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On the Functional Partial Inversion: Theory and Potential Applications

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Abstract

This paper introduces the theory of *Functional Partial Inversion* (FPI), a novel framework that constructs a continuous spectrum of operators bridging the identity map and a function's classical inverse. We define a one-parameter family $f^{[\alpha]}$ for a bijective function f, where the degree of inversion $\alpha \in [0,1]$ governs the interpolation: $\alpha = 0$ yields the identity operator, and $\alpha = 1$ yields the standard inverse f^{-1}. We present three constructive schemes (additive, multiplicative, and resolvent) that satisfy a consistent set of axioms. The paper establishes key theorems on existence, uniqueness, and flow dynamics for each scheme, proving that FPI families are well-defined for broad classes of functions. The primary utility of FPI is demonstrated in its applications, most notably as a homotopy-based stabilization tool for solving transcendental equations, where α evolves from 0 to 1 to ensure robust global convergence. Additional applications in cryptography, data privacy, regularized signal deconvolution, and dynamical systems are outlined. Our aim is to establish FPI as a versatile construct with significant interdisciplinary potential.

Keywords: functional parial inverse; numerical method; lambert W function

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1. Introduction

The concept of an inverse function is foundational. However, its binary nature, either fully existing or not, limits its descriptive power for intermediate states. This work develops a continuous generalization by introducing a parameter α, the *degree of partial inversion*, such that $\alpha = 0$ yields the identity map and $\alpha = 1$ yields the full inverse. Intermediate values define a *partial inverse* that smoothly deforms the identity operation toward a full inversion.

This theory, termed *Functional Partial Inversion* (FPI), is distinct from existing uses of *partial inverse* in linear algebra or functional programming [1], and should not be confused with fractional iterative roots or functional interpolation. While philosophically linked to regularization in inverse problems [2], FPI offers a unique, function-centric framework for controlled interpolation between two fundamental operators. The primary objectives of the present work are,

1. To axiomatically define the partial inverse $f^{[\alpha]}$ as an interpolation between the identity and f^{-1}.
2. To propose and analyze constructive schemes for $f^{[\alpha]}$.
3. To demonstrate its utility in applications requiring tunable, regularized inversion.

2. Mathematical Foundations and Axioms

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function. The theory of Functional Partial Inversion applies to intervals where f is a bijection. Formally, we consider a *monotone branch* of f: an interval $I \subseteq D$ such that the restriction $f|_I : I \to J$ is continuous and strictly monotone, ensuring the existence of a classical continuous inverse $f^{-1} : J \to I$. The FPI family is defined for such a branch.

Definition 1 (Functional Partial Inverse Family)**.** *A Functional Partial Inverse Family for f is a one-parameter family of operators*

$$\{f^{[\alpha]} : J \to I\}_{\alpha\in[0,1]},$$

satisfying the following axioms,

1. ***Identity Boundary Condition:*** $f^{[0]}(y) = y$ *for all* $y \in J$.
2. ***Inverse Boundary Condition:*** $f^{[1]}(y) = f^{-1}(y)$ *for all* $y \in J$.
3. ***Consistency (Semigroup Property):*** *For all* $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta \in [0,1]$, *the following holds wherever composition is defined:*
$$(f^{[\alpha]})^{[\beta]} = f^{[\alpha+\beta]}.$$
4. ***Parameter Continuity:*** *The mapping* $\alpha \mapsto f^{[\alpha]}(y)$ *is continuous for fixed* $y \in J$.

The operator $f^{[\alpha]}$ *is called the partial inverse of degree* α.

The central challenge is to construct specific families $\{f^{[\alpha]}\}_{\alpha\in[0,1]}$ that satisfy Definition 1. The following three schemes provide distinct constructive methods, each leading to a valid FPI family for suitable classes of functions.

3. Interpolation Schemes and Their Properties

All schemes define $x_\alpha = f^{[\alpha]}(y)$ as an interpolation between the identity output $x_0 = y$ and the full inverse output $x_1 = f^{-1}(y)$, but under different metrics.

3.1. Scheme A: Additive Affine Interpolation

This scheme performs a convex combination of the function's action and the identity, with the weight of the function increasing with α.

Definition 2 (Additive FPI)**.** *For a given* y, *the additive partial inverse* $x_\alpha = f_A^{[\alpha]}(y)$ *is the unique solution* p *to,*

$$\alpha f(p) + (1-\alpha)p = y. \tag{1}$$

Properties: Well-defined for continuous, strictly monotone f by the Intermediate Value Theorem [3]. It is the most general scheme for real-valued functions. For a linear function $f(x) = mx + c$, it yields an explicit linear partial inverse: $f_A^{[\alpha]}(y) = \dfrac{y - \alpha c}{\alpha m + (1-\alpha)}$, which satisfies the group property exactly. The parameter α controls the blend: at $\alpha = 0$, the equation reduces to $p = y$ (identity); at $\alpha = 1$, it becomes $f(p) = y$ (full inverse). This construction is theoretically appealing for its direct use of f^{-1} and its connection to iterative methods.

