Article

Not peer-reviewed version

On Generalized Fibospinomials: Generalized Fibonacci Polynomial Spinors

Ece Gulsah Colak * , Nazmiye Gonul Bilgin , Yuksel Soykan

Posted Date: 16 February 2024

doi: 10.20944/preprints202402.0910.v1

Keywords: fibonacci numbers; fibonacci polynomials; fibonacci spinors; generalized fibonacci spinors; generalized fibonacci polynomial spinors

Article

On Generalized Fibospinomials: Generalized Fibonacci Polynomial Spinors

Ece Gulsah Colak *, Nazmiye Gonul Bilgin and Yuksel Soykan

Department of Mathematics, Zonguldak Bulent Ecevit University, Merkez, Zonguldak 67100, Turkey; nazmiyegonul@beun.edu.tr; yuksel_soykan@hotmail.com
* Correspondence: egsavkli@beun.edu.tr

Abstract: In this paper, we introduce and investigate a new family of sequences called the generalized Fibospinomials (or the generalized Fibonacci polynomial spinors or Horadam polynomial spinors). Being particular cases, we handle with (r,s)-Fibonacci and (r,s)-Lucas polynomial spinors. After a short history on spinors and quaternions, we present Binet's formulas, generating functions and the summation formulas for these polynomials. In addition, we obtain some identities of generalized Fibonacci polynomial spinors, (r,s)-Fibonacci polynomial spinors and (r,s)-Lucas polynomial spinors. Moreover, we give some special identities such as Catalan's and Cassini's identities and we present matrices related with these polynomials.

Keywords: Fibonacci numbers; Fibonacci polynomials; Fibonacci spinors; Generalized Fibonacci spinors; Generalized Fibonacci polynomial spinors

MSC: 11B37; 11B39; 11R52; 15A66

1. Introduction

Classical mechanics, whose laws were given by Newton in the late 1600s, was not sufficient to interpret the discoveries concerning the electronic structure of atoms and the nature of light. It has become a necessity to develop a different type of mechanics known as quantum mechanics or wave mechanics to explain such situations [19]. Heisenberg and Schrödinger discovered matrix mechanics and wave mechanics, respectively. Later, Schördinger and Eckart tried to prove that these two theories are mathematically equivalent. In Heisenberg's mechanics, a physical quantity is represented by a matrix, while in Schrödinger's mechanics it is represented by a linear operator [1]. In quantum mechanics, three different equations, often called Dirac, Pauli, and Schrödinger equations, are used to describe the movement of an electron [16]. Schrödinger defined a wave equation in 1925 based on the suggestion of the famous physicist Peter Debye and the work of de Broglie. The equation did not match the real atom observations, as Schrödinger did not include electron spin in his work, which began with trying to find wave equation that would characterize the behaviour of an electron in hydrogen [2]. In 1926, Schrödinger presented his papers in which he wrote the full mathematical basis of non-relativistic quantum mechanics [19]. Klein, Fock, and Gordon in 1926 independently discovered the relativistic Schrödinger equation for the free spin-*zero* particle, while Dirac discovered for the free electron the relativistic wave equation, spin-$1/2$ particle, in 1928 [15]. According to Bandyopadhyay and Cahay, the eigenvectors of Pauli spin matrices are examples of spinor, which are 2×1 column vectors that represent the spin state of an electron [1]. Spinors were first used in the field of quantum mechanics by physicists under this name. But this concept in their most general mathematical form, were defined much earlier, by Cartan in 1913. In four-dimensional space, spinors appear in Dirac's famous electron equations, and the components of a spinor are four wave functions indicated by Cartan [5]. The most important reason for spinors to enter physics is the existence of spin. Spinors, a concept that is not yet fully understood [13] and needs to be studied a lot, is a concept that mathematicians focus on algebraic and geometrical studies, and physicists carry out deep studies on quantum physics. When the literature is examined, the fact that spinors have been studied without any geometrical meaning has led to the complexity of attempts to extend Dirac's equations to general relativity and spinors to be an incomprehensible concept [5]. Spinors and Dirac equations in general relativity theory on Riemannian spaces have been studied independently by Weyl [38], Schrödinger [31] and Fock [14], and then many studies have been done in terms of space-time geometry [9]. Vaz and Rocha advocated three fundamentally different definitions of spinor, each of which was defined by different researchers, each emphasizing a different perspective. Two of these are more widely accepted, and the third is just beginning to be recognized in the literature [35]. They made this classification into algebraic, classical and operatorial.

Now the definition of $SU(2)$ as given in Westra's notes [37] will be reminded. Let U be a 2×2 type matrix and let $U^{\dagger}$ represent the conjugate transpose of U. $SU(2)$ is the group that provides the following properties:

$$U=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, U^{\dagger}U=UU^{\dagger}=I \text{ and } \det U=1 \text{ where } a,b,c,d\in\mathbb{C}.$$

Then the most general element of $SU(2)$ is written as

$$U(x,y)=\begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix}, |x|^2+|y|^2=1.[37]$$

The elements of the representation space of $SU(2)$, obtained with the help of Cayley-Klein parameters, which are the inputs of a unitary matrix A belonging to the $SU(2)$, are called classical spinors [35]. In Quantum Mechanics, particle spin, defined in Lie Group theory as $n-$dimensional $Spin(n)$ with elements known as spinors, is used to represent quaternions. Because spinors change sign when rotated 360°, it is advantageous to use spinors instead of vectors and tensors to describe the spin angular characteristic of the electron. Three-dimensional spinors of group $Spin(3)$ is $SU(2)$. The most important application of spinors in quantum physics is to provide mathematical representations of energy transfer in EM fields [30].

Now, let's give the definition of spinor in this sense with the notations used in the book written by Nagashima [25]. In two-dimensional complex variable space, the spinor is defined as the base vector of the group representation $SU(2)$.

Two-component column vector $\varphi=\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$, where φ_1,φ_2 are in general complex numbers is a notation for spinors. On the other hand, the representation matrices are stated as 2×2 unitary matrices with unit determinants. φ transforms under $SU(2)$ as follows:

$$\varphi\longrightarrow\varphi^{'}=U\varphi, \text{ i.e., } \begin{bmatrix} \varphi_1^{'} \\ \varphi_2^{'} \end{bmatrix}=U(x,y)\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \tag{1.1}$$

where $U(x,y)\in SU(2)$ and

$$\varphi_1^{'}=x\varphi_1+y\varphi_2, \varphi_2^{'}=-\bar{y}\varphi_1+\bar{x}\varphi_2. \tag{1.2}$$

Furthermore, there are three independent parameters in $SU(2)$ [25].

In the algebraic definition, in physics, the spinor space is defined as a member of the minimal left ideal of Clifford's algebra [35]. Cartan created the mathematical form of spinors while investigating linear representations of simple groups [5].

The concept, which was first called by the quantum physicist P. Ehrenfest, has been an important tool in many physical theories, especially in the mechanics of solids [3].

In addition, there are important studies in which spinors are used in the applications of mathematics in the field of physics. On the other hand, spinors, also studied geometrically by E. Cartan, are elements of complex vector space and are used in mathematics and physics to extend the

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

concepts of rotation and space vector. Spinors, which consist of two complex components in terms of vectors, were obtained in three-dimensional Euclidean space by Cartan [5]. The properties of spinors have been studied in different dimensions by different authors. In 2004, Castillo and Barrales gave some main properties of spinors in three-dimensional real space [9]. Later, Castillo defined spinor formulation in four-dimensional space [10].

Let's explain this concept better with the representation of spinors established with the help of orthonormal base. The homomorphic groups $SO(3)$ and $SU(2)$ are the rotation group around the origin in $\mathbb{R}^3$ and unitary complex 2×2 matrices group with unit determinant, respectively. Here, the elements of $SU(2)$ move on two vectors with complex structure named spinors [9].

Every spinor $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ defines vectors $d, e, f \in \mathbb{R}^3$ through $d + ie = \varphi^t \sigma \varphi$, $f = \widehat{\varphi}\sigma\varphi$. Here σ is a vector whose $\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3$ components are the complex symmetric 2×2 matrices and $\widehat{\varphi}$ denote conjugate of φ where

$$\widetilde{\sigma}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \widetilde{\sigma}_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \widetilde{\sigma}_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{1.3}$$

and

$$\widehat{\varphi} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{\varphi} \equiv - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \end{bmatrix} = \begin{bmatrix} -\overline{\varphi_2} \\ \overline{\varphi_1} \end{bmatrix}. \tag{1.4}$$

On the other hand $d, e, f \in \mathbb{R}^3$ vectors are defined with

$$d + ie = \left(\varphi_1^2 - \varphi_2^2, i\left(\varphi_1^2 + \varphi_2^2\right), -2\varphi_1\varphi_2\right) \tag{1.5}$$

and

$$f = \left(\overline{\varphi_2}\varphi_1 + \overline{\varphi_1}\varphi_2, i\overline{\varphi_2}\varphi_1 - i\overline{\varphi_1}\varphi_2, |\varphi_1|^2 - |\varphi_2|^2\right). \tag{1.6}$$

Also, $|d| = |e| = |f| = \overline{\varphi^t}\varphi$ and $d \times e.f > 0$. Let φ and ϕ be two arbitrary spinors and d, e be complex numbers. In this case, $\varphi^t\sigma\phi = -\widehat{\varphi}^t\sigma\widehat{\phi}$, $\widehat{(d\varphi + e\phi)} = \overline{d}\widehat{\phi} + \overline{e}\widehat{\varphi}$ and $\widehat{\widehat{\varphi}} = -\varphi$. Furthermore for nonzero spinor φ, $\{\varphi, \widehat{\varphi}\}$ is linear independent and the spinors corresponding to $\{d, e, f\}$, $\{e, f, d\}$, $\{f, e, d\}$ are different [9].

Now let's give place to one of the important concepts in this field: Pauli matrices. Hermitian, involutory and unitary Pauli matrices are 2×2 type matrices. However, all of the Pauli matrices can be compacted into a single expression. In addition, every 2×2 Hermitian matrix is uniquely written as a linear combination of Pauli matrices where all coefficients are real numbers. Now, we remember the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.7}$$

Clearly, $\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3$ matrices in (1.3) are obtained by multiplying $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ matrix and the Pauli matrices. Pauli used spinors, thought to be elements of $\mathbb{C}^2$, to reveal the behavior of an electron by taking the spin of the electron into calculate in quantum mechanics. In physics, spinors arose as a product of Pauli's theory of non-relativistic quantum mechanics (1926) and Dirac's (1928) theory of relativistic quantum mechanics [11,28]. These matrices, which appear in the Pauli equation, that takes into calculus the interaction of a particle's spin with an external electromagnetic field, are named after the physicist Wolfgang Pauli[28] and these matrices have a very important place in nuclear physics studies. Dirac gave important formulas about Pauli matrices [11]. The space of Dirac spinors is a complex four-dimensional vector space, and turns out to split as the sum of a complex two-dimensional vector space S, called a spin space, also its complex conjugate **S**. Since spinors are complex objects, both the space of complex conjugate spinors and its dual must be given. Vivarelli [36] involved in this area in geometrical aspect. He showed a injective and linear correspondence between spinors and quaternions and in three-dimensional Euclidean space he gave spinor representations of rotations. Thus, a more concise and simpler depiction of quaternions can be reached by the concept of spinors. Quaternions being applied to the fields of mathematics, physics, robotics, engineering and chemistry can be worked through spinors with the help of the correspondence given by Vivarelli [36].

Recent studies using special numbers like Fibonacci and Lucas numbers have brought a different perspective to the use of spinors in mathematics. Erişir and Güngör [12], in 2020, introduced Fibonacci and Lucas spinors with the help of the Fibonacci quaternions. Later, in 2023, Kumari et. al [24]. examined the $k-$Fibonacci and $k-$Lucas spinors via $k-$Fibonacci and $k-$Lucas quaternions which has been worked by Ramirez [29]. Horadam [20] first introduce the Fibonacci and Lucas quaternions which are worked in several areas. Soykan [32] exhibited the generalized Fibonacci polynomials in many aspects. Trying to see the beauty of these sequences in quantum mechanics will be interesting and worth to be examined. In our paper, we will work with generalized Fibonacci polynomial spinors through the correspondence between generalized Fibonacci spinors with generalized Fibonacci quaternions as Erişir and Güngör [12] did. We will investigate several properties of this new polynomial spinor sequence such as Binet's formula, etc. In addition, as particular cases, we will obtain this features for $(r,s)-$Fibonacci polynomial spinors, $(r,s)-$Fibonacci-Lucas polynomial spinors, Fibonacci polynomial spinors and Fibonacci-Lucas polynomial spinors and we will reveal the relations between these polynomial spinors.

2. Preliminaries

The Horadam polynomial sequence, or the generalized Fibonacci polynomial sequence, $\{W_n(x)\}$ was introduced by Horadam [21] with

$$W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x), \ \ n \geq 2 \tag{2.1}$$

where $W_0(x), W_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0, s(x) \neq 0$. See also the paper [32].

Binet's formula of generalized Fibonacci (Horadam) polynomials can be calculated using its characteristic equation which is given as

$$z^2 - r(x)z - s(x) = 0. \tag{2.2}$$

The roots of characteristic equation are

$$\alpha(x) = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \ \ \beta(x) = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2} \tag{2.3}$$

and the sum and product of the these roots are as follows:

$$\alpha(x) + \beta(x) = r(x), \tag{2.4}$$

$$\alpha(x)\beta(x) = -s(x). \tag{2.5}$$

If $\alpha(x) \neq \beta(x)$ then $r^2(x) + 4s(x) \neq 0$ and if $\alpha(x) = \beta(x)$ then (2.2) can be written as

$$z^2 - r(x)z - s(x) = (z - \alpha(x))^2 = z^2 - 2\alpha(x)z + \alpha^2(x) = 0 \tag{2.6}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

and, in this case,

$$\begin{aligned} r(x) &= 2\alpha(x), & (2.7)\\ s(x) &= -\alpha^2(x) = -\frac{r^2(x)}{4}, & (2.8)\\ r^2(x)+4s(x) &= 0. & (2.9)\end{aligned}$$

The Horadam polynomial sequence can be expanded to negative subscripts through defining

$$W_{-n}(x) = -\frac{r(x)}{s(x)}W_{-(n-1)}(x) + \frac{1}{s(x)}W_{-(n-2)}(x) \tag{2.10}$$

for $n = 1, 2, 3, ...$ where $s(x) \neq 0$. Thus, recurrence (2.1) holds for all integers n. Soykan examined the Horadam polynomials in detail with many aspects [32].

