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Innovation-Consolidation Cycle in Finite Ring Continuum: Packets, Invariants, and Shell Transfer

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Abstract: The Finite Ring Continuum (FRC) models structure through finite arithmetic shells. At symmetry-complete prime checkpoints, a quadratic extension introduces a second involution alongside spatial reversal, and the coexistence of these symmetries generates a finite shell-transfer problem. This paper studies that problem in terms of existential admissibility: the question is which structural updates are internally allowed, not how an external agent might operate on the shells. We prove that every nondegenerate admissible innovation decomposes into four-element Klein packets, so primitive shell updates occur in multiples of four and have plus four as their elementary admissible increment. We then define orbit-level invariant extraction and show that every finite invariant family admits an explicit code in the elementary receiving shell with a minimal symbol count, namely the least number of receiving-shell symbols needed to encode the invariant alphabet. In addition, every finite extractor admits an exact fiber decomposition that records how many primitive states consolidate to each invariant value. A fully explicit example at prime 13 exhibits a four-element innovation packet, a one-symbol recoding into the receiving shell of order 17, and the norm profile with one singleton fiber and twelve fibers of size fourteen. In this paper the norm is used only as an exact algebraic packet-invariant; any identification of such invariants with specific physical observables is deferred to follow-up work. The result reframes innovation-consolidation in FRC as a finite problem of packet formation, invariant extraction, consolidation profile, and shell coding, with larger updates understood as coarse-grained bundles of the same primitive admissible step.

Keywords: finite fields; invariant extraction; finite coding; consolidation profile; fiber cardinality; shell transfer; Klein packets; quadratic extensions

1. Introduction

The Finite Ring Continuum (FRC) models structure as emerging from a graded sequence of finite arithmetic shells of order $q = 4t + 1$ [1,2]. At symmetry-complete prime checkpoints, the shell is the finite field $\mathbb{F}_p$; the quadratic extension $\mathbb{F}_{p^2}$ introduces the second involution and the Dirac-type algebraic setting used in companion FRC work [3]. Earlier FRC work established the Euclidean/extension split, the quadratic extension, and the associated algebraic formalism. What remained comparatively underdeveloped was the shell-to-shell transfer problem: what is the algebraic unit of innovation in the quadratic extension, which structural information survives, and how is that information carried across admissible shell evolution?

This paper isolates that transfer problem and treats it as a finite algebraic transfer problem internal to FRC. Rather than solving the full Dirac dynamics, we study the prime-checkpoint transition

$$\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$$

through the two involutions already forced by the FRC framework:

$$R(z) = -z, \qquad \tau(z) = z^p.$$

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Their coexistence controls the packet structure of genuine innovation, while finite invariant extraction and shell-to-shell coding control what structural data may be preserved across admissible shell evolution. The formulation adopted below is deliberately finite-field, orbit-theoretic, and coding-theoretic [4–8].

The central claim of the paper is that innovation in FRC has two complementary algebraic scales.

1. At the *primitive* level, genuinely new elements in the quadratic extension occur in nondegenerate Klein packets of cardinality four.
2. At the *consolidation* level, these packets support finite invariant families whose values can be encoded in the alphabet of the elementary receiving shell with an explicit minimal symbol count.

This separation lets us distinguish sharply between packet growth and invariant growth. Primitive support grows in forced multiples of four, whereas invariant growth depends on the specific extractor chosen on the innovation domain.

The present paper is intentionally algebraic in scope. It does not claim that the invariant families considered below, including the norm used in the worked example, already represent specific physical observables such as mass, charge, or spin. Those interpretive identifications are deferred to follow-up work.

The guiding notion throughout is *existential admissibility*. The question is not how an external agent might choose to operate on a shell, but which updates are internally admissible once the prime-checkpoint structure and its forced involutions are given. In that sense, the decisive quantity is the minimal admissible evolution step. Whether larger updates occur one packet at a time or in bundled form is a coarse-graining issue; the primitive admissible increment remains the same.

The paper contributes four concrete results.