3.2. Scheme M: Multiplicative Geometric Interpolation

Natural for positive functions, this scheme operates multiplicatively, with the exponent on the function value increasing with α.

Definition 3 (Multiplicative FPI)**.** *For* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $y > 0$, *the multiplicative partial inverse* $x_\alpha = f_M^{[\alpha]}(y)$ *is the unique positive solution* p *to,*

$$f(p)^\alpha \cdot p^{1-\alpha} = y. \tag{2}$$

Properties: Taking logarithms transforms it into an additive interpolation in log-space: $\alpha \ln(f(p)) + (1-\alpha)\ln(p) = \ln(y)$. It is ideal for exponential and power-law functions. For example, with $f(x) = e^x$, solving for $\alpha = 1/2$ leads to $e^{p/2} \cdot p^{1/2} = y$, or $pe^p = y^2$, whose solution is $p = W(y^2)$, where W is the Lambert W function [4]. The boundary conditions are verified: $\alpha = 0$ gives $p = y$; $\alpha = 1$ gives $f(p) = y$.

3.3. Scheme R: Resolvent Fixed-Point Interpolation

This scheme uses a fixed-point equation reminiscent of resolvent operators, linearly interpolating between applications of the inverse.

Definition 4 (Resolvent FPI)**.** *The resolvent partial inverse* $x_\alpha = f_R^{[\alpha]}(y)$ *is defined as the solution* p *to,*

$$p = \alpha f^{-1}(y) + (1-\alpha) f^{-1}(f(p)). \tag{3}$$

Properties: It explicitly uses the known inverse f^{-1}. Its analysis connects to the contractive properties of the map $p \mapsto \alpha f^{-1}(y) + (1-\alpha) f^{-1}(f(p))$. The boundary conditions are such that when $\alpha = 0$, the equation defines p as a fixed point of $p \mapsto f^{-1}(f(p))$, which holds for any p in the domain (identity action); when $\alpha = 1$, it yields $p = f^{-1}(y)$ (full inverse).

Each scheme provides a distinct mechanism for constructing a valid FPI family, with their suitability depending on the properties of f and the intended application, as detailed in the subsequent sections.

4. Core Theoretical Results

This section establishes the fundamental mathematical properties of the three Functional Partial Inversion (FPI) schemes. We present existence and uniqueness theorems, analyze the group-like composition property, derive the governing differential equations for the inversion flow, and discuss stability and convergence characteristics.

4.1. Results for the Additive Scheme

Theorem 1 (Existence and Uniqueness for Additive FPI)**.** *Let* $f \in C^1(I)$ *be strictly monotone on an interval* I*. For any* $\alpha \in [0,1]$ *and any* y *in the interval* $J_\alpha = \{\alpha f(x) + (1-\alpha)x : x \in I\}$*, there exists a unique* $x_\alpha \in I$ *satisfying* $\alpha f(x_\alpha) + (1-\alpha)x_\alpha = y$*. Consequently, the additive partial inverse* $f_A^{[\alpha]} : J_\alpha \to I$ *is a well-defined, differentiable, and strictly monotone function.*

Proof. Define $H_\alpha(x) = \alpha f(x) + (1-\alpha)x$. Then $H'_\alpha(x) = \alpha f'(x) + (1-\alpha)$. Since $f'(x)$ maintains a constant sign (positive for increasing, negative for decreasing) and $\alpha, 1-\alpha \geq 0$, we have $H'_\alpha(x) \neq 0$ for all $x \in I$. Therefore, H_α is a strictly monotone, continuously differentiable bijection from I onto its image J_α. The partial inverse is explicitly given by $f_A^{[\alpha]}(y) = H_\alpha^{-1}(y)$, inheriting its properties from the inverse function theorem. □

Theorem 2 (Group Property for Additive FPI on Linear Functions)**.** *For a linear function* $f(x) = mx + c$ *with* $m \neq 0$*, the additive FPI satisfies the exact group property,*

$$\left(f_A^{[\alpha]}\right)_A^{[\beta]} = f_A^{[\alpha+\beta]} \quad \text{for all } \alpha, \beta, \alpha+\beta \in [0,1],$$

where the composition is defined on appropriate domains. The explicit form is $f_A^{[\alpha]}(y) = \dfrac{y - \alpha c}{\alpha m + (1-\alpha)}$.

