi + 0.9636) + \Omega(3.4328\,\sin\varphi \\ &- 2.5336\,\sin 2\varphi + 0.552\,\sin 3\varphi) - 1.7358\cos\varphi - 3.606\cos 2\varphi + 1.7358\cos 3\varphi + 3.606. \end{aligned} \tag{20}$$

Note that $v^{-1}(0) = \{(0,0), (0,\pi)\}$ is an invariant set. Also, $\rho(\Omega, \varphi) = 1/v(\Omega, \varphi)$ is integrable sufficiently far away from $v^{-1}(0)$ as per Remark 1. Thus, $\rho(\Omega, \varphi) = 1/v(\Omega, \varphi)$ is a Lyapunov density for the system (19) and almost all solutions of (19) converge to $v^{-1}(0)$ by Theorem 1. This can also be verified through Figure 2 (left). However, $(0, \pi)$ is a saddle and attracts only a zero measure of initial conditions, hence almost all initial conditions of (19) converge to $(0, 0)$. Thus, the original system exhibits almost global phase synchronization.

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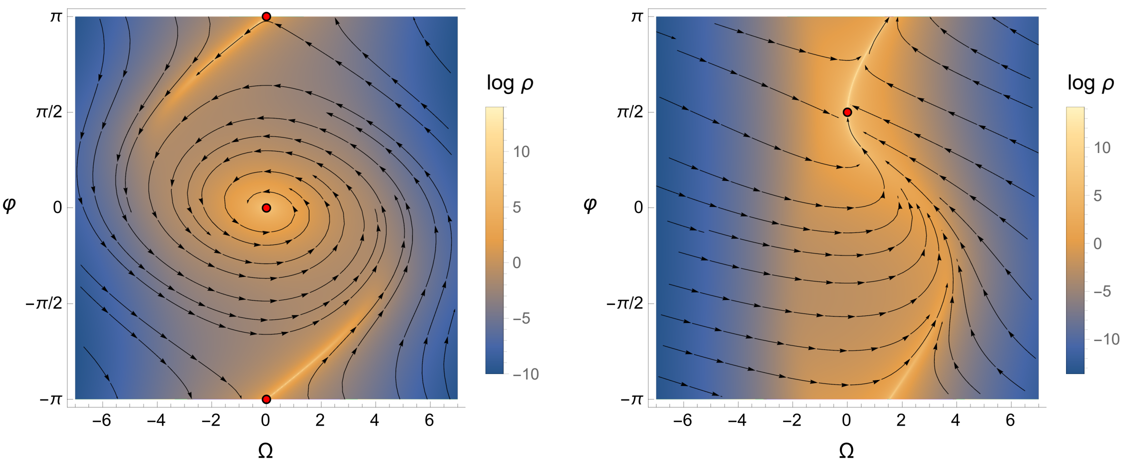

Figure 2. Phase portrait of the phase-difference system in Example 2 (left) and Example 3 (right), and the logarithmic plot of the density $\rho = 1/v$ where v given in (20) and (22), respectively, is obtained using Theorem 1 via `vSOS-hybrid` [25]. The density blows up at $\{(0,0),(0,\pi)\}$ in the left figure and at $(0,\pi/2)$ in the right figure.

When all the natural frequencies are not equal, the coupled oscillator cannot exhibit almost global phase synchronization. In such a case, the system might exhibit almost global phase locking, meaning $\lim_{t\to\infty}(\theta_i(t)-\theta_j(t)) = \theta_{ij}$ for all $i,j \in \{1,\dots,d\}$ and $\lim_{t\to\infty}(\omega_i(t)-\omega_j(t)) = 0$ for all $i,j\in\{1,\dots,c\}$.

Example 3 (Almost global phase-locking)**.** *Consider the non-uniform second-order Kuramoto model with* $\omega_1^0 = 1.2$*, and* $\omega_2^0 = 0.4$*, that is*

$$\begin{aligned}\ddot\theta_1(t) &= -5\,\dot\theta_1(t) + 3 + 5\,\sin(\theta_2(t)-\theta_1(t))\\ \ddot\theta_2(t) &= -5\,\dot\theta_2(t) - 7 + 5\,\sin(\theta_1(t)-\theta_2(t)).\end{aligned}$$