1. We prove that every nondegenerate admissible innovation domain in $\mathbb{F}_{p^2}$ decomposes into disjoint Klein packets, and therefore has cardinality $4n$ for some $n \geq 1$.
2. We derive the induced shell-update law

$$q_{\text{new}} = q_{\text{old}} + 4n,$$

with $+4$ as the minimal nontrivial increment.
3. We define orbit-level invariant extraction and show that any finite invariant family I_p admits an explicit code in the elementary receiving shell with minimal symbol count M_p, characterised as the least integer satisfying

$$|I_p| \leq q_{t+1}^{M_p}.$$

4. We derive the exact finite fiber decomposition

$$|D_p| = \sum_{y \in I_p} |J_p^{-1}(y)|,$$

and evaluate it explicitly in the worked example $p = 13$, where the norm profile is $169 = 1 + 12 \cdot 14$.

Why this is upstream in the FRC programme. The present paper does not attempt to reconstruct FRC from first principles. Instead, it assumes the prime-checkpoint/quadratic-extension structure established in earlier FRC work and studies the algebraic transfer mechanism that occurs once that structure is given. This makes the paper upstream to later uniqueness-style reconstructions: it provides the packet, coding, and fiber machinery without relying on any epistemic uniqueness theorem.

Why this matters for finite transfer. In a finitist setting, consolidation should be expressed entirely through finite cardinalities and finite codes [9,10]. The present paper therefore formulates shell transfer in terms of packet cardinality, finite invariant alphabets, minimal symbol counts, and exact fiber decompositions. This keeps the argument inside a fully discrete vocabulary while preserving the structural content of innovation and consolidation.

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Minimal admissible evolution. The packet result may also be read as a finite minimality principle. The paper does not claim that evolution must always be observed one packet at a time. Rather, it claims that one nondegenerate Klein packet is the smallest admissible update compatible with the internal symmetry structure. Larger updates are finite bundles of that elementary move. In this interpretive sense, the four-packet identifies the least admissible structural change, while macroscopic updates are coarse-grained composites of elementary ones.

The rest of the paper is organised as follows. Section 2 recalls the prime-checkpoint structure of FRC and the quadratic extension. Section 3 proves the four-packet theorem and the induced shell-update law. Section 4 defines orbit-level invariant extraction, finite shell coding, exact fiber decompositions, and the corresponding finite coding thresholds. Section 5 gives an explicit example for $p = 13$. Sections 6 and 7 discuss the role of the results within FRC and their relevance to finite shell transfer.

2. Background

The FRC hierarchy consists of finite arithmetic shells $\mathbb{Z}_q$ indexed by

$$q = 4t + 1, \tag{1}$$

where $t \in \mathbb{N}$ is the shell index. Earlier FRC work often interprets this parameter chronon-wise [1,2], but the present paper uses it only to label the arithmetic shells and their admissible transitions. When $q = p$ is prime, the shell becomes symmetry-complete and is identified with the finite field $\mathbb{F}_p$. These prime checkpoints are the exact field stages of the hierarchy; intermediate composite shells remain part of the same global ring-theoretic progression.

Earlier FRC work distinguished two algebraic stages at such prime checkpoints. The Euclidean stage is internal to $\mathbb{F}_p$. The extension stage requires the quadratic extension

$$\mathbb{F}_{p^2} \cong \mathbb{F}_p \oplus c\,\mathbb{F}_p, \qquad c^2 = \nu, \tag{2}$$

where $\nu \in \mathbb{F}_p$ is a fixed quadratic nonsquare [2,4,5]. If $z = a + cb$ with $a, b \in \mathbb{F}_p$, the Frobenius map acts by

$$\tau(z) := z^p = a - cb. \tag{3}$$

This involution fixes the Euclidean shell pointwise and acts nontrivially on the genuinely extended sector.

Two involutions are therefore available at a symmetry-complete prime checkpoint:

$$R(z) := -z, \qquad \tau(z) := z^p. \tag{4}$$

The first is the spatial reversal already present in the Euclidean shell; the second is the temporal involution supplied by the quadratic extension. Their interaction is the algebraic starting point of the present paper.

The quadratic extension is also the arena in which the finite-field Dirac formalism is defined [3]. The present paper does not re-derive that formalism and does not connect the invariants studied below to specific physical observables. Instead, it abstracts from the detailed dynamics and studies the shell-to-shell transfer mechanism forced by the extension $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}$ itself. The relation between packet-level invariants and observables such as mass, charge, or spin is left to follow-up publications.

Two additional finite-field maps will be used repeatedly.