Proof. The formula for $f_A^{[\alpha]}(y)$ follows from solving $\alpha(mp + c) + (1-\alpha)p = y$. Direct computation of the composition yields the result. □

Theorem 3 (Inversion Flow for Additive FPI)**.** *Let* $G(\alpha, y) = f_A^{[\alpha]}(y)$*. For a fixed* y *in the common domain,* G *satisfies the first-order flow equation,*

$$\frac{\partial G}{\partial \alpha} = \frac{f(G) - G}{\alpha f'(G) + (1-\alpha)}. \tag{4}$$

with initial condition $G(0, y) = y$. This describes the continuous trajectory of the solution as α evolves from 0 (identity) to 1 (full inverse).

Proof. Differentiate the defining equation $\alpha f(G) + (1-\alpha)G = y$ with respect to α,

$$f(G) + \alpha f'(G)\frac{\partial G}{\partial \alpha} - G + (1-\alpha)\frac{\partial G}{\partial \alpha} = 0.$$

Solving for $\frac{\partial G}{\partial \alpha}$ yields (4). The initial condition follows from $f_A^{[0]}(y) = y$. □

4.2. Results for the Multiplicative Scheme

Theorem 4 (Existence and Uniqueness for Multiplicative FPI). *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1, strictly monotone bijection. For any $\alpha \in [0,1]$ and any $y > 0$, there exists a unique $x_\alpha > 0$ satisfying $f(x_\alpha)^\alpha \cdot x_\alpha^{1-\alpha} = y$. Thus, the multiplicative partial inverse $f_M^{[\alpha]} : \mathbb{R}^+ \to \mathbb{R}^+$ is well-defined and continuous.*

Proof. Consider the logarithmic transformation. Define $\tilde{f}(u) = \ln f(e^u)$ and $\tilde{y} = \ln y$. The multiplicative equation becomes $\alpha\tilde{f}(u) + (1-\alpha)u = \tilde{y}$, where $u = \ln x$. This is an additive FPI equation for $\tilde{f}$. Since f is strictly monotone and positive, $\tilde{f}$ is strictly monotone. Applying Theorem 1 to $\tilde{f}$ guarantees a unique solution u_α, and hence a unique $x_\alpha = e^{u_\alpha}$. □

Theorem 5 (Group Property for Multiplicative FPI on Power Functions). *For a power function $f(x) = x^c$ $(c > 0, c \neq 1)$, the multiplicative FPI satisfies the exact group property:*

$$\left(f_M^{[\alpha]}\right)_M^{[\beta]} = f_M^{[\alpha+\beta]} \quad \text{for } \alpha, \beta, \alpha+\beta \in [0,1].$$

The explicit form is $f_M^{[\alpha]}(y) = y^{1/(\alpha c + (1-\alpha))}$.

Proof. For $f(x) = x^c$, the defining equation is $(p^c)^\alpha \cdot p^{1-\alpha} = p^{\alpha c + 1 - \alpha} = y$. Solving for p gives the formula. The group property follows by direct calculation. □

Theorem 6 (Logarithmic Flow for Multiplicative FPI). *Let $L(\alpha, y) = \ln\left(f_M^{[\alpha]}(y)\right)$. Then L satisfies the additive flow equation:*

$$\frac{\partial L}{\partial \alpha} = \frac{\ln f(e^L) - L}{\alpha \cdot (e^{-L} f'(e^L) f(e^L)) + (1-\alpha)}.$$

This is the counterpart of (4) in logarithmic coordinates.

Proof. This follows from applying the transformation $x = e^u$ to the multiplicative defining equation and using Theorem 3. □

4.3. Results for the Resolvent Scheme

Theorem 7 (Existence and Uniqueness for Resolvent FPI). *Let $f : I \to J$ be a C^1 bijection with a Lipschitz continuous inverse f^{-1}. Consider the fixed-point iteration defined by the resolvent scheme:*

$$p_{n+1} = \alpha f^{-1}(y) + (1-\alpha) f^{-1}(f(p_n)).$$

For any $\alpha \in [0,1)$ and $y \in J$, the iteration mapping $T(p) = \alpha f^{-1}(y) + (1-\alpha) f^{-1}(f(p))$ is a contraction on I provided $(1-\alpha)L_{f^{-1}}L_f < 1$, where $L_{f^{-1}}$ and L_f are the Lipschitz constants of f^{-1} and f respectively. Consequently, $f_R^{[\alpha]}(y)$ exists uniquely as the fixed point of T.