Now, we define two special cases of the generalized Fibonacci polynomials $W_n(x)$ in according to their first two values denoted by $G_n(x)$ and $H_n(x)$, respectively, via the second-order recurrence relations

$$\begin{aligned} G_{n+2}(x) &= r(x)G_{n+1}(x)+s(x)G_n(x),\ \ G_0(x)=0, G_1(x)=1, & (2.11)\\ H_{n+2}(x) &= r(x)H_{n+1}(x)+s(x)H_n(x),\ \ H_0(x)=2, H_1(x)=r(x). & (2.12)\end{aligned}$$

The (sequences of polynomials) $\{G_n(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ are named $(r,s)-$Fibonacci polynomial sequence and $(r,s)-$Fibonacci-Lucas polynomial sequence. We can obtain Fibonacci polynomial sequence $\{F_n(x)\}_{n\geq 0}$ coming from $\{G_n(x)\}_{n\geq 0}$ and Fibonacci-Lucas polynomial sequence $\{L_n(x)\}_{n\geq 0}$ coming from $\{H_n(x)\}_{n\geq 0}$ when $r(x) = x$ and $s(x) = 1$ as a special case as follows:

$$\begin{aligned} F_{n+2}(x) &= xF_{n+1}(x)+F_n(x), \quad F_0(x)=0, F_1(x)=1, & (2.13)\\ L_{n+2}(x) &= xL_{n+1}(x)+L_n(x), \quad L_0(x)=2, L_1(x)=x. & (2.14)\end{aligned}$$

Furthermore, the sequence of polynomials $\{W_n(x)\}_{n\geq 0}$ which is the generalized Fibonacci polynomial sequence becomes the sequence of numbers $\{W_n\}_{n\geq 0}$ which we call with the generalized Fibonacci numbers if $r(x)$ and $s(x)$ are the sequence of numbers as in the following:

$$W_n = rW_{n-1} + sW_{n-2},\ \ n \geq 2 \tag{2.15}$$

with initial values W_0, W_1 not all being zero integers where r, s are integers with $r \neq 0$, $s \neq 0$. As a special case, (r,s) $-$Fibonacci numbers denoted by $\{G_n\}_{n\geq 0}$ and $(r,s)-$Fibonacci-Lucas numbers denoted by $\{H_n\}_{n\geq 0}$ are given by the following recurrence relations with regarding inital values.

$$\begin{aligned} G_{n+2} &= rG_{n+1}+sG_n, \quad G_0=0, G_1=1, & (2.16)\\ H_{n+2} &= rH_{n+1}+sH_n, \quad H_0=2, H_1=r. & (2.17)\end{aligned}$$

In particular, the sequence of polynomials $\{F_n(x)\}_{n\geq 0}$ and $\{L_n(x)\}_{n\geq 0}$ are the extensions of Fibonacci and Fibonacci-Lucas numbers, respectively, so that if $r(x) = 1$ and $s(x) = 1$ then we have Fibonacci numbers $\{F_n\}$ and Fibonacci-Lucas numbers $\{L_n\}$ as follows:

$$\begin{aligned} F_{n+2} &= F_{n+1}+F_n, \quad F_0=0, F_1=1, & (2.18)\\ L_{n+2} &= L_{n+1}+L_n, \quad L_0=2, L_1=1. & (2.19)\end{aligned}$$

As we indicated above, the sequence of polynomials generalize the sequence of numbers. Now, let us have a look at the generalized Fibonacci quaternion polynomials and the generalized Fibonacci quaternions. For this, let us take a quick look at the definiton of quaternions.

Quaternions extend the complex numbers and are defined in the form $q = q_0 + iq_1 + jq_2 + kq_3$, where q_0, q_1, q_2, q_3 are real numbers and $1, i, j, k$ are basis vectors satisfying $i^2 = j^2 = k^2 = ijk = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. We denote the set of quaternions by $\mathbb{H}$. Multiplication of quaternions is not commutative. Hamilton defined that quaternions consist of a scalar part given above as q_0 above and and a vector part given above as $iq_1 + jq_2 + kq_3$. The conjugate and norm of a quaternion q is given with $q^* = q_0 - iq_1 - jq_2 - kq_3$ and $\|q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$, respectively.

Catarino defined the $h(x)-$Fibonacci quaternion polynomials generalizing the $k-$Fibonacci quaternion numbers and examined this polynomial sequence with its several properties [6,7]. Then, Özkoç and Porsuk [26] examined the generalized Fibonacci quaternion polynomials generalizing the generalized Fibonacci quaternion numbers defined by

$$QW_n(x) = W_n(x) + iW_{n+1}(x) + jW_{n+2}(x) + kW_{n+3}(x) \tag{2.20}$$

where $\{W_n(x)\}$ is a Horadam polynomial sequence and they presented the Binet's formula, generating function and some identities for this polynomial sequence.

From this point of view, it can be easily seen that the generalized Fibonacci quaternion polynomials sequence $\{QW_n(x)\}$ can be written with second order recurrence relation as in the following:

$$QW_n(x) = r(x)QW_{n-1}(x) + s(x)QW_{n-2}(x), \quad n \geq 2. \tag{2.21}$$

It can be noted that the characteristic equation for generalized Fibonacci (Horadam) polynomial quaternions will be the same that of generalized Fibonacci (Horadam) polynomial since they have same linear recurrence relation. In addition, while changing the coefficients r and s, we obtain that a generalization of different sequences such as Fibonacci quaternion polynomials, Fibonacci-Lucas quaternion polynomials, Pell quaternion polynomials, Pell-Lucas quaternion polynomials, Jacobsthal quaternion polynomials, Jacobsthal-Lucas quaternion polynomials which are among the most well-known sequences.

The Horadam quaternions polynomial sequence generalize also the sequence of numbers which we will recall for Horadam quaternions below.

Horadam [20] defined the Fibonacci quaternions $\{QF_n\}$ and Fibonacci-Lucas quaternions $\{QL_n\}$ and showed a few relations regarding the Fibonacci quaternions. Later, Iyer had exhibited some relations between Fibonacci quaternions and Fibonacci-Lucas quaternions [22] and Swamy had found new properties between the generalized Fibonacci quaternion sequence and Fibonacci quaternion sequence [33]. Then, Halıcı introduced the Binet's formulas for the Fibonacci and Fibonacci-Lucas quaternions, and also exhibited generating functions and some sum formulas for these sequences [17]. Later on, İpek introduced $(r, s)-$Fibonacci quaternions [23] and Patel and Ray [27] introduced $(r, s)-$Fibonacci-Lucas quaternions and exhibited some identities about $(r, s)-$Fibonacci quaternions and $(r, s)-$Fibonacci-Lucas quaternions such as Catalan's identity, d'Ocagne's identity etc. Then, Cerda-Morales presented these well-known identities for $(r, s)-$Fibonacci quaternions and $(r, s)-$Fibonacci-Lucas quaternions using their Binet's formula [8] and Szynal-Liana and Wloch worked on generalized commutative Fibonacci quaternions [34]. Halıcı and Karataş gave the most generalized version of these series as follows:

The nth generalized Fibonacci quaternion was exhibited by Halıcı and Karataş [18] such that

$$QW_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} \tag{2.22}$$

where $\{W_n\}$ is a generalized Fibonacci or Horadam sequence satisyfing (2.15). The generalized Fibonacci quaternion sequence, in another way of saying Horadam quaternion sequence is a second order linear recurrence relation so that for $n \geq 0$

$$QW_{n+2} = rQW_{n+1} + sQW_n \tag{2.23}$$

where $r, s \in \mathbb{Z}$.

For the sake of simplicity throughout the rest of the paper, for all integers n we use

$$W_n, G_n, H_n, F_n, L_n, r, s, \alpha, \beta$$

instead of

$$W_n(x), G_n(x), H_n(x), F_n(x), L_n(x), r(x), s(x), \alpha(x), \beta(x)$$

respectively, unless otherwise is stated.

We can see first few values of the sequence of polynomials $\{W_n\}, \{G_n\}, \{H_n\}, \{F_n\}$ and $\{L_n\}$ in Table 1.

Table 1. Some values of generalized Fibonacci, $(r,s)-$ Fibonacci, $(r,s)-$ Fibonacci Lucas, Fibonacci and Fibonacci-Lucas polynomials.

n	0	1	2	3
W_n	W_0	W_1	$rW_0 + sW_1$	$r^2W_0 + s(r+1)W_1$
W_{-n}		$\dfrac{(W_1 - rW_0)}{s}$	$\dfrac{((r^2+s)W_0 - rW_1)}{s^2}$	$\dfrac{(-(r^3+2rs)W_0 + (r^2+s)W_1)}{s^3}$
G_n	0	1	r	$s + r^2$
G_{-n}		$\dfrac{1}{s}$	$-\dfrac{r}{s^2}$	$\dfrac{1}{s^3}(s+r^2)$
H_n	2	r	$2s + r^2$	$r(3s + r^2)$
H_{-n}		$-\dfrac{r}{s}$	$\dfrac{2s+r^2}{s^2}$	$-r\dfrac{3s+r^2}{s^3}$
F_n	0	1	x	$1 + x^2$
F_{-n}		1	$-x$	$1 + x^2$
L_n	2	x	$2 + x^2$	$x^3 + 3x$
L_{-n}		$-x$	$2 + x^2$	$-x^3 - 3x$

Next, we can present the first few values of the sequence of polynomials $\{QW_n\}$ as follows:

$$\begin{aligned} QW_0 &= W_0 + iW_1 + j(rW_0 + sW_1) + k\left(r^2W_0 + s(r+1)W_1\right), \\ QW_1 &= W_1 + i(rW_0 + sW_1) + j\left(r^2W_0 + s(r+1)W_1\right) \\ &\quad +k\left((r^3+rs)W_0 + (s^2 + rs + r^2s)W_1\right), \\ QW_2 &= rQW_1 + sQW_0. \end{aligned} \tag{2.24}$$

We now recall the spinors obtained by the help of the quaternions. Consider a spinor φ given by

$$\varphi = (\varphi_1, \varphi_2) \cong \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \tag{2.25}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

where $\varphi_1, \varphi_2 \in \mathbb{C}$. We denote the set of spinors by $\mathbb{S}$. Vivarelli [36] pointed out that there is a correspondence between any quaternion $q = q_0 + iq_1 + jq_2 + kq_3$ and a spinor $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ such that

$$f : \mathbb{H} \to \mathbb{S}, f(q_0 + iq_1 + jq_2 + kq_3) = \begin{bmatrix} q_3 + iq_0 \\ q_1 + iq_2 \end{bmatrix} \equiv \varphi. \tag{2.26}$$

Since $f(q+p) = f(q) + f(p)$, $f(\lambda q) = \lambda f(q)$ where $\lambda \in \mathbb{C}$, $p, q \in \mathbb{H}$ and $\ker f = \{0\}$, f is linear and injective. Under this correspondence f, the conjugate of q i.e., q^* is mapped to φ^* as below:

$$f(q^*) = f(q_0 - iq_1 - jq_2 - kq_3) = \begin{bmatrix} -q_3 + iq_0 \\ -q_1 - iq_2 \end{bmatrix} \equiv \varphi^*. \tag{2.27}$$

Given two spinors $\varphi_a = \begin{bmatrix} \varphi_{1_a} \\ \varphi_{2_a} \end{bmatrix}$ and $\varphi_b = \begin{bmatrix} \varphi_{1_b} \\ \varphi_{2_b} \end{bmatrix}$, $\varphi_a = \varphi_b$ are equal if and only if $\varphi_{1_a} = \varphi_{1_b}$ and $\varphi_{2_a} = \varphi_{2_b}$.

The product of quaternions $q \times p$ has been shown by Vivarelli [36] in relation to a spinor matrix product as follows:

$$q \times p \to -i\widehat{U} f(p) = -i \begin{bmatrix} q_3 + iq_0 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 + iq_0 \end{bmatrix} \begin{bmatrix} p_3 + ip_0 \\ p_1 + ip_2 \end{bmatrix} \tag{2.28}$$

where $\widehat{U}$ is complex, unitary 2×2 type matrix taking in $SU(2)$. Here, $\widehat{U}$ can be written by means of Pauli matrices (1.7):

$$\widehat{U} = \begin{bmatrix} q_3 + iq_0 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 + iq_0 \end{bmatrix} = q_0 iI + q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3 \tag{2.29}$$

where σ_1, σ_2 and σ_3 are Pauli matrices and I is the unit square matrix of type 2×2. The connection between a spinor φ and a 2×2 matrix $\widehat{U}$ is given [36] by

$$\varphi = \widehat{U} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.30}$$

Conjugate of a spinor φ is given by Cartan [5] with $\widetilde{\varphi}$ and the mate of a spinor φ is presented by Castillo and Barrales [9] with $\overset{\vee}{\varphi}$ as in the following identites

$$\widetilde{\varphi} = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{\varphi}, \ \overset{\vee}{\varphi} = - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{\varphi} \tag{2.31}$$

respectively, where $\overline{\varphi}$ is the complex conjugate of φ.

3. Generalized Fibonacci (Horadam) Polynomial Spinors

Definition 3.1. For integer n, the nth generalized Fibonacci polynomial spinor sequence of W_n is defined by

$$SW_n = \begin{bmatrix} W_{n+3} + iW_n \\ W_{n+1} + iW_{n+2} \end{bmatrix} \tag{3.1}$$

where W_n is the nth generalized Fibonacci polynomial.