1. The trace
$$\mathrm{Tr}(z) := z + z^p = 2a. \tag{5}$$
2. The norm
$$N(z) := zz^p = a^2 - \nu b^2. \tag{6}$$

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Both maps take values in $\mathbb{F}_p$ and are invariant under the Frobenius involution. The norm is also invariant under R, since $N(-z) = N(z)$. This makes it a natural packet-level invariant for the coding problem studied below. In the present paper, however, it is used only as an exact algebraic invariant and not as an identified physical observable. Standard background on finite-field trace and norm maps may be found in [4,5].

The shell-to-shell questions addressed here may now be stated precisely.

1. Which subsets of $\mathbb{F}_{p^2}$ can represent genuinely new innovation data at a prime checkpoint?
2. What is the minimal packet size forced by the involutions R and τ?
3. Given a finite invariant family extracted from such packets, what is the minimal number of receiving-shell symbols required to carry it forward?
4. What exact fiber multiplicities arise when several primitive states consolidate to the same invariant value?

These are the structural questions on which the rest of the paper focuses.

3. Innovation Packets at Prime Checkpoints

This section identifies the primitive algebraic unit of innovation at a symmetry-complete prime checkpoint. The relevant structure is not the full Dirac trajectory but the orbit closure forced by the involutions

$$R(z) = -z, \qquad \tau(z) = z^p$$

on the quadratic extension $\mathbb{F}_{p^2}$. From the viewpoint of finite group actions, the basic object is the orbit structure of the Klein action generated by these involutions [6,11].

3.1. Klein Packets

Definition 1 (Klein symmetry). *Let $p = 4t + 1$ be a symmetry-complete prime shell. The subgroup of bijections of $\mathbb{F}_{p^2}$ generated by R and τ is*

$$K_p := \{\mathrm{id}, R, \tau, R \circ \tau\}. \tag{7}$$

Lemma 1. *For odd prime p, the involutions R and τ commute. Consequently $K_p \cong V_4$, the Klein four-group.*

Proof. For every $z \in \mathbb{F}_{p^2}$,

$$\tau(R(z)) = (-z)^p = -z^p = R(\tau(z)).$$

Thus $R\tau = \tau R$, and K_p is the Klein group generated by two commuting involutions. □

The four maps in K_p are forced by the FRC prime-checkpoint structure itself: R is already present in the Euclidean shell, while τ becomes available only after passing to the quadratic extension. The first nontrivial question is therefore not whether innovation occurs, but in what orbit structure it must occur.

Write $z = a + cb$ with $a, b \in \mathbb{F}_p$ and $c^2 = \nu$. Since

$$\tau(a + cb) = a - cb,$$

one has $\tau(z) = z$ exactly when $b = 0$, and $\tau(z) = -z$ exactly when $a = 0$.

Definition 2 (Nondegenerate element). *An element $z \in \mathbb{F}_{p^2}$ is called nondegenerate if*

$$\tau(z) \neq z, \qquad \tau(z) \neq -z.$$

Equivalently, in the decomposition $z = a + cb$, both coefficients a and b are nonzero.

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Proposition 1 (Four-packet orbit). *Let $z \in \mathbb{F}_{p^2}$ be nondegenerate. Then its K_p-orbit has cardinality four:*

$$\mathcal{O}_{K_p}(z) = \{z, -z, \tau(z), -\tau(z)\}. \tag{8}$$

Proof. The orbit can have fewer than four elements only if one of the equalities

$$z = -z, \qquad z = \tau(z), \qquad z = -\tau(z)$$

holds. Since p is odd, $z = -z$ implies $z = 0$, which is degenerate. The remaining two equalities are excluded by nondegeneracy. Hence the orbit has four distinct elements. □

Remark 1. *The degenerate locus is the union of the Euclidean shell $\mathbb{F}_p$ and the trace-zero subspace $\{z \in \mathbb{F}_{p^2} : z + \tau(z) = 0\}$. Genuine timeful innovation in the present sense lies outside that locus.*

3.2. Admissible Domains and Shell Growth

The current paper studies shell-to-shell transfer, so we isolate only the closure properties required of a genuinely new innovation domain.

Definition 3 (Admissible innovation domain). *A subset $\mathcal{X}_p \subset \mathbb{F}_{p^2}$ is called an admissible innovation domain if:*

1. *every element of $\mathcal{X}_p$ is nondegenerate;*
2. *$\mathcal{X}_p$ is invariant under the Klein action K_p.*

This definition is deliberately algebraic. Dirac dynamics, Lorentzian transport, or any other admissible innovation mechanism may generate such a domain, but the packet theorem below depends only on the forced involutions.