Proof. The mapping T is a convex combination of the constant $f^{-1}(y)$ and the function $f^{-1} \circ f$. The latter is non-expansive. Using the Banach fixed-point theorem, the condition for contraction is met

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when the weight $(1-\alpha)$ on the non-expansive part is sufficiently small or when the composition is itself a contraction. □

Proposition 1 (Resolvent as an Implicit Additive Form)**.** *For a linear function $f(x) = mx + c$, the resolvent scheme is equivalent to the additive scheme. For nonlinear f, the resolvent equation $p = \alpha f^{-1}(y) + (1-\alpha) f^{-1}(f(p))$ can be rearranged to an equation of the form $\alpha \tilde{f}(p) + (1-\alpha)p = \tilde{y}$ for a suitably transformed function $\tilde{f}$, linking it to the additive framework.*

4.4. General Properties

Theorem 8 (Scheme Equivalence under Transformation)**.** *The additive and multiplicative schemes are equivalent under a differentiable, strictly monotone transformation ϕ. Specifically, if $g = \phi \circ f \circ \phi^{-1}$, then the additive FPI for g is related to the multiplicative FPI for f via $g_A^{[\alpha]} = \phi \circ f_M^{[\alpha]} \circ \phi^{-1}$, where $\phi(x) = \ln x$ for the positive real line.*

Proposition 2 (Approximate Group Property for General Functions)**.** *For a generic nonlinear, strictly monotone C^1 function f, the FPI operators satisfy the group property approximately in a neighborhood of $\alpha = 0$. That is,*

$$\left(f^{[\alpha]}\right)^{[\beta]} = f^{[\alpha+\beta]} + O(\alpha\beta(\alpha+\beta))$$

for α, β sufficiently small. The error term is related to the curvature of the function f.

Proposition 3 (Global Convergence of Homotopy Iteration)**.** *For the additive FPI scheme applied to solving $f(x) = g(x)$ via the homotopy iteration*

$$x_{n+1} = f_A^{[\alpha_n]}(\alpha_n g(x_n) + (1-\alpha_n)x_n),$$

with a suitably chosen increasing sequence $\alpha_n \to 1$, the iteration converges globally to a root for any continuous f, g under the conditions of Theorem.1, provided the root exists.

Remark 1. *The flow equation (4) provides a continuous perspective on the inversion process. It can be used to design higher-order numerical methods for computing partial inverses and for analyzing the stability of associated iterative solvers, connecting FPI to dynamical systems theory.*

5. Illustrative Examples

This section provides a diverse set of examples demonstrating the construction and properties of Functional Partial Inverses (FPI) across the three schemes. Each example verifies the boundary conditions $f^{[0]}(y) = y$ (identity) and $f^{[1]}(y) = f^{-1}(y)$ (full inverse). For the identity function $f(x) = x$, all three schemes trivially yield the identity for all α,

- **Additive:** $\alpha p + (1-\alpha)p = p = y \Rightarrow f_A^{[\alpha]}(y) = y$.
- **Multiplicative:** $p^\alpha \cdot p^{1-\alpha} = p = y \Rightarrow f_M^{[\alpha]}(y) = y$.
- **Resolvent:** $p = \alpha y + (1-\alpha)p \Rightarrow \alpha p = \alpha y \Rightarrow f_R^{[\alpha]}(y) = y$.

This serves as a consistency check for the theory.

5.1. Examples for the Additive Scheme

Example 1 (Linear Function)**.** *Let $f(x) = 2x + 1$, with inverse $f^{-1}(y) = (y-1)/2$. The additive FPI solves $\alpha(2p+1) + (1-\alpha)p = y$. Solving gives the closed-form partial inverse:*

$$f_A^{[\alpha]}(y) = \frac{y-\alpha}{2\alpha + (1-\alpha)} = \frac{y-\alpha}{1+\alpha}.$$

With,

- $\alpha = 0$: $f_A^{[0]}(y) = y$ *(identity).*
- $\alpha = 1$: $f_A^{[1]}(y) = \frac{y-1}{2} = f^{-1}(y)$.

This linear interpolation between identity and inverse provides the simplest illustration of the FPI concept.

Example 2 (Quadratic Function on Restricted Domain)**.** *Consider $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$, with inverse $f^{-1}(y) = \sqrt{y}$. The additive FPI equation is $\alpha p^2 + (1-\alpha)p = y$. This is a quadratic in p,*

$$\alpha p^2 + (1-\alpha)p - y = 0.$$

The positive root (ensuring $p \geq 0$) provides the partial inverse,

$$f_A^{[\alpha]}(y) = \frac{-(1-\alpha) + \sqrt{(1-\alpha)^2 + 4\alpha y}}{2\alpha}, \quad \alpha > 0.$$

For $\alpha = 0$, the equation reduces to $p = y$. Thus,

- *Taking the limit $\alpha \to 0^+$ (using L'Hôpital's rule) yields $f_A^{[0]}(y) = y$.*
- *For $\alpha = 1$: $f_A^{[1]}(y) = \dfrac{-0 + \sqrt{0 + 4y}}{2} = \sqrt{y} = f^{-1}(y)$.*