The generalized Fibonacci polynomial spinor sequence $\{SW_n\}$ satisfy the second order linear recurrence sequence from the recurrence (2.1). We can see this in the next lemma.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

Lemma 1. *The generalized Fibonacci polynomial spinors $\{SW_n\}$ has the following identity for all integers n:*

$$SW_{n+2} = rSW_{n+1} + sSW_n \tag{3.2}$$

where $SW_0 = \begin{bmatrix} W_3 + iW_0 \\ W_1 + iW_2 \end{bmatrix}$, $SW_1 = \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix}$ *are arbitrary polynomial spinors with real coefficients.*

Proof. For all integers n, by using the recurrence (2.1), we can easily have the required identity:

$$\begin{aligned} rSW_{n+1} + sSW_n &= r\begin{bmatrix} W_{n+4} + iW_{n+1} \\ W_{n+2} + iW_{n+3} \end{bmatrix} + s\begin{bmatrix} W_{n+3} + iW_n \\ W_{n+1} + iW_{n+2} \end{bmatrix} \\ &= \begin{bmatrix} W_{n+5} + iW_{n+2} \\ W_{n+3} + iW_{n+4} \end{bmatrix} = SW_{n+2}. \end{aligned} \tag{3.3}$$

□

Theorem 1. *Note that first we can define the generalized Fibonacci polynomial spinor sequence $\{SW_n\}$ as (3.2) then we get (3.1).*

We can see a correspondence between the generalized Fibonacci polynomial quaternions and the generalized Fibonacci polynomial spinors by adapting from the transformation between quaternions and spinors with the following linear and injective transformation:

$$\begin{aligned} f &: \mathbb{W} \to \mathbb{S}, \\ f(QW_n) &= f(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}) \qquad (3.4) \\ &= \begin{bmatrix} W_{n+3} + iW_n \\ W_{n+1} + iW_{n+2} \end{bmatrix} = SW_n. \qquad (3.5) \end{aligned}$$

If $QW_n^* = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$ is the conjugate of the nth-generalized Fibonacci quaternion polynomial QW_n, then the nth-generalized Fibonacci polynomial spinor SW_n corresponding to QW_n^* is

$$SW_n^* = \begin{bmatrix} -W_{n+3} + iW_n \\ -W_{n+1} - iW_{n+2} \end{bmatrix}. \tag{3.6}$$

We can write the complex conjugate of SW_n as

$$\overline{SW_n} = \begin{bmatrix} W_{n+3} - iW_n \\ W_{n+1} - iW_{n+2} \end{bmatrix}, \tag{3.7}$$

the spinor conjugate to the SW_n as

$$\widetilde{SW_n} = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} W_{n+3} - iW_n \\ W_{n+1} - iW_{n+2} \end{bmatrix} = \begin{bmatrix} W_{n+2} + iW_{n+1} \\ -W_n - iW_{n+3} \end{bmatrix}, \tag{3.8}$$

and the mate of SW_n as

$$\check{SW_n} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} W_{n+3} - iW_n \\ W_{n+1} - iW_{n+2} \end{bmatrix} = \begin{bmatrix} -W_{n+1} + iW_{n+2} \\ W_{n+3} - iW_n \end{bmatrix}. \tag{3.9}$$

Now, we define two special cases of the polynomial spinors SW_n. $(r,s)-$
Fibonacci polynomial spinors or shortly SG_n and $(r,s)-$Fibonacci-Lucas polynomial spinors or shortly SH_n are the special cases of (3.1).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

Definition 3.2. For integer n, the nth sequence of polynomials SG_n are defined by

$$SG_n = \begin{bmatrix} G_{n+3} + iG_n \\ G_{n+1} + iG_{n+2} \end{bmatrix} \tag{3.10}$$

where G_n is a nth $(r,s)-$Fibonacci polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SG_{n+2} = rSG_{n+1} + sSG_n, \tag{3.11}$$

with

$$SG_0 = \begin{bmatrix} r^2 + s \\ 1 + ir \end{bmatrix}, SG_1 = \begin{bmatrix} r^3 + 2rs + i \\ r + i\left(r^2 + s\right) \end{bmatrix}, \tag{3.12}$$

and the nth sequence of polynomials SH_n are defined by

$$SH_n = \begin{bmatrix} H_{n+3} + iH_n \\ H_{n+1} + iH_{n+2} \end{bmatrix} \tag{3.13}$$

where H_n is a nth $(r,s)-$Fibonacci-Lucas polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SH_{n+2} = rSH_{n+1} + sSH_n, \tag{3.14}$$

with

$$SH_0 = \begin{bmatrix} r\left(3s + r^2\right) + 2i \\ r + i\left(2s + r^2\right) \end{bmatrix}, SH_1 = \begin{bmatrix} 4r^2s + r^4 + 2s^2 + ir \\ 2s + r^2 + ir\left(3s + r^2\right) \end{bmatrix}. \tag{3.15}$$

When $r = x$ and $s = 1$, we have the special case of generalized Fibonacci polynomial spinors, which is called by Fibonacci polynomial spinor denoting with $\{SF_n\}$ and Fibonacci-Lucas polynomial spinors denoting with $\{SL_n\}$ as in the next corollary.

Definition 3.3. For integer n, the nth Fibonacci polynomial spinors SF_n are defined by

$$SF_n = \begin{bmatrix} F_{n+3} + iF_n \\ F_{n+1} + iF_{n+2} \end{bmatrix} \tag{3.16}$$

where F_n is the nth Fibonacci polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SF_{n+2} = xSF_{n+1} + SF_n, \tag{3.17}$$

with

$$SF_0 = \begin{bmatrix} x^2 + 1 \\ 1 + ix \end{bmatrix}, SF_1 = \begin{bmatrix} x^3 + 2x + i \\ x + i\left(x^2 + 1\right) \end{bmatrix}. \tag{3.18}$$

and the nth Fibonacci-Lucas polynomial spinors SL_n are defined by

$$SL_n = \begin{bmatrix} L_{n+3} + iL_n \\ L_{n+1} + iL_{n+2} \end{bmatrix} \tag{3.19}$$

where L_n is the nth Fibonacci-Lucas polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SL_{n+2} = xSL_{n+1} + SL_n, \tag{3.20}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

with

$$SL_0 = \begin{bmatrix} x + x^3 + 2i \\ x + i\left(2 + x^2\right) \end{bmatrix}, SL_1 = \begin{bmatrix} x^4 + 4x^2 + 2 + ix \\ 2 + x^2 + i\left(3x + x^3\right) \end{bmatrix}. \tag{3.21}$$

If we take $r = x = 1$ then the sequence of Fibonacci polynomial spinors $\{SF_n\}$ and $\{SL_n\}$ becomes the number sequence of Fibonacci spinors and Fibonacci-Lucas spinors, respectively. Erişir and Güngör exhibited some algebraic definitions for Fibonacci and Fibonacci-Lucas spinors besides giving some significant formulas like Binet's and Cassini's formulas for this sequence of numbers [12]. One can also see Cartan [5] and Vivarelli [36] for some algebraic properties of spinors.

Now, we list a few values of generalized fibospinomials in Table 2.

Table 2. The first few values of generalized Fibonacci polynomial spinors with negative and positive subscripts.

n	0	1	2
SW_n	$\begin{bmatrix} \left(r^2+s\right)W_1 \\ +rsW_0+iW_0 \\ \\ W_1 \\ +i\left(rW_1+sW_0\right) \end{bmatrix}$	$\begin{bmatrix} \left(r^3+2rs\right)W_1 \\ +\left(r^2s+s^2\right)W_0 \\ +iW_1 \\ \\ rW_1+sW_0 \\ +i(\left(r^2+s\right)W_1 \\ +rsW_0) \end{bmatrix}$	$\begin{bmatrix} (r^4+3r^2s+s^2)W_1 \\ +(r^3s+2rs^2)W_0 \\ +(irW_1+sW_0) \\ \\ \left(r^2+s\right)W_1+rsW_0 \\ +i(\left(r^3+2rs\right)W_1 \\ +\left(r^2s+s^2\right)W_0) \end{bmatrix}$
SW_{-n}		$\begin{bmatrix} rW_1+sW_0 \\ +i\frac{1}{s}\left(W_1-rW_0\right) \\ \\ W_0+iW_1 \end{bmatrix}$	$\begin{bmatrix} W_1+i\frac{1}{s}(W_0 \\ -\frac{r}{s}\left(W_1-rW_0\right)) \\ \\ \frac{1}{s}\left(W_1-rW_0\right) \\ +iW_0 \end{bmatrix}$

Next, we present the first few values of the special polynomial spinors of second order with negative and positive subscripts:

Table 3. The first few values of $(r,s)-$ Fibonacci and (r,s) -Lucas polynomial spinors with negative and positive subscripts.

n	0	1	2
SG_n	$\begin{bmatrix} r^2+s \\ 1+ir \end{bmatrix}$	$\begin{bmatrix} r^3+2rs+i \\ r+i\left(r^2+s\right) \end{bmatrix}$	$\begin{bmatrix} r^4+3r^2s \\ +s^2+ir \\ \\ r^2+s \\ +i\left(r^3+2rs\right) \end{bmatrix}$
SG_{-n}		$\begin{bmatrix} r+i\frac{1}{s} \\ i \end{bmatrix}$	$\begin{bmatrix} 1-i\frac{r}{s^2} \\ \frac{1}{s} \end{bmatrix}$
SH_n	$\begin{bmatrix} r\left(3s+r^2\right)+2i \\ r+i\left(2s+r^2\right) \end{bmatrix}$	$\begin{bmatrix} 4r^2s+r^4 \\ +2s^2+ir \\ \\ 2s+r^2 \\ +ir\left(3s+r^2\right) \end{bmatrix}$	$\begin{bmatrix} r\left(r^4+5r^2s+5s^2\right) \\ +i\left(2s+r^2\right) \\ \\ r\left(3s+r^2\right) \\ +i\left(4r^2s+r^4+2s^2\right) \end{bmatrix}$
SH_{-n}		$\begin{bmatrix} \left(2s+r^2\right)-i\frac{r}{s} \\ 2+ir \end{bmatrix}$	$\begin{bmatrix} r+i\frac{2s+r^2}{s^2} \\ -\frac{r}{s}+2i \end{bmatrix}$

Using the recurrence relation of Fibonacci and Fibonacci-Lucas polynomial spinor $\{SF_n\}$ and $\{SL_n\}$, we can write the first few terms of these sequences of polynomials, respectively. See Table 4.

Table 4. The first few values of Fibonacci and Lucas polynomial spinors with positive and negative subscripts.

n	0	1	2
SF_n	$\begin{bmatrix} x^2+1 \\ 1+ix \end{bmatrix}$	$\begin{bmatrix} x^3+2x+i \\ x+i\left(x^2+1\right) \end{bmatrix}$	$\begin{bmatrix} x^4+3x^2 \\ +1+ix \\ \\ x^2+1 \\ +i\left(x^3+2x\right) \end{bmatrix}$
SF_{-n}		$\begin{bmatrix} x+i \\ i \end{bmatrix}$	$\begin{bmatrix} 1-ix \\ 1 \end{bmatrix}$
SL_n	$\begin{bmatrix} x\left(3+x^2\right)+2i \\ x+i\left(2+x^2\right) \end{bmatrix}$	$\begin{bmatrix} 4x^2+x^4 \\ +2+ix \\ \\ 2+x^2 \\ +ix\left(3+x^2\right) \end{bmatrix}$	$\begin{bmatrix} x\left(x^4+5x^2+5\right) \\ +i\left(2+x^2\right) \\ \\ x\left(3+x^2\right) \\ +i\left(4x^2+x^4+2\right) \end{bmatrix}$
SL_{-n}		$\begin{bmatrix} \left(2+x^2\right)-ix \\ 2+ix \end{bmatrix}$	$\begin{bmatrix} x+i\left(2+x^2\right) \\ -x+2i \end{bmatrix}$

We can show the product of unitary complex matrix $\widehat{SW_n}$ obtained by SW_n with a generalized Fibonacci polynomial spinor sequence SW_n as follows:

$$\begin{aligned} \widehat{SW_n}SW_n &= \begin{bmatrix} W_{n+3}+iW_n & W_{n+1}-iW_{n+2} \\ W_{n+1}+iW_{n+2} & -W_{n+3}+iW_n \end{bmatrix} \begin{bmatrix} W_{n+3}+iW_n \\ W_{n+1}+iW_{n+2} \end{bmatrix} \\ &= \begin{bmatrix} W_{n+1}^2+W_{n+2}^2+W_{n+3}^2+i\left(W_nW_{n+3}+W_n\right) \\ -2W_nW_{n+2}+i2W_nW_{n+1} \end{bmatrix}. \end{aligned} \tag{3.22}$$

Considering this product it can be easily seen the following identities:

Lemma 2. *For all integers n, the next identities hold:*

(i)
$$\widehat{SW_n}\ SW_n^* = \widehat{SW_n^*}\ SW_n. \tag{3.23}$$

(ii)
$$\widehat{SW_n^*}\ SW_n^* = -\left(\widehat{SW_n}\ SW_n\right)^*. \tag{3.24}$$

(iii)
$$\overline{\widehat{SW_n}\ SW_n} = \widehat{\overline{SW_n}}\ \overline{SW_n}. \tag{3.25}$$

(iv)
$$\widetilde{\widehat{SW_n}\ SW_n} = -\widehat{SW_n}\ \widetilde{SW_n}. \tag{3.26}$$

(v)
$$\overset{\vee}{\widehat{SW_n}SW_n} = -\widehat{SW_n}\overset{\vee}{SW_n}. \tag{3.27}$$

(vi)
$$\widehat{SW_n}\overset{\vee}{SW_n} = i\widehat{SW_n}\widetilde{SW_n}. \tag{3.28}$$

Lemma 3. *The following equalities are true:*

(i)
$$s\left(\widehat{SG_0}SG_1-\widehat{SG_1}SG_0\right) = \widehat{SG_2}SG_1-\widehat{SG_1}SG_2. \tag{3.29}$$

(ii)
$$r\left(\widehat{SG_0}SG_1-\widehat{SG_1}SG_0\right) = \widehat{SG_0}SG_2-\widehat{SG_2}SG_0. \tag{3.30}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

(iii)

$$s\left(\widehat{SH_0}SH_1 - \widehat{SH_1}SH_0\right) = \widehat{SH_2}SH_1 - \widehat{SH_1}SH_2. \tag{3.31}$$

(iv)

$$r\left(\widehat{SH_0}SH_1 - \widehat{SH_1}SH_0\right) = \widehat{SH_0}SH_2 - \widehat{SH_2}SH_0. \tag{3.32}$$

Proof. Once we take the value of n as $0, 1$ and 2 in Table 3, we can easily obtain the required equalities by (3.22). □

It can be noted that the characteristic equation for generalized Fibonacci (Horadam) polynomial spinors is the same that of generalized Fibonacci (Horadam) polynomial.