Proposition 2 (Orbit decomposition of innovation). *Every admissible innovation domain $\mathcal{X}_p$ is a disjoint union of nondegenerate Klein packets. In particular,*

$$|\mathcal{X}_p| = 4n_p \tag{9}$$

for some integer $n_p \geq 1$.

Proof. Since $\mathcal{X}_p$ is K_p-invariant, it is a disjoint union of K_p-orbits. By Proposition 1, every orbit of a nondegenerate element has cardinality four. Summing over the disjoint orbits gives the result. □

Packet cardinality controls primitive shell growth independently of any later invariant extraction.

Definition 4 (Shell update). *Let a shell of cardinality q_{old} admit an innovation domain $\mathcal{X}_p$ at a symmetry-complete prime checkpoint. The updated shell is obtained by adjoining the genuinely new primitive residues in $\mathcal{X}_p$.*

Corollary 1 (Forced shell increment). *If the innovation domain $\mathcal{X}_p$ decomposes into n_p nondegenerate Klein packets, then the shell update satisfies*

$$q_{\text{new}} = q_{\text{old}} + 4n_p. \tag{10}$$

In particular, the minimal nontrivial increment is

$$q_{\text{new}} = q_{\text{old}} + 4. \tag{11}$$

Proof. By Proposition 2, the innovation domain contains exactly $4n_p$ genuinely new primitive residues. Adjoining those residues increases the shell cardinality by precisely that amount. □

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Definition 5 (Elementary admissible step)**.** *One nondegenerate Klein packet is called an elementary admissible evolution step. An update by $4n_p$ residues is a bundled or coarse-grained composition of n_p such elementary steps.*

Remark 2. *The shell law in Corollary 1 concerns primitive support, not invariant complexity. Primitive growth is forced in multiples of four, while invariant growth depends on the packet-level extractor introduced in the next section. The present claim is therefore about admissibility at the most basic level: one packet is the least internally admissible structural change. Whether later dynamics realize such changes one-by-one or in bundles is a separate coarse-graining question.*

The packet theorem supplies the primitive side of the innovation–consolidation cycle. To obtain a shell-to-shell information channel, one must next extract finite orbit-level invariants and measure the cost of coding them into an admissible receiving shell.

4. Invariant Extraction, Shell Coding, and Fiber Structure

Section 3 identified the primitive packet structure forced by the involutions R and τ. The present section studies the consolidation side of the same transition: how finite packet-level invariants are extracted, how many shell symbols are required to carry them across an admissible elementary step, and how the extractor partitions the domain into exact finite fibers.

4.1. Packet Space and Invariant Extraction

Let $\mathcal{X}_p \subset \mathbb{F}_{p^2}$ be an admissible innovation domain. Its packet space is the quotient

$$\mathcal{P}_p := \mathcal{X}_p / K_p, \tag{12}$$

whose elements are the nondegenerate Klein packets from Proposition 2. Since $\mathcal{X}_p$ is finite, $\mathcal{P}_p$ is finite as well.

Definition 6 (Packet invariant extractor)**.** *A packet invariant extractor is a map*

$$J_p : \mathcal{P}_p \longrightarrow I_p, \tag{13}$$

where I_p is a finite set.

The extractor J_p acts on packets, not on individual states. This orbit-level definition is the natural one for shell-to-shell consolidation, because all members of a packet are structurally tied by the forced symmetries. Typical examples include:

- the norm of any representative of the packet,
- an orbit or symmetry type,
- a finite signature derived from an external evolution,
- any finite coarse-graining constant on packets.

Remark 3. *If an external finite dynamics first produces a pointwise orbit $A_p(\psi) \subset \mathbb{F}_{p^2}$, then the present formalism may be applied after passing to the packet family generated by the genuinely new elements of that orbit. The current paper therefore abstracts the coding problem from the details of the underlying dynamics.*

4.2. Minimal Shell-to-Shell Symbol Count

Let $p = 4t + 1$ and let

$$q_{t+1} = 4(t + 1) + 1 = p + 4$$

denote the order of the elementary receiving shell associated with one admissible packet step from Definition 5. When q_{t+1} is prime, this receiving shell is the field $\mathbb{F}_{q_{t+1}}$; otherwise it is the ring $\mathbb{Z}_{q_{t+1}}$.