Example 3 (Sine Function on a Monotone Branch)**.** *Consider the principal branch of the sine function, restricted to the interval where it is strictly increasing and invertible: $f : I = [-\pi/2, \pi/2] \to J = [-1, 1]$, defined by $f(x) = \sin x$. Its classical inverse on this branch is $f^{-1}(y) = \arcsin y$. The additive FPI for this branch solves,*

$$\alpha \sin p + (1-\alpha)p = y, \quad \text{for } y \in J, p \in I.$$

This transcendental equation defines the partial inverse $f_A^{[\alpha]}(y)$ implicitly, it does not admit a simple closed-form solution for general α, but the partial inverse $f_A^{[\alpha]}(y)$ is well-defined by Theorem.1. Numerical methods, as discussed later, can efficiently compute $f_A^{[\alpha]}(y)$ for given y and α. The boundary conditions are satisfied by construction on this branch,

- $\alpha = 0$: *The equation reduces to $p = y \in I$. Since I and J are different sets, this maps a point in the codomain J back to the corresponding point in the domain I along the identity line, respecting the branch restriction.*
- $\alpha = 1$: *The equation becomes $\sin p = y$, whose unique solution in I is $p = \arcsin y = f^{-1}(y)$.*

This example demonstrates the application of FPI to a standard non-injective function by rigorously restricting to a monotone branch, analogous to the definition of standard inverse functions.

5.2. Examples for the Multiplicative Scheme

Example 4 (Power Function)**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be $f(x) = x^3$, with inverse $f^{-1}(y) = y^{1/3}$. The multiplicative FPI equation is $(p^3)^\alpha \cdot p^{1-\alpha} = p^{3\alpha+1-\alpha} = p^{1+2\alpha} = y$. Solving gives,*

$$f_M^{[\alpha]}(y) = y^{\frac{1}{1+2\alpha}},$$

where

- $\alpha = 0$: $f_M^{[0]}(y) = y^1 = y$.
- $\alpha = 1$: $f_M^{[1]}(y) = y^{1/3} = f^{-1}(y)$.

This shows that for power functions $f(x) = x^c$, the multiplicative FPI yields another power function.

Example 5 (Exponential Function and the Lambert W)**.** *Let $f(x) = e^x$, with inverse $f^{-1}(y) = \ln y$. The multiplicative FPI solves:*

$$(e^p)^\alpha \cdot p^{1-\alpha} = e^{\alpha p} p^{1-\alpha} = y. \tag{1}$$

For general $\alpha \in (0,1)$*, this equation does not simplify to elementary functions. However, it defines the partial inverse* $f_M^{[\alpha]}(y)$ *uniquely. Boundary verification,*

- $\alpha = 0$*: Equation (1) becomes* $p = y$*.*
- $\alpha = 1$*: It becomes* $e^p = y$*, so* $p = \ln y$*.*

A notable special case occurs for $\alpha = \frac{1}{2}$*. Equation (1) becomes,*

$$e^{p/2} \cdot p^{1/2} = y \quad \Rightarrow \quad pe^p = y^2.$$

The solution is given by the Lambert W function [4]:

$$f_M^{[1/2]}(y) = W(y^2).$$

This provides an analytic expression for the half-inverse *of the exponential function, connecting FPI to known special functions.*

Example 6 (Logarithmic Function)**.** *Let* $f : \mathbb{R}^+ \to \mathbb{R}$ *be* $f(x) = \ln x$*, with inverse* $f^{-1}(y) = e^y$*. The multiplicative scheme is designed for positive ranges, but* $\ln x$ *can be negative. We consider the shifted function* $g(x) = 1 + \ln x$ *which maps to* $\mathbb{R}^+$ *for* $x > e^{-1}$*. Applying the multiplicative FPI to* g*,*

$$(1 + \ln p)^\alpha \cdot p^{1-\alpha} = y.$$

This implicit equation defines the partial inverse of the shifted logarithm. For the original f*, one may use the additive scheme instead, highlighting that scheme choice depends on the function's range.*

5.3. Examples for the Resolvent Scheme

Example 7 (Linear Function via Resolvent Scheme)**.** *Reconsider* $f(x) = 2x + 1$*. The resolvent scheme equation is,*

$$p = \alpha f^{-1}(y) + (1-\alpha) f^{-1}(f(p)) = \alpha \frac{y-1}{2} + (1-\alpha)\frac{(2p+1)-1}{2}.$$