Now, we can give the Binet's formula of SW_n using the roots α, β in (2.3) and recurrence relation (3.2) as follows:

Theorem 2. *For all integers n, the Binet's formula for the generalized Fibonacci (Horadam) polynomial spinor SW_n is given by the following formula:*

$$SW_n = \begin{cases} \dfrac{1}{\alpha-\beta}\left((SW_1 - \beta SW_0)\,\alpha^n - (SW_1 - \alpha SW_0)\beta^n\right), & \alpha \neq \beta \\ (SW_0 + (\dfrac{1}{\alpha}\,(SW_1 - \alpha SW_0)\,n)\alpha^n, & \alpha = \beta. \end{cases} \tag{3.33}$$

Proof. When the roots α, β of the characteristic equation (2.2) are distinct, one can write the general formula of SW_n as follows:

$$SW_n = p_1\alpha^n + q_1\beta^n \tag{3.34}$$

where the coefficients p_1 and q_1 are determined by the system of linear equations

$$\begin{aligned} SW_0 &= p_1 + q_1 \\ SW_1 &= p_1\alpha + q_1\beta. \end{aligned} \tag{3.35}$$

Solving these two simultaneous equations for SW_0 and SW_1, we obtain

$$\begin{aligned} p_1 &= \frac{1}{\alpha-\beta}(SW_1 - \beta SW_0) \\ q_1 &= \frac{-1}{\alpha-\beta}(SW_1 - \alpha SW_0). \end{aligned} \tag{3.36}$$

If the roots α, β are equal, then we can write SW_n as follows:

$$SW_n = (p_2 + q_2 n)\,\alpha^n \tag{3.37}$$

where the coefficients p_2 and q_2 are the polynomials whose values are determined by SW_0 and any other known value of the sequence. By using the values SW_0 and SW_1, we obtain

$$\begin{aligned} SW_0 &= p_2 \\ SW_1 &= (p_2 + q_2)\,\alpha. \end{aligned} \tag{3.38}$$

Solving these two simultaneous equations for SW_0 and SW_1, we obtain

$$\begin{aligned} p_2 &= SW_0 \\ q_2 &= \frac{1}{\alpha}(SW_1 - \alpha SW_0). \end{aligned} \tag{3.39}$$

□

 doi:10.20944/preprints202402.0910.v1

Now, let us calculate the values of $SW_1 - \beta SW_0$ and $SW_1 - \alpha SW_0$, which are in the Binet's formula, by using (2.1), (2.4) and (2.5) as in the following:

$$\begin{aligned} SW_1 - \beta SW_0 &= \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix} - \beta \begin{bmatrix} W_3 + iW_0 \\ W_1 + iW_2 \end{bmatrix} \\ &= \begin{bmatrix} \alpha^3 (W_1 - \beta W_0) + i(W_1 - \beta W_0) \\ \alpha (W_1 - \beta W_0) + i\alpha^2 (W_1 - \beta W_0) \end{bmatrix} \\ &= (W_1 - \beta W_0) \begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} SW_1 - \alpha SW_0 &= \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix} - \alpha \begin{bmatrix} W_3 + iW_0 \\ W_1 + iW_2 \end{bmatrix} \\ &= \begin{bmatrix} \beta^3 (W_1 - \alpha W_0) + i(W_1 - \alpha W_0) \\ \beta (W_1 - \alpha W_0) + i\beta^2 (W_1 - \alpha W_0) \end{bmatrix} \\ &= (W_1 - \alpha W_0) \begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix}. \end{aligned} \tag{3.41}$$

We can also find the Binet's formula of the generalized Fibonacci polynomial spinors $\{SW_n\}$ by using the Binet's formula of the generalized Fibonacci polynomial $\{W_n\}$ given by Soykan [32] as

$$W_n = \begin{cases} \frac{1}{\alpha - \beta}(c_1\alpha^n - c_2\beta^n)), & \alpha \neq \beta \\ (d_1 + d_2 n)\alpha^n, & \alpha = \beta. \end{cases} \tag{3.42}$$

where

$$c_1 = W_1 - \beta W_0, \; c_2 = W_1 - \alpha W_0, \tag{3.43}$$

and

$$d_1 = W_0, \; d_2 = \frac{1}{\alpha}(W_1 - \alpha W_0). \tag{3.44}$$

Hence, we present an alternative method for finding the Binet's formula of SW_n as follows: For $\alpha \neq \beta$, we obtain that

$$\begin{aligned} SW_n &= \frac{1}{\alpha - \beta} \begin{bmatrix} (c_1\alpha^{n+3} - c_2\beta^{n+3}) + i(c_1\alpha^n - c_2\beta^n) \\ (c_1\alpha^{n+1} - c_2\beta^{n+1}) + i(c_1\alpha^{n+2} - c_2\beta^{n+2}) \end{bmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{bmatrix} c_1\alpha^{n+3} + ic_1\alpha^n \\ c_1\alpha^{n+1} + ic_1\alpha^{n+2} \end{bmatrix} - \frac{1}{\alpha - \beta} \begin{bmatrix} c_2\beta^{n+3} + ic_2\beta^n \\ c_2\beta^{n+1} + ic_2\beta^{n+2} \end{bmatrix} \\ &= \frac{1}{\alpha - \beta}((W_1 - \beta W_0)\alpha^n \begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} - (W_1 - \alpha W_0)\beta^n \begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix}) \end{aligned} \tag{3.45}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

and for $\alpha = \beta$ by using (2.1), (2.7), (2.8) and (2.9) we obtain that

$$SW_n = \begin{bmatrix} (d_1 + d_2(n+3))\alpha^{n+3} + i(d_1 + d_2 n)\alpha^n \\ (d_1 + d_2(n+1))\alpha^{n+1} + i(d_1 + d_2(n+2))\alpha^{n+2} \end{bmatrix} \tag{3.46}$$

$$= \alpha^n \begin{bmatrix} (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)(n+3))\alpha^3 + i(W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)n) \\ (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)(n+1))\alpha + i(W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)(n+2))\alpha^2 \end{bmatrix}$$

$$= \alpha^n((\frac{1}{\alpha}W_1 - W_0)n \begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} + SW_0). \tag{3.47}$$

We have the next corollary for special cases of generalized Fibonacci polynomial spinors after the previous theorem immediately.

Corollary 3.4. *For all integers n, the Binet's formula for $(r,s)-$Fibonacci polynomial spinors $\{SG_n\}$, $(r,s)-$Fibonacci-Lucas polynomial spinors $\{SH_n\}$, Fibonacci polynomial spinors $\{SF_n\}$ and Fibonacci-Lucas polynomial spinors $\{SL_n\}$ is given by the following formulas:*

(i)

$$SG_n = \begin{cases} \frac{1}{\alpha - \beta}(\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n), & \alpha \neq \beta \\ (SG_0 + (\frac{1}{\alpha}\widehat{\beta}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{3.48}$$

where

$$\widehat{\alpha} = \begin{bmatrix} \alpha(r^2 + s) + rs + i \\ \alpha + i(\alpha r + s) \end{bmatrix}, \widehat{\beta} = \begin{bmatrix} \beta(r^2 + s) + rs + i \\ \beta + i(\beta r + s) \end{bmatrix} \text{ and}$$

$$SG_0 = \begin{bmatrix} r^2 + s \\ 1 + ir \end{bmatrix}.$$

(ii)

$$SH_n = \begin{cases} \frac{1}{\alpha - \beta}(\widetilde{\alpha}\alpha^n - \widetilde{\beta}\beta^n), & \alpha \neq \beta \\ (SH_0 + (\frac{1}{\alpha}\widetilde{\beta}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{3.49}$$

where

$$\widetilde{\alpha} = \begin{bmatrix} \alpha(r^3 + 3rs) + r^2 s + 2s^2 + i(-r + 2\alpha) \\ \alpha r + 2s + i(\alpha(r^2 + 2s) + rs) \end{bmatrix},$$

$$\widetilde{\beta} = \begin{bmatrix} \beta(r^3 + sr + 2rs) + r^2 s + 2s^2 + i(-r + 2\beta) \\ \beta r + 2s + i(\beta(r^2 + 2s) + rs) \end{bmatrix}$$

and $SH_0 = \begin{bmatrix} r^3 + 3rs + 2i \\ r + i(r^2 + 2s) \end{bmatrix}.$

(iii)

$$SF_n = \begin{cases} \frac{1}{\alpha - \beta}(\overline{\alpha}\alpha^n - \overline{\beta}\beta^n), & \alpha \neq \beta \\ (SF_0 + \frac{1}{\alpha}\overline{\beta}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{3.50}$$

where

$$\overline{\alpha} = \begin{bmatrix} \alpha x^2 + x + \alpha + i \\ \alpha + i(\alpha x + 1) \end{bmatrix},$$

$$\overline{\beta} = \begin{bmatrix} \beta x^2 + x + \beta + i \\ \beta + i(\beta x + 1) \end{bmatrix} \text{ and } SF_0 = \begin{bmatrix} x^2 + 1 \\ 1 + ix \end{bmatrix}.$$

(iv)

$$SL_n = \begin{cases} \frac{1}{\alpha - \beta}(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n), & \alpha \neq \beta \\ (SL_0 + \frac{1}{\alpha}\underline{\alpha}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{3.51}$$

where

$$\underline{\alpha} = \begin{bmatrix} \alpha(x^3+3x) + x^2 + 2 + i(-x+2\alpha) \\ \alpha x + 2 + i(\alpha(x^2+2) + x) \end{bmatrix},$$

$$\underline{\beta} = \begin{bmatrix} \beta(x^3+3x) + x^2 + 2 + i(-x+2\beta) \\ \beta x + 2 + i(\beta(x^2+2) + x) \end{bmatrix}$$

and $SL_0 = \begin{bmatrix} x^3+3x+2i \\ x+i(x^2+2) \end{bmatrix}$.

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} SW_n y^n$ of the sequence $\{SW_n\}$.

Lemma 4. *Suppose that $f_{SW_n}(y) = \sum_{n=0}^{\infty} SW_n y^n$ is the ordinary generating function of the generalized Fibonacci (Horadam) polynomial spinors $\{SW_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} SW_n y^n$ is given by*

$$\sum_{n=0}^{\infty} SW_n y^n = \frac{SW_0 + (SW_1 - rSW_0)\,y}{1 - ry - sy^2}. \tag{3.52}$$

Proof. Using the definition of generalized Fibonacci polynomials spinors, and substracting $ry\sum_{n=0}^{\infty} SW_n y^n$ and $sy^2\sum_{n=0}^{\infty} SW_n y^n$ from $\sum_{n=0}^{\infty} SW_n y^n$ we obtain

$$\begin{aligned} (1 - ry - sy^2)\sum_{n=0}^{\infty} SW_n y^n &= \sum_{n=0}^{\infty} SW_n y^n - ry\sum_{n=0}^{\infty} SW_n y^n - sy^2\sum_{n=0}^{\infty} SW_n y^n \\ &= \sum_{n=0}^{\infty} SW_n y^n - r\sum_{n=0}^{\infty} SW_n y^{n+1} - s\sum_{n=0}^{\infty} SW_n y^{n+2} \\ &= \sum_{n=0}^{\infty} SW_n y^n - r\sum_{n=1}^{\infty} SW_{n-1} y^n - s\sum_{n=2}^{\infty} SW_{n-2} y^n \\ &= (SW_0 + SW_1 y) - rSW_0 y \\ &\quad + \sum_{n=2}^{\infty}(SW_n - rSW_{n-1} - sSW_{n-2})y^n \\ &= SW_0 + (SW_1 - rSW_0)y. \end{aligned} \tag{3.53}$$

Rearranging the above equation, we obtain (3.52). □

Lemma 4 gives the following results as particular examples.

Corollary 3.5. *Generating functions of (r,s)−Fibonacci, (r,s)−Fibonacci-Lucas, Fibonacci and Fibonacci-Lucas polynomials spinors are given by the following formulas:*

(i)

$$\sum_{n=0}^{\infty} SG_n y^n = \frac{1}{1 - ry - sy^2}\begin{bmatrix} r^2 + s + rsy + iy \\ 1 + i(r + sy) \end{bmatrix}. \tag{3.54}$$

(ii)

$$\sum_{n=0}^{\infty} SH_n y^n = \frac{1}{1 - ry - sy^2}\begin{bmatrix} r^3 + 3rs + (2s^2 + r^2 s)\,y + i(2 - ry) \\ r + 2sy + i(r^2 + 2s + rsy) \end{bmatrix}. \tag{3.55}$$

(iii)

$$\sum_{n=0}^{\infty} SF_n y^n = \frac{1}{1-xy-y^2}\begin{bmatrix} x(x+y)+1+iy \\ 1+i(x+y) \end{bmatrix}. \tag{3.56}$$

(iv)

$$\sum_{n=0}^{\infty} SL_n y^n = \frac{1}{1-xy-y^2}\begin{bmatrix} x^3+3x+\left(2+x^2\right)y+i(2-xy) \\ x+2y+i(x(x+y)+2) \end{bmatrix}. \tag{3.57}$$

respectively.

Proof. In Lemma 4, take SW_n as SG_n, SH_n, SF_n and SL_n, respectively. Use the first two terms of these sequences of polynomial by taking $n=0,1$ in Table 3 and Table 4 for the formula. □

4. Simson's Formulas

We start with by defining the generalized Fibonacci polynomial spinor matrix with the help of Fibonacci spinor matrix definition given by Erişir and Güngör (8). In order to define required matrix, we need to recall the Fibonacci quaternion matrix defined by Halıcı (8) as follows:

$$Q = \begin{bmatrix} QF_2 & QF_1 \\ QF_1 & QF_0 \end{bmatrix} \tag{4.1}$$

where

$$\begin{aligned} QF_0 &= F_0+iF_1+jF_2+kF_3 = i+j+2k, \\ QF_1 &= F_1+iF_2+jF_3+kF_4 = 1+i+2j+3k, \\ QF_2 &= F_2+iF_3+jF_4+kF_5 = 1+2i+3j+5k. \end{aligned} \tag{4.2}$$

From this point of view, we can define generalized Fibonacci quaternion polynomial matrix as follows:

$$QW(x) = \begin{bmatrix} QW_2 & QW_1 \\ QW_1 & QW_0 \end{bmatrix}. \tag{4.3}$$

In addition, we can define the using pattern of the determinant of a given matrix as follows:

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{22}a_{11} - a_{12}a_{21}. \tag{4.4}$$

We prefer to use this formula of determinant of a 2×2 type matrix throughout the paper.

Erişir and Güngör (8) gave the Fibonacci spinor matrix for Fibonacci spinors helping to obtain the Simson's identity. Now, we will present the generalized Fibonacci polynomial spinor matrix in the next theorem and we will denote this matrix with S_W. Then, we will use the matrix S_W to find the several identities such as Simson's identity, and so on.