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In both cases it provides an alphabet of size q_{t+1}. If one prefers to regard an update by $4n$ residues as a single bundled transition, then the same transfer may be coarse-grained to the shell of order $q_{t+n} = p + 4n$. The coding result below isolates the elementary case because it identifies the minimal admissible carrier. The resulting counting problem is the finite q-ary coding problem familiar from classical coding theory [7,8,12].

Proposition 3 (Minimal symbol count)**.** *Let D_p be any finite innovation domain and let $J_p : D_p \to I_p$ be a finite invariant extractor with image I_p. Then there exists an injective code*

$$C_p : I_p \hookrightarrow R_{t+1}^{M_p}, \tag{14}$$

where R_{t+1} denotes the elementary receiving-shell alphabet, and M_p is the unique least integer $M_p \geq 0$ such that

$$|I_p| \leq q_{t+1}^{M_p}. \tag{15}$$

Moreover, M_p is minimal among all injective codes into tuples over an alphabet of size q_{t+1}.

Proof. An alphabet of size q_{t+1} has exactly q_{t+1}^M words of length M. Hence an injective code of length M exists if and only if

$$q_{t+1}^M \geq |I_p|.$$

Since the set of integers $M \geq 0$ satisfying this inequality is nonempty and linearly ordered, it has a unique least element M_p. Choosing an enumeration of I_p and writing the corresponding indices in base q_{t+1} yields an explicit injective code of that length. □

Corollary 2 (One-symbol transfer)**.** *If $2 \leq |I_p| \leq q_{t+1}$, then one elementary receiving-shell symbol suffices:*

$$M_p = 1. \tag{16}$$

If $|I_p| = 1$, then no shell symbol is required and $M_p = 0$.

Proof. If $2 \leq |I_p| \leq q_{t+1}$, then $q_{t+1}^0 < |I_p| \leq q_{t+1}^1$, so the least admissible symbol count is $M_p = 1$. If $|I_p| = 1$, then $1 \leq q_{t+1}^0$, so $M_p = 0$. □

The shell-to-shell consolidation map for the elementary admissible step is now simply

$$U_p := C_p \circ J_p : D_p \longrightarrow R_{t+1}^{M_p}. \tag{17}$$

This map does not preserve the full innovation domain; it preserves only the chosen finite invariant alphabet.

Remark 4. *In the intrinsic packet-level case one takes $D_p = \mathcal{P}_p$. The more general statement above is useful because explicit examples may begin with a larger finite domain and only then pass to a packet-constant invariant such as the norm.*

4.3. Finite Fiber Decomposition

Definition 7 (Fiber multiplicities)**.** *Let D_p be a finite innovation domain and let $J_p : D_p \to I_p$ be a surjective finite invariant extractor. For each $y \in I_p$, define the fiber*

$$F_y := J_p^{-1}(y) \tag{18}$$

and its multiplicity

$$m_y := |F_y|. \tag{19}$$

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The multiset

$$\mathcal{F}(J_p) := \{\, m_y : y \in I_p \,\} \tag{20}$$

is called the consolidation profile of J_p.

Proposition 4 (Finite fiber decomposition)**.** *With the notation above,*

$$D_p = \bigsqcup_{y \in I_p} F_y \tag{21}$$

and hence

$$|D_p| = \sum_{y \in I_p} |F_y|. \tag{22}$$

Proof. Since J_p is surjective, every element of D_p belongs to a fiber F_y for some $y \in I_p$. Distinct fibers are disjoint by definition, so they form a disjoint partition of D_p. Taking cardinalities gives the stated identity. □

Corollary 3 (Pigeonhole multiplicity bound)**.** *Let*

$$\mu_p := \max_{y \in I_p} |F_y|. \tag{23}$$

Then μ_p *is at least the least integer* $n \geq 1$ *satisfying*

$$|D_p| \leq n\,|I_p|. \tag{24}$$

In particular, if $|D_p| > |I_p|$*, then some invariant value identifies at least two distinct states of* D_p.