Its phase-difference system (18) *takes the form*

$$\begin{bmatrix}\dot\Omega\\ \dot\varphi\end{bmatrix} = \begin{bmatrix}-5\,\Omega + 10 + 5\,i\,(e^{i\,\varphi} - e^{-i\,\varphi})\\ \Omega\end{bmatrix}. \tag{21}$$

The `vSOS-hybrid` *program* [25] *returns a* 24×24 *matrix* V *satisfying* $\psi_{5,3}(\Omega_0,\varphi_0)^\dagger\, V\,\psi_{5,3}(\Omega_0,\varphi_0) = 0$ *for* $(\Omega_0,\varphi_0) = (0,\pi/2)$*, which is the unique equilibrium point of the system. The expression of* v *is given by*

$$\begin{aligned}v(\Omega,\varphi) = {} & \Omega^2(0.1227\sin\varphi - 0.1053\sin 2\varphi + 0.2446\cos\varphi) + \Omega(1.5342\sin\varphi + 0.1459\sin 2\varphi - 0.2534\cos\varphi\\ &+ 0.2149\cos 2\varphi - 1.3244) - 5.6995\sin\varphi - 0.5793\sin 2\varphi + 0.2824\sin 3\varphi + 0.7224\cos\varphi\\ &- 2.0605\cos 2\varphi - 0.1453\cos 3\varphi + 3.9215.\end{aligned} \tag{22}$$

Also, $\rho(\Omega,\varphi) = 1/v(\Omega,\varphi)$ *is integrable sufficiently far away from* $(0,\pi/2)$ *as per Remark 1. Thus, almost all solutions satisfy* $\varphi\to\pi/2$ *and* $\dot\varphi\to 0$ *as can be verified by Figure 2 (right). This implies that* $\theta_1-\theta_2\to\pi/2$ *and* $\dot\theta_1-\dot\theta_2\to 0$*, hence the original coupled oscillator system is almost globally phase-locked.*

In the next example, we obtain local and global LaSalle's function for an oscillatory system using Theorem 2.

Example 4 (Local phase synchronization)**.** *Consider the second-order Kuramoto model in Example 2. Consider the Putinar domain* (12) *with* $q_1(\Omega,\varphi) = \cos\varphi$ *and* $q_2(\Omega,\varphi) = 16-\Omega^2$*. Both matrices can be represented by Gram matrices of common size* $[\![(1,1)]\!] = 4$*. Note that* $\mathcal{D} = \{(\Omega,\varphi)\in\mathbb{T}\times\mathbb{R}\colon |\Omega|\le 4,\ |\varphi|\le \pi/2\}$*. Solving Theorem 2 for* $c=1$*,* $d=1$ *and* $\boldsymbol{n}_v = (1,1) = \boldsymbol{n}_q$*, we obtained a feasible solution through the* `vSOS-hybrid` *program,* [25]*. The program returned positive definite matrices* S_0^w*,* S_1^w *and* S_2^w *of sizes* 9×9*,*

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4×4 and 4×4, respectively, each having nullity 1; a matrix V of size 4×4; satisfying the hypothesis of Theorem 2. Using (13), we obtained the hybrid polynomial

$$\begin{aligned} v(\Omega, \varphi) &= 0.237\,\Omega^2 + 0.115\,\Omega\,\sin\varphi + 5.041 - 4.642\,\cos\varphi, \\ w(\Omega, \varphi) &= \Omega^2\,(-0.118\cos\varphi + 0.380) + 0.208\,\Omega\,\sin\varphi + 0.576\,(1 - \cos 2\varphi) \end{aligned}$$

satisfying $-\operatorname{grad} v(\Omega, \varphi) \cdot \boldsymbol{f}(\Omega, \varphi) = w(\Omega, \varphi) \geq 0$ in $\mathcal{D}$. The local stability of the phase-difference system in the largest invariant subset $\mathfrak{K}_{inv}$ of $\mathcal{D}$ follows, which is depicted in Figure 3 (left).