Theorem 3. *Let $QW(x)$ be the generalized Fibonacci quaternion polynomial matrix. Then, the following equality holds from quaternion products via spinors:*

$$\det QW(x) \equiv i \det S_W$$

where

$$S_W = -i\begin{bmatrix} SW_2 & \widehat{SW}_1 \\ SW_1 & \widehat{SW}_0 \end{bmatrix} \tag{4.5}$$

with

$$\widehat{SW}_0 = \begin{bmatrix} W_3 + iW_0 & W_1 - iW_2 \\ W_1 + iW_2 & -W_3 + iW_0 \end{bmatrix}, \widehat{SW}_1 = \begin{bmatrix} W_4 + iW_1 & W_2 - iW_3 \\ W_2 + iW_3 & -W_4 + iW_1 \end{bmatrix} \tag{4.6}$$

and

$$SW_1 = \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix}, SW_2 = \begin{bmatrix} W_5 + iW_2 \\ W_3 + iW_4 \end{bmatrix}. \tag{4.7}$$

Proof. Let nth generalized Fibonacci polynomial spinor SW_n correspond to nth generalized Fibonacci quaternion polynomial QW_n. Given a generalized Fibonacci quaternion polynomial matrix $QW(x)$, we obtain by (2.28) that

$$\begin{aligned} \det QW(x) &= QW_0 QW_2 - QW_1 QW_1 \\ &\equiv -i\widehat{SW}_0 SW_2 + i\widehat{SW}_1 SW_1 \equiv -i(\widehat{SW}_0 SW_2 - \widehat{SW}_1 SW_1) \\ &= -i \det \begin{bmatrix} SW_2 & \widehat{SW}_1 \\ SW_1 & \widehat{SW}_0 \end{bmatrix} = i \det S_W. \end{aligned} \tag{4.8}$$

Hence, we can write the generalized Fibonacci polynomial spinor matrix corresponding to quaternion version by using the formula of determinant as follows:

$$S_W = -i \begin{bmatrix} SW_2 & \widehat{SW}_1 \\ SW_1 & \widehat{SW}_0 \end{bmatrix} \tag{4.9}$$

where $\widehat{SW}_0$, $\widehat{SW}_1$, SW_1 and SW_2 are given in (4.6) and (4.7). Thus, we have the following matrix:

$$S_W = -i \begin{bmatrix} \begin{bmatrix} W_5 + iW_2 \\ W_3 + iW_4 \end{bmatrix} & \begin{bmatrix} W_4 + iW_1 & W_2 - iW_3 \\ W_2 + iW_3 & -W_4 + iW_1 \end{bmatrix} \\ \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix} & \begin{bmatrix} W_3 + iW_0 & W_1 - iW_2 \\ W_1 + iW_2 & -W_3 + iW_0 \end{bmatrix} \end{bmatrix}. \tag{4.10}$$

□

From Theorem 3, we can present the Simson's identity in two different forms in the next theorem.

Theorem 4. *(Simson's Identity) For all integers n, we have*

$$\widehat{SW}_{n-1} SW_{n+1} - \widehat{SW}_n SW_n = (-s)^n (\widehat{SW}_{-1} SW_1 - \widehat{SW}_0 SW_0). \tag{4.11}$$

i.e.,

$$\det \begin{bmatrix} SW_{n+1} & \widehat{SW}_n \\ SW_n & \widehat{SW}_{n-1} \end{bmatrix} = (-s)^n \det \begin{bmatrix} SW_1 & \widehat{SW}_0 \\ SW_0 & \widehat{SW}_{-1} \end{bmatrix}.$$

Proof. For $\alpha \neq \beta$, using (3.45), the Binet's formula of SW_n, and (2.5) we obtain the following identities:

$$\begin{aligned} &\widehat{SW}_{n-1} SW_{n+1} - \widehat{SW}_n SW_n \\ &= -(-s)^{n-1} c_1 c_2 \begin{bmatrix} (\beta - \alpha - \alpha^3\beta^4 + \alpha^4\beta^3 - \alpha^2\beta^3 \\ +\alpha^3\beta^2 - \alpha\beta^2 + \alpha^2\beta + i(\alpha^4 - \beta^4)) \\ \\ (\beta^3 - \alpha^3 + \alpha^4\beta - \alpha\beta^4 + \alpha^2\beta - \alpha\beta^2 \\ +\alpha^3\beta^2 - \alpha^2\beta^3 + i(\alpha^2 - \beta^2 - \alpha^2\beta^4 + \alpha^4\beta^2)) \end{bmatrix} \end{aligned} \tag{4.12}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

where $c_1 = W_1 - \beta W_0$ and $c_2 = W_1 - \alpha W_0$. On the other hand,

$$\begin{aligned} & \widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0 \\ = & -\frac{1}{s}c_1c_2 \begin{bmatrix} \begin{array}{c} (\beta - \alpha - \alpha^3\beta^4 + \alpha^4\beta^3 - \alpha^2\beta^3 \\ +\alpha^3\beta^2 - \alpha\beta^2 + \alpha^2\beta + i\left(\alpha^4 - \beta^4\right)) \end{array} \\ \\ \begin{array}{c} (\beta^3 - \alpha^3 + \alpha^4\beta - \alpha\beta^4 + \alpha^2\beta - \alpha\beta^2 + \alpha^3\beta^2 \\ -\alpha^2\beta^3 + i\left(\alpha^2 - \beta^2 - \alpha^2\beta^4 + \alpha^4\beta^2\right)) \end{array} \end{bmatrix}. \end{aligned} \tag{4.13}$$

From (4.12) and (4.13), we have the required identity:

$$\begin{aligned} \widehat{SW_{n-1}}SW_{n+1} - \widehat{SW_n}SW_n &= (-s)^{n-1}s\left(\widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0\right) \\ &= (-s)^n(\widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0). \end{aligned}$$

Now, let us see when the roots are equal, i.e., for $\alpha = \beta$, (4.11) holds for all integers n. Using (3.46), we obtain that

$$\begin{aligned} & \widehat{SW_{n-1}}SW_{n+1} - \widehat{SW_n}SW_n \\ = & -\alpha^{2n}d_2^2 \begin{bmatrix} \left(\alpha^6 + \alpha^4 + \alpha^2 - 1 + 4i\alpha^3\right) \\ -4\alpha^4 + 2\alpha^2 - 2\alpha i\left(\alpha^4 + 1\right) \end{bmatrix} \end{aligned} \tag{4.14}$$

where $d_1 = W_0$ and $d_2 = \frac{1}{\alpha}\left(W_1 - \alpha W_0\right)$. On the other hand,

$$\begin{aligned} & \widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0 \\ = & -d_2^2 \begin{bmatrix} \left(\alpha^6 + \alpha^4 + \alpha^2 - 1 + 4i\alpha^3\right) \\ 2\alpha\left(i\alpha^4 + 2\alpha^3 - \alpha + i\right) \end{bmatrix}. \end{aligned} \tag{4.15}$$

From (4.14), (4.15) and (2.8) we get the required identity as follows:

$$\begin{aligned} \widehat{SW_{n-1}}SW_{n+1} - \widehat{SW_n}SW_n &= \alpha^{2n}\left(\widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0\right) \\ &= (-s)^n\left(\widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0\right). \end{aligned}$$

□

The previous theorem gives the following results as particular examples.

Corollary 4.1. *For all integers n, Simson's formula of $(r,s)-$Fibonacci, $(r,s)-$Fibonacci-Lucas, Fibonacci and Fibonacci- Lucas polynomials spinors are given as*

(i)

$$\begin{aligned} & \widehat{SG_{n+1}}SG_{n-1} - \widehat{SG_n}SG_n \\ = & (-1)^n s^n(\widehat{SG_{-1}}SG_1 - \widehat{SG_0}SG_0) \\ = & (-1)^n s^{n-1} \begin{bmatrix} \left(ir^3 + 2irs - s^3 + s^2 - s - 1\right) \\ \left(-r^2s - r^2 + irs^2 + ir - 2s\right) \end{bmatrix}. \end{aligned} \tag{4.16}$$

(ii)

$$\begin{aligned}
& \widehat{SH_{n+1}}SH_{n-1}-\widehat{SH_n}SH_n \\
= \;& (-1)^n s^n (\widehat{SH_{-1}}SH_1-\widehat{SH_0}SH_0) \\
= \;& (-1)^{n-1}s^{n-1}\left(r^2+4s\right)\left[\begin{array}{c}\left(ir^3+2irs-s^3+s^2-s-1\right)\\ \left(-r^2s-r^2+irs^2+ir-2s\right)\end{array}\right].
\end{aligned} \tag{4.17}$$

(iii)

$$\widehat{SF_{n+1}}SF_{n-1}-\widehat{SF_n}SF_n=(-1)^n\left[\begin{array}{c}\left(-2+i\left(x^3+2x\right)\right)\\ \left(-2x^2-2+2ix\right)\end{array}\right]. \tag{4.18}$$

(iv)

$$\widehat{SL_{n+1}}SL_{n-1}-\widehat{SL_n}SL_n=(-1)^{n-1}\left(x^2+4\right)\left[\begin{array}{c}\left(-2+i\left(x^3+2x\right)\right)\\ \left(-2x^2-2+2ix\right)\end{array}\right]. \tag{4.19}$$

respectively.

Proof. The proof can be obtained by Equation 3.22, Table 3 and Table 4. □

If one compare the results (i) and (ii) in Corollary 4.1, it can be seen the quick result presenting the relation between Simson's identities for SG_n and SH_n as follows:

Corollary 4.2. *For all integers n, the next identities hold:*

(i)

$$\widehat{SH_{n+1}}SH_{n-1}-\widehat{SH_n}SH_n=\left(r^2+4s\right)\left(\widehat{SG_{n+1}}SG_{n-1}-\widehat{SG_n}SG_n\right). \tag{4.20}$$

(ii)

$$\widehat{SL_{n+1}}SL_{n-1}-\widehat{SL_n}SL_n=\left(x^2+4\right)\left(\widehat{SF_{n+1}}SF_{n-1}-\widehat{SF_n}SF_n\right). \tag{4.21}$$

Next theorem exhibits the relation of generalized Fibonacci polynomial spinor transforms with different terms.

Theorem 5. *For all integers n, the following identities hold:*

(i)

$$\widehat{SW_{-n}}SW_n-\widehat{SW_0}SW_0=(-s)^{-n}\left(\widehat{SW_0}SW_{2n}-\widehat{SW_n}SW_n\right). \tag{4.22}$$

(ii)

$$\begin{aligned}
& \widehat{SW_{-n}}SW_n-\widehat{SW_0}SW_0 \\
= \;& \left(W_1^2-sW_0^2-rW_0W_1\right)\left(\widehat{SG_{-n}}SG_n-\widehat{SG_0}SG_0\right).
\end{aligned} \tag{4.23}$$

(iii)

$$\begin{aligned}
& \widehat{SW_{-n}}SW_n-\widehat{SW_0}SW_0 \\
= \;& \left(W_1^2-sW_0^2-rW_0W_1\right)(-s)^{-n}\left(\widehat{SG_0}SG_{2n}-\widehat{SG_n}SG_n\right).
\end{aligned} \tag{4.24}$$

(iv)

$$\widehat{SH_{-n}}SH_n-\widehat{SH_0}SH_0=-\left(r^2+4s\right)\left(\widehat{SG_{-n}}SG_n-\widehat{SG_0}SG_0\right). \tag{4.25}$$

(v)

$$\widehat{SH_0}SH_{2n}-\widehat{SH_n}SH_n=-(-s)^n\left(r^2+4s\right)\left(\widehat{SG_{-n}}SG_n-\widehat{SG_0}SG_0\right). \tag{4.26}$$

Proof. (i) For $\alpha \neq \beta$, using (3.45) and (2.5) we obtain the following identities:

$$\begin{aligned} & \widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \\ = & -\frac{1}{(-s)^n}c_1c_2(\alpha^n-\beta^n)\left[\begin{array}{c} (\beta^n-\alpha^n-\alpha\beta^{n+1}+\alpha^{n+1}\beta-\alpha^2\beta^{n+2} \\ +\alpha^{n+2}\beta^2-\alpha^3\beta^{n+3}+\alpha^{n+3}\beta^3 \\ -i\alpha\beta^{n+2}+i\alpha^{n+2}\beta-i\alpha^3\beta^n+i\alpha^n\beta^3 \\ +i\alpha^{n+3}-i\beta^{n+3}+i\alpha^2\beta^{n+1}-i\alpha^{n+1}\beta^2) \\ \\ (\alpha^{n+3}\beta-\alpha\beta^{n+3}+\alpha^2\beta^n-\alpha^n\beta^2+\beta^{n+2} \\ -\alpha^{n+2}+\alpha^3\beta^{n+1}-\alpha^{n+1}\beta^3+i\alpha^{n+1} \\ -i\beta^{n+1}-i\alpha^2\beta^{n+3}+i\alpha^3\beta^{n+2} \\ -i\alpha^{n+2}\beta^3+i\alpha^{n+3}\beta^2-i\alpha\beta^n+i\alpha^n\beta) \end{array}\right]. \end{aligned} \tag{4.27}$$

where $c_1 = W_1 - \beta W_0$ and $c_2 = W_1 - \alpha W_0$. On the other hand,

$$\begin{aligned} & \widehat{SW_0}SW_{2n} - \widehat{SW_n}SW_n \\ = & -c_1c_2\left(\alpha^n-\beta^n\right)\left[\begin{array}{c} (\beta^n-\alpha^n-\alpha\beta^{n+1}+\alpha^{n+1}\beta-\alpha^2\beta^{n+2} \\ +\alpha^{n+2}\beta^2-\alpha^3\beta^{n+3}+\alpha^{n+3}\beta^3 \\ -i\alpha\beta^{n+2}+i\alpha^{n+2}\beta-i\alpha^3\beta^n+i\alpha^n\beta^3 \\ +i\alpha^{n+3}-i\beta^{n+3}+i\alpha^2\beta^{n+1}-i\alpha^{n+1}\beta^2) \\ \\ (\alpha^{n+3}\beta-\alpha\beta^{n+3}+\alpha^2\beta^n-\alpha^n\beta^2 \\ +\alpha^3\beta^{n+1}-\alpha^{n+2}+\beta^{n+2}-\alpha^{n+1}\beta^3 \\ +i\alpha^{n+1}-i\beta^{n+1}-i\alpha^2\beta^{n+3}+i\alpha^3\beta^{n+2} \\ -i\alpha^{n+2}\beta^3+i\alpha^{n+3}\beta^2-i\alpha\beta^n+i\alpha^n\beta) \end{array}\right]. \\ = & (-s)^n\left(\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0\right). \end{aligned} \tag{4.28}$$