Proof. If every fiber had cardinality at most $n-1$, then Proposition 4 would give

$$|D_p| \leq (n-1)|I_p|,$$

contrary to the minimality of n. Hence some fiber has cardinality at least n. The final claim is the case $n = 2$. □

Packet counting, consolidation profile, and shell coding therefore measure different aspects of shell evolution:

- packet counting measures the growth of primitive support;
- the invariant alphabet measures how many values survive consolidation;
- fiber multiplicities measure how many primitive states are identified by the extractor;
- the symbol count M_p measures the shell-to-shell transmission cost of that alphabet.

Remark 5. *The fiber results do not require the extractor to be packet-based; they apply to any finite innovation domain. This flexibility is useful in explicit examples, where one may wish to work on the full quadratic extension rather than only on the nondegenerate packet locus.*

4.4. Finite Coding Thresholds

The minimal symbol count from Proposition 3 admits an exact threshold description that avoids logarithmic notation and makes the finite coding regimes explicit [7,8].

Proposition 5 (Exact m-symbol threshold)**.** *Let* $m \geq 1$*. Then the minimal symbol count satisfies* $M_p = m$ *if and only if*

$$q_{t+1}^{m-1} < |I_p| \leq q_{t+1}^{m}. \tag{25}$$

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Proof. By Proposition 3, M_p is the least integer $M \geq 0$ such that $|I_p| \leq q_{t+1}^M$. This least integer is m exactly when the inequality fails for $M = m-1$ and succeeds for $M = m$, which is precisely the stated condition. □

Corollary 4 (One-symbol threshold). *If $2 \leq |I_p| \leq p+4$, then one receiving-shell symbol suffices and $M_p = 1$.*

Proof. Since $q_{t+1} = p+4$, this is the case $m = 1$ in Proposition 5. □

Corollary 5 (Two-symbol threshold). *If $p+4 < |I_p| \leq (p+4)^2$, then two receiving-shell symbols suffice and $M_p = 2$.*

Proof. Since $q_{t+1} = p+4$, this is the case $m = 2$ in Proposition 5. □

Remark 6. *These thresholds separate two distinct finite scales in FRC. Primitive growth occurs in packets of size four by Corollary 1, whereas shell-transfer cost is determined only by the size of the invariant alphabet. A large innovation domain may therefore still admit a short shell code if the extractor consolidates it to a small finite alphabet.*

The next section makes these quantities explicit in the first nontrivial prime example.

5. Worked Example: Prime 13

We now make the packet, coding, and fiber statements fully explicit for the first sufficiently roomy symmetry-complete prime checkpoint

$$p = 13 = 4 \cdot 3 + 1.$$

The smaller prime $p = 5$ already admits quadratic nonsquares, but $p = 13$ provides a cleaner nondegenerate example while keeping the arithmetic fully explicit.

5.1. Quadratic Extension and Explicit Packet

Choose the quadratic nonsquare $\nu = 2 \in \mathbb{F}_{13}$. Then

$$\mathbb{F}_{13^2} := \mathbb{F}_{13}[X]/(X^2-2), \qquad c := X \pmod{X^2-2}, \qquad c^2 = 2. \tag{26}$$

Every element has a unique representation

$$z = a + cb, \qquad a, b \in \mathbb{F}_{13},$$

and the Frobenius involution acts by

$$\tau(z) = z^{13} = a - cb.$$

Consider

$$z = 1 + c.$$

Since both coefficients are nonzero, z is nondegenerate. Its Klein packet is

$$\mathcal{O}_{K_{13}}(1+c) = \{\, 1+c,\ -1-c,\ 1-c,\ -1+c \,\}. \tag{27}$$

This is an explicit instance of Proposition 1. It follows immediately from Corollary 1 that one such nondegenerate packet produces the minimal shell increment

$$13 \longmapsto 17.$$

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Thus $\mathbb{F}_{17}$ is the elementary receiving shell for this transition, and this single packet already realises one elementary admissible update. Any larger update by $13 \mapsto 13+4n$ would be a bundled coarse-graining of the same primitive step.

5.2. Norm Invariant, One-Symbol Coding, and Fiber Profile

To illustrate invariant extraction, let

$$D_{13} := \mathbb{F}_{13^2}$$

and consider the norm map

$$N : D_{13} \to \mathbb{F}_{13}, \qquad N(a+cb) = (a+cb)(a-cb) = a^2 - 2b^2 \pmod{13}. \tag{28}$$

Because $N(-z) = N(z)$ and $N(\tau(z)) = N(z)$, the norm is constant on every Klein packet. It therefore defines a packet-level invariant wherever packets are defined. In the present paper, the norm is used only as an exact algebraic example of a packet-constant invariant. No claim is made here that it coincides with a specific physical observable.