In fact, Theorem 2 with $\boldsymbol{n}_v = (1,1)$ and $\{Q_\ell : \ell = 1, \ldots, \mathscr{L}\} = \emptyset$, returns a feasible solution with

$$\begin{aligned} v(\Omega, \varphi) &= 1.5205\,\Omega^2 + 0.8727\,\Omega\sin\varphi - 29.2313\cos\varphi + 35.1297 \geq 0, \\ w(\Omega, \varphi) &= \Omega^2(-0.8717\cos\varphi + 2.4328) + 1.8769\,\Omega\,\sin\varphi + 4.3639\,(1 - \cos 2\varphi) \geq 0. \end{aligned}$$

Computations show that $w^{-1}(0) = \{(0,0), (0,\pi)\}$. Thus, by LaSalle's invariance principle, global attraction to this set can be established. However, since $(0,\pi)$ is a saddle, only a measure zero set of trajectories converge to it. This gives an alternate way to prove almost global stability of the origin, which was earlier proved using densities in Example 2.

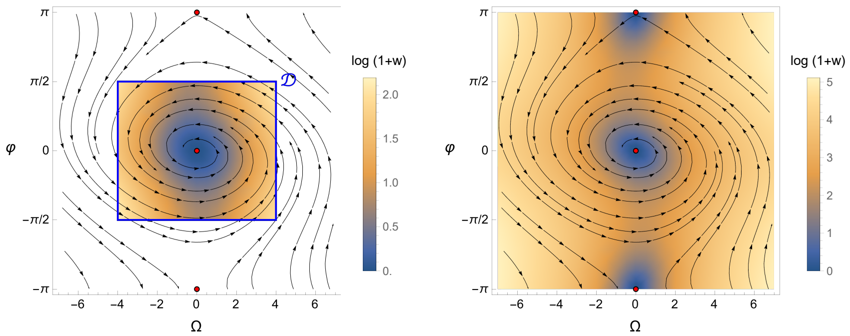

Figure 3. Phase portrait of the phase-difference system in Example 4. The existence of a local Lyapunov function in the region $\mathcal{D}$ (left) is established using Theorem 2 via `vSOS-hybrid`, [25]. Existence of a global LaSalle's function (right) is established which implies attraction to $\{(0,0), (0,\pi)\}$. The plot of lie derivative is provided.

In the following section, we obtain a result which allows us to find a swing-up control for a class of inverted pendulum system on a massless cart using the technique developed in the paper.

6. Application to Swing-Up Control of Inverted Pendulums

Consider the inverted pendulum system inspired by [4], given by $\ddot{\theta} = a\,\sin\theta + u\,k(\theta)$ for some $a > 0$. The system can be rewritten as a first-order system

$$\begin{bmatrix} \dot{\omega} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} a\,\sin\theta + u\,k(\theta) \\ \omega \end{bmatrix} := \boldsymbol{f}(\omega, \theta). \tag{23}$$

The uncontrolled system (when $u \equiv 0$) is a Hamiltonian system with Hamiltonian $e(\omega,\theta) = 0.5\omega^2 + a(\cos\theta - 1)$. Since e does not change along system trajectories, the level curves of e are invariant sets under the flow. More specifically, the level curve $e(\omega,\theta) = 0$ is a heteroclinic cycle containing the unstable equilibrium $(0,0)$ when $u \equiv 0$. The swing-up of the inverted pendulum can be established if u is constructed such that almost global stability to the zero level curve is established. For the same, we can look for control of the form $u(\omega,\theta) = e(\omega,\theta)\,\tilde{u}(\omega,\theta)$ such that $\rho(\omega,\theta) = 1/e(\omega,\theta)^2$ is a Lyapunov density for the system.

Proposition 3. *Given the control system (23) such that $e(\omega,\theta)$ satisfies $e(0,0) = 0$, and is the Hamiltonian of the system with $u \equiv 0$; S and R are Gram representations of $s(\omega,\theta) = e(\omega,\theta)\,k(\theta)$ and $r(\omega,\theta) = -\omega\,k(\theta)$,*

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respectively, of sizes $[\![\boldsymbol{n_s}]\!]$ *each. There exists a swing-up control* $\tilde{u}(\omega,\theta)$ *if, for some* $\boldsymbol{n_{\tilde{u}}} \in \mathbb{Z}^2_{\geq 0}$*, there exist Hermitian matrices* $\widetilde{W} \geq 0$ *and* $\widetilde{U}$ *of sizes* $[\![\boldsymbol{n_s}+\boldsymbol{n_{\tilde{u}}}]\!]$ *and* $[\![\boldsymbol{n_{\tilde{u}}}]\!]$*, respectively, such that*

$$\mathscr{H}_{\eta,k}\Big(X^\dagger\Big(S\otimes\Big(D_\omega^\dagger\widetilde{U}+\widetilde{U}D_\omega\Big)+R\otimes\widetilde{U}\Big)X\Big)=\mathscr{H}_{\eta,k}\Big(\widetilde{W}\Big), \tag{24}$$

for each $0 \leq \eta \leq n_{s_\omega}+n_{\tilde{u}_\omega}$ *and* $|k| \leq n_{s_\theta}+n_{\tilde{u}_\theta}$*.*