Now, let us see when the roots are equal, i.e., for $\alpha = \beta$, (4.22) holds for all integers n. Using (3.46), we obtain that

$$\begin{aligned} & \widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \\ = & -nd_2^2\left[\begin{array}{c} (-n+n\alpha^2+2in\alpha^3+n\alpha^4+n\alpha^6+2i\alpha^3) \\ 2\alpha\left(i\alpha^4+2\alpha^3-n\alpha+in\right) \end{array}\right] \end{aligned} \tag{4.29}$$

where $d_1 = W_0$ and $d_2 = \frac{1}{\alpha}\left(W_1 - \alpha W_0\right)$. On the other hand,

$$\begin{aligned} & \widehat{SW_0}SW_{2n} - \widehat{SW_n}SW_n \\ = & -n\alpha^{2n}d_2^2\left[\begin{array}{c} (-n+n\alpha^2+2in\alpha^3+n\alpha^4+n\alpha^6+2i\alpha^3) \\ 2\alpha\left(i\alpha^4+2\alpha^3-n\alpha+in\right) \end{array}\right]. \end{aligned} \tag{4.30}$$

Therefore, (4.29) and (4.30) give us the next required equality:

$$\begin{aligned} \widehat{SW_0}SW_{2n} - \widehat{SW_n}SW_n &= \alpha^{2n}\left(\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0\right) \\ &= (-s)^n\left(\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0\right). \end{aligned} \tag{4.31}$$

(ii) For $\alpha \neq \beta$, since $c_1 = G_1 - \beta G_0 = 1$ and $c_2 = G_1 - \alpha G_0 = 1$, we can immediately the following identity from using (4.27) and taking $W_n = G_n$:

$$\begin{aligned}&\widehat{SG_{-n}}SG_n-\widehat{SG_0}SG_0\\&=-\frac{1}{(-s)^n}(\alpha^n-\beta^n)\begin{bmatrix}(\beta^n-\alpha^n-\alpha\beta^{n+1}+\alpha^{n+1}\beta-\alpha^2\beta^{n+2}\\+\alpha^{n+2}\beta^2-\alpha^3\beta^{n+3}+\alpha^{n+3}\beta^3\\-i\alpha\beta^{n+2}+i\alpha^{n+2}\beta-i\alpha^3\beta^n+i\alpha^n\beta^3\\+i\alpha^{n+3}-i\beta^{n+3}+i\alpha^2\beta^{n+1}-i\alpha^{n+1}\beta^2)\\ \\(\alpha^{n+3}\beta-\alpha\beta^{n+3}+\alpha^2\beta^n-\alpha^n\beta^2\\+\beta^{n+2}-\alpha^{n+2}+\alpha^3\beta^{n+1}-\alpha^{n+1}\beta^3\\+i\alpha^{n+1}-i\beta^{n+1}-i\alpha^2\beta^{n+3}+i\alpha^3\beta^{n+2}\\-i\alpha^{n+2}\beta^3+i\alpha^{n+3}\beta^2-i\alpha\beta^n+i\alpha^n\beta)\end{bmatrix}.\end{aligned}\tag{4.32}$$

Hence, the product c_1c_2 in (4.27) equals $\left(W_1^2-sW_0^2-rW_0W_1\right)$ by (2.4) and (2.5) and by comparing the identities (4.27) and (4.32) have the desired identity:

$$\begin{aligned}&\widehat{SW_{-n}}SW_n-\widehat{SW_0}SW_0\\&=\left(W_1^2-sW_0^2-rW_0W_1\right)\left(\widehat{SG_{-n}}SG_n-\widehat{SG_0}SG_0\right).\end{aligned}$$

We now prove the identity holds for all integers n for equal roots, i.e, for $\alpha=\beta$. By taking $W_n=G_n$ in (4.29), since $d_1=0$ and $d_2=\frac{1}{\alpha}$ we arrive the following identity.

$$\begin{aligned}&\widehat{SG_{-n}}SG_n-\widehat{SG_0}SG_0\\&=-\frac{n}{\alpha^2}\begin{bmatrix}(-n+n\alpha^2+2in\alpha^3+n\alpha^4+n\alpha^6+2i\alpha^3)\\2\alpha\left(i\alpha^4+2\alpha^3-n\alpha+in\right)\end{bmatrix}.\end{aligned}\tag{4.33}$$

Therefore, the product d_2^2 in (4.29) equals $\left(W_1^2-rW_0W_1-sW_0^2\right)$ by (2.7) and (2.8) and by comparing the identities (4.29) and (4.33), we have the desired identity.

(iii) It is clear from (i) and (ii).
(iv) Considering $SW_n=SH_n$ for all integers n and setting the value of $\left(W_1^2-sW_0^2-rW_0W_1\right)$ in (ii) from Table 1, we reach the desired identity.
(v) It is obvious from (i) and (iii).

□

Through Theorem 5 by taking $SW_n=SG_n$ and $SW_n=SH_n$, we can see a more general formula of $(r,s)-$Fibonacci polynomial spinors SG_n and $(r,s)-$Fibonacci-Lucas polynomial spinors SH_n in the next corollary.

Corollary 4.3. *For all integers n, the following equalities hold:*

(i)
$$\widehat{SG_{-n}}SG_n-\widehat{SG_0}SG_0=(-s)^{-n}\left(\widehat{SG_0}SG_{2n}-\widehat{SG_n}SG_n\right).\tag{4.34}$$

(ii)
$$\widehat{SH_{-n}}SH_n-\widehat{SH_0}SH_0=(-s)^{-n}\left(\widehat{SH_0}SH_{2n}-\widehat{SH_n}SH_n\right).\tag{4.35}$$

Next corollary is the result of Corollary 4.3 for special cases.

Corollary 4.4. *For all integers n, the following equalities hold:*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

(i)
$$\widehat{SF_{-n}}SF_n-\widehat{SF_0}SF_0=(-1)^n\left(\widehat{SF_0}SF_{2n}-\widehat{SF_n}SF_n\right). \tag{4.36}$$

(ii)
$$\widehat{SL_{-n}}SL_n-\widehat{SL_0}SL_0=(-1)^n\left(\widehat{SL_0}SL_{2n}-\widehat{SL_n}SL_n\right). \tag{4.37}$$

We can rewrite the Simson's formula in five different way by previous corollaries and theorem.

Theorem 6. *(Simson's Identity) For all integers n, Simson's identity of generalized Fibonacci polynomial spinors can be given with the next five different formulas:*

$$\begin{aligned}&\widehat{SW_{n-1}}SW_{n+1}-\widehat{SW_n}SW_n\\ =\;&(-s)^{n-1}\left(\widehat{SW_0}SW_2-\widehat{SW_1}SW_1\right)\\ =\;&(-s)^n\left(W_1^2-sW_0^2-rW_0W_1\right)\left(\widehat{SG_{-1}}SG_1-\widehat{SG_0}SG_0\right)\\ =\;&(-s)^{n-1}\left(W_1^2-sW_0^2-rW_0W_1\right)\left(\widehat{SG_0}SG_2-\widehat{SG_1}SG_1\right).\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}&\left(r^2+4s\right)\left(\widehat{SW_{n-1}}SW_{n+1}-\widehat{SW_n}SW_n\right)\\ =\;&(-s)^{n-1}\left(W_1^2-sW_0^2-rW_0W_1\right)\left(\widehat{SH_{-1}}SH_1-\widehat{SH_0}SH_0\right)\\ =\;&-\left(W_1^2-sW_0^2-rW_0W_1\right)\left(\widehat{SH_0}SH_2-\widehat{SH_1}SH_1\right).\end{aligned} \tag{4.39}$$

5. Some Identities

In this section, we obtain some identities of generalized Fibonacci (Horadam) polynomials spinors, (r,s)-Fibonacci polynomials spinors and (r,s)-Lucas polynomials spinors. Firstly, we can give a few basic relations between $\{G_n\}$ and $\{SW_n\}$.

Theorem 7. *For all integers m,n we have*

$$SW_{n+m}=SW_nG_{m+1}+sSW_{n-1}G_m \tag{5.1}$$

i.e.,

$$SW_{n+m}=SW_mG_{n+1}+sSW_{m-1}G_n \tag{5.2}$$

Proof. For $m\geq 1$ and $m\leq 0$, we use induction on m. First we assume that $m\geq 1$.

The equation is true for $m=1$ since

$$\begin{aligned}SW_{n+1}&=rSW_n+sSW_{n-1}\\ &=SW_nG_2+sSW_{n-1}G_1\end{aligned} \tag{5.3}$$

where $G_2=r$ and $G_1=1$. For $m=2$, the equation is also true which we can see below, because using definition of SW_n and the values $G_2=r, G_3=s+r^2$, we get

$$\begin{aligned}SW_{n+2}&=rSW_{n+1}+sSW_n=r(rSW_n+sSW_{n-1})+sSW_n\\ &=(s+r^2)SW_n+rsSW_{n-1}=SW_nG_3+sSW_{n-1}G_2.\end{aligned} \tag{5.4}$$

 doi:10.20944/preprints202402.0910.v1

Assume now that the equation holds for all m with $0 \leq m \leq k+1$. Then, by assumption, for $m = k$ and $m = k+1$ we have, respectively,

$$sSW_{n+k} = s(SW_n G_{k+1} + sSW_{n-1} G_k), \tag{5.5}$$

and

$$rSW_{n+k+1} = r(SW_n G_{k+2} + sSW_{n-1} G_{k+1}). \tag{5.6}$$

By adding up these two equations we get

$$rSW_{n+k+1} + sSW_{n+k} = r(SW_n G_{k+2} + sSW_{n-1} G_{k+1}) + s(SW_n G_{k+1} + sSW_{n-1} G_k), \tag{5.7}$$

i.e.,

$$\begin{aligned} SW_{n+k+2} &= SW_n \left(rG_{k+2} + sG_{k+1}\right) + sSW_{n-1} \left(rG_{k+1} + sG_k\right) \\ &= SW_n G_{k+3} + sSW_{n-1} G_{k+2} \end{aligned} \tag{5.8}$$

which yields the equation for $m = k+2$.

Now, we proceed by induction on $|m| = -m = v$ when $m \leq 0$. For $v = 0$, that is $m = 0$, the equation is true because

$$SW_n = SW_n G_1 + sSW_{n-1} G_0 \tag{5.9}$$

where $G_0 = 0$ and $G_1 = 1$. For $v = 1$, that is $m = -1$, it is true because

$$SW_{n-1} = SW_n G_0 + sSW_{n-1} G_{-1} \tag{5.10}$$

where $G_0 = 0$ and $G_{-1} = \frac{1}{s}$. Suppose that it holds for all $v = |m| = -m$ with $1 \leq v \leq k+1$. Then, by assumption, for $v = k$ and $v = k+1$ we have, respectively,

$$\frac{1}{s} SW_{n-k} = \frac{1}{s}(SW_n G_{-k+1} + sSW_{n-1} G_{-k}) \tag{5.11}$$

and

$$\frac{-r}{s} SW_{n-k-1} = \frac{-r}{s}(SW_n G_{-k} + sSW_{n-1} G_{-k-1}). \tag{5.12}$$

By adding up these two equations we get

$$\begin{aligned} & \frac{-r}{s} SW_{n-k-1} + \frac{1}{s} G_{n-k} \\ = & \frac{-r}{s}(SW_n G_{-k} + sSW_{n-1} G_{-k-1}) + \frac{1}{s}(SW_n G_{-k+1} + sSW_{n-1} G_{-k}), \end{aligned} \tag{5.13}$$

i.e.,

$$\begin{aligned} SW_{n-k-2} &= SW_n(-\frac{r}{s} G_{-k} + \frac{1}{s} G_{-k+1}) + sSW_{n-1}(-\frac{r}{s} G_{-k-1} + \frac{1}{s} G_{-k}) \\ &= SW_n G_{-k-1} + sSW_{n-1} G_{-k-2}; \end{aligned} \tag{5.14}$$

thus we obtain the equation for $v = |m| = k+2$. □

We then have next result via Theorem 7 as a corollary.

Corollary 5.1. *For $n \in \mathbb{Z}$, the following equalities are true:*

(i)

$$SW_n = SW_0 G_{n+1} + (SW_1 - rSW_0) G_n \tag{5.15}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

(ii)
$$(4s+r^2)SW_n = (2SW_1 - rSW_0)H_{n+1} + (-rSW_1 + (2s+r^2)SW_0)H_n \tag{5.16}$$
(iii)
$$SW_{n+m} = G_n SW_{m+1} + sG_{n-1}SW_m \tag{5.17}$$
(iv)
$$SG_{n+m} = SG_n G_{m+1} + sSG_{n-1}G_m \tag{5.18}$$
(v)
$$SH_{n+m} = SH_n G_{m+1} + sSH_{n-1}G_m \tag{5.19}$$
(vi)
$$SG_n = SG_0 G_{n+1} + (SG_1 - rSG_0)G_n \tag{5.20}$$
(vii)
$$SH_n = SH_0 G_{n+1} + (SH_1 - rSH_0)G_n \tag{5.21}$$
(viii)
$$(4s+r^2)SH_n = (2SH_1 - rSH_0)H_{n+1} + (-rSH_1 + (2s+r^2)SH_0)H_n \tag{5.22}$$
(ix)
$$(4s+r^2)SG_n = (2SG_1 - rSG_0)H_{n+1} + (-rSG_1 + (2s+r^2)SG_0)H_n \tag{5.23}$$
(x)
$$SG_{n+m} = G_n SG_{m+1} + sG_{n-1}SG_m \tag{5.24}$$
(xi)
$$SH_{n+m} = G_n SH_{m+1} + sG_{n-1}SH_m \tag{5.25}$$

Proof. (i) If we take m equals 0 in (5.2) then we obtain the required equation.

(ii) If we use the following equations in a**)** coming from [32]

$$\begin{aligned}(r^2+4s)G_n &= 2H_{n+1} - rH_n, \\ (r^2+4s)G_n &= rH_n + 2sH_{n-1}\end{aligned} \tag{5.26}$$

then we have the required equation.

(iii) Take $n-1$ instead of n and $m+1$ instead of m in (5.2).

(iv)-(v) Take SG_{n+m} and SH_{n+m} instead of SW_{n+m} in (5.1).

(vi)-(vii) Take SG_n and SH_n instead of SW_n in (5.15).

(viii)-(ix) Replace $SW_n = SG_n$ and $SW_n = SH_n$, respectively in (5.16).

(x)-(xi) Replace $SW_{n+m} = SG_{n+m}$ and $SW_{n+m} = SH_{n+m}$, respectively in (5.17). □

We obtain the next corollary from Theorem 7 and Corollary 5.1 (iii). Next identities will be useful for us in the last section.