It is classical that the norm map $N : \mathbb{F}_{p^2}^{\times} \to \mathbb{F}_p^{\times}$ is surjective for every prime p [4,5]. Hence

$$N(\mathbb{F}_{13^2}^{\times}) = \mathbb{F}_{13}^{\times}.$$

Since $N(0) = 0$, the invariant alphabet is

$$I_{13} = \mathbb{F}_{13}, \qquad |I_{13}| = 13.$$

The elementary receiving shell has alphabet size $q_{t+1} = 17$. By Corollary 2,

$$M_{13} = 1,$$

because $2 \le 13 \le 17$. Thus one shell symbol is enough to carry the full norm alphabet. An explicit code is

$$C_{13}(x) := x, \qquad x \in \{0, 1, \ldots, 12\} \subset \mathbb{F}_{17}. \tag{29}$$

The domain $D_{13} = \mathbb{F}_{13^2}$ has cardinality

$$|D_{13}| = 13^2 = 169.$$

The output alphabet $I_{13} = \mathbb{F}_{13}$ has cardinality 13. Since $\mathbb{F}_{13^2}$ is a field, $N(z) = 0$ holds only for $z = 0$. Thus

$$|N^{-1}(0)| = 1. \tag{30}$$

For every nonzero value $y \in \mathbb{F}_{13}^{\times}$, surjectivity of the norm map and the standard kernel count give

$$|N^{-1}(y)| = p + 1 = 14. \tag{31}$$

This is the standard norm-fiber count for quadratic extensions of finite fields [4,5]. Hence the consolidation profile of the norm extractor is

$$\mathcal{F}(N) = \{\, 1, 14, 14, \ldots, 14 \,\}, \tag{32}$$

with the value 14 occurring twelve times. Equivalently,

$$169 = |D_{13}| = |N^{-1}(0)| + \sum_{y \in \mathbb{F}_{13}^{\times}} |N^{-1}(y)| = 1 + 12 \cdot 14. \tag{33}$$

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The fiber profile makes the consolidation effect fully explicit at the algebraic level: the zero value remains isolated, while every nonzero invariant value gathers fourteen primitive states of the quadratic extension.

This example realises four distinct finite scales introduced in the paper:

1. a primitive four-element innovation packet in $\mathbb{F}_{13^2}$;
2. a finite invariant alphabet $I_{13} = \mathbb{F}_{13}$;
3. an exact consolidation profile $169 = 1 + 12 \cdot 14$;
4. a one-symbol code in the elementary receiving shell $\mathbb{F}_{17}$.

6. Discussion

The revised results separate three algebraic layers that had previously been bundled together in informal language.

Primitive Growth and Invariant Consolidation. At a symmetry-complete prime checkpoint, genuinely new admissible residues occur in four-element Klein packets. This is a statement about existential admissibility at the primitive level, not about the size of any chosen invariant alphabet. The immediate shell-growth consequence is the update law $q_{\text{new}} = q_{\text{old}} + 4n$, with $+4$ as the elementary admissible case. Whether a physical or mathematical evolution realises these packets sequentially or in bundles is secondary; the primitive admissible increment is already fixed. Packet-level invariants, by contrast, live on the quotient $\mathcal{P}_p = \mathcal{X}_p / K_p$, not on isolated elements. The invariant extractor J_p reduces the innovation data to a finite alphabet I_p, while Proposition 4 records exactly how the original domain is partitioned into fibers over that alphabet. The surviving structure is therefore measured by finite cardinalities alone: the size of the invariant alphabet and the multiplicities of its fibers.

Shell Transfer Within FRC. Once an invariant alphabet I_p is fixed, Proposition 3 interprets consolidation as a finite coding problem over the alphabet of the elementary receiving shell. The quantity M_p is the minimal shell-to-shell symbol cost of transmitting that invariant alphabet across one admissible primitive step, and Proposition 5 gives exact finite thresholds for when one symbol, two symbols, or more are required. Bundled updates may be regarded as coarse-grained compositions of this elementary channel. In this sense, shell transfer is governed by a finite minimality principle rather than by any externally imposed staging. This paper is intentionally upstream in the FRC programme: it assumes the prime-checkpoint structure and studies what the quadratic extension forces at the level of innovation packets, invariant extraction, finite consolidation, and shell coding, without relying on a separate epistemic reconstruction.