Proof. We will show that the conditions imply that almost all solutions approach the level curve $e(\omega,\theta)=0$ containing $(0,0)$. Note that (24) can be rewritten as

$$\begin{aligned}\tilde{w}(\omega,\theta) &= \psi_{\boldsymbol{n_s}+\boldsymbol{n_{\tilde{u}}}}(\omega,\theta)^\dagger\,\widetilde{W}\,\psi_{\boldsymbol{n_s}+\boldsymbol{n_{\tilde{u}}}}(\omega,\theta)\\ &= \psi_{\boldsymbol{n_s}+\boldsymbol{n_{\tilde{u}}}}(\omega,\theta)^\dagger\Big(X^\dagger\Big(S\otimes\Big(D_\omega^\dagger\widetilde{U}+\widetilde{U}D_\omega\Big)+R\otimes\widetilde{U}\Big)X\Big)\,\psi_{\boldsymbol{n_s}+\boldsymbol{n_{\tilde{u}}}}(\omega,\theta).\end{aligned}$$

Using Propositions 1 and 2, we have

$$\begin{aligned}\tilde{w}(\omega,\theta) &= s(\omega,\theta)\,\tilde{u}_\omega(\omega,\theta)+r(\omega,\theta)\,\tilde{u}(\omega,\theta)\\ &= e(\omega,\theta)\,k(\theta)\,\tilde{u}_\omega(\omega,\theta)-\omega\,k(\theta)\,\tilde{u}(\omega,\theta)\\ &= k(\theta)(u_\omega(\omega,\theta)-e_\omega(\omega,\theta)\,\tilde{u}(\omega,\theta))-\omega\,k(\theta)\,\tilde{u}(\omega,\theta)\\ &= k(\theta)\,u_\omega(\omega,\theta)-2\,\omega\,k(\theta)\,\frac{u(\omega,\theta)}{e(\omega,\theta)}.\end{aligned}$$

It follows that

$$\begin{aligned}w(\omega,\theta):=\frac{\tilde{w}(\omega,\theta)}{e(\omega,\theta)^2} &= \frac{k(\theta)}{e(\omega,\theta)^2}\left(u_\omega(\omega,\theta)-2\,\omega\,\frac{u(\omega,\theta)}{e(\omega,\theta)}\right)\\ &= \frac{e(\omega,\theta)^2\,\mathrm{div}\,\boldsymbol{f}(\omega,\theta)-2\,e(\omega,\theta)\,(\mathrm{grad}\,e(\omega,\theta)\cdot\boldsymbol{f}(\omega,\theta))}{e(\omega,\theta)^4}=\mathrm{div}\left(\frac{1}{e(\omega,\theta)^2}\boldsymbol{f}(\omega,\theta)\right).\end{aligned}$$

For $v(\omega,\theta):=e(\omega,\theta)^2\geq 0$ and $w(\omega,\theta)\geq 0$, $v^{-1}(0)$ is forward-invariant under the flow of the control system (23); and satisfies (9) for almost all $(\omega,\theta)\in(\mathbb{R}\times\mathbb{T})\setminus v^{-1}(0)$. Also, $1/v(\omega,\theta)$ is integrable by Remark 1 as $v(\omega,\theta)$ has degree 4. The result follows from Lemma 3. □

Example 5 (Swing-up of an inverted pendulum)**.** *Consider the inverted pendulum control system (23) with* $a=1$ *and* $k(\theta)=\cos\theta$ *as in [30–32]. Then the Hamiltonian* $e(\omega,\theta)=0.5\omega^2+\cos\theta-1$ *satisfies* $e(0,0)=0$*. Our program* `vSOS-hybrid`*, [25], confirms the feasibility of (24) for* $\boldsymbol{n_{\tilde{u}}}=(1,1)$ *with Gram matrices S and R corresponding to* $s(\omega,\theta)=(0.5\omega^2+\cos\theta-1)\cos\theta$ *and* $r(\omega,\theta)=-\omega\cos\theta$*. The program returns a* 12×12 *matrix* $\widetilde{W}$ *and a* 4×4 *matrix* $\widetilde{U}$ *with* $\widetilde{U}_{1,4}=\widetilde{U}_{1,4}=-8.4328$ *and* 0 *elsewhere, which corresponds to the hybrid polynomial* $\tilde{u}(\omega,\theta)=-4.2164\,\omega\cos\theta$ *as per (4). Substituting* $u(\omega,\theta)=e(\omega,\theta)\tilde{u}(\omega,\theta)$ *in (23), the phase portrait is given as in Figure 4, which confirms that almost all trajectories of the controlled system approach the level set* $e(\omega,\theta)=0$*.*