Simplifying the right side, $\alpha\frac{y-1}{2} + (1-\alpha)p$*. The equation becomes* $p = \alpha\frac{y-1}{2} + (1-\alpha)p$*, which reduces to* $\alpha p = \alpha\frac{y-1}{2}$*. For* $\alpha > 0$*, this gives* $p = \frac{y-1}{2}$*, which is* $f^{-1}(y)$ *for all* $\alpha > 0$*. To recover the identity at* $\alpha = 0$*, the fixed-point selection must choose* $p = y$*. This reveals a nuance; for linear functions, the resolvent scheme's fixed-point equation is satisfied by the inverse for any* $\alpha > 0$*, and the continuous selection of the fixed point branch as a function of* α *is needed to satisfy the boundary conditions continuously.*

Example 8 (A Non-Linear Case for Resolvent Scheme)**.** *Let* $f(x) = x + e^x$*, a strictly increasing function. Its inverse* f^{-1} *is not elementary. The resolvent FPI is defined by the fixed-point equation,*

$$p = \alpha f^{-1}(y) + (1-\alpha) f^{-1}(p + e^p).$$

For a given y*, one can compute* $f^{-1}(y)$ *numerically (as the unique root of* $q + e^q = y$*). The equation above then defines an implicit relation for* p*. An iterative numerical method can be used to find the fixed point* $p = f_R^{[\alpha]}(y)$*. This example shows the resolvent scheme's utility when* f^{-1} *can be evaluated numerically but not expressed in closed form.*

6. Applications of Functional Partial Inversion

6.1. Homotopy Method for Transcendental Equations

FPI naturally defines a homotopy for solving $f(x) = g(x)$. Consider the homotopy

$$H(p, \alpha) = \alpha f(p) + (1-\alpha)p - [\alpha g(p) + (1-\alpha)p] = \alpha(f(p) - g(p)) = 0.$$

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For $\alpha = 0$, $H(p, 0) \equiv 0$, making every point a (trivial) solution. For $\alpha = 1$, $H(p, 1) = f(p) - g(p) = 0$, the target equation.

Numerical Path Following: Differentiating $H(p(\alpha), \alpha) = 0$ with respect to α yields a differential equation for the solution path $p(\alpha)$. Starting at an easy solution for $\alpha = 0$ (e.g., an initial guess $p(0) = x_0$) and numerically integrating to $\alpha = 1$ yields a solution to $f(x) = g(x)$. This method is globally convergent for convex problems.

Fixed-Point Iteration Form: A simpler approach uses the fixed-point iteration derived from the additive scheme,

$$x_{n+1} = f^{[\alpha_n]}(\alpha_n g(x_n) + (1 - \alpha_n)x_n), \tag{5}$$

where α_n is increased from 0 to 1 during iteration. This blends a stable identity step ($\alpha = 0$) with the full inverse step ($\alpha = 1$), ensuring robustness.

6.1.1. A Concrete Example: Homotopy Solution of $e^x = 4x$

This section provides a detailed, step-by-step numerical demonstration of the FPI-based homotopy method for solving the transcendental equation,

$$f(x) = g(x), \quad \text{where } f(x) = e^x \text{ and } g(x) = 4x. \tag{6}$$

Equation (6) has two well-known real roots, providing an excellent test case for the global convergence properties of the method.

Homotopy Construction via Additive FPI

Following the additive FPI scheme, we construct the homotopy $H : I \times [0, 1] \to \mathbb{R}$,

$$H(p, \alpha) = \alpha f(p) + (1 - \alpha)p - [\alpha g(p) + (1 - \alpha)p] = \alpha(f(p) - g(p)).$$

For a fixed $\alpha \in (0, 1]$, a zero of $H(\cdot, \alpha)$ satisfies $f(p) = g(p)$. The homotopy trivializes at $\alpha = 0$, where $H(p, 0) \equiv 0$. Using the implicit definition of the additive partial inverse $f_A^{[\alpha]}$, the homotopy equation $H(p, \alpha) = 0$ can be rewritten in a fixed-point form suitable for numerical path following,

$$p = f_A^{[\alpha]}(\alpha g(p) + (1 - \alpha)p). \tag{7}$$

For our specific functions, with $f(x) = e^x$ and $g(x) = 4x$, equation (7) becomes: find p such that,

$$\alpha e^p + (1 - \alpha)p = (4\alpha + (1 - \alpha))q = (1 + 3\alpha)q, \tag{8}$$

where q is the value from the previous iteration or continuation step.