Corollary 5.2. *Let $n \in \mathbb{Z}$. The following equalities are true:*

(i)
$$SW_n = G_n SW_1 + sG_{n-1}SW_0. \tag{5.27}$$
(ii)
$$SW_{n+1} = G_n SW_2 + sG_{n-1}SW_1. \tag{5.28}$$
(iii)
$$SW_{n+1} = G_{n+1}SW_1 + sG_n SW_0. \tag{5.29}$$
(iv)
$$SW_{n+2} = G_{n+1}SW_2 + sG_n SW_1. \tag{5.30}$$

Theorem 8. *(Catalan's identity) Let n and m be any integers. Then the following identity is true:*

$$\widehat{SW_{n-m}}SW_{n+m}-\widehat{SW_{n}}SW_{n}=(-s)^{n}\left(\widehat{SW_{-m}}SW_{m}-\widehat{SW_{0}}SW_{0}\right) \tag{5.31}$$

i.e.,

$$\begin{vmatrix} SW_{n+m} & \widehat{SW_{n}} \\ SW_{n} & \widehat{SW_{n-m}} \end{vmatrix}=(-s)^{n}\begin{vmatrix} SW_{m} & \widehat{SW_{0}} \\ SW_{0} & \widehat{SW_{-m}} \end{vmatrix}. \tag{5.32}$$

Proof. Let n and m be any integers. Using the Binet's formula of SW_n, SW_{n-m} and SW_{n+m}, we obtain the desired identitiy by (2.5) and Equation 4.22 in Theorem 4.3 (i).

$$\begin{aligned}
&\widehat{SW_{n-m}}SW_{n+m}-\widehat{SW_{n}}SW_{n}\\
=\;& -\alpha^{n-m}\beta^{n-m}c_1c_2\left(\alpha^{m}-\beta^{m}\right)\left[\begin{array}{c}
(\beta^{m}-\alpha^{m}-\alpha\beta^{m+1}+\alpha^{m+1}\beta-\alpha^{2}\beta^{m+2}\\
+\alpha^{m+2}\beta^{2}-\alpha^{3}\beta^{m+3}+\alpha^{m+3}\beta^{3}\\
-i\alpha\beta^{m+2}+i\alpha^{m+2}\beta-i\alpha^{3}\beta^{m}+i\alpha^{m}\beta^{3}\\
+i\alpha^{m+3}-i\beta^{m+3}+i\alpha^{2}\beta^{m+1}-i\alpha^{m+1}\beta^{2})\\
\\
(\alpha^{m+3}\beta-\alpha\beta^{m+3}+\alpha^{2}\beta^{m}-\alpha^{m}\beta^{2}\\
+\beta^{m+2}-\alpha^{m+2}+\alpha^{3}\beta^{m+1}-\alpha^{m+1}\beta^{3}\\
+i\alpha^{m+1}-i\beta^{m+1}-i\alpha^{2}\beta^{m+3}+i\alpha^{3}\beta^{m+2}\\
-i\alpha^{m+2}\beta^{3}+i\alpha^{m+3}\beta^{2}-i\alpha\beta^{m}+i\alpha^{m}\beta)
\end{array}\right]\\
=\;& \alpha^{m}\beta^{m}\alpha^{n-m}\beta^{n-m}\left(\widehat{SW_{-m}}SW_{m}-\widehat{SW_{0}}SW_{0}\right)\\
=\;& (-s)^{n}\left(\widehat{SW_{-m}}SW_{m}-\widehat{SW_{0}}SW_{0}\right).
\end{aligned} \tag{5.33}$$

When we take SG_n,SH_n, SF_n,SL_n instead of SW_n, we have the Catalan's identity for $(r,s)-$Fibonacci, $(r,s)-$Fibonacci-Lucas polynomial spinors, Fibonacci polynomial spinors and Fibonacci-Lucas polynomial spinors, respectively. □

Corollary 5.3. *Let n and m be any integers. Then the following identities are true:*

(i)

$$\widehat{SG_{n-m}}SG_{n+m}-\widehat{SG_{n}}SG_{n}=(-s)^{n}\left(\widehat{SG_{-m}}SG_{m}-\widehat{SG_{0}}SG_{0}\right) \tag{5.34}$$

i.e.,

$$\begin{vmatrix} SG_{n+m} & \widehat{SG_{n}} \\ SG_{n} & \widehat{SG_{n-m}} \end{vmatrix}=(-s)^{n}\begin{vmatrix} SG_{m} & \widehat{SG_{0}} \\ SG_{0} & \widehat{SG_{-m}} \end{vmatrix}. \tag{5.35}$$

(ii)

$$\widehat{SH_{n-m}}SH_{n+m}-\widehat{SH_{n}}SH_{n}=(-s)^{n}\left(\widehat{SH_{-m}}SH_{m}-\widehat{SH_{0}}SH_{0}\right) \tag{5.36}$$

i.e.,

$$\begin{vmatrix} SH_{n+m} & \widehat{SH_{n}} \\ SH_{n} & \widehat{SH_{n-m}} \end{vmatrix}=(-s)^{n}\begin{vmatrix} SH_{m} & \widehat{SH_{0}} \\ SH_{0} & \widehat{SH_{-m}} \end{vmatrix}. \tag{5.37}$$

(iii)

$$\widehat{SF_{n-m}}SF_{n+m}-\widehat{SF_{n}}SF_{n}=(-1)^{n}\left(\widehat{SF_{-m}}SF_{m}-\widehat{SF_{0}}SF_{0}\right) \tag{5.38}$$

i.e.,

$$\begin{vmatrix} SF_{n+m} & \widehat{SF_{n}} \\ SF_{n} & \widehat{SF_{n-m}} \end{vmatrix}=(-1)^{n}\begin{vmatrix} SF_{m} & \widehat{SF_{0}} \\ SF_{0} & \widehat{SF_{-m}} \end{vmatrix}. \tag{5.39}$$

(iv)

$$\widehat{SL_{n-m}}SL_{n+m}-\widehat{SL_{n}}SL_{n}=(-1)^{n}\left(\widehat{SL_{-m}}SL_{m}-\widehat{SL_{0}}SL_{0}\right) \tag{5.40}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

i.e.,

$$\begin{vmatrix} SL_{n+m} & \widehat{SL_n} \\ SL_n & \widehat{SL_{n-m}} \end{vmatrix} = (-1)^n \begin{vmatrix} SL_m & \widehat{SL_0} \\ SL_0 & \widehat{SL_{-m}} \end{vmatrix}. \quad (5.41)$$

We next take an example of Catalan's formula for a speacial n and m.

Theorem 9. *We have the Catalan's identity for $n = 2$ and $m = 1$,*

$$\widehat{SW_1}SW_3 - \widehat{SW_2}SW_2 = s^2\left(\widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0\right). \quad (5.42)$$

We can also exhibit the Catalan's identity in other forms via Theorem 4.3 as in the next corollary.

Corollary 5.4. *(Catalan's Identity) For all integers n and m, Catalan's identity of generalized Fibonacci polynomial spinors can be given with the next five different formulas:*

$$\begin{aligned} & \widehat{SW_{n-m}}SW_{n+m} - \widehat{SW_n}SW_n \\ = & (-s)^{n-m}\left(\widehat{SW_0}SW_{2m} - \widehat{SW_m}SW_m\right) \\ = & (-s)^n\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_{-m}}SG_m - \widehat{SG_0}SG_0\right) \\ = & \left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_0}SG_{2m} - \widehat{SG_m}SG_m\right) \end{aligned} \quad (5.43)$$

and

$$\begin{aligned} & \left(r^2+4s\right)\left(\widehat{SW_{n-m}}SW_{n+m} - \widehat{SW_n}SW_n\right) \\ = & -(-s)^n\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SH_{-m}}SH_m - \widehat{SH_0}SH_0\right) \\ = & -\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SH_0}SH_{2m} - \widehat{SH_m}SH_m\right). \end{aligned} \quad (5.44)$$

Theorem 10. *Due to Theorem 4.3, Catalan's identity of $(r,s)-$Fibonacci polynomial spinors, $(r,s)-$Fibonacci-Lucas polynomial spinors, Fibonacci polynomial spinors and Fibonacci-Lucas polynomial spinors can be written in many different ways as a result of Corollary 5.4.*

(i)

$$\begin{aligned} & \widehat{SG_{n-m}}SG_{n+m} - \widehat{SG_n}SG_n \\ = & (-s)^{n-m}\left(\widehat{SG_0}SG_{2m} - \widehat{SG_m}SG_m\right) \end{aligned} \quad (5.45)$$

$$\begin{aligned} & \left(r^2+4s\right)\left(\widehat{SG_{n-m}}SG_{n+m} - \widehat{SG_n}SG_n\right) \\ = & -(-s)^n\left(\widehat{SH_{-m}}SH_m - \widehat{SH_0}SH_0\right) \end{aligned} \quad (5.46)$$

$$\begin{aligned} & \left(r^2+4s\right)\left(\widehat{SG_{n-m}}SG_{n+m} - \widehat{SG_n}SG_n\right) \\ = & -\left(\widehat{SH_0}SH_{2m} - \widehat{SH_m}SH_m\right). \end{aligned} \quad (5.47)$$

(ii)

$$\begin{aligned} &\widehat{SH_{n-m}}SH_{n+m} - \widehat{SH_n}SH_n \\ = &(-s)^{n-m}\left(\widehat{SH_0}SH_{2m} - \widehat{SH_m}SH_m\right) \end{aligned} \tag{5.48}$$

$$\begin{aligned} &\widehat{SH_{n-m}}SH_{n+m} - \widehat{SH_n}SH_n \\ = &-(-s)^n\left(r^2+4s\right)\left(\widehat{SG_{-m}}SG_m - \widehat{SG_0}SG_0\right) \end{aligned} \tag{5.49}$$

$$\begin{aligned} &\widehat{SH_{n-m}}SH_{n+m} - \widehat{SH_n}SH_n \\ = &-\left(r^2+4s\right)\left(\widehat{SG_0}SG_{2m} - \widehat{SG_m}SG_m\right). \end{aligned} \tag{5.50}$$

(iii)

$$\begin{aligned} &\widehat{SF_{n-m}}SF_{n+m} - \widehat{SF_n}SF_n \\ = &(-1)^{n-m}\left(\widehat{SF_0}SF_{2m} - \widehat{SF_m}SF_m\right) \end{aligned} \tag{5.51}$$

$$\begin{aligned} &\left(x^2+4\right)\left(\widehat{SF_{n-m}}SF_{n+m} - \widehat{SF_n}SF_n\right) \\ = &-(-1)^n\left(\widehat{SL_{-m}}SL_m - \widehat{SL_0}SL_0\right) \end{aligned} \tag{5.52}$$

$$\begin{aligned} &\left(x^2+4\right)\left(\widehat{SF_{n-m}}SF_{n+m} - \widehat{SF_n}SF_n\right) \\ = &-\left(\widehat{SL_0}SL_{2m} - \widehat{SL_m}SL_m\right). \end{aligned} \tag{5.53}$$

(iv)

$$\begin{aligned} &\widehat{SL_{n-m}}SL_{n+m} - \widehat{SL_n}SL_n \\ = &(-1)^{n-m}\left(\widehat{SL_0}SL_{2m} - \widehat{SL_m}SL_m\right) \end{aligned} \tag{5.54}$$

$$\begin{aligned} &\widehat{SL_{n-m}}SL_{n+m} - \widehat{SL_n}SL_n \\ = &-(-1)^n\left(x^2+4\right)\left(\widehat{SF_{-m}}SF_m - \widehat{SF_0}SF_0\right) \end{aligned} \tag{5.55}$$

$$\begin{aligned} &\widehat{SL_{n-m}}SL_{n+m} - \widehat{SL_n}SL_n \\ = &-\left(x^2+4\right)\left(\widehat{SF_0}SF_{2m} - \widehat{SF_m}SF_m\right). \end{aligned} \tag{5.56}$$

6. Sum Formulas

In this section, we present sum formulas of generalized Fibonacci (Horadam) polynomial spinors.

Theorem 11. *For generalized Fibonacci (Horadam) polynomial spinors, we have the following sum formula: If $-s-r+1 \neq 0$ then*

$$\sum_{k=0}^{n} SW_k = \frac{(-s-r)SW_n - sSW_{n-1} + SW_1 + (1-r)SW_0}{-s-r+1}. \tag{6.1}$$

Proof. Using the recurrence relation

$$SW_n = rSW_{n-1} + sSW_{n-2} \tag{6.2}$$

i.e.

$$-sSW_{n-2} = rSW_{n-1} - SW_n \tag{6.3}$$

we obtain

 doi:10.20944/preprints202402.0910.v1

$$\begin{aligned} -sSW_0 &= rSW_1 - SW_2 \\ -sSW_1 &= rSW_2 - SW_3 \\ -sSW_2 &= rSW_3 - SW_4 \\ &\vdots \\ -sSW_{n-2} &= rSW_{n-1} - SW_n \end{aligned} \tag{6.4}$$

If we add the equations side by side, we get

$$-s\sum_{k=0}^{n-2} SW_k = r\sum_{k=1}^{n-1} SW_k - \sum_{k=2}^{n} SW_k. \tag{6.5}$$

Since

$$\begin{aligned} & -s\left(\sum_{k=0}^{n} SW_k - SW_{n-1} - SW_n\right) \\ = \;& r\left(\sum_{k=0}^{n} SW_k - SW_n - SW_0\right) - \left(\sum_{k=0}^{n} SW_k - SW_0 - SW_1\right) \end{aligned} \tag{6.6}$$

we get

$$\begin{aligned} & (-s-r+1)\sum_{k=0}^{n} SW_k \\ = \;& -sSW_{n-1} - sSW_n - rSW_n - rSW_0 + SW_0 + SW_1 \end{aligned} \tag{6.7}$$

and so

$$\sum_{k=0}^{n} SW_k = \frac{1}{-s-r+1}\left(-(r+s)SW_n - sSW_{n-1} + SW_1 + (1-r)SW_0\right). \tag{6.8}$$

□

Corollary 6.1. *For $(r,s)-$Fibonacci, $(r,s)-$Fibonacci-Lucas, Fibonacci and Fibonacci-Lucas polynomials spinors, we have the following sum formulas: If $-s-r+1 \neq 0$ and $x \neq 0$ then*

(i)

$$\begin{aligned} & \sum_{k=0}^{n} SG_k \\ = \;& \frac{1}{-s-r+1}((-s-r)SG_n - sSG_{n-1} + \begin{bmatrix} r^2+sr+s+i \\ 1+i(r+s) \end{bmatrix}). \end{aligned} \tag{6.9}$$

(ii)

$$\begin{aligned} & \sum_{k=0}^{n} SH_k \\ = \;& \frac{1}{-s-r+1}((-s-r)SH_n - sSH_{n-1} \\ & + \begin{bmatrix} r^3+r^2s+3rs+2s^2+i(r-2) \\ r+2s+i(rs+r^2+2s) \end{bmatrix}). \end{aligned} \tag{6.10}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

(iii)

$$\sum_{k=0}^{n} SF_k = -\frac{1}{x}((-1-x)SF_n - SF_{n-1} + \begin{bmatrix} x^2+x+1+i \\ 1+i(x+1) \end{bmatrix}). \tag{6.11}$$

(iv)

$$\sum_{k=0}^{n} SL_k = -\frac{1}{x}((-1-x)SL_n - SL_{n-1} + \begin{bmatrix} x^3+x^2+3x+2+i(x-2) \\ x+2+i(x+x^2+2) \end{bmatrix}). \tag{6.12}$$

7. Matrices associated with Generalized Fibonacci Polynomial Spinors

Let $A = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}$ and $N_{SW} = \begin{pmatrix} SW_2 & SW_1 \\ SW_1 & SW_0 \end{pmatrix}$. Then, we know by Soykan[32] that

$$A^n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}. \tag{7.1}$$

Hence, we obtain the next theorem from matrix product by using the identities in Corollary 5.2.