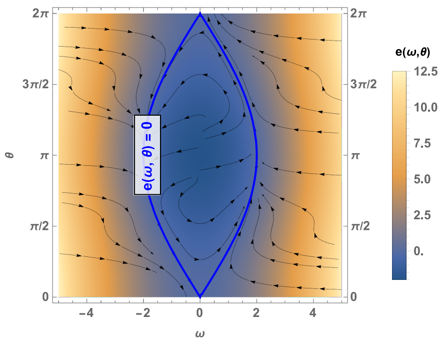

Figure 4. Phase portrait of the inverted pendulum with the swing-up control found in Example 5. Almost all trajectories approach $e(\omega,\theta)=0$.

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A control law for the system (23) was earlier found in [32] by using a recasting of the system, which increases the system dimension in the first step, performs a Sum of Squares (SOS) programming in the second step, and then removes the extra variables produced by recasting in the third step. The technique developed in this paper, on the other hand, skips the first and the third steps altogether.

Example 6 (Swing-up control on a cart)**.** *The model is analogous to balancing a stick on a finger. The model for a planar inverted pendulum on a cart [4] is given by*

$$\begin{bmatrix} M+m & ml_c\cos\theta \\ ml_c\cos\theta & J \end{bmatrix}\begin{bmatrix}\ddot r\\ \ddot\theta\end{bmatrix} + \begin{bmatrix} 0 & -ml_c\dot\theta\sin\theta \\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot r\\ \dot\theta\end{bmatrix} = \begin{bmatrix} F \\ mgl_c\sin\theta\end{bmatrix},$$

where M and m are the mass of the cart and the pendulum, respectively, l_c is the distance between the joint and the center of mass of the pendulum, J is the rotational moment of inertia of the pendulum, and F is the input control force acting on the cart in the horizontal plane (refer Figure 5). The system can be rewritten in terms of $[r,s,\omega,\theta]^\top \in \mathbb{R}^3\times\mathbb{T}$ as

$$\begin{aligned} \dot s &= \frac{ml_cJ\omega^2\sin\theta + JF - m^2l_c^2g\sin\theta\cos\theta}{(M+m)J - m^2l_c^2\cos^2\theta} \\ \dot r &= s \\ \dot\omega &= \frac{ml_cg}{J}\sin\theta - \frac{ml_c}{J}\dot s\cos\theta \\ \dot\theta &= \omega. \end{aligned} \tag{25}$$

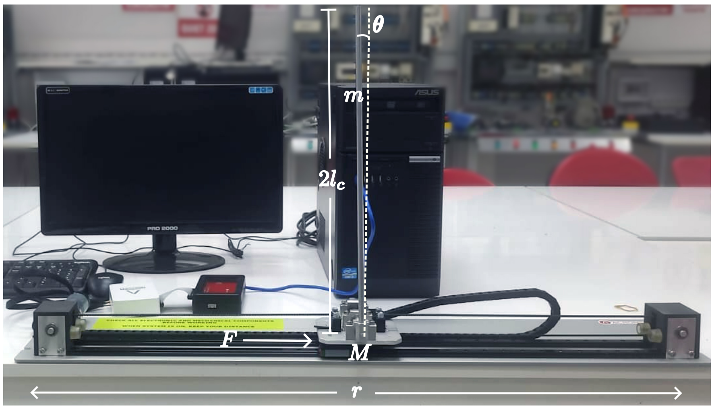

Figure 5. The cart-and-pendulum setup used in Example 6.