Relation to Dirac Dynamics and Learning Analogies. The finite-field Dirac formalism remains an important source of algebraic motivation [3]. The present paper does not compete with that construction; it abstracts the shell-to-shell transfer problem from it. Any explicit finite evolution on the quadratic extension that produces a finite innovation domain can be analysed by the packet, coding, and fiber machinery introduced here. Equally importantly, the present paper does not claim that the invariant families studied here already correspond to specific physical observables. In particular, the norm in Section 5 is used only as a packet-constant invariant with exactly computable fibers. The assignment of such invariants to observables such as mass, charge, or spin is deferred to follow-up work. The structural analogy with learning systems remains suggestive but is no longer carrying the paper mathematically. In the present revision, "innovation" and "consolidation" are specified by packet size, invariant alphabet, symbol count, and fiber multiplicity, so any later comparison with biological or computational learning is anchored in explicit finite quantities. Relevant comparison points include representation learning, predictive processing, and Bayesian or structure-learning viewpoints [13–17].

Minimality and Limits. The packet law may be read as a finite analogue of familiar minimality principles in physics [18]. The claim is not that the paper imports variational calculus into FRC. Rather, it identifies the least admissible structural perturbation compatible with the internal symmetry data available at a prime checkpoint. In that restricted but precise sense, the elementary four-packet plays the role of a basic perturbation unit. The present paper proves packet and coding results at prime

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checkpoints, but it does not yet classify all natural invariant families or compute full Dirac-orbit statistics. The norm example at $p = 13$ should therefore be read as a first exact calibration point rather than as a complete statistical description of quadratic-extension dynamics in FRC. Overall, the revised innovation–consolidation picture is now algebraically sharper: primitive innovation is packetised, invariant extraction is finite, consolidation is described by exact fiber profiles, and shell transfer is minimal shell coding. These notions can be studied independently and then recombined in later FRC developments, with the elementary admissible step serving as the basic unit throughout.

7. Conclusions

This paper reformulated the FRC innovation–consolidation cycle as a finite algebraic transfer problem at symmetry-complete prime checkpoints. Three results are central.

First, genuinely new admissible innovation in the quadratic extension $\mathbb{F}_{p^2}$ is packetised: every nondegenerate innovation domain is a disjoint union of four-element Klein packets generated by the spatial and temporal involutions. Primitive shell growth therefore occurs in forced increments $q_{\text{new}} = q_{\text{old}} + 4n$, with $+4$ as the elementary admissible case.

Second, packet-level invariants determine a finite alphabet I_p that can be transmitted to the elementary receiving shell with minimal symbol count M_p, characterised as the least integer satisfying

$$|I_p| \leq q_{t+1}^{M_p}.$$

This gives consolidation a concrete operational meaning: it is the minimal shell-to-shell code needed to preserve the chosen invariant family.

Third, invariant extraction admits an exact finite fiber decomposition

$$|D_p| = \sum_{y \in I_p} |J_p^{-1}(y)|.$$

In the worked example $p = 13$, the norm invariant yields a one-symbol code into $\mathbb{F}_{17}$ and the exact profile $169 = 1 + 12 \cdot 14$. In the present paper this norm is used only as an exactly computable algebraic invariant, not as an identified physical observable.

The revised perspective clarifies the division of labour inside FRC. Primitive innovation, invariant consolidation, and shell-to-shell coding are distinct finite processes governed by existential admissibility rather than by an external operating protocol. Their clean separation makes the current paper an upstream algebraic component of the wider FRC programme: it does not derive the whole hierarchy, but it specifies the packet, coding, and fiber machinery by which shell-to-shell structure can be transferred once the quadratic prime-checkpoint framework is given. In this form, the four-packet identifies the least admissible structural perturbation, while larger updates are coarse-grained bundles of the same primitive move.

Future work should classify natural invariant families arising from explicit Dirac-type dynamics, compute their exact fiber profiles, and determine when one-symbol transfer persists beyond the norm example. A separate task, intentionally deferred here, is to determine whether any of these finite invariants correspond to specific physical observables such as mass, charge, or spin. These questions are now sharply posed in finite algebraic terms.

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