Theorem 12. *For all integers n, we have*

$$A^n N_{SW} = \begin{pmatrix} SW_{n+2} & SW_{n+1} \\ SW_{n+1} & SW_n \end{pmatrix}. \tag{7.2}$$

Proof. For all integers n, we product the required matrices and use the identities in Corollary 5.2.

$$\begin{aligned} A^n N_{SW} &= \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \begin{pmatrix} SW_2 & SW_1 \\ SW_1 & SW_0 \end{pmatrix} \\ &= \begin{pmatrix} G_{n+1}SW_2 + sG_nSW_1 & G_{n+1}SW_1 + sG_nSW_0 \\ G_nSW_2 + sG_{n-1}SW_1 & G_nSW_1 + sG_{n-1}SW_0 \end{pmatrix} \\ &= \begin{pmatrix} SW_{n+2} & SW_{n+1} \\ SW_{n+1} & SW_n \end{pmatrix}. \end{aligned} \tag{7.3}$$

□

Note that by taking SW_n as SG_n, SH_n, SF_n as SL_n we get the following matrices.

$$\begin{aligned} N_{SG} &= \begin{pmatrix} SG_2 & SG_1 \\ SG_1 & SG_0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} r^4+3r^2s+s^2+ir \\ r^2+s+i\left(r^3+2rs\right) \end{bmatrix} & \begin{bmatrix} r^3+2rs+i \\ r+i\left(r^2+s\right) \end{bmatrix} \\ \begin{bmatrix} r^3+2rs+i \\ r+i\left(r^2+s\right) \end{bmatrix} & \begin{bmatrix} r^2+s \\ 1+ir \end{bmatrix} \end{pmatrix}, \end{aligned} \tag{7.4}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 February 2024 doi:10.20944/preprints202402.0910.v1

$$
\begin{aligned}
N_{SH} &= \begin{pmatrix} SH_2 & SH_1 \\ SH_1 & SH_0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} 2r^4+r^3s+5r^2s \\ +2rs^2+s^2+i\,(2r+s) \\ \\ \left(2r^2+rs+s\right) \\ +i\left(2r^3+r^2s+3rs+s^2\right) \end{bmatrix} & \begin{bmatrix} 2r^3+r^2s \\ +3rs+s^2+i \\ \\ (2r+s) \\ +i\left(2r^2+rs+s\right) \end{bmatrix} \\ \begin{bmatrix} 2r^3+r^2s \\ +3rs+s^2+i \\ \\ (2r+s) \\ +i\left(2r^2+rs+s\right) \end{bmatrix} & \begin{bmatrix} 2r^2+rs+s+2i \\ 1+i\,(2r+s) \end{bmatrix} \end{pmatrix},
\end{aligned} \tag{7.5}
$$

$$
\begin{aligned}
N_{SF} &= \begin{pmatrix} SF_2 & SF_1 \\ SF_1 & SF_0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} x^4+3x^2+1+ix \\ x^2+1+i\left(x^3+2x\right) \end{bmatrix} & \begin{bmatrix} x^3+2x+i \\ x+i\left(x^2+1\right) \end{bmatrix} \\ \begin{bmatrix} x^3+2x+i \\ x+i\left(x^2+1\right) \end{bmatrix} & \begin{bmatrix} x^2+1 \\ 1+ix \end{bmatrix} \end{pmatrix},
\end{aligned} \tag{7.6}
$$

$$
\begin{aligned}
N_{SL} &= \begin{pmatrix} SL_2 & SL_1 \\ SL_1 & SL_0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} 2x^4+x^3+5x^2+2x \\ +1+i\,(2x+1) \\ \\ \left(2x^2+x+1\right) \\ +i\left(2x^3+x^2+3x+1\right) \end{bmatrix} & \begin{bmatrix} 2x^3+x^2+3x \\ +1+i \\ \\ (2x+1) \\ +i\left(2x^2+x+1\right) \end{bmatrix} \\ \begin{bmatrix} 2x^3+x^2 \\ +3x+1+i \\ \\ (2x+1) \\ +i\left(2x^2+x+1\right) \end{bmatrix} & \begin{bmatrix} 2x^2+x+1+2i \\ \\ 1+i\,(2x+1) \end{bmatrix} \end{pmatrix}.
\end{aligned} \tag{7.7}
$$

Theorem 12 gives the following results as particular examples.

Corollary 7.1. *For all integers n, we have*

(i)

$$
A^n N_{SG} = \begin{pmatrix} SG_{n+2} & SG_{n+1} \\ SG_{n+1} & SG_n \end{pmatrix}. \tag{7.8}
$$

(ii)

$$
A^n N_{SH} = \begin{pmatrix} SH_{n+2} & SH_{n+1} \\ SH_{n+1} & SH_n \end{pmatrix}. \tag{7.9}
$$

(iii)

$$A^n N_{SF} = \begin{pmatrix} SF_{n+2} & SF_{n+1} \\ SF_{n+1} & SF_n \end{pmatrix}. \tag{7.10}$$

(iv)

$$A^n N_{SL} = \begin{pmatrix} SL_{n+2} & SL_{n+1} \\ SL_{n+1} & SL_n \end{pmatrix}. \tag{7.11}$$

8. Conclusions

Fibonacci numbers and the golden ratio associated with these numbers have been the focus of attention among mathematicians for many years. Over the years, new number and polynomial sequences have also been defined and examined their several properties. The most prominent ones are Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers and polynomials. On the other part, the starting point of this study is spinors, which scientists initially abstained from, but recently showed great interest in, and are a tool for explaining complex situations in physics through concepts in mathematics. This work not only introduces generalized Fibonacci polynomial spinors, (r,s) −Fibonacci polynomial spinors, (r,s) −Fibonacci-Lucas polynomial spinors, Fibonacci polynomial spinors and Fibonacci-Lucas polynomial spinors but also gives a wide perspective for sequences of polynomial spinors. While changing r and s, known different sequences of polynomial spinors are found. For instance, taking $r = 2x$ and $s = 1$ in (3.2), generalized Pell polynomial spinors can be defined and taking $r = 1$ and $s = 2x$ in (3.2), generalized Jacobsthal polynomial spinors can be described. Thus, as a corollary, generalized Pell spinors and generalized Jacobsthal spinors can be introduced. Aesthetic results of the sequences we worked on in quantum mechanics seem interesting.

References

1. Bandyopadhyay S., Cahay M.: *Introduction to spintronics*. CRC press, (2008)
2. Barley K., Vega-Guzmán J., Ruffing A., Suslov S. K.: Discovery of the relativistic Schrödinger equation. Physics-Uspekhi **65** (1), 90-103 (2022). https://doi.org/10.3367/ufne.2021.06.039000
3. Budinich P., Trautman A.: *The Spinorial Chessboard*. Springer-Verlag, (1988)
4. Cartan É.: Les groupes projectifs qui ne laissent invariante aucune multiplicité plane. Bulletin de la Société Mathématique de France **41,** 53-96 (1913). 10.24033/bsmf.916. http://www.numdam.org/articles/10.24033/bsmf.916/
5. Cartan É.: *The Theory of Spinors*. Dover Publications, (1981)
6. Catarino P.: A note on $h(x)-$Fibonacci quaternion polynomials. Chaos, Solitons and Fractals **77** (C), 1-5 (2015). https://doi.org/10.1016/j.chaos.2015.04.017
7. Catarino P.: The $h(x)-$Fibonacci Quaternion Polynomials: Some Combinatorial Properties. Advanced in Applied Clifford Algebras **26,** 71-79 (2016). https://doi.org/10.1007/s00006-015-0606-1
8. Cerda-Morales G.: Some identities involving (p,q)-Fibonacci and Lucas quaternions. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69** (2), 1104-1110 (2020). https://doi.org/10.31801/cfsuasmas.696617
9. Del Castillo G. F. T., Barrales G. S.: Spinor Formulation of the Differential Geometry of Curves. Revista Colombiana de Matematicas **38** (1), 27-34 (2004)
10. Del Castillo G. F. T.: *Spinors in Four-Dimensional Spaces*. Birkauser, (2010).
11. Dirac P. A. M.: The quantum theory of the electron. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **117** (778) ,610-624 (1928). https://doi.org/10.1098/rspa.1928.0023
12. Erişir T., Güngör M. A.: On Fibonacci spinors. International Journal of Geometric Methods in Modern Physics **17** (4), (2020). https://doi.org/10.1142/S0219887820500656
13. Farmelo G.: *The Strangest Man: The Hidden Life of Paul Dirac, Quantum Genius*. Faber and Faber, p. 430 (2009)
14. Fock V.: Geometrisierung der Diracschen Theorie des Elektrons. Zeitschrift für Physik **57,** 261–277 (1929). http://dx.doi.org/10.1007/BF01339714
15. Greiner W.: *Relativistic Quantum Mechanics*. Springer, Berlin (2000)

16. Gurtler R., Hestenes D.: Consistency in the formulation of the Dirac, Pauli, and Schrödinger theories. Journal of Mathematical Physics **16**, 3 (1975). https://doi.org/10.1063/1.522555
17. Halıcı S.: On Fibonacci Quaternions. Advanced in Applied Clifford Algebras **22,** 321-327 (2012). https://doi.org/10.1007/s00006-011-0317-1
18. Halıcı S., Karataş A.: On a generalization for Fibonacci quaternions. Chaos, Solitons and Fractals **98** (C), 178-182 (2017). https://doi.org/10.1016/j.chaos.2017.03.037
19. Hameka H. F.: *Quantum Mechanics: A Conceptual Approach*. John Wiley & Sons, (2004)
20. Horadam A. F.: Complex Fibonacci numbers and Fibonacci quaternions. Amer. Math. Mon. **70** (3), 289-291 (1963). https://doi.org/10.2307/2313129
21. Horadam A. F.: Extension of a synthesis for a class of polynomial sequences. Fibonacci Quart. **34**, 68-74 (1996).
22. Iyer M. Y.: A Note On Fibonacci Quaternions. Fibonacci Quart. **7** (3), 225-229 (1969)
23. İpek A.: On $(p,q)-$Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums. Advanced in Applied Clifford Algebras **27** (2), 1343-1351 (2017). https://doi.org/10.1007/s00006-016-0704-8
24. Kumari M., Prasad K., Frontczak R.: On the $k-$Fibonacci and $k-$Lucas spinors. Notes on Number Thoery and Discrete Mathematics **29** (2), 322–335 (2023). https://doi.org/10.7546/nntdm.2023.29.2.322-335
25. Nagashima Y.: *Elementary Particle Physics, Volume 1: Quantum Field Theory and Particles*. Wiley, Weinheim, 803-812 (2010)
26. Özkoç A., Porsuk A.: A Note for the $(p,q)-$Fibonacci and Lucas Quaternions Polynomials. Konuralp Journal of Mathematics **5** (2), 36-46 (2017)
27. Patel B. K., Ray P. K.: On the properties of $(p,q)-$Fibonacci and $(p,q)-$Lucas quaternions. Romanian Academy Mathematical Reports **21** (1), 15-25 (2019)
28. Pauli W.: Zur Quantenmechanik des magnetischen Elektrons. Zeitschrift fur Physik **43 (**1), 601-623 (1927). http://dx.doi.org/10.1007/BF01397326
29. Ramirez J. L.: Some Combinatorial Properties of the $k-$Fibonacci and the $k-$Lucas Quaternions. Analele Stiintifice ale Universitatii Ovidius Constanta Seria Mathematica **23 (2)**, 201-212 (2015). https://doi.org/10.1515/auom-2015-0037
30. Reed L.: *Quantum Wave Mechanics*. ch 25. Spinors. 4th ed., Booklocker, 267-268 (2022)
31. Schrödinger E.: Diracsches Elektron im Schwerefeld I. SSitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.), 105-128 (1932)
32. Soykan Y.: On Generalized Fibonacci Polynomials: Horadam Polynomials. Eartline Journal of Mathematical Sciences **11** (1), 23-114 (2023). https://doi.org/10.34198/ejms.11123.23114
33. Swamy M. N. S.: On Generalized Fibonacci Quaternions. Fibonacci Quart. **5**, 547-550 (1973)
34. Szynal-Liana, A., Wloch, I., Generalized Commutative Quaternions of the Fibonacci type. Bol. Soc. Mat. Mex. **28** (1), (2022). https://doi.org/10.1007/s40590-021-00386-4
35. Vaz J., Rocha R.: *An Introduction to Clifford Algebras and Spinors*, Oxford University Press, Oxford (UK) (2016)
36. Vivarelli M. D.: Developement of Spinor Descriptions of Rotational Mechanics from Euler's Rigid Body Displacement Theorem. Celestial mechanics **32 (**3), 193-207 (1984). https://doi.org/10.1007/BF01236599
37. Westra D. B.: *$SU(2)$ and $SO(3)$*, University of Groningen (2008). https://www.mat.univie.ac.at/~westra/so3su2.pdf
38. Weyl H.: Elektron und Gravitation I. Zeitschrift für Physik **56,** 330–352 (1929). http://dx.doi.org/10.1007/BF01339504