It was established in [33] that if a stabilizing law is known for the two dimensional system (ω,θ) with basin of attraction $\mathscr{A}$, then it can be used to obtain a stabilizing law on $\mathbb{R}^2\times\mathscr{A}$ for an inverted pendulum system on a moving cart. Let $e(\omega,\theta) = 0.5\omega^2 + a(\cos\theta - 1)$. Finding control of the form

$$F = \frac{\left((M+m)J - m^2l_c^2\cos^2\theta\right)ve}{J} + \frac{m^2l_c^2g}{J}\sin\theta\cos\theta - ml_c\omega^2\sin\theta \tag{26}$$

reduces the (ω,θ) system to

$$\begin{aligned} \dot\omega &= \frac{ml_cg}{J}\sin\theta - \frac{ml_c}{J}ve\cos\theta \\ \dot\theta &= \omega, \end{aligned} \tag{27}$$

which again has the form of the system (23) with $a = ml_cg/J$ and $k(\theta) = -\cos\theta$ and $u = \tilde u e = (ml_c/J)ve$, and Proposition 3 can be utilized to establish almost convergence of solutions to a level set of the Hamiltonian. For instance, for the inverted pendulum system in [34], we have $M = 0.25$ kg, $m = 0.125$ kg, $l_c = 0.16\,m$ (half the length of rod), $g = 9.81\,m/s^2$, $J = 1.06\times10^{-3}$, hence $a = 185.094$ and $\tilde u = 18.868\,v$. Also $s(\omega,\theta)$ and $r(\omega,\theta)$ appearing in Proposition 3 take the form $s(\omega,\theta) = -\cos\theta\,(0.5\omega^2 + 185.094(\cos\theta - 1))$ and

$r(\omega,\theta) = \omega\cos\theta$. Our `vSOS-hybrid` program [25] returns a solution for (24) with $\boldsymbol{n}_{\tilde{u}} = (1,1)$, where $\widetilde{W}$ is a 12×12 matrix and $\widetilde{U}$ satisfies $\widetilde{U}_{1,4} = \widetilde{U}_{1,4} = 15.698$ and 0 elsewhere, which corresponds to the hybrid polynomial $\tilde{u}(\omega,\theta) = 7.849\,\omega\cos\theta$. Thus, for the control input F in (26) with $v(\omega,\theta) = 0.416\,\omega\cos\theta$, the dynamics of the pendulum approaches the zero level set of $e(\omega,\theta)$.

A control scheme that performs swing-up and smoothly blends into near upright stabilization will now be implemented. The experimental platform is the ACROME cart–and–pendulum system (Figure 5) composed of an Arduino Mega 2560-based drive board, a 1600 RPM Pololu DC motor, a 360° potentiometric angle sensor, and power/interface units [35]. The state variable r [m] denotes cart position, s [m/s] the cart velocity, θ [rad] the pendulum angle, and ω [rad/s] the angular rate. The upright equilibrium is set to $\theta = 0$ while the downward position $\theta = \pi$. Near the top, the pendulum position error is defined as

$$\theta_{\text{err}} \;=\; \operatorname{atan2}(\sin\theta,\;\cos\theta)\;\in[-\pi,\pi], \tag{28}$$

and s is obtained from a low-delay derivative estimate of r. The energy-shaping swing–up force $F(\theta,\omega)$ is softened near the top by

$$\begin{aligned} w_{\text{tap}}(\theta_{\text{err}},\omega) \;&=\; \operatorname{clip}_{[0,1]}\left(\max\left\{\left|\frac{\theta_{\text{err}}}{\theta_{\text{tap}}}\right|^2,\;\left|\frac{\omega}{\omega_{\text{tap}}}\right|\right\}\right) \\ F_{\text{sw}}^{\text{eff}} \;&=\; w_{\text{tap}}\,F, \end{aligned} \tag{29}$$

where $\theta_{\text{tap}} \approx 22°$, and $\omega_{\text{tap}} \approx 2.5$ rad/s are used in experimental study. Thus, the stabilization algorithm takes place when the pivot arm enters the region $\pm\theta_{tap}$ with an angular velocity ω_{tap} (or smaller). During the upright position pass of the pendulum arm, momentum is consumed through a brief brake phase

$$u_{\text{cb}}(t) = \begin{cases} -k_b\,\omega(t), & \text{if } |\theta_{\text{err}}(t)| < \theta_{\text{brk}} \\ 0, & \text{otherwise} \end{cases}, \tag{30}$$