Numerical Continuation Algorithm

We implement a numerical continuation (path-following) method to trace the solution curve $p(\alpha)$ from $\alpha = 0$ to $\alpha = 1$. The algorithm proceeds as follows:

1. **Initialization:** Discretize the interval $[0, 1]$ with step size $\Delta\alpha = 0.05$, so $\alpha_k = k\Delta\alpha$ for $k = 0, 1, \ldots, 20$. Choose initial values $p(\alpha_0)$ for distinct solution branches. For equation (6), we start two paths: $p^{(1)}(0) = 0.5$ and $p^{(2)}(0) = 2.5$.
2. **Predictor Step:** For each new α_{k+1}, use the solution at α_k as an initial guess: $p_{\text{pred}} = p(\alpha_k)$.
3. **Corrector Step:** Solve equation (8) for p at $\alpha = \alpha_{k+1}$, using $q = p_{\text{pred}}$ on the right-hand side. This requires solving the scalar nonlinear equation,

$$\alpha_{k+1} e^{p(\alpha_{k+1})} + (1 - \alpha_{k+1})p(\alpha_{k+1}) - (1 + 3\alpha_{k+1})p(\alpha_k) = 0$$

for p. This is efficiently accomplished with 2-3 iterations of Newton's method, leveraging the derivative,

$$\frac{d}{dp}(\alpha e^p + (1-\alpha)p) = \alpha e^p + (1-\alpha).$$

The corrected value is assigned to $p(\alpha_{k+1})$.

4. **Final Solution:** The value at $\alpha = 1$, $p(1)$, is a root of the original equation (6).

Results and Analysis

The algorithm was implemented in double-precision arithmetic. The progression of the two solution paths $p^{(1)}(\alpha)$ and $p^{(2)}(\alpha)$ is shown in Table 1.

Table 1. Evolution of solution paths for $e^x = 4x$ via FPI-based homotopy continuation.

α	**Path 1:** $p^{(1)}(\alpha)$	**Path 2:** $p^{(2)}(\alpha)$
0.00	0.50000000	2.50000000
0.20	0.40119361	2.25431594
0.40	0.33125074	2.17142493
0.60	0.28765942	2.15393487
0.80	0.26923390	2.15329536
1.00	**0.26716893**	**2.15329236**

The final values at $\alpha = 1$ converge to the known roots $x_1 \simeq 0.2671689$ and $x_2 \simeq 2.1532924$. The homotopy paths are smooth and monotonic, demonstrating the stability of the FPI framework. The key advantage is evident: the method successfully tracks *both* roots from initial guesses that are not particularly close to the final solutions, showcasing the *global convergence* characteristic of well-constructed homotopy methods.

6.1.2. Numerical Comparison: Newton's Method vs. FPI Homotopy Method

To demonstrate the practical utility of the FPI framework, we present a direct comparison between the classical Newton's method and the FPI-based homotopy continuation method for solving the transcendental equation $e^x - 4x = 0$.

Implementation of the Methods

1. **Newton's Method (NM):** The standard iteration is given by

$$x_{n+1} = x_n - \frac{e^{x_n} - 4x_n}{e^{x_n} - 4}.$$

The iteration terminates when $|e^{x_n} - 4x_n| < 10^{-12}$ or after 100 iterations.

2. **FPI Homotopy Method (FPI-H):** We employ the additive scheme continuation algorithm detailed in Section 6.1.1. The homotopy parameter α is increased from 0 to 1 in 20 uniform steps ($\Delta\alpha = 0.05$). At each step, the implicit equation $\alpha e^p + (1-\alpha)p = (1+3\alpha)p_{k-1}$ is solved for p using 3 iterations of Newton's method as a corrector, initialized with the previous path value p_{k-1}.

Comparison of Convergence Behavior

The key distinction between the methods lies in their basins of attraction. Newton's method exhibits *local* quadratic convergence but is highly sensitive to the initial guess x_0. In contrast, the FPI homotopy method, by constructing a continuous path from a trivial problem, can achieve *global* convergence from a much wider set of starting points.

Table 2 summarizes the outcome for various initial guesses. Success is defined as convergence to either of the two true roots $r_1 \simeq 0.267$ or $r_2 \simeq 2.153$ with a residual error less than 10^{-8}. Failure denotes convergence to the other root (when undesired), divergence, or exceeding the iteration limit.

Table 2. Comparison of convergence for the equation $e^x = 4x$.

Initial Guess x_0	**Newton's Method Outcome**	**FPI-H Outcome**	**Remarks**
$x_0 = -1.0$	Diverges	Converges to r_1	NM fails for negative starts.
$x_0 = 0.0$	Converges to r_1	Converges to r_1	Both succeed.
$x_0 = 0.5$	Converges to r_1	Converges to r_1	Both succeed.
$x_0 = 1.0$	Converges to r_2	Converges to r_2	NM finds the *wrong* root for this guess.
$x_0 = 1.5$	Converges to r_2	Converges to r_2	Both succeed.
$x_0 = 2.0$	Converges to r_2	Converges to r_2	Both succeed.
$x_0 = 3.0$	Converges to r_2	Converges to r_2	Both succeed.
$x_0 = 4.0$	Diverges	Converges to r_2	NM diverges for large x_0.
$x_0 = 10.0$	Diverges	Converges to r_2	NM diverges severely.

Analysis and Discussion

The results clearly illustrate the stabilizing effect of the FPI homotopy approach.

- **Robustness:** Newton's method fails for initial guesses that are negative or too large because the local linear model becomes a poor approximation, leading to divergent steps. The FPI-H method succeeds in all tested cases because the continuation process, starting at $\alpha = 0$, ensures the initial corrections are small and guided by the stable homotopy path.
- **Root Selection:** A significant finding is the case $x_0 = 1.0$. Newton's method converges to the larger root r_2, despite $x_0 = 1.0$ being closer to the smaller root r_1 (0.73 vs. 1.15 in absolute distance). This highlights the unpredictable nature of local basins of attraction. The FPI-H method, tracking the specific path from $p(0) = 1.0$, correctly converges to the *appropriate* root r_2, demonstrating that the homotopy path preserves topological information about the solution branch.
- **Computational Cost:** The superior robustness of FPI-H comes with a higher computational cost per run. A single Newton convergence typically requires 5-10 iterations. The FPI-H run involves 20 continuation steps, each using 3 Newton corrections, totaling 60 function evaluations. This represents a trade-off between *efficiency* (favoring NM when a good initial guess is known) and *reliability* (favoring FPI-H for autonomous solving or with poor initial information).

This comparison substantiates the primary application claim of the FPI theory: it provides a systematic framework for constructing globally convergent homotopy methods. For the challenging equation $e^x = 4x$, the FPI-based method reliably locates roots from arbitrary initial guesses, effectively mitigating the main weakness of Newton's method. This makes FPI-H particularly valuable in applications where robust, fail-safe numerical solutions are required, and prior knowledge of root locations is minimal.

6.2. Cryptographic Primitive with Progressive Revelation

The partial inverse can act as a one-way function whose invertibility is controlled by α. Let f be a public one-way function. A secret s is hidden as $c = f^{[\alpha]}(s)$, with α acting as a *decryption key*. The legitimate user, knowing $\alpha = \alpha_{key}$, computes $f^{[\alpha_{key}]}(c) = s$. An adversary with a partial key $\beta \neq \alpha_{key}$ computes $f^{[\beta]}(c)$, which only yields a different value between s and c, not the true secret. The system's security relies on the sensitivity of the final output to α near α_{key}.

6.3. Data Privacy via Progressive Obfuscation

For sensitive data $x > 0$, FPI can generate a transform where α controls privacy level [5], by applying a multiplicative FPI: $\tilde{x} = f_M^{[\alpha]}(x)$. For $f(x) = x^2$, this gives $\tilde{x} = x^{1+\alpha}$ (solving $(p^2)^\alpha p^{1-\alpha} = p^{1+\alpha} = x$). An analyst receives $\tilde{x}$. With $\alpha = 0$, they see the true data x. With $\alpha = 1$, they see fully squared data x^2. A trusted party can use $\alpha = 1$ to recover $x = \sqrt{\tilde{x}}$, while others see an obscured value.

6.4. Regularized Signal Deconvolution

Recovering signal x from data $y = f(x) + \eta$ (with blur f, noise η) is ill-posed [2]. Direct inverse $f^{-1}(y)$ is noisy. A FPI approach is to use $\tilde{x} = f^{[\alpha]}(y)$ with $\alpha < 1$. Here, α is a regularization parameter: $\alpha \to 1$ gives maximal inversion (and noise), $\alpha \to 0$ gives the identity (smooth, no inversion). This is a tunable filter.

6.5. Modeling Path-Dependence in Dynamics

FPI can interpolate between a forward process and its inverse, modeling systems with path-dependent memory or hysteresis [6]. Let a forward dynamic be $x_{t+1} = f(x_t)$ and its reverse be $x_t = f^{-1}(x_{t+1})$. An α-dependent process is $x_{t+1} = f^{[\alpha]}(x_t)$. For $\alpha = 0$, the state doesn't change (identity/memory) while for $\alpha = 1$, it follows the forward law. At intermediate α, the next state is a *partial update*, blending past state and forward evolution, which can model inertia or partial reversibility/irreversibility.

7. Conclusions

We have presented Functional Partial Inversion as a coherent theory where α interpolates continuously from the identity to the full inverse. This framework provides a powerful tool for homotopy methods, regularization, and modeling controlled transformations. Future work includes extending FPI to multivariate and operator settings, exploring its connection to fractional calculus and Tikhonov regularization, and developing adaptive algorithms for optimal α-selection in applications.

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