$t \in [t_0,\, t_0 + T_{\text{coast}}]$, with the choices $\theta_{\text{brk}} \approx 12°$, $k_b = 1.2$ Ns/rad, and $T_{\text{coast}} \approx 0.12$ s. This brake maneuver of the cart slows the pendulum's pass down while it is approaching the top position. The total softened control input is given by

$$u_{total}(t) \;=\; (1-\alpha)\,F_{\text{sw}}^{\text{eff}}(\theta,\theta_{err},\omega) \;+\; \alpha\,u_{\text{PD}}(r,s,\theta_{\text{err}},\omega), \tag{31}$$

where u_{PD} (or LQR) is designed from a local linearization around the upright position, $\theta = 0$ [35]. The blending variable $\alpha \in [0,1]$ follows a first-order filter, which also known as Q-blending [36]

$$\alpha^\star(t) = \begin{cases} 1, & \text{if } |\theta_{\text{err}}| < \theta_{\text{in}} \text{ and } |\omega| < \omega_{\text{cap}} \\ 0, & \text{otherwise}, \end{cases} \tag{32}$$

$$\dot{\alpha}(t) = \frac{1}{\tau(\alpha^\star)}\big(\alpha^\star(t) - \alpha(t)\big), \quad \tau(\alpha^\star) = \begin{cases} \tau_{\text{on}}, & \alpha^\star = 1, \\ \tau_{\text{off}}, & \alpha^\star = 0, \end{cases} \tag{33}$$

where $\alpha(t) \equiv 1$ swing–up effect is removed in total control input. In experiments, $\theta_{\text{in}} \approx 6°$, $\omega_{\text{cap}} \approx 2.0$ rad/s, $\tau_{\text{on}} \approx 60$ ms, and $\tau_{\text{off}} \approx 120$ ms are used. Control input (31) is implemented on MATLAB/Simulink 2020b environment with a fixed sampling rate 0.01 seconds (Figure 6).

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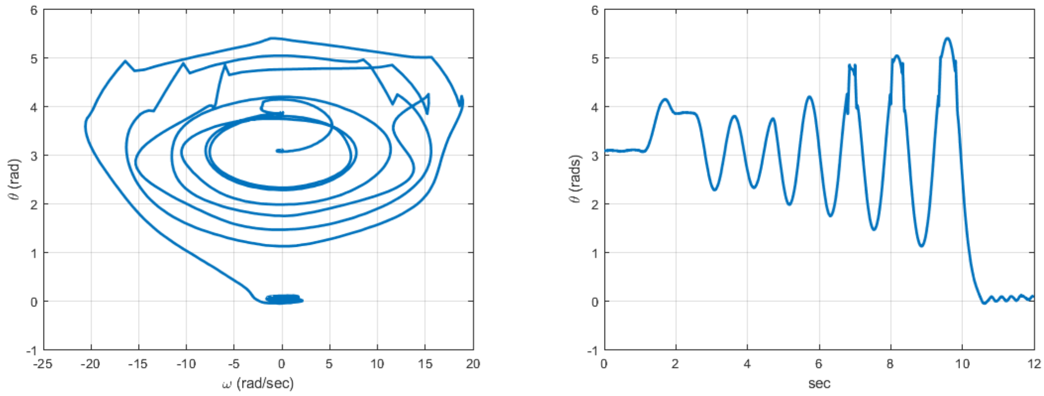

Figure 6. Phase portrait and evolution of $\theta(t)$ of the system in Example 6 after applying the global control (26) followed by the local control (31).

7. Conclusions

We develop a novel technique to obtain Lyapunov density and local Lyapunov-like functions for vector fields having a hybrid polynomial structure. Using Gram matrix representation of hybrid polynomials, we obtain semidefinite programming certificates for almost global stability and local stability of oscillatory systems. The results were used to establish synchronization of second-order Kuramoto models, as well as to develop a swing-up control for a class of inverted pendulums. The methodology introduced in this paper could yield results for switched oscillatory systems and for the design of feedback controllers for nonlinear oscillatory systems.

As with most SDP-based approaches, the method is sensitive to the problem dimension. In higher dimensions, the number of decision variables increases significantly, which may result in increased computational burden. These considerations motivate future research to develop more scalable formulations or to exploit problem structure to reduce computational complexity.

Data Availability Statement: The program supporting the findings of this study is openly available on GitHub [25].

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Conflicts of Interest: The authors have no conflicts of interest to declare that are relevant to the content of this article